Role of the Unruh effect in Bremsstrahlung

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(Received 16 January 2020; accepted 4 March 2020; published 20 March 2020)

An equivalence is demonstrated, by an explicit first order quantum calculation, between the Minkowski photon emission rate in the inertial frame for an accelerating charge moving on a Rindler trajectory with additional transverse drift motion and the combined Rindler photon emission and absorption rate of the same charge in the Rindler frame in the presence of the Davies-Unruh thermal bath. The equivalence also extends, for the Bremsstrahlung emitted by the same charge as calculated using the machinery of classical electrodynamics. The equivalence is shown to also hold for the case of accelerating charges moving on a Rindler trajectory with additional arbitrary transverse motion. Our results generalize those of Higuchi *et al.* (1992) and of Cozzella *et al.* (2017) for accelerated trajectories with circular transverse motion. Related issues and experimental implications are discussed.

DOI: 10.1103/PhysRevD.101.065012

I. INTRODUCTION

The Unruh effect is one of the most interesting predictions of quantum field theory when observer dependent relativistic motions are taken into account. It is well known that an accelerated detector with constant proper acceleration a in Minkowski vacuum detects a thermal bath of particles at temperature, $T_U = \hbar a/2\pi c k_B$ [1]. The physics of the Unruh effect in accelerated frames offers insights into those for Hawking radiation in curved spacetime. A direct experimental confirmation of the Unruh effect is difficult to achieve in practice since the magnitude of the constant acceleration required to observe the Unruh radiation, say at temperature 1 K, is of the order of $10^{20}\ m/s^2$ [2]. Numerous experimental proposals whose outcomes can be indirectly explained due to the existence of the Unruh effect have been proposed in the literature such as the electron depolarization in storage rings [3], decay of accelerated protons [4], interaction of ultraintense lasers with electrons [5] etc. (see also [2] and references therein).

Within the framework of classical field theory, by suitably defining a classical particle number as the energy associated with a timelike Killing field divided by frequency, the number of Minkowski particles emitted by a source moving on an arbitrary trajectory was shown to be related to the number of Rindler particles emitted by the same source but with an additional thermally weighted factor to account for the presence of the Unruh bath in the Rindler frame [6]. Even though the Unruh temperature has a purely quantum origin, the quantity $\hbar\omega/k_B T_U = 2\pi c\omega/a$ which appears in the weight factor is classical. In a quantum treatment up to linear order perturbations, the Minkowski photons emission rate by a uniformly accelerated point charge moving in a flat space-time as seen by the inertial observer in Minkowski vacuum state equals the emission (including absorption) rate of zero frequency Rindler photons in a thermal bath, with temperature being the Unruh temperature, when viewed from the corresponding Rindler frame of the point charge [7,8]. The classical Larmor radiation of the uniformly accelerated point charge is built from zero energy Rindler modes and relates to the classical retarded solution obtained from the field expectation value in the coherent state as seen by inertial observers in the asymptotic future [9]. Also see references [10–12] for related work on the relation between the classical and quantum counterparts in the context of the Unruh effect.

In a recent proposal by Cozzella *et al.* [13], this correspondence between the first order quantum emission rate in the inertial and Rindler frames (with an appropriate thermally weighted factor) as well as with the classical Larmor radiation was shown to hold for a point charge moving on a uniformly linearly accelerated trajectory but with an additional circular motion in the remaining two transverse directions. In particular, the rate of emission of photons with Minkowski frequencies in the inertial frame, when the charge is linearly coupled to the electromagnetic field in the Minkowski vacuum state turns out to be proportional to the emission (and absorption) rate of photons with Rindler frequencies in the coaccelerated Rindler frame, provided the charge is immersed in a thermal bath with Unruh temperature. The benefit of additional circular motion in the transverse directions, in this case, causes even the nonzero Rindler frequencies to contribute in the upward and downward transitions. A key

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observation made by Unruh and Wald in [14] vis-a-vis, the absorption of Rindler particles in the accelerated frame causing the emission of Minkowski particles in the inertial frame, is crucial to understand the correct thermal weight factor appearing in the quantum calculation in the Rindler frame. For the same trajectory, a classical electrodynamics calculation of the spectrum of Larmor radiation by the charge in the transverse directions also matches with that of the first order perturbative quantum treatment, provided one adopts a prescribed regularization procedure. Using these correspondences, Cozzella *et al.* [13] claim that the existence of the classical radiation then indirectly proves, through virtual confirmation, the existence of Unruh radiation and further propose an experimental setup to detect this classical radiation. The challenges involved to perform the experiment are discussed in [15].

One could then ask how robust is this correspondence for trajectories other than the special trajectories assumed in the previous work mentioned above. Does the claim of virtual confirmation of the Unruh effect hold for a more general motion of the point charge thus relaxing some of the more taxing requirements of the experimental setup. We address these issues in the present work.

In Sec. II, we first investigate the case of point charge moving with uniform linear acceleration in the t-z plane, where t, z are the usual Minkowski coordinates, and additionally with a drift velocity along one of the transverse coordinates x. We show, by an explicit calculation, the said correspondence under investigation holds between the emission rate in the inertial frame and in the Rindler frame (in the presence of the Davies-Unruh bath) as well as the spectrum of Larmor radiation expected from classical electrodynamics. In Sec. III, we generalize our results for trajectories having Rindler motion in the t-z plane and an arbitrary motion in the remaining transverse directions. The conclusions are discussed in Sec. IV.

The signature adopted is $(+,-,-,-)$ and the natural units, $k_B = c = G = \hbar = 1$ are used throughout the paper.

II. RINDLER WITH TRANSVERSE DRIFT

In this section, we analyze the quantum and classical radiation by a point charge coupled to the electromagnetic field on a trajectory having uniform linear acceleration in the t-z plane, where t, z are the usual Minkowski coordinates, and additionally with a drift velocity v along one of the transverse coordinates x.

The trajectory can also be defined in terms of the conformal Rindler coordinates (λ, ξ, x, y) covering the right wedge of the Minkowski spacetime in which the Rindler metric takes the form

$$ds^2 = e^{2a\xi}(d\lambda^2 - d\xi^2) - dx^2 - dy^2, \tag{2.1}$$

where λ is the usual Rindler time coordinate, the acceleration four vector is along the ξ direction and a is the proper acceleration of the comoving Rindler observer who is at rest at $\xi = 0 = x = y$. The Rindler coordinates (λ, ξ) are related to Minkowski coordinates (t, z) by,

$$t = a^{-1}e^{a\xi}\sinh(a\lambda), \qquad z = a^{-1}e^{a\xi}\cosh(a\lambda), \tag{2.2}$$

Note the transverse coordinates x and y are same for both Minkowski and Rindler frame. For a point charge q drifting in the x-direction with a constant velocity v in the Rindler frame, its worldline is described as $\xi = y = 0$ and $x = v\lambda$. The four velocity of the charge q in terms of the Rindler coordinates is then

$$u_R^\mu = \gamma(1, 0, v, 0) \tag{2.3}$$

with the normalization constant being $\gamma = 1/\sqrt{1-v^2}$. Then using the transformations given by Eq. (2.2), the four velocity can be written in terms of the Minkowski coordinates (t, z, x, y) as

$$u_M^\mu = \gamma(\cosh(a\lambda), \sinh(a\lambda), v, 0) \tag{2.4}$$

where $\lambda = \gamma\tau$ along the trajectory and γ is still the same normalization constant as defined in Eq. (2.3). It is a function of only the drift velocity parameter v in the x direction through $\gamma = 1/\sqrt{1-v^2}$. The instantaneous velocity v_z along the z direction is $v_z = u_M^z/u_M^t = \tanh\gamma a\tau$ with the range $-1 < v_z < +1$ for $-\infty < \tau < \infty$. The instantaneous velocity v_z of the charge in the z direction is relativistic in the large $|\tau|$ regime for the motion to be restricted in the (right) Rindler wedge and one could further choose the drift velocity v in the x direction to lie anywhere in the range $-1 < v < +1$. The results derived in this Sec. II and in Sec. III are valid for the complete range of the velocities in the z and transverse direction.

The acceleration four vector for the trajectory in Eq. (2.4) turns out to be,

$$a_M^\mu = \gamma^2(a\sinh(a\lambda), a\cosh(a\lambda), 0, 0), \tag{2.5}$$

with the corresponding magnitude of four acceleration vector being equal to $|a| = \sqrt{-g_{\mu\nu}a_M^\mu a_M^\nu} = a\gamma^2$. One can note as a consistency check, the expressions in Eq. (2.4) and Eq. (2.5) reduce to those in the usual Rindler case when v is set to zero.

Having defined the trajectory of interest, we proceed to evaluate the quantum emission rates of photons in Rindler and Minkowski frames and also the classical radiation emitted by the accelerating charge using classical electrodynamics.

A. Quantum calculation in Rindler frame

We consider a point charge q to move on the classical Rindler trajectory with an additional transverse drift as described by Eq. (2.3) and coupled to the background quantized electromagnetic field in the Davies-Unruh bath.

The conserved four current vector for the point charge in the Rindler frame is defined as

$$j_R^\mu = q\frac{u_R^\mu}{u_R^0}\delta(\xi)\delta(x - v\lambda)\delta(y) \tag{2.6}$$

with the four velocity u_R^μ as in Eq. (2.3). The Lagrangian density for the background electromagnetic field is given by

$$\mathcal{L} = -\sqrt{-g}\left[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2\alpha}(\nabla^\mu A_\mu)^2\right] \tag{2.7}$$

which leads to the field equation $\nabla_\mu\nabla^\mu A_\nu = 0$, working in Feynman gauge $\alpha = 1$. Out of the four independent mode solutions of the field equation, $A_\mu^{(\ell,\omega,\mathbf{k}_\perp)}$, two are pure gauge modes labeled by $\ell = 0$, 3 while the remaining two physical modes are not pure gauge and labeled by $\ell = 1$, 2 satisfy the Lorenz gauge condition, $\nabla_\mu A^\mu = 0$ [2,7]. The two physical modes for the Rindler metric in Eq. (2.1) can be expressed as

$$A_\mu^{(1,\omega,\mathbf{k}_\perp)}(x^\nu) = \frac{1}{2\pi^2 k_\perp}\left(\frac{\sinh(\pi\omega/a)}{a}\right)^{1/2} \times (0, 0, k_y f^{(\omega,\mathbf{k}_\perp)}, -k_x f^{(\omega,\mathbf{k}_\perp)}) \tag{2.8}$$

$$A_\mu^{(2,\omega,\mathbf{k}_\perp)}(x^\nu) = \frac{1}{2\pi^2 k_\perp}\left(\frac{\sinh(\pi\omega/a)}{a}\right)^{1/2} \times (\partial_\xi f^{(\omega,\mathbf{k}_\perp)}, -i\omega f^{(\omega,\mathbf{k}_\perp)}, 0, 0) \tag{2.9}$$

where, ω is the frequency of Rindler photon, $\mathbf{k}_\perp$ is transverse momentum vector with magnitude $k_\perp = ({k_x}^2 + {k_y}^2)^{1/2}$ and the function $f^{(\omega,\mathbf{k}_\perp)}$ defined as

$$f^{(\omega,\mathbf{k}_\perp)} = K_{i\omega/a}\left(\frac{k_\perp}{a}e^{a\xi}\right)\exp\left[i(k_x x + k_y y - \omega\lambda)\right] \tag{2.10}$$

where $K_\nu(z)$ is the modified Bessel function [16]. The quantized electromagnetic field operator can then be expressed as

$$\hat{A}_\mu(x^\nu) = \int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_0^\infty d\omega\sum_{\ell=0}^{3}(\mathbf{a}^{(i)}A_\mu^{(i)}(x^\nu) + \mathrm{H.c.}) \tag{2.11}$$

where the label (i) represents $(i) \equiv (\ell, \omega, \mathbf{k}_\perp)$ and $(\mathbf{a}_{(i)}^\dagger, \mathbf{a}_{(i)})$ are the corresponding creation and annihilation operators respectively satisfying the commutation relations,

$$[\mathbf{a}_{(\ell,\omega,\mathbf{k}_\perp)}, \mathbf{a}^\dagger_{(\ell',\omega',\mathbf{k}'_\perp)}] = \delta_{\ell\ell'}\delta(\omega - \omega')\delta(\mathbf{k}_\perp - \mathbf{k}'_\perp) \tag{2.12}$$

for ℓ and ℓ' corresponding to only the physical modes. The interaction between the charged particle and the electromagnetic field can now be described by the Lagrangian density,

$$\mathcal{L}_{\rm int} = \sqrt{-g}j_R^\mu\hat{A}_\mu. \tag{2.13}$$

Now, to lowest order in perturbation, the amplitude for absorption of a Rindler photon to the Rindler vacuum where the photon is described by the single photon state $|\ell, \omega, \mathbf{k}_\perp\rangle_R = \mathbf{a}^\dagger_{(\ell,\omega,\mathbf{k}_\perp)}|0\rangle_R$ is

$$\mathcal{A}^{\rm abs}_{(\ell,\omega,\mathbf{k}_\perp)} = i\int d^4x\sqrt{-g}j^\mu_R{}_R\langle 0|\hat{A}_\mu|\ell,\omega,\mathbf{k}_\perp\rangle_R. \tag{2.14}$$

Here, the Rindler vacuum $|0\rangle_R$ is taken to be the state annihilated by all annihilation operators $\mathbf{a}_{(i)}$, that is defined as $\mathbf{a}_{(\ell,\omega,\mathbf{k}_\perp)}|0\rangle_R = 0$. Then for the four current vector j_R^μ in Eq. (2.6) for the case of Rindler trajectory with a transverse drift, the absorption amplitudes for the physical modes $\ell = 1$, 2 are obtained to be,

$$\mathcal{A}^{\rm abs}_{(1,\omega,\mathbf{k}_\perp)} = i\int d^4x\sqrt{-g}j_R^x A_x^{(1,\omega,\mathbf{k}_\perp)} \tag{2.15}$$

$$= \frac{iq}{\pi}\left(\frac{\sinh(\pi\omega/a)}{a}\right)^{1/2}\left(\frac{vk_y}{k_\perp}\right) \times K_{i\omega/a}\left(\frac{k_\perp}{a}\right)\delta(\omega - k_x v) \tag{2.16}$$

$$\mathcal{A}^{\rm abs}_{(2,\omega,\mathbf{k}_\perp)} = i\int d^4x\sqrt{-g}j_R^\lambda A_\lambda^{(2,\omega,\mathbf{k}_\perp)} \tag{2.17}$$

$$= \frac{iq}{\pi}\left(\frac{\sinh(\pi\omega/a)}{a}\right)^{1/2}K'_{i\omega/a}\left(\frac{k_\perp}{a}\right) \times \delta(\omega - k_x v) \tag{2.18}$$

where, prime denotes the derivative with respect to the argument of Bessel function.

We shall assume that the state of the quantized electromagnetic field is the Minkowski vacuum state or equivalently the Davies-Unruh thermal bath in the Rindler wedge with temperature $T = a/2\pi$. In such a case, the probability of absorption is additionally weighed by the thermal factor $1/[\exp(\omega/T) - 1]$ corresponding to the number of photons already present in the background thermal bath, for each Rindler photon frequency ω. Thus the total rate of absorption of Rindler photons is then

$$P_R^{\rm abs} = \sum_{\ell=1,2}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_0^\infty d\omega\frac{|\mathcal{A}^{\rm abs}_{(\ell,\omega,\mathbf{k}_\perp)}|^2}{\Delta\tau}\left(\frac{1}{e^{\omega/T} - 1}\right) \tag{2.19}$$

where $\Delta\tau$ is the total proper time interval of Rindler observer during which the interaction remains switched on. Since we also have, $|\mathcal{A}^{\rm abs}_{(\ell,\omega,\mathbf{k}_\perp)}| = |\mathcal{A}^{\rm emi}_{(\ell,\omega,\mathbf{k}_\perp)}|$, the total emission rate of Rindler photons is found by a similar procedure to be

$$P_R^{\rm emi} = \sum_{\ell=1,2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \times \int_0^\infty d\omega \frac{|\mathcal{A}^{\rm abs}_{(\ell,\omega,\mathbf{k}_\perp)}|^2}{\Delta\tau}\left(1+\frac{1}{e^{\omega/T}-1}\right) \tag{2.20}$$

where the factor of unity in the last term in the bracket corresponds to spontaneous emission. Thus the total rate which includes the emission rate as well as the absorption rate is given by,

$$P_R^{\rm total} = \sum_{\ell=1,2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_0^\infty d\omega \frac{|\mathcal{A}^{\rm abs}_{(\ell,\omega,\mathbf{k}_\perp)}|^2}{\Delta\tau}\coth\left(\frac{\omega}{2T}\right) \tag{2.21}$$

The emission and absorption rates in Eq. (2.20) and (2.19) are added to arrive at the $\coth(\omega/2T)$ factor in the total rate $P_R^{\rm total}$. Such a reasoning, as elaborated in [7,13], is based on Unruh and Wald's observation in [14] that the absorption of Rindler particles in the presence of the background Davies-Unruh bath in the accelerated frame is seen by the inertial observer as the emission of Minkowski particles in the inertial frame.

In the expression of $P_R^{\rm total}$ in Eq. (2.21), the amplitude of absorption is proportional to $\delta(\omega - k_x v)$ and one can simply evaluate the integral over ω. The Rindler photon energy ω can only have non-negative values in range $[0,\infty)$, which restricts k_x to be non-negative since we can choose v to be positive (a particular direction of drift which in this case is the positive x direction). Thus, the total rate is obtained as,

$$P_R^{\rm total} = \frac{q^2}{2\pi^3 a}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \Theta(k_x)\sinh\left(\frac{\pi k_x v}{a}\right)\coth\left(\frac{k_x v}{2T}\right) \times\left(|K'_{ik_x v/a}(k_\perp/a)|^2 + \left(\frac{v k_y}{k_\perp}\right)^2 |K_{ik_x v/a}(k_\perp/a)|^2\right) \tag{2.22}$$

where, $\Theta(k_x)$ is the Heaviside step function and we have identified $\Delta\tau = 2\pi\delta(0)$ as per the regularisation procedure adopted in [7,13]. In the above expression, we now set the temperature T of the background bath to be equal to the Unruh bath temperature $T_U = a/2\pi$ to finally get

$$P_R^{\rm total} = \frac{q^2}{2\pi^3 a}\int_0^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \cosh\left(\frac{\pi k_x v}{a}\right) \times\left(|K'_{ik_x v/a}(k_\perp/a)|^2 + \left(\frac{v k_y}{k_\perp}\right)^2 |K_{ik_x v/a}(k_\perp/a)|^2\right) \tag{2.23}$$

Substituting $v = 0$, one can check the consistency of the above expression with the total rate obtained in the case of the zero frequency modes in [7].

B. Quantum calculation in Minkowski frame

In this subsection, we calculate the emission rate in the inertial frame for the same point charge q as described by the trajectory in Eq. (2.3) and coupled to the background quantized electromagnetic field in the Minkowski vacuum state. In terms of the Minkowski coordinates, the corresponding conserved four vector current is

$$j_M^\mu = q\frac{u_M^\mu}{u_M^0}\delta(z - a^{-1}\cosh(a\lambda))\delta(x - v\lambda)\delta(y). \tag{2.24}$$

The quantized electromagnetic field in the inertial frame is expressed in terms of the standard Minkowski plane wave mode solutions and given by [8],

$$\hat{A}_\mu(x) = \int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\sum_{\ell=1}^{4}[a^{(\ell)}\epsilon_\mu^{(\ell)} e^{-ik_\nu x^\nu} + \text{H.c.}] \tag{2.25}$$

where $k_0 = \sqrt{k_z^2 + k_\perp^2}$ is the energy of Minkowski photon and $\epsilon_\mu^{(\ell)}$ are polarization vectors. As in the Rindler frame case, we label the two pure gauge modes out of the four independent mode solutions by $\ell = 0$, 3, and the two physical modes by $\ell = 1, 2$. The two physical modes satisfy the Lorenz gauge condition, $\nabla_\mu A^\mu = 0$. Accordingly the polarization vectors $\epsilon_\mu^{(\ell)}$ are chosen in a Cartesian frame such that $k^\mu = (|k|, 0, 0, |k|)$ as in [8].

The amplitude of emission of a single Minkowski photon with momentum $\mathbf{k}$ and polarization ℓ to the background Minkowski vacuum state can be computed as

$$\mathcal{A}^{\rm em}_{(\ell,\mathbf{k})} = i\int d^4x\, j_M^\mu(x)_M\langle \mathbf{k},\ell|\hat{A}_\mu(x)|0\rangle_M. \tag{2.26}$$

Hence the rate of emission of photons with transverse momentum $\mathbf{k}_\perp$ then becomes,

$$P^{\rm em}_{M,\mathbf{k}_\perp} = \frac{1}{\Delta\tau}\sum_{\ell=1}^{2}\int_{-\infty}^{\infty}\frac{dk_z}{2k_0(2\pi)^3}|\mathcal{A}^{\rm em}_{(\ell,\mathbf{k})}|^2 \tag{2.27}$$

$$= \frac{-1}{\Delta\tau}\int_{-\infty}^{\infty}\frac{dk_z}{2k_0(2\pi)^3}\int d^4x \times\int d^4x'\, j^\mu(x) j_\mu(x') e^{ik_0(t-t') - i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tag{2.28}$$

where, we have dropped the subscript M of the current j_M^μ in the above expression for calculational simplicity. Substituting the current j^μ from Eq. (2.24) and performing the spatial integrals, we get,

$$P^{\rm em}_{M,\mathbf{k}_\perp} = \frac{-q^2}{2\Delta\tau}\int_{-\infty}^{\infty}\frac{dk_z}{2k_0(2\pi)^3}\int_{-\infty}^{\infty}d\sigma \times\int_{-\infty}^{\infty}d\rho(\cosh(a\sigma)-v^2)e^{-ik_x v\sigma} \times\exp\left[\frac{2i}{a}\sinh\left(\frac{a\sigma}{2}\right)\left(k_0\cosh\left(\frac{a\rho}{2}\right)-k_z\sinh\left(\frac{a\rho}{2}\right)\right)\right] \tag{2.29}$$

where, $\sigma = \lambda - \lambda'$ and $\rho = \lambda + \lambda'$. To evaluate the above integral, we proceed in a similar way to [8]. We define new variables $\bar{k_0}$ and $\bar{k_z}$ by using the following transformations,

$$\bar{k_0} = k_0\cosh(a\rho/2) - k_z\sinh(a\rho/2) \tag{2.30}$$

$$\bar{k_z} = k_z\cosh(a\rho/2) - k_0\sinh(a\rho/2) \tag{2.31}$$

with $\bar{k_0} = \sqrt{\bar{k_z}^2 + k_\perp^2}$, which essentially boosts back the momentum variables. Doing so, essentially makes the integral independent of ρ and we can factor out infinite integral $\int_{-\infty}^{\infty} d(\lambda+\lambda')/2 = \int_{-\infty}^{\infty} d\rho/2$ by identifying $\int_{-\infty}^{\infty} d\rho/2 = \Delta\tau$, the total proper time interval as the Minkowski regularisation adopted in [8]. The resulting simplified expression is

$$P^{\rm em}_{M,\mathbf{k}_\perp} = -q^2\int_{-\infty}^{\infty}\frac{d\bar{k_z}}{2\bar{k_0}(2\pi)^3}\int_{-\infty}^{\infty}d\sigma(\cosh(a\sigma)-v^2)e^{-ik_x v\sigma} \times\exp\left[\frac{2i\bar{k_0}}{a}\sinh\left(\frac{a\sigma}{2}\right)\right]. \tag{2.32}$$

We next define new variables S_+ and S_- as per the following relations

$$S_\pm = \frac{\bar{k_0}+\bar{k_z}}{k_\perp}e^{\pm a\sigma/2}. \tag{2.33}$$

In terms of these variables, one can express the integral as

$$P^{\rm em}_{M,\mathbf{k}_\perp} = \frac{-q^2}{4a(2\pi)^3}\int_0^{\infty}dS_+ \times\int_0^{\infty}dS_-\left(\frac{1}{S_+^2}+\frac{1}{S_-^2}-\frac{2v^2}{S_+S_-}\right)\left(\frac{S_+}{S_-}\right)^{-ik_x v/a} \times\exp\left[\frac{ik_\perp}{2a}\left(S_+-\frac{1}{S_+}\right)\right]\exp\left[\frac{-ik_\perp}{2a}\left(S_--\frac{1}{S_-}\right)\right]. \tag{2.34}$$

Using the integral representation of modified Bessel function [16] and their recurrence relations,

$$\int_0^{\infty}t^{\nu-1}\exp\left[\frac{ix}{2}\left(t-\frac{z^2}{t}\right)\right]dt = 2z^\nu e^{i\pi\nu/2}K_\nu(xz) \tag{2.35}$$

the integrals are evaluated to get the emission rate for fixed transverse momenta k_x and k_y to be

$$P^{\rm em}_{M,\mathbf{k}_\perp} = \frac{q^2}{4\pi^3 a}\exp\left[\frac{\pi k_x v}{a}\right]\left[|K'_{ik_x v/a}(k_\perp/a)|^2 + \left(\frac{vk_y}{k_\perp}\right)^2|K_{ik_x v/a}(k_\perp/a)|^2\right]. \tag{2.36}$$

Hence total emission rate is written by integrating over all the transverse modes to get

$$P^{\rm em}_{M} = \frac{q^2}{2\pi^3 a}\int_0^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\cosh\left(\frac{\pi k_x v}{a}\right) \times\left(|K'_{ik_x v/a}(k_\perp/a)|^2 + \left(\frac{vk_y}{k_\perp}\right)^2|K_{ik_x v/a}(k_\perp/a)|^2\right) \tag{2.37}$$

which is equivalent to the total rate computed in the Rindler frame in the presence of the Davies-Unruh bath in Eq. (2.23) as expected.

C. Radiation using classical electrodynamics

In this subsection, we perform a simple classical electrodynamics calculation involving the spectral angular distribution of radiation from an accelerating charge on the Rindler trajectory with transverse drift. The number of emitted photons, each of energy k_0, is given by the expression

$$N_M = \int_0^{\infty}\frac{dk_0}{k_0}\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\,I(k_0,\theta,\phi) \tag{2.38}$$

where, $I(k_0,\theta,\phi)$ is the spectral angular distribution which is the energy emitted through classical radiation by the accelerating charge for a particular frequency k_0 in a particular direction θ, ϕ. Then changing the variables (k_0,θ,ϕ) from spherical coordinates to the Cartesian ones (k_x,k_y,k_z) and noting that the volume element changes as $dk_0 d(\cos\theta)d\phi = k_0^{-2}dk_x dk_y dk_z$ with $k_0 = \sqrt{k_x^2+k_y^2+k_z^2}$, the total number of emitted photons with the fixed transverse momenta k_x, k_y can be written as,

$$N_{M,\mathbf{k}_\perp} = \int_{-\infty}^{\infty}\frac{dk_z}{k_0^3}I(k_x,k_y,k_z). \tag{2.39}$$

We shall demonstrate below that the photon number $N_{M,\mathbf{k}_\perp}$ is proportional to the total emission rate $P^{\rm em}_{M,\mathbf{k}_\perp}$ obtained through the quantum calculation in the inertial frame as expressed in Eq. (2.36). For an accelerated point charge q, the spectral angular distribution $I(k_0,\theta,\phi)$ using standard classical electrodynamics is given by [17],

$$I(k_0,\theta,\phi) = \frac{q^2 k_0^2}{4\pi^2} |\bar{F}(k_0,\theta,\phi)|^2 \tag{2.40}$$

where the function $\bar{F}(k_0,\theta,\phi)$ is defined as

$$\bar{F}(k_0,\theta,\phi) = \hat{r} \times \int_{-\infty}^{\infty} d\lambda \frac{d\bar{r}_q}{d\lambda} \exp\left[-ik_0\hat{r}\cdot\bar{r}_q(\lambda) + ik_0 t(\lambda)\right] \tag{2.41}$$

where, $\hat{r} = \bar{k}/k_0$ is the unit vector pointing in the observed direction and $\bar{r}_q = v\lambda\hat{i} + a^{-1}\cosh(a\lambda)\hat{k}$, is the trajectory, in the spatial Cartesian coordinates, of the charge q having Rindler motion with additional transverse drift and the Minkowski time co-ordinate $t(\lambda) = a^{-1}\sinh(a\lambda)$ is expressed in terms of other parameter λ (which coincides with the Rindler time coordinate). Substituting the trajectory in the above expression, we get,

$$\begin{aligned}\bar{F}(k_x,k_y,k_z) &= \int_{-\infty}^{\infty} d\lambda \left[\frac{k_y\sinh(a\lambda)}{k_0}\hat{i} + \left(\frac{k_z v}{k_0} - \frac{k_x\sinh(a\lambda)}{k_0}\right)\hat{j} - \frac{k_y v}{k_0}\hat{k}\right] \\ &\quad\times \exp\left[\frac{ik_0\sinh(a\lambda)}{a} - \frac{ik_z\cosh(a\lambda)}{a} - ik_x v\lambda\right].\end{aligned} \tag{2.42}$$

Using the integral representation of Bessel function as given in Eq. (2.35), the integral in above expression can be simplified to arrive at the following form for $|\bar{F}(k_x,k_y,k_z)|^2$,

$$\begin{aligned}|\bar{F}(k_x,k_y,k_z)|^2 &= \frac{4}{a^2}\exp\left[\frac{\pi k_x v}{a}\right] \\ &\quad\times \left(|K'_{ik_x v/a}(k_\perp/a)|^2 + \left(\frac{vk_y}{k_\perp}\right)^2 |K_{ik_x v/a}(k_\perp/a)|^2\right).\end{aligned} \tag{2.43}$$

Using the above expression in Eq. (2.40), the number of emitted photons $N_{M,\mathbf{k}_\perp}$ with the fixed transverse momentum $\mathbf{k}_\perp$ in Eq. (2.39) is found to be

$$\begin{aligned}N_{M,\mathbf{k}_\perp} &= \frac{q^2}{\pi^2 a^2}\exp\left(\frac{\pi k_x v}{a}\right) \\ &\quad\times \left(|K'_{ik_x v/a}(k_\perp/a)|^2 + \left(\frac{vk_y}{k_\perp}\right)^2 |K_{ik_x v/a}(k_\perp/a)|^2\right) \\ &\quad\times \int_{-\infty}^{\infty} \frac{dk_z}{(k_\perp^2 + k_z^2)^{1/2}}.\end{aligned} \tag{2.44}$$

One can note here that the number of emitted photons with a fixed transverse momentum $N_{M,\mathbf{k}_\perp}$ is proportional to the quantum emission rate with a fixed transverse momentum $P^{\mathrm{em}}_{M,\mathbf{k}_\perp}$ given by Eq. (2.36) in the inertial frame with a divergent proportionality constant.

We can next identify the divergent integral over k_z with the infinite amount of time $\Delta\tau$, the charge is assumed to be accelerating for [13]. In such a case, the number of photons emitted with a fixed $\mathbf{k}_\perp$ will be infinite. The total number of emitted photons is then expressed as

$$\begin{aligned}N_{M,\mathbf{k}_\perp} &= \frac{q^2}{\pi^2 a}\Delta\tau\exp\left(\frac{\pi k_x v}{a}\right) \\ &\quad\times \left(|K'_{ik_x v/a}(k_\perp/a)|^2 + \left(\frac{vk_y}{k_\perp}\right)^2 |K_{ik_x v/a}(k_\perp/a)|^2\right)\end{aligned} \tag{2.45}$$

$$= 4\pi\Delta\tau P^{\mathrm{em}}_{M,\mathbf{k}_\perp} \tag{2.46}$$

and is proportional to the emission rate computed in the inertial frame in Eq. (2.36) with proportionality constant equal to $4\pi\Delta\tau$.

III. RINDLER WITH ARBITRARY TRANSVERSE MOTION

In this section, we generalize the results derived in the previous section for Rindler trajectories having an additional arbitrary transverse motion.

Below, we describe the trajectory of the point charge q. We consider the charge to have the usual Rindler motion in t-z plane while having an arbitrary motion in transverse directions x and y. We again define the trajectory first in terms of the conformal coordinates with the Rindler metric in the right Rindler wedge,

$$ds^2 = e^{2a\xi}(d\lambda^2 - d\xi^2) - dx^2 - dy^2, \tag{3.1}$$

where λ is the usual Rindler time co-ordinate, a is the proper acceleration of the co-moving Rindler observer at rest at $\xi = 0 = x = y$ and whose acceleration four vector is along the ξ direction.

We define the class of trajectories with Rindler motion in the λ-ξ plane, $\xi(\lambda) = 0$ and arbitrary motion in the transverse directions with $x(\lambda)$ and $y(\lambda)$ by the four velocity vector,

$$\tilde{u}^\mu_R = \gamma(1, 0, u_x(\lambda), u_y(\lambda)) = \gamma u^\mu_R \tag{3.2}$$

where the normalization factor $\gamma = 1/\sqrt{1 - u_x^2 - u_y^2}$ is, in general, not a constant. Thus the proper time τ along the trajectory considered may not be proportional to the Rindler time coordinate λ as in the earlier case and satisfies $d\lambda/d\tau = \gamma$. In terms of the Minkowski coordinates in the inertial frame, the corresponding four velocity reads as,

$$\tilde{u}^\mu_M = \gamma(\cosh(a\lambda), \sinh(a\lambda), u_x(\lambda), u_y(\lambda)) = \gamma u^\mu_M. \tag{3.3}$$

The magnitude of proper acceleration for this trajectory is, $|a|^2 = \gamma^4(a^2 + a_x^2 + a_y^2) - 3\gamma^6(u_x a_x + u_y a_y)^2$, where $a_{(x,y)} = du_{(x,y)}/d\lambda$. Thus the proper acceleration for these trajectories is in general proper time dependent.

The transverse components of the four velocity are, in general, arbitrary smooth functions of λ. In the special case, when $u_x = \sinh(a\lambda)$, the charge will also undergo a corresponding Rindler motion in the t-x plane. Then, due to the explicit symmetry in the motion of the charge in the z and x directions, one could also perform an explicit calculation of the quantum rates in the rest frame of the Rindler motion in the t-x plane [instead of the rest frame of the Rindler motion in the t-z plane as chosen in Eq. (3.1)], with z now being one of the transverse directions. The roles of z and x would be simply exchanged. In such a case, one would again have an Unruh effect due to the Rindler motion in the t-x plane which allows to define the background thermal state for a quantum calculation. The conclusions regarding the equivalence in the corresponding rates in this case will be exactly same as those obtained in the sections below.

With the trajectory defined, we now proceed to obtain the expressions for the photon emission rates in the inertial frame and Rindler frame with a thermal bath and also the bremsstrahlung using classical electrodynamics.

A. Quantum calculation in the Minkowski frame

In this subsection, we calculate the emission rate in the inertial frame for a point charge q as described by the trajectory in Eq. (3.3) and coupled to the background quantised electromagnetic field in the Minkowski vacuum state. In terms of the Minkowski coordinates, the corresponding conserved four vector current is

$$j^\mu_M = q\frac{\tilde{u}^\mu_M}{\tilde{u}^0_M}\delta(z - a^{-1}\cosh(a\lambda))\delta(x - x(\lambda))\delta(y - y(\lambda)) \tag{3.4}$$

$$= q\frac{u^\mu_M}{u^0_M}\delta(z - a^{-1}\cosh(a\lambda))\delta(x - x(\lambda))\delta(y - y(\lambda)) \tag{3.5}$$

The rate of emission of photons with a fixed transverse momentum $\mathbf{k}_\perp$ as measured in an inertial frame is then given by Eq. (2.28). Substituting the four current j^μ from Eq. (3.5) in Eq. (2.28) and performing the spatial integrals, we get,

$$P^{\mathrm{total}}_{M,\mathbf{k}_\perp} = \frac{-q^2}{\Delta\tau}\int_{-\infty}^{\infty}\frac{d\bar{k}_z}{2\bar{k}_0(2\pi)^3}\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}d\lambda'[u^\mu_M(\lambda)u_{M\mu}(\lambda')] \times e^{-i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)}\exp\left(\frac{2i\bar{k}_0}{a}\sinh\left(\frac{a\sigma}{2}\right)\right) \tag{3.6}$$

where, $\mathbf{x}_\perp \equiv (x(\lambda), y(\lambda))$ and $\bar{k}_0$ is as defined in Eq. (2.30) with $\sigma = \lambda - \lambda'$ and $\rho = \lambda + \lambda'$. The overall emission rate, then can be obtained to be,

$$P^{\mathrm{total}}_M = \int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y P^{\mathrm{total}}_{M,\mathbf{k}_\perp} \tag{3.7}$$

$$= \frac{-q^2}{\Delta\tau}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y \times\int_{-\infty}^{\infty}\frac{d\bar{k}_z}{2\bar{k}_0(2\pi)^3}\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}d\lambda' \times\left[[u^\mu(\lambda)u_\mu(\lambda')]e^{-i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)}\exp\left(\frac{2i\bar{k}_0}{a}\sinh\left(\frac{a\sigma}{2}\right)\right)\right]. \tag{3.8}$$

Here we have dropped the label M for the four velocity u^μ_M for notational simplicity. From here onwards u^μ shall represent the Minkowski four velocity as specified in Eq. (3.3) unless otherwise stated. Now, we proceed to get the expression for the total rate for a Rindler detector.

B. Quantum calculation in the Rindler frame

In this subsection, we calculate the emission rate in the Rindler frame for a point charge q moving on the Rindler trajectory with additional arbitrary transverse motion as described by Eq. (3.2) and coupled to the background quantized electromagnetic field in the Davies-Unruh bath. The conserved four current vector j^μ_R for the point charge in the Rindler frame is defined as

$$j^\mu_R = q\frac{\tilde{u}^\mu_R}{\tilde{u}^0_R}\delta(\xi)\delta(x - x(\lambda))\delta(y - y(\lambda)) \tag{3.9}$$

$$= q\frac{u^\mu_R}{u^0_R}\delta(\xi)\delta(x - x(\lambda))\delta(y - y(\lambda)). \tag{3.10}$$

For the above four current, the amplitude of absorption of a single Rindler photon as defined by Eq. (2.14) can be computed by using the mode solutions in Rindler frame, Eq. (2.8) and Eq. (2.9). The amplitudes of absorption corresponding to the two physical modes $\ell = 1, 2$, then turn out to be,

$$\mathcal{A}^{\mathrm{abs}}_{(1,\omega,\mathbf{k}_\perp)} = i\int d^4x\sqrt{-g}\,j^\mu_R A^{(1,\omega,\mathbf{k}_\perp)}_\mu \tag{3.11}$$

$$= \frac{iq}{2\pi^2 k_\perp}\left(\frac{\sinh(\pi\omega/a)}{a}\right)^{1/2}K_{i\omega/a}\left(\frac{k_\perp}{a}\right) \times\int_{-\infty}^{\infty}d\lambda[u_x(\lambda)k_y - u_y(\lambda)k_x]e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp(\lambda) - i\omega\lambda} \tag{3.12}$$

$$\mathcal{A}^{\mathrm{abs}}_{(2,\omega,\mathbf{k}_\perp)} = i\int d^4x\sqrt{-g}\,j^\mu_R A^{(2,\omega,\mathbf{k}_\perp)}_\mu \tag{3.13}$$

$$= \frac{iq}{2\pi^2}\left(\frac{\sinh(\pi\omega/a)}{a}\right)^{1/2}K'_{i\omega/a}\left(\frac{k_\perp}{a}\right) \times\int_{-\infty}^{\infty}d\lambda\, e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp(\lambda) - i\omega\lambda} \tag{3.14}$$

As in the special case discussed in Sec. II A, the rate $P^{\rm total}_{R,\mathbf{k}_\perp}$ for Rindler photons with fixed transverse momentum $\mathbf{k}_\perp$ in the Rindler frame with a background thermal bath at Unruh temperature, $T_U = a/2\pi$ is

$$P^{\rm total}_{R,\mathbf{k}_\perp} = \sum_{\ell=1,2}\int_0^\infty d\omega \frac{|\mathcal{A}^{\rm abs}_{(\ell,\omega,\mathbf{k}_\perp)}|^2}{\Delta\tau}\coth\left(\frac{\pi\omega}{a}\right). \quad (3.15)$$

Now, we write the total rate corresponding to the polarizations $\ell = 1$ and $\ell = 2$ separately as,

$$P^{\rm total}_1 = \frac{q^2}{\Delta\tau}\int_{-\infty}^\infty d\lambda \int_{-\infty}^\infty d\lambda' [u_x k_y - u_y k_x][u'_x k_y - u'_y k_x] \times \frac{e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)}}{4\pi^4 k_\perp^2 a}\int_0^\infty d\omega\, e^{-i\omega(\lambda-\lambda')}\cosh\left(\frac{\pi\omega}{a}\right) \times \left[K_{i\omega/a}\left(\frac{k_\perp}{a}\right)\right]^2 \quad (3.16)$$

$$P^{\rm total}_2 = \frac{q^2}{\Delta\tau}\int_{-\infty}^\infty d\lambda \int_{-\infty}^\infty d\lambda' \frac{e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)}}{4\pi^4 a} \times \int_0^\infty d\omega\, e^{-i\omega(\lambda-\lambda')}\cosh\left(\frac{\pi\omega}{a}\right)\left[K'_{i\omega/a}\left(\frac{k_\perp}{a}\right)\right]^2 \quad (3.17)$$

where, u'_x, u'_y, and $\mathbf{x}'_\perp$ are functions of λ' whereas the prime over the Bessel function K denotes the derivative with respect to its argument. Now to obtain the total rate completely in terms of the variables in inertial frames, as in Eq. (3.8), we first eliminate the Rindler photon frequency ω from above expressions by performing the ω integrals. These integrations lead to the Bessel functions of second kind $Y_n(z)$. Then writing these Bessel functions in their integral representations and changing the variable of integration to $\bar{k}_z$, we arrive at the expression of the rate of emission of photons in inertial frame as given in Eq. (3.8). The detailed calculation is shown below:

The integral over ω in Eq. (3.16) can be simply evaluated to be,

$$\int_0^\infty d\omega\, e^{-i\omega\sigma}\cosh\left(\frac{\pi\omega}{a}\right)\left[K_{i\omega/a}\left(\frac{k_\perp}{a}\right)\right]^2 = -\frac{\pi^2 a}{4} Y_0\left[\frac{2k_\perp}{a}\sinh(a\sigma/2)\right] \quad (3.18)$$

where, $\sigma = \lambda - \lambda'$ and $Y_n(x)$ is the Bessel function of second kind [16]. Now using the relation,

$$[K'_\nu(z)]^2 = \frac{1}{2}\left[\frac{d^2}{dz^2}[K_\nu(z)]^2 + \frac{1}{z}\frac{d}{dz}[K_\nu(z)]^2 - 2\left(1+\frac{\nu^2}{z^2}\right)[K_\nu(z)]^2\right] \quad (3.19)$$

the integral over ω in Eq. (3.17) can be obtained to be,

$$\int_0^\infty d\omega\, e^{-i\omega\sigma}\cosh\left(\frac{\pi\omega}{a}\right)\left[K'_{i\omega/a}\left(\frac{k_\perp}{a}\right)\right]^2 = \frac{\pi^2 a}{8}\cosh(a\sigma) Y_0\left[\frac{2k_\perp}{a}\sinh(a\sigma/2)\right] + \frac{\pi^2 a}{8} Y_2\left[\frac{2k_\perp}{a}\sinh(a\sigma/2)\right] \quad (3.20)$$

Substituting Eq. (3.18) and (3.20) in Eq. (3.16) and (3.17), the rate $P^{\rm total}_{R,\mathbf{k}_\perp}$ of Rindler photons with fixed transverse momentum $\mathbf{k}_\perp$ becomes,

$$P^{\rm total}_{R,\mathbf{k}_\perp} = \frac{q^2}{32\pi^2\Delta\tau}\int_{-\infty}^\infty d\lambda\int_{-\infty}^\infty d\lambda' e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\mathbf{x}'_\perp)} \times \Big[\cosh(a\sigma)Y_0(z) + Y_2(z) - \frac{2}{k_\perp^2}[u_x k_y - u_y k_x][u'_x k_y - u'_y k_x] Y_0(z)\Big] \quad (3.21)$$

where $z = 2k_\perp \sinh(a\sigma/2)/a$. Now we write $Y_0(z)$ as an integral using the integral representation of the Bessel function [16],

$$Y_\nu(x) = -\frac{2}{\pi}\int_0^\infty \cos\left(x\cosh t - \frac{\nu\pi}{2}\right)\cosh(\nu t)dt \qquad [|\mathrm{Re}(\nu)| < 1] \quad (3.22)$$

and using the series expansion for the derivative of Bessel function,

$$Y^{(k)}_\nu(x) = \frac{1}{2^k}\sum_{n=0}^k (-1)^n \binom{k}{n} Y_{\nu-k+2n}(x) \quad (3.23)$$

we express $Y_2(x)$ in terms of $Y_0(x)$ as,

$$Y_2(x) = Y_0(x) + 2\frac{d^2}{dx^2}Y_0(x). \quad (3.24)$$

Then identifying $\cosh t$ as $\bar{k}_0/k_\perp$, where $\bar{k}_0 = \sqrt{\bar{k}_z{}^2 + k_\perp^2}$, with $\bar{k}_0$ given by Eq. (2.30) and the corresponding $\bar{k}_z$ given by Eq. (2.31), we get the Bessel functions $Y_0(z)$ and $Y_2(z)$ as the following integrals with $\bar{k}_z$ as integration variable.

$$Y_0\left[\frac{2k_\perp}{a}\sinh(a\sigma/2)\right] = \frac{-1}{\pi}\int_{-\infty}^\infty \cos\left(\frac{2\bar{k}_0}{a}\sinh(a\sigma/2)\right)\frac{d\bar{k}_z}{\bar{k}_0} \quad (3.25)$$

$$Y_2\left[\frac{2k_\perp}{a}\sinh(a\sigma/2)\right] = \frac{1}{\pi}\int_{-\infty}^\infty \cos\left(\frac{2\bar{k}_0}{a}\sinh(a\sigma/2)\right)\left(\frac{\bar{k}_z{}^2 + \bar{k}_0{}^2}{k_\perp^2}\right)\frac{d\bar{k}_z}{\bar{k}_0} \quad (3.26)$$

Substituting these integral representations of $Y_0(z)$ and $Y_2(z)$ in Eq. (3.21), we get for the rate $P^{\mathrm{total}}_{R,\mathbf{k}_\perp}$,

$$P^{\mathrm{total}}_{R,\mathbf{k}_\perp} = \frac{q^2}{32\pi^3 k_\perp^2 \Delta\tau}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)} \times \int_{-\infty}^{\infty}\frac{d\bar{k}_z}{\bar{k}_0}\cos\left(\frac{2\bar{k}_0}{a}\sinh(a\sigma/2)\right) \times [-k_\perp^2\cosh(a\sigma) + \bar{k}_z{}^2 + \bar{k}_0{}^2 + 2[u_x k_y - u_y k_x][u'_x k_y - u'_y k_x]]. \tag{3.27}$$

Now, as the current j^μ is conserved, i.e., $\partial_\mu j^\mu = 0$, one can write for its Fourier transform,

$$\int d^4x\sqrt{-g}e^{ik_\nu x^\nu} j^\mu(x)k_\mu = 0. \tag{3.28}$$

In the Minkowski frame for the current given by Eq. (3.5), this equation simplifies to

$$\int_{-\infty}^{\infty} d\lambda\, e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)}\exp\left(\frac{2i\bar{k}_0}{a}\sinh(a\sigma/2)\right)u^\mu(\lambda)k_\mu = 0. \tag{3.29}$$

The above constraint on $u^\mu(\lambda)$ with the definitions of $\bar{k}_0$ and $\bar{k}_z$ simplifies the expression of total rate with the term in the bracket reducing to $-2k_\perp^2 u^\mu(\lambda)u_\mu(\lambda')$. The total rate is then given by,

$$P^{\mathrm{total}}_{R,\mathbf{k}_\perp} = \frac{q^2}{\Delta\tau}\int_{-\infty}^{\infty}\frac{d\bar{k}_z}{(2\pi)^3 2\bar{k}_0}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda'[-u^\mu(\lambda)u_\mu(\lambda')] \times e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)}\cos\left(\frac{2\bar{k}_0}{a}\sinh(a\sigma/2)\right) \tag{3.30}$$

The overall rate is then given by integrating over the transverse momenta k_x and k_y. Using the symmetry in λ and λ' in the above expression and interchanging the limits of the k_x, k_y integrals, it can be expressed as

$$\begin{aligned} P^{\mathrm{total}}_{R} &= \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y P^{\mathrm{total}}_{R,\mathbf{k}_\perp} \\ &= \frac{-q^2}{\Delta\tau}\int_{-\infty}^{\infty} dk_x\int_{-\infty}^{\infty} dk_y\int_{-\infty}^{\infty}\frac{d\bar{k}_z}{(2\pi)^3 2\bar{k}_0}\int_{-\infty}^{\infty} d\lambda\int_{-\infty}^{\infty} d\lambda'[u^\mu(\lambda)u_\mu(\lambda')]\cos(\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp))\exp\left(\frac{2i\bar{k}_0}{a}\sinh(a\sigma/2)\right) \\ &= \frac{-q^2}{\Delta\tau}\int_{-\infty}^{\infty} dk_x\int_{-\infty}^{\infty} dk_y\int_{-\infty}^{\infty}\frac{d\bar{k}_z}{(2\pi)^3 2\bar{k}_0}\int_{-\infty}^{\infty} d\lambda\int_{-\infty}^{\infty} d\lambda'[u^\mu(\lambda)u_\mu(\lambda')]e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)}\exp\left(\frac{2i\bar{k}_0}{a}\sinh(a\sigma/2)\right) \end{aligned} \tag{3.31}$$

which matches with the expression of emission rate obtained in inertial frame in Eq. (3.8). We proceed to calculate the number of emitted photons with the classical framework in the next section.

C. Radiation using classical electrodynamics

In this subsection, we perform a straightforward classical electrodynamics calculation involving the spectral angular distribution of radiation from an accelerating charge on the Rindler trajectory with arbitrary transverse motion as described by Eq. (3.2).

The expression for number of emitted Minkowski photons presented in Eq. (2.38) is, in general, valid for any arbitrary trajectory of charge q. The vector $\bar{F}(k_x, k_y, k_z)$ defined in Eq. (2.41), then can be reexpressed as,

$$F^l(k_x, k_y, k_z) = \epsilon^{lmn}\hat{r}_m\int_{-\infty}^{\infty} d\lambda\left(\frac{dr_q}{d\lambda}\right)_n \exp(-ik_0[\hat{r}\cdot\bar{r}_q(\lambda) - t(\lambda)]) \tag{3.32}$$

where, ϵ^{lmn} is the completely antisymmetric Levi-Civita symbol in three dimensions. Then for the magnitude of vector $\bar{F}(k_x, k_y, k_z)$, we get

$$\begin{aligned} |\bar{F}(k_x, k_y, k_z)|^2 &= \delta_{al}F^a F^{l*} \\ &= \delta_{al}\epsilon^{abc}\epsilon^{lmn}\hat{r}_b\hat{r}_m\int_{-\infty}^{\infty} d\lambda\int_{-\infty}^{\infty} d\lambda'\left(\frac{d\bar{r}_q(\lambda)}{d\lambda}\right)_c\left(\frac{d\bar{r}_q(\lambda')}{d\lambda'}\right)_n \exp(-ik_0[\hat{r}\cdot(\bar{r}_q(\lambda) - \bar{r}_q(\lambda')) - (t(\lambda) - t(\lambda'))]) \\ &= \int_{-\infty}^{\infty} d\lambda\int_{-\infty}^{\infty} d\lambda' \exp(-ik_0[\hat{r}\cdot(\bar{r}_q(\lambda) - \bar{r}_q(\lambda')) - (t(\lambda) - t(\lambda'))]) \\ &\quad\times\left[(\hat{r}\cdot\hat{r})\left(\frac{d\bar{r}_q(\lambda)}{d\lambda}\cdot\frac{d\bar{r}_q(\lambda')}{d\lambda'}\right) - \left(\hat{r}\cdot\frac{d\bar{r}_q(\lambda)}{d\lambda}\right)\left(\hat{r}\cdot\frac{d\bar{r}_q(\lambda')}{d\lambda'}\right)\right] \end{aligned} \tag{3.33}$$

Here we have used the identity, $\delta_{al}\epsilon^{abc}\epsilon^{lmn} = (\delta^{bm}\delta^{cn} - \delta^{bn}\delta^{cm})$ to arrive at the final equality. Now, using the constraint on u^μ as defined in Eq. (3.29) and the definition of unit direction vector $\hat{r} = \bar{k}/k_0$, the term $(d\bar{r}_q/d\lambda)\cdot\hat{r}$ can be replaced by $(dt/d\lambda)$. The magnitude $|\bar{F}(k_x,k_y,k_z)|^2$ then becomes,

$$\begin{aligned}
&|\bar{F}(k_x,k_y,k_z)|^2 \\
&\quad = \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' \\
&\quad \times \exp(-i[\bar{k}\cdot(\bar{r}_q(\lambda)-\bar{r}_q(\lambda')) - k_0(t(\lambda)-t(\lambda'))]) \\
&\quad \times \left[\frac{d\bar{r}_q(\lambda)}{d\lambda}\cdot\frac{d\bar{r}_q(\lambda')}{d\lambda'} - \frac{dt(\lambda)}{d\lambda}\frac{dt(\lambda')}{d\lambda'}\right]
\end{aligned} \tag{3.34}$$

Now, substituting the trajectory, $\bar{r}_q(\lambda) = x(\lambda)\hat{i} + y(\lambda)\hat{j} + a^{-1}\cosh(a\lambda)\hat{k}$, with the time co-ordinate as $t(\lambda) = a^{-1}\sinh(a\lambda)$, we get,

$$\begin{aligned}
|\bar{F}(k_x,k_y,k_z)|^2 &= \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' [-u^\mu(\lambda)u_\mu(\lambda')] e^{-i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)} \\
&\quad \times \exp\left[\frac{2i\bar{k}_0}{a}\sinh\left(\frac{a\sigma}{2}\right)\right]
\end{aligned} \tag{3.35}$$

where $\bar{k}_0$ is as defined in Eq. (2.30). With this expression for $|\bar{F}(k_x,k_y,k_z)|^2$, the total number of emitted photons N_M is obtained to be,

$$\begin{aligned}
N_M &= \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} \frac{dk_z}{k_0}\frac{q^2}{4\pi^2} |\bar{F}(k_x,k_y,k_z)|^2 \\
&= \frac{-q^2}{4\pi^2}\int_{-\infty}^{\infty}\frac{dk_z}{k_0}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' [u^\mu(\lambda)u_\mu(\lambda')] e^{-i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)} \exp\left(\frac{2i\bar{k}_0}{a}\sinh\left(\frac{a\sigma}{2}\right)\right) \\
&= 4\pi(-q^2)\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty}\frac{dk_z}{(2\pi)^3 2k_0}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' [u^\mu(\lambda)u_\mu(\lambda')] e^{-i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)} \exp\left(\frac{2i\bar{k}_0}{a}\sinh\left(\frac{a\sigma}{2}\right)\right).
\end{aligned} \tag{3.36}$$

With a comparison of the above expression with Eq. (3.31), one can simply write the number of emitted photons in terms of total rate of a Rindler detector as,

$$N_M = 4\pi\Delta\tau P_R^{\text{total}} \tag{3.37}$$

As discussed in the previous Sec. II C, the proportionality constant is $4\pi\Delta\tau$, which will be finite for a charge accelerating for a finite time.

IV. DISCUSSION

An equivalence is demonstrated, by an explicit quantum calculation till linear order in perturbations, between the Minkowski photon emission rate, in the inertial frame, for an accelerating charge moving on a Rindler trajectory with additional arbitrary transverse motion and the combined Rindler photon emission and absorption rate of the same charge in the Rindler frame in the presence of the Davies-Unruh bath. The equivalence also extends, between the Bremsstrahlung emitted by the same charge as calculated using the machinery of classical electrodynamics and the expectation values from the quantum calculations mentioned above.

Two immediate observations follow: (i) as noted in [13], the presence of the Unruh bath in the Rindler frame is utmost necessary for the quantum calculation to match with the classical one or with the quantum calculation in the inertial frame. The particular $\cosh(\omega/2T)$ factor in Eqs. (2.21) and (3.15) which arises through the combined emission and absorption rate is crucial for the equivalence to work. The $\cosh(\omega/2T)$ factor is universal in the sense that it is common for all trajectories under consideration and does not require any fudging. In retrospect, it reaffirms earlier observations that absorption of Rindler particles in the Rindler frame appears to the inertial observer as emission of Minkowski particles in the inertial frame. Additionally, one needs to clarify that even though the Unruh effect is purely quantum in its origin, the said equivalence works due to the cancellation of the $\hbar$ in the thermal Planckian factor (leading to the $\cosh(\omega/2T)$ factor) since both the photon energies and Unruh temperature are linearly proportional to $\hbar$. Any other value of the thermal bath temperature chosen shall not suffice. In fact, once the choice is made, the form of classical mode solutions in the Minkowski and Rindler frames are sufficient (mathematically) to arrive at the results. This strongly suggests that the Unruh effect is necessary to explain certain classical features of electromagnetic radiation when two different frames are involved. (ii) The correspondence principle in quantum mechanics holds for many particle systems in the limit when the number of particles involved are large. In the present scenario, the charge is taken to be classical and moving on a well defined trajectory. It is however coupled to the quantized electromagnetic field. In linear order perturbation theory, only a single photon excitation or deexcitation is admissible, although it holds true for every (continuum of) energy eigenstates of the quantized Maxwell fields. In principle, the point charge does excite an infinite number of photons of the whole frequency spectrum due to its motion and hence one could argue that the large particle limit is inherently built into the case under investigation. Then, it should be no surprise that quantum calculation in inertial frame or the Rindler frame agrees with that of Larmor radiation using classical electrodynamics. However, it would be interesting to go further than the first order approximation to analyze the quantum effects of the Davies-Unruh effect beyond the classicality of Maxwell's equations. It would be interesting to test the equivalence for a more broad class of trajectories wherein additional motion along the special direction ξ is also allowed.

In terms of experimental prospects, our results have generalized the proposal of Cozzella *et al.* [13] for the virtual confirmation of the Unruh effect using classical Bremsstrahlung. The requirement of circular motion in the transverse directions could be relaxed depending on experimental setup constraints etc. or any slight deviation from the circularity will not affect the main conclusions of the proposal since the equivalence is shown to hold for arbitrary transverse motion.

ACKNOWLEDGMENTS

S. K. and K. P. thank the Department of Science and Technology, India, for financial support.

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