

Relativistic compressibility conditions

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The constraints imposed by the relativistic compressibility hypothesis on the square of the speed of sound in a medium are obtained. This result allows us to obtain purely hydrodynamic conditions for the physical reality of a perfect energy tensor representing the energetic evolution of a perfect fluid in local thermal equilibrium. The results are applied to the paradigmatic case of the generic ideal gases. Then the physical reality of the ideal gas Stephani models is analyzed and the Rainich-like theory for ideal gas solutions is built.

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I. INTRODUCTION

In relativity, the gravitational field g is related by Einstein equations to the energy content T of the spacetime. We know that not all energy tensors T represent physically admissible energy contents, so that they have to be constrained by suitable *causal* and *energy conditions* which, in turn, restrict the physically admissible gravitational fields.

In addition, the energy content of particular physical media is described by specific energy tensors. For example, a *perfect fluid* imposes T to be of the specific form:

$$T = (\rho + p)u \otimes u + pg, \quad (1)$$

where u , ρ , and p denote the *unit velocity* of the fluid, its *energy density*, and its *pressure*, respectively. Obviously, such a particular choice of physical media additionally restricts the physically admissible gravitational fields.

We are concerned here with thermodynamic perfect fluids in local thermal equilibrium. A *thermodynamic perfect fluid* is a perfect fluid characterized, in addition to the above quantities, by its conserved *matter density* n , *internal energy* ϵ , *specific entropy* s , and *temperature* Θ . And it is in *local thermal equilibrium* (LTE) if it verifies the energy balance:

$$\Theta ds = d\epsilon + pdv, \quad (2)$$

where $v = 1/n$ is the *specific volume*. It was shown a long time ago that, in order to be physically admissible,

relativistic thermodynamic perfect fluids in LTE have to verify the *relativistic compressibility conditions*¹: which may be written as follows:

$$\begin{aligned} H_1: & \quad (\tau'_p)_s < 0, \quad (\tau''_p)_s > 0, \\ H_2: & \quad (\tau'_s)_p > 0, \end{aligned}$$

where the *dynamic volume* $\tau = fv$ is the specific volume v weighted by the *enthalpy index*,² $f = 1 + \epsilon/c^2$, the *specific enthalpy* i being given by $i = \epsilon + pv$.

In Newtonian gravitation, whatever the thermodynamic characterization of a perfect fluid may be, the Poisson equation states that the only physical quantity of a volume element of the fluid that causes or undergoes directly the gravitational field is its matter density.

Correspondingly, in relativity, Einstein equations tell us that there is the particular combination of the three *hydrodynamic quantities*, u , ρ and p , in the *perfect energy tensor* T given by (1) which directly causes or undergoes the gravitational field.

The above relativistic compressibility conditions involve thermodynamic, not only hydrodynamic, quantities, leading us to ask the following question: Do the relativistic compressibility conditions impose constraints on the energy tensor T of a thermodynamic perfect fluid in LTE?

Note that, via Einstein equations, this question is the equivalent of asking whether or not the relativistic

¹See Sec. III for details and nuances. For a function $f(x, y)$, $(f'_x)_y$ denotes the partial derivative of f with respect to x , at constant y .

²In this work the units are such that $c = 1$. We make explicit it here in order to make clear that in the classical limit $c^2 \rightarrow \infty$, the dynamical volume τ reduces to the specific volume v .

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compressibility conditions restrict the physically admissible gravitational fields.

The analysis of this question is interesting for both, conceptual and practical reasons:

- (1) From a conceptual point of view, a relativistic thermodynamic perfect fluid may be considered as a conservative and deterministic hydrodynamic perfect flow endowed with *subsidiary thermic quantities*.³ Thus, the relativistic compressibility conditions could impose restrictions on these subsidiary, nongravitational quantities, on its hydrodynamic flow creating or undergoing the gravitational field or on both ingredients. And it is important to determine which of these three possibilities is correct and to explicitly obtain the form of the corresponding constraints. This is the first objective of the present work.
- (2) From a practical point of view, the solution to the *inverse problem* for perfect fluids (namely, the determination of all the physical thermodynamic perfect fluids whose evolution in LTE is described by a given perfect fluid energy tensor T) is tantamount to the acquisition of all characteristic equations of state and thermodynamic quantities compatible with that given tensor T . It becomes crucial to know the degree of freedom or constraints to which these unknown thermodynamic relations and quantities are subject. The second objective of the present work is to analyze and practically illustrate this situation. We shall do this for the particular case of the generic ideal gases.

In Sec. II we introduce the concept of hydrodynamic flow of a thermodynamic perfect fluid, which gathers a significant result [1,2]: the hydrodynamic quantities $\{u, \rho, p\}$ can be subject to complementary hydrodynamic constraints that ensure the existence of the full set of thermodynamic quantities. In fact, these additional conditions state that the indicatrix of local thermal equilibrium, $\chi \equiv \dot{p}/\dot{\rho}$, is a function of state, $\chi = \chi(\rho, p)$, and then it coincides with the square of the speed of sound.

In Sec. III we address the main goal of this paper, namely, the hydrodynamic flow approach to the relativistic compressibility conditions. First, we summarize the results by Israel [3] and Lichnerowicz [4], and then we analyze separately two cases: the intrinsic barotropic media and the nonbarotropic ones. We show that the compressibility conditions H_1 can be stated in terms of the indicatrix function $\chi(\rho, p)$; that is, they impose constraints on the hydrodynamic flow. Nevertheless, condition H_2 generically only imposes constraints on the thermodynamic subsidiary quantities.

In Sec. IV we study in detail the specific case of a generic ideal gas. We start from our hydrodynamic characterization

of a generic ideal gas [2], which requires that the indicatrix function $\chi \equiv \dot{p}/\dot{\rho}$ depends on the hydrodynamic variable $\pi = p/\rho$, $\chi = \chi(\pi)$. Then, the compressibility condition H_2 on the ideal gas thermodynamic quantities can also be stated in terms of the function $\chi(\pi)$.

In Sec. V the results in the previous section enable us to acquire the Rainich approach for the perfect fluid solutions of the Einstein equations performing an ideal gas which fulfills the relativistic compressibility conditions.

In Sec. VI we apply our results to analyze the physical reality of the ideal gas Stephani models obtained in [5]. In addition to the full compressibility conditions, we impose additional constraints ensuring a good performance at high and low temperatures, i.e., that its behavior is similar to that of a monoatomic relativistic gas.

Finally, in Sec. VII we comment on the conceptual, practical, and numerical relevance of the results obtained here.

In this paper we work on an oriented spacetime with a metric tensor g of signature $\{-, +, +, +\}$. We denote with the same symbol any tensor and its associated ones by raising and lowering indexes with the metric tensor. For the metric product of two vectors, we write $(x, y) = g(x, y)$, and we put $x^2 = g(x, x)$. If S is a 2-tensor, $S(x)$ denotes the vector with covariant components $S_{\alpha\beta}x^\beta$, $S(x, y) = S_{\alpha\beta}x^\alpha y^\beta$, and S^2 the 2-tensor with components $(S^2)_{\alpha\beta} = S_\alpha^\lambda S_{\lambda\beta}$.

II. HYDRODYNAMIC FLOW OF A THERMODYNAMIC PERFECT FLUID IN LTE

The divergence-free condition for the perfect energy tensor T given by (1),

$$\nabla \cdot ((\rho + p)u \otimes u + pg) = 0, \quad (3)$$

leads to a system of four equations for the five hydrodynamic quantities $\{u, \rho, p\}$. For thermodynamic perfect fluids in LTE the *deterministic closure*⁴ to this system is obtained [6] by considering the matter density $n = n(\rho, p) > 0$ subject to the matter conservation equation:

$$\dot{n} + n\theta = 0, \quad (4)$$

$\theta = \nabla_\alpha u^\alpha$ being the expansion of the fluid, and where a dot denotes the directional derivative, with respect to u , of a quantity q , $\dot{q} = u(q) = u^\alpha \partial_\alpha q$. Moreover, the specific internal energy ϵ is defined by the relation:

$$\rho = n(1 + \epsilon). \quad (5)$$

³See Sec. II for details.

⁴That is to say, the set of complementary equations to be added to the system in order that it admits unicity of the Cauchy problem.

Then, according to a classical argument, it is always possible to identify each integral divisor of the one-form $\Lambda \equiv d\epsilon + p d(1/n)$ with the (absolute) temperature Θ , allowing us to define the specific entropy s by the energy balance equation (2), which can be written as:

$$\Theta ds = (1/n)d\rho + (\rho + p)d(1/n). \quad (6)$$

As a consequence, only two thermodynamic quantities are independent and they define a thermodynamic plane.⁵ Equations (3), (4), and (6) imply the equivalence between matter conservation and a local adiabatic evolution, a condition which can be stated as

$$\dot{s} = 0. \quad (7)$$

The set of equations $\mathcal{F} \equiv \{(3), (4), (5), (6)\}$ constitutes the deterministic fundamental system of the relativistic hydrodynamics.

Any solution $\{u, \rho, p, n, \epsilon, s, \Theta\}$ of the fundamental system \mathcal{F} defines a thermodynamic fluid in local thermal equilibrium, and it involves, in addition to the hydrodynamic quantities $\{u, \rho, p\}$, a set of *thermodynamic* quantities $\{n, \epsilon, s, \Theta\}$. This fact generates the impression that any LTE closure to the conservation equations (3) necessarily needs the introduction of these new, thermodynamic quantities. Such an impression is incorrect, as already shown by a previous result [1] (see also the recent paper [2]).

Theorem 1. The necessary and sufficient condition for a conservative perfect energy tensor $T = (\rho + p)u \otimes u + pg$ to represent the energetic evolution of a thermodynamic perfect fluid in LTE is that its hydrodynamic quantities $\{u, \rho, p\}$ fulfill the *hydrodynamic sonic condition*:

$$(\dot{\rho}d\dot{p} - \dot{p}d\dot{\rho}) \wedge d\rho \wedge dp = 0. \quad (8)$$

The hydrodynamic sonic condition (8) is a deterministic closure for the conservative system (3) so that the evolution of the thermodynamic perfect fluid in LTE is uniquely determined by the differential system $\mathcal{H} \equiv \{(3), (8)\}$ in the hydrodynamic quantities $\{u, \rho, p\}$. We shall call this system \mathcal{H} the *hydrodynamic flow* of the thermodynamic perfect fluid in LTE.

Note that the above theorem states the following: (i) if $\{u, \rho, p, n, \epsilon, s, \Theta\}$ is a solution of the fundamental system $\mathcal{F} \equiv \{(3), (4), (5), (6)\}$, then $\{u, \rho, p\}$ is a solution of the hydrodynamic flow system $\mathcal{H} \equiv \{(3), (8)\}$, and conversely, (ii) if $\{u, \rho, p\}$ is a solution of the hydrodynamic flow system $\mathcal{H} \equiv \{(3), (8)\}$, then a solution

⁵In fluid hydrodynamics, the thermodynamic relation (6) has a dual interpretation: (i) as a spacetime equation that constrains the scalar thermodynamic quantities, and (ii) as a relation in the thermodynamic plane that restricts the thermodynamic states.

$\{u, \rho, p, n, \epsilon, s, \Theta\}$ of the fundamental system $\mathcal{F} \equiv \{(3), (4), (5), (6)\}$ exists.⁶

This means that the solutions of the system \mathcal{H} characterize *the evolution* of the thermodynamic perfect fluid in LTE. But they do not characterize the thermodynamic perfect fluid in LTE *itself*, because they may also be verified by other, different continuous media.

As follows from the above considerations, a thermodynamic perfect fluid in LTE is determined by its hydrodynamic flow \mathcal{H} endowed with two quantities n and s depending on the two hydrodynamic variables ρ and p and subject to Eqs. (4) and (7), respectively.

It is important to observe that, when the hydrodynamic sonic condition (8) is included in the hydrodynamic flow \mathcal{H} as a deterministic closure, the hydrodynamic quantities determine completely the evolution of the fluid, meanwhile the thermodynamic quantities appear as subsidiary variables specifically allowing the characterization of the sole *thermodynamic properties* of the fluid. In this sense, Eqs. (4) and (7) no longer play the role of deterministic closures for the conservative system (3) but are “reduced” to simple *evolution equations* for the subsidiary thermodynamic quantities n and s .

The interest of the hydrodynamic flow approach to the LTE of a thermodynamic perfect fluid has been widely pointed out in [2] and some applications have recently been developed [7,8]. From this approach, the indicatrix of local thermal equilibrium [2], $\chi \equiv \dot{p}/\dot{\rho}$, plays a central role. We remember here that, when $\dot{\rho} \neq 0$, χ is a function of state, $\chi = \chi(\rho, p)$, and that it then coincides with the square of the speed of sound and the hydrodynamic LTE characterization (8) identically holds; for this reason we also call this characterization the hydrodynamic sonic condition.

III. RELATIVISTIC COMPRESSIBILITY CONDITIONS

In his study of shock waves in a relativistic thermodynamic perfect fluid in LTE, Israel [3] was lead to impose the above relativistic compressibility conditions H_1 and H_2 in order to obtain a coherent theory.⁷ Later, Lichnerowicz [4] showed that, in *relativistic magnetohydrodynamics*, these relativistic compressibility conditions lead to analogous results.

The relativistic compressibility conditions H_1 and H_2 in terms of the dynamic specific volume τ have the same form as the Weyl classical ones [9], $(v'_p)_s < 0$, $(v''_p)_s > 0$,

⁶All the conditions involved in the fundamental system \mathcal{F} and in the hydrodynamic flow system \mathcal{H} are local like all the results and statements presented in this paper.

⁷Namely, one in which (i) shock velocities are always less than the light constant c , (ii) shocks are compressive supersonic waves with increasing entropy across them, and (iii) the state after the shock is univocally determined by the state before the shock.

$(v'_s)_p > 0$, in terms of the specific volume v and, of course, both conditions coincide in the classical limit $c^2 \rightarrow \infty$ (see footnote 2). These facts suggest that the dynamic specific volume τ is a leading variable in relativistic thermodynamics.

Following this idea, the relativistic compressibility conditions H_1 and H_2 , and a suitable relativistic definition of the exothermic reaction in terms of the dynamical volume τ , allowed one of us, Coll [10,11], to construct the relativistic theory of deflagrations and detonations in a magnetohydrodynamic fluid.⁸

From Synge's results [13] on the relativistic Boltzmann gas, it follows that this gas verifies the relativistic compressibility conditions H_1 and H_2 . Israel [14] showed that the first inequality of the compressibility conditions H_1 remains true also for relativistic Bose and Fermi gases. And Lucquiaud [15,16] proved that both relativistic compressibility conditions H_1 and H_2 are equally satisfied by relativistic Boltzmann, Bose, and Fermi gases.

In [17], Thorne extended (for vanishing magnetic field) the work of Israel and Lichnerowicz, showing that their essential results could be obtained in the absence of the compressibility condition H_2 . Subsequently, Lichnerowicz [18,19] reworked the relativistic theory of shock waves in magnetohydrodynamics under both hypothesis, H_2 and its inverse, that we denote \bar{H}_2 :

$$\bar{H}_2: \quad (\tau'_s)_p < 0.$$

For any medium in LTE one has⁹ $(v'_\tau)_p = (\Theta'_p)_s / (\tau'_s)_p$. Consequently, if the medium has a classical limit, $\tau \rightarrow v$ and $(v'_\tau)_p \rightarrow 1$, then we have, necessarily, $(\Theta'_p)_s < 0$. Thus, for a medium that holds \bar{H}_2 , any isentropic evolution implies that its temperature decreases when pressure increases. In the scientific literature such media seem scarce. Anyway, their possible existence leads us, in what follows, to take into account also this condition \bar{H}_2 .

We want now to analyze to what extent the relativistic compressibility conditions H_1 and H_2 or \bar{H}_2 concern the hydrodynamic flow of the perfect fluid in LTE or only its subsidiary thermodynamic quantities. For this task, we must consider separately the barotropic and nonbarotropic cases.

A. Compressibility conditions for barotropic media

Let us consider a barotropic fluid subject to the barotropic equation of state $p = \phi(\rho)$.¹⁰ In this case

⁸An abridged English version may be found in [12], Chap. 8, Sec. III.

⁹This expression follows from the integrability condition of (2), $(v'_s)_p = (\Theta'_p)_s$, and the relation $(v'_s)_p = (v'_\tau)_p (\tau'_s)_p$ deduced from $v = v(\tau(s, p), p)$.

¹⁰We avoid the case $\rho = \rho_0$, which leads to an unphysical thermodynamic scheme since it does not meet any of the above relativistic compressibility conditions.

the characteristic equation $n = n(\rho, s)$ takes the expression [2]:

$$n(\rho, s) = \frac{G(\rho)}{R(s)}, \quad G(\rho) = \exp \left[\int \frac{d\rho}{\rho + \phi(\rho)} \right]. \quad (9)$$

The dynamic specific volume τ may be written

$$\tau = \frac{1}{n^2}(\rho + p), \quad (10)$$

and by derivation taking into account (9) we obtain

$$(\tau'_p)_s = \frac{1}{n^2 \phi'}(\phi' - 1), \quad (11)$$

$$(\tau''_p)_s = \frac{1}{n^2(\rho + p)\phi'^3}[\phi''(\rho + p) + 2\phi'(1 - \phi')], \quad (12)$$

$$(\tau'_s)_p = \frac{2\tau}{R(s)}R'(s). \quad (13)$$

Then, replacing these expressions in the compressibility conditions H_1 and H_2 , it results in the following:

Theorem 2. For a barotropic perfect fluid in LTE with equation of state $p = \phi(\rho)$, the relativistic compressibility conditions H_1 and H_2 (respectively, \bar{H}_2) may be written as

$$H_1: \quad 0 < \phi' < 1, \quad (\rho + p)\phi'' + 2\phi'(1 - \phi') > 0, \quad (14)$$

and

$$H_2: \quad R'(s) > 0 \quad (\text{respect., } \bar{H}_2: \quad R'(s) < 0), \quad (15)$$

where $R(s)$ is the entropy denominator of the characteristic equation $n = n(\rho, s)$ given by (9).

Note that a barotropic perfect energy tensor, $d\rho \wedge dp = 0$, always meets the hydrodynamic sonic condition (1). Moreover, the entropy denominator $R(s)$ is arbitrary. Then, taking into account these facts, from Theorems 1 and 2, it follows:

Theorem 3. The necessary and sufficient condition for a barotropic perfect fluid energy tensor T to represent the evolution in LTE of a barotropic fluid verifying H_1 and H_2 or \bar{H}_2 is H_1 . Any such energy tensor T may be endowed with a characteristic equation $n = n(\rho, s)$ verifying either H_2 or \bar{H}_2 .

Before studying the nonbarotropic case, let us note three points:

- (i) The expression (14) of the conditions H_1 is known and can be found in the literature (see for example [12]). Note that ϕ' is the square of the speed of sound, $c_s^2 = (p'_\rho)_s = \phi'(\rho)$.

- (ii) Conditions (14) exclusively involve hydrodynamic quantities. Thus, the relativistic compressibility conditions H_1 only concern the hydrodynamic flow of the perfect fluid in LTE. Nevertheless, constraint H_2 (or \bar{H}_2) imposes an additional condition on its subsidiary thermodynamic quantities and it does not restrict the perfect energy tensor.
- (iii) It is worth remarking that Theorem 2 only applies for *intrinsic* barotropic fluids, that is, for barotropic perfect fluid energy tensors where $p = \phi(\rho)$ is considered to be an equation of state.¹¹ Otherwise, a barotropic relation $p = \phi(\rho)$ that does not fulfill the expressions (14) could be an admissible evolution path of a nonbarotropic fluid which fulfills the (general) compressibility conditions H_1 . Conversely, a nonbarotropic perfect fluid which does not fulfill the compressibility hypothesis H_1 could admit a barotropic evolution that fulfills the conditions (14). We will return to this fact at the end of the following subsection.

B. Compressibility conditions for nonbarotropic media

Under the nonbarotropic assumption, we can consider the thermodynamic quantities (ρ, p) as coordinates in the thermodynamic plane.¹² As we have seen, a thermodynamics is determined by two functions $n(\rho, p)$ and $s(\rho, p)$ that are solution to Eqs. (4) and (7), and are constrained by the thermodynamic relation (6). If one of these functions of state is known, the square of the speed of sound can be obtained as [2]

$$c_s^2 = \chi(\rho, p) \equiv -\frac{s'_\rho}{s'_p} = \frac{1}{n'_p} \left[\frac{n}{\rho + p} - n'_\rho \right]. \quad (16)$$

And conversely, it is worth remarking the simplicity with which the square of the speed of sound $\chi(\rho, p)$ allows us to control the different thermodynamic schemes with which a hydrodynamic flow may be endowed. Indeed, as seen in [2], the mass density $n = n(\rho, p)$ and the entropy $s = s(\rho, p)$ are of the form $n = \bar{n}R(\bar{s})$ and $s = s(\bar{s})$, where $\bar{n}(\rho, p)$ is any particular solution of Eq. (16) in n , and $R(\bar{s})$ and $s(\bar{s})$ are arbitrary functions of any particular solution $\bar{s}(\rho, p)$ of Eq. (16) in s .

From (16) we obtain that, for any function of state $z = z(s, p) = z(\rho, p)$, we have

$$(z'_s)_p = \frac{z'_\rho}{s'_\rho}, \quad (z'_p)_s = z'_p - z'_\rho \frac{s'_p}{s'_\rho} = z'_p + \frac{1}{\chi} z'_\rho. \quad (17)$$

Moreover, from (10) we obtain

¹¹And not an evolution path in the thermodynamic plane of a more general equation of state.

¹²For a function of state $z(\rho, p)$ we write $z'_\rho = (z'_\rho)_p$, $z'_p = (z'_p)_\rho$.

$$\tau'_\rho = -\frac{2}{n^3} n'_\rho (\rho + p) + \frac{1}{n^2}, \quad \tau'_p = -\frac{2}{n^3} n'_p (\rho + p) + \frac{1}{n^2}. \quad (18)$$

Then, from (17) (with $z = \tau$) and (18) we have

$$(\tau'_p)_s = \xi(\rho, p) \equiv \frac{1}{n^2 \chi} (\chi - 1). \quad (19)$$

From the function of state $\xi(\rho, p)$ given above we derive

$$\xi'_\rho = \frac{1}{n^2 \chi^2} \left[\chi'_\rho - \frac{2}{n} \chi (\chi - 1) n'_\rho \right], \quad (20)$$

$$\xi'_p = \frac{1}{n^2 \chi^2} \left[\chi'_p - \frac{2}{n} \chi (\chi - 1) n'_p \right]. \quad (21)$$

Then, making use of (17) (with $z = \xi$), (18), and (16), we have

$$(\tau''_p)_s = \frac{(\rho + p)(\chi \chi'_p + \chi'_\rho) + 2\chi(1 - \chi)}{n^2 (\rho + p) \chi^3}. \quad (22)$$

On the other hand, from (17) (with $z = \tau$) and (18) we obtain

$$(\tau'_s)_p = \frac{1}{n^2 s'_p} \left[1 - \frac{2(\rho + p)}{n} n'_p \right]. \quad (23)$$

Then, replacing expressions (19), (22) and (23) in the compressibility conditions H_1 and H_2 , it results in the following:

Theorem 4. For a nonbarotropic perfect fluid in LTE, the relativistic compressibility conditions H_1 and H_2 (respectively, \bar{H}_2) may be written as

$$H_1: \quad \begin{cases} 0 < \chi < 1, \\ (\rho + p)(\chi \chi'_p + \chi'_\rho) + 2\chi(1 - \chi) > 0, \end{cases} \quad (24)$$

and

$$H_2: \quad I > 0 \quad (\text{respect., } \bar{H}_2: \quad I < 0), \quad (25)$$

where χ is the square of the speed of sound as a function of state depending on (ρ, p) , $\chi = \chi(\rho, p)$, and $I = I(\rho, p)$ is the function that depends on $n = n(\rho, p)$ and $s = s(\rho, p)$ as

$$I(\rho, p) \equiv \frac{1}{s'_p} [n - 2(\rho + p)n'_p]. \quad (26)$$

If we take into account the expression of the temperature $\Theta(\rho, p)$ in terms of the functions $n(\rho, p)$ and $s(\rho, p)$ [2], the function of state $I(\rho, p)$ given in (26) can be written as $I = n[2n\Theta - 1/s'_p]$. Then, we can obtain an equivalent

expression for the condition H_2 which can be useful in subsequent applications [8]:

Corollary 1. The compressibility condition H_2 (respectively, \bar{H}_2) can be written as:

$$H_2: 2n\Theta > \frac{1}{s'_\rho} \left(\text{respect.}, \bar{H}_2: 2n\Theta < \frac{1}{s'_\rho} \right). \quad (27)$$

Note that, under the physical requirements $\Theta > 0$ and $n > 0$, a sufficient condition for H_2 is $s'_\rho < 0$.

The equation of state $\chi(\rho, p)$, being not a characteristic equation, needs to be complemented by another one in order to determine completely the thermodynamic properties of the fluid. We know that two different pairs (\bar{n}, \bar{s}) and (n, s) , which are the solutions of Eq. (16), are related by $n = \bar{n}R(\bar{s})$ and $s = s(\bar{s})$. Then, the corresponding functions of state $\bar{I} \equiv \bar{I}(\rho, p)$ and $I \equiv I(\rho, p)$ given in (26) are related by

$$I = \frac{1}{s'(\bar{s})} [R(\bar{s})\bar{I} - 2\bar{n}(\rho + p)R'(\bar{s})]. \quad (28)$$

This expression implies that we can take the arbitrary function $R(\bar{s})$ such that I does not vanish, and thus either H_2 or \bar{H}_2 is fulfilled. Then, any of them may be selected by choosing the arbitrary function $s = s(\bar{s})$ such that $s'(\bar{s})$ has the appropriate sign. Consequently, we have the following:

Lemma 1. Any function of state $\chi(\rho, p)$ is compatible with a thermodynamic scheme verifying either the relativistic compressibility condition H_2 or \bar{H}_2 . It is the nature of the physical fluid to be described that will dictate what compressibility condition, H_2 or \bar{H}_2 , has to be chosen.

Let us consider a nonbarotropic ($d\rho \wedge dp \neq 0$) and nonisoenergetic ($\dot{\rho} \neq 0$) energy tensor T . Then, we can define the *indicatrix function* $\chi \equiv \dot{p}/\dot{\rho}$. When χ is a function of state ($d\chi \wedge d\rho \wedge dp = 0$), Theorem 1 implies that the hydrodynamic sonic condition holds, and then χ coincides with the square of the speed of sound, $\chi = \chi(\rho, p)$, of the fluids for which T is a possible evolution. Theorem 4 and Lemma 1 apply for these thermodynamics and, taking into account Theorem 1, we have the following:

Theorem 5. The necessary and sufficient condition for a nonbarotropic and nonisoenergetic perfect energy tensor T to represent the evolution in LTE of a perfect fluid verifying the relativistic compressibility conditions H_1 and H_2 or \bar{H}_2 is that the indicatrix function $\chi \equiv \dot{p}/\dot{\rho}$ is a function of state, $\chi = \chi(\rho, p)$, verifying expression (24) of the relativistic compressibility conditions H_1 . Any such energy tensor T may be endowed with thermodynamic quantities $n = n(\rho, p)$ and $s = s(\rho, p)$ solutions of (16) such that one the relativistic compressibility conditions H_2 or \bar{H}_2 holds.

When $\dot{\rho} = 0$ and $\dot{p} \neq 0$ we have necessarily [2] $n = n(\rho)$ and $s = s(\rho)$, and then the compressibility conditions H_1 do not hold. If we have an isobaroenergetic evolution, $\dot{\rho} = \dot{p} = 0$, the associated thermodynamic schemes are defined by an arbitrary matter density $n = n(\rho, p)$ and a specific entropy $s = s(\rho, p)$ solution to the last equality in (16) [2]. But, alternatively, we can give an arbitrary speed of sound, namely its square $\chi = \chi(\rho, p)$, which constraints $n = n(\rho, p)$ and $s = s(\rho, p)$ by means of (16). Then, the results in [2], and Theorem 4 and Lemma 1 imply the following:

Theorem 6. A nonbarotropic and isoenergetic perfect energy tensor T represents the evolution in LTE of a perfect fluid subject to the relativistic compressibility conditions H_1 and H_2 or \bar{H}_2 if, and only if, it is isobaroenergetic, $\dot{\rho} = \dot{p} = 0$.

For it, the admissible general thermodynamic quantities $n = n(\rho, p)$ and $s = s(\rho, p)$ are those for which (16) defines a square of the speed of sound $\chi = \chi(\rho, p)$ verifying expression (24) of the relativistic compressibility condition H_1 . Then, (28) may generate new quantities $n = n(\rho, p)$ and $s = s(\rho, p)$ verifying either H_2 or \bar{H}_2 .

We finish this section with some suitable comments on the above results.

- (i) Note that Theorems 5 and 6 provide a purely hydrodynamic characterization of the perfect energy tensors T which are the evolution in LTE of a thermodynamic perfect fluid subject to the compressibility conditions H_1 and H_2 or \bar{H}_2 . In other words, Theorems 5 and 6 characterize all the hydrodynamic flows able to correspond to thermodynamic perfect fluids subject to the relativistic compressibility conditions. This characterization imposes conditions (24) on the square of the speed of sound $\chi(\rho, p)$. In the nonisoenergetic case, this function is determined by the hydrodynamic flow $\mathcal{H} \equiv \{(3), (8)\}$ through the indicatrix function $\chi(\rho, p) = \chi \equiv \dot{p}/\dot{\rho}$. In the isoenergetic case, $\dot{\rho} = 0$, we have necessarily $\dot{p} = 0$, the indicatrix function is indeterminate, and the compressibility conditions does not impose any condition on the hydrodynamic flow \mathcal{H} . The thermodynamics associated with any of these isobaroenergetic flows are constrained by conditions (24) that now define the square of the speed of sound $\chi(\rho, p)$.
- (ii) In the inverse problem [2] under consideration, H_1 imposes the constraint (24) on the function $\chi(\rho, p)$ and H_2 or \bar{H}_2 imposes the constraint (25)–(26) to the, nonhydrodynamic, thermodynamic quantities n and s . And, as stated in Lemma 1, for *any* given hydrodynamic function $\chi(\rho, p)$, there *always* exist thermodynamic functions n and s that fulfill H_2 or \bar{H}_2 . Nevertheless, in *restricted* inverse problems [2] for specific families of perfect fluids, characterized by a relation involving hydrodynamic as well as thermodynamic quantities, the compressibility

conditions H_2 or \bar{H}_2 may restrict strongly the hydrodynamic function $\chi \equiv \dot{p}/\dot{\rho}$ (this is the case, for example, of the ideal gas. See next section).

- (iii) From (28), the thermodynamic fluids that do not fulfill neither H_2 nor \bar{H}_2 are those verifying $I = 0$, where I is given in (26). It follows from this expression of I that the matter density is of the form $n = n(\rho, p) = f(p)(\rho + p)^{1/2}$, where $f(p)$ is any arbitrary positive function. Then, from (16), the square of the speed of sound is given by $\chi(\rho, p) = [1 + (\rho + p)F(p)]^{-1}$, with $F \equiv 2f'(p)/f(p)$, and it verifies H_1 if and only if $F(p) > 0$ and $F'(p) < F^2(p)$. In fact, from (10), the function $f(p)$ and the dynamic volume τ are related by $\tau = f^{-2}(p)$, which allows us to directly see that $(\tau'_s)_p = 0$, i.e., that neither H_2 nor \bar{H}_2 are verified.
- (iv) For a nonbarotropic fluid, a barotropic evolution can be defined by a relation $s = s(\rho)$. Then, if $p = p(\rho, s)$, the barotropic relation between ρ and p is given by $p = \phi(\rho) \equiv p(\rho, s(\rho))$. We have $c_s^2 = (p'_\rho)_s \neq \phi'(\rho)$. Thus, $\phi'(\rho)$ is not the square of the speed of sound in this case. Consequently, Theorem 2 concerns barotropic energy tensors that describe the evolution of intrinsic barotropic fluids, and Theorem 4 concerns nonbarotropic energy tensors describing the evolution of nonbarotropic fluids. The cases of barotropic energy tensors that describe barotropic evolutions of nonbarotropic fluids require specific individual studies. Note that Theorem 4 remains valid in this case and it gives restrictions on the square of the speed of sound.
- (v) We know [2] that every barotropic and isobaronegetic ($\dot{\rho} = \dot{p} = 0$) perfect energy tensor represents the evolution in LTE of *any* perfect fluid. Consequently, perfect fluids verifying the constraints (24) and (25)–(26) could have a particular barotropic evolution of the form $p = \phi(\rho)$ that is not subjected to constraints (14). For example, this is the case of any physical fluid evolving at constant pressure, $\phi(\rho) = p_0$. On the other hand, nonbarotropic fluids failing to verify H_1 , i.e., that do not fulfill the constraints (24), might have a barotropic evolution $p = \phi(\rho)$ satisfying (14) (see at the end of next section for a specific example).

IV. COMPRESSIBILITY CONDITIONS FOR A GENERIC IDEAL GAS

A *generic ideal gas*¹³ is described by the equation of state:

¹³The equation of state (29) not being a *characteristic* equation of state in the Gibbs' sense, is insufficient to define completely the thermodynamics of the fluid (it is usually completed with a particular expression of the internal energy ϵ as a function of the temperature Θ). It does not define *an* ideal gas, but anyone of the whole class. To reflect this fact, we call it a generic ideal gas.

$$p = kn\Theta, \quad k \equiv \frac{k_B}{m}. \quad (29)$$

In [2] we have studied the expression of the speed of sound and the other thermodynamic quantities of a generic ideal gas in terms of the hydrodynamic quantities (ρ, p) . We present in the next lemma a slightly different version of these results, which will be useful in order to analyze the relativistic compressibility conditions.

Lemma 2. The square of the speed of sound χ of a nonbarotropic generic ideal gas depends on the sole hydrodynamic variable $\pi \equiv p/\rho$, $\chi = \chi(\pi) \neq \pi$. Then, in terms of the hydrodynamic quantities (ρ, p) , the specific internal energy ϵ , the temperature Θ , the matter density n , and the specific entropy s are given, respectively, by

$$\epsilon(\rho, p) = e(\pi) - 1, \quad \Theta(\rho, p) = \frac{\pi}{k} e(\pi), \quad (30)$$

$$n(\rho, p) = \frac{\rho}{e(\pi)}, \quad s(\rho, p) = k \ln \frac{f(\pi)}{\rho}, \quad (31)$$

the generating functions $e(\pi)$ and $\phi(\pi)$ being, respectively,

$$e(\pi) = e_0 \exp \left\{ \int \psi(\pi) d\pi \right\}, \quad \psi(\pi) \equiv \frac{\pi}{(\chi(\pi) - \pi)(\pi + 1)}; \quad (32)$$

$$f(\pi) = f_0 \exp \left\{ \int \phi(\pi) d\pi \right\}, \quad \phi(\pi) \equiv \frac{1}{\chi(\pi) - \pi}. \quad (33)$$

The compressibility conditions H_1 and H_2 or \bar{H}_2 have been stated, for general perfect fluids, in Theorem 4. Now, in order to express H_1 for generic ideal gases, where $\chi(\rho, p) = \chi(\pi)$, we have

$$\chi'_\rho = \frac{\chi'(\pi)}{\rho}, \quad \chi'_p = -\frac{\chi'(\pi)}{\rho} \pi, \quad (34)$$

$$\chi\chi'_\rho + \chi'_p = \frac{\chi'(\pi)}{\rho} (\chi(\pi) - \pi). \quad (35)$$

Then, from the expression (24) of the compressibility conditions H_1 , we arrive to the following:

Lemma 3. A generic ideal gas fulfills the relativistic compressibility condition H_1 if, and only if, the square of the speed of sound $\chi = \chi(\pi)$ satisfies

$$H_1: \quad 0 < \chi < 1, \quad (1 + \pi)(\chi - \pi)\chi' + 2\chi(1 - \chi) > 0. \quad (36)$$

On the other hand, H_2 or \bar{H}_2 have been stated in Theorem 4 in terms of $n(\rho, p)$ and $s(\rho, p)$. For a generic ideal gas, where we have the expressions (31) for these functions of state, we obtain

$$s'_\rho = -\frac{k\chi}{\rho(\chi - \pi)}, \quad \frac{n'_\rho}{n} = \frac{\chi(\pi + 1) - \pi}{\rho(\chi - \pi)(\pi + 1)}.$$

Consequently, if we substitute these expressions in (26), we have

$$I(\rho, p) = \frac{\rho^2}{k\chi e(\pi)} [(2\pi + 1)\chi - \pi]. \quad (37)$$

Taking into account that the first factor in (37) is positive, from expressions (34), (35), (37) and Theorem 4, it follows:

Lemma 4. A generic ideal gas fulfills the relativistic compressibility condition H_2 (respectively, \bar{H}_2) if, and only if, the square of the speed of sound $\chi = \chi(\pi)$ satisfies

$$H_2: \quad \chi > \frac{\pi}{2\pi + 1} \quad \left(\text{respect., } \bar{H}_2: \quad \chi < \frac{\pi}{2\pi + 1} \right). \quad (38)$$

Here we are interested in current generic ideal gases as defined by (29). Although we have seen that, from a formal point of view, Eq. (29) is compatible with compressibility condition \bar{H}_2 , from now on we will consider only the current compressibility conditions H_1 and H_2 for ideal gases suggested by the mentioned Lucquiaud's results. Thus, from the two lemmas above we obtain the following:

Theorem 7. A generic ideal gas fulfills the relativistic compressibility conditions H_1 and H_2 if, and only if, the square of the speed of sound $\chi = \chi(\pi)$ satisfies:

$$H_1, H_2: \quad \begin{cases} \frac{\pi}{2\pi + 1} < \chi < 1, \\ (1 + \pi)(\chi - \pi)\chi' + 2\chi(1 - \chi) > 0. \end{cases} \quad (39)$$

This result allows us to characterize the perfect energy tensors that represent the evolution in LTE of an ideal gas fulfilling the corresponding compressibility conditions. Now, for similar reasons made before Theorems 5 and 6, we analyze separately the isoenergetic and nonisoenergetic evolutions. Then, if we take into account Lemma 2 and Theorem 7, we arrive at the following result, which considers both cases together.

Theorem 8. The necessary and sufficient condition for a nonbarotropic perfect energy tensor T to represent the evolution in LTE of a generic ideal gas subject to the relativistic compressibility conditions H_1 and H_2 is

- (i) If it is nonisoenergetic, that the indicatrix function $\chi \equiv \dot{p}/\dot{\rho}$ depends on the sole hydrodynamic variable $\pi \equiv p/\rho$, $\chi = \chi(\pi) \neq \pi$, and it satisfies the constraints (39).
- (ii) If it is isoenergetic, that it has be isobaroenergetic, $\dot{p} = \dot{\rho} = 0$. Then, the generic ideal gas thermodynamic quantities verifying these compressibility conditions are those for which the square of the

speed of sound, $\chi = \chi(\pi)$, determined by these quantities, fulfills the constraints (39).

The above two theorems concern the compressibility condition H_2 but, from Lemma 4, two similar theorems can be stated for condition \bar{H}_2 by changing the first inequalities in (39) to

$$0 < \chi < \frac{\pi}{2\pi + 1}. \quad (40)$$

It is worth presenting some remarks about the results in this section.

- (i) In the general inverse problem for a perfect energy tensor T , the inequalities (36) guarantee H_1 if the square of the speed of sound is a function of the sole variable π , $\chi = \chi(\pi)$. Then, Lemma 1 states that we can complete the thermodynamics with functions $n(\rho, p)$, $s(\rho, p)$ ensuring H_2 or \bar{H}_2 . But not *all* the resulting fluids will be generic ideal gases. The latter are the solution to the restricted inverse problem for generic ideal gases; they are given by Lemma 2 and satisfy H_2 or \bar{H}_2 if χ fulfills the corresponding inequality in (38). Note that now, by means of the function χ , H_2 or \bar{H}_2 also restrict the hydrodynamic flow in this restricted inverse problem.
- (ii) We have seen that the hydrodynamic variable $\pi = p/\rho$ plays a central role in the study of the properties of a generic ideal gas. In fact, for a general perfect fluid, the Plebański [20] energy conditions state, $-p \leq \rho < p$, that is, they limit the range of physically admissible values of π to $-1 \leq \pi \leq 1$. Under the reasonable assumption of positive matter density and temperature, the equation of state (29) implies a positive pressure and thus, the energy conditions for a generic ideal gas state that $0 < \pi < 1$. On the other hand, this is the range where the compressibility conditions H_1 are fulfilled for the thermodynamic fluids in which neither condition H_2 nor condition \bar{H}_2 are met. From (38), they are those whose square of the velocity of sound, say χ_* , is $\chi_* = \pi/(2\pi + 1)$. Thus, condition (38) states that the graphic of $\chi(\pi)$ for the generic ideal gases verifying the relativistic compressibility conditions H_1 and H_2 (respectively, \bar{H}_2) is over (respectively, under) the graphic of $\chi_*(\pi)$ in the region $[0, 1] \times (0, 1)$.
- (iii) Observe that Theorem 8 only concerns the nonbarotropic case. Barotropic generic ideal gases have, necessarily [2,21], the equation of state¹⁴ $p = (\gamma - 1)\rho$, and the compressibility conditions H_1 given by (14) hold if, and only if, $1 < \gamma < 2$.

¹⁴We see here that in the present restricted inverse problem for generic ideal gases, the hypothesis of barotropicity completely determines the thermodynamics of the ideal gas.

But these ideal gases are physically meaningless in the common domain, because for them $\epsilon = c_v \Theta - 1$ and the internal energy takes negative values at low temperatures.

- (iv) The expressions (36) of the compressibility conditions H_1 have been obtained from the corresponding general expressions (24) under the hypothesis $\chi(\rho, p) = \chi(\pi)$. Thus, constraints (36) also apply for a fluid with a speed of sound $c_s^2 = p/\rho$, that is $\chi = \pi$, and it is verified by this function for $\pi \in [0, 1]$. Nevertheless the expressions (38) of the compressibility conditions H_2 or \bar{H}_2 do not apply in this case because expressions (31) are only valid for ideal gases ($\chi \neq \pi$). An open problem that will be considered elsewhere is to solve the inverse problem in this case, and to study the compressibility condition H_2 or \bar{H}_2 for the associated thermodynamic schemes.
- (v) As we mentioned in comment (v) at the end of previous section, a nonbarotropic media failing to fulfill the expression (24) of the compressibility conditions H_1 might have a barotropic evolution with a barotropic relation $p = \phi(\rho)$ satisfying the expression (14) of H_1 for barotropic fluids. As an example, let us consider a generic ideal gas with a square of the speed of sound $\chi(\pi)$ which does not fulfill the expression (36) of the compressibility conditions H_1 . Let $e(\pi)$ be the specific energy given in (32), and let $\pi_0 \in [0, 1]$ be a value of the hydrodynamic variable $\pi = p/\rho$. Then, from the results in [2] we obtain that an evolution at constant temperature $\Theta_0 = \frac{1}{\chi} \pi_0 e(\pi_0)$ leads to a barotropic evolution $p = \pi_0 \rho$ which fulfills the expression (14) of H_1 .

V. RAINICH APPROACH TO THE IDEAL GAS SOLUTIONS

The necessary and sufficient conditions for a metric g to be a non-null Einstein-Maxwell solution were studied by Rainich [22]. We can distinguish the algebraic Rainich conditions, which impose that the Ricci tensor is of (non-null) electromagnetic type, and the differential ones, which guarantee that the electromagnetic field is a solution of the Maxwell equations.

A similar approach for the perfect fluid implies, in a first step, that we obtain the algebraic conditions for a Ricci tensor to have a triple eigenvalue, and a simple one with an associated timelike eigenvector. These conditions and the additional ones assuring that the energy tensor fulfills the Plebański [20] energy conditions were obtained years ago [1,23]. A more recent version [24] states the following:

Lemma 5. Consider the following scalar and tensor functions of the Ricci tensor R :

$$t \equiv \text{tr}R, \quad N \equiv R - \frac{1}{4}tg, \quad (41)$$

$$q \equiv -2\sqrt[3]{\frac{\text{tr}N^3}{3}}, \quad Q \equiv N - \frac{1}{4}qg. \quad (42)$$

A spacetime is a perfect fluid solution fulfilling the energy conditions if, and only if, the Ricci tensor R satisfies

$$Q^2 + qQ = 0, \quad Q(x, x) > 0, \quad q > 0, \quad t + q > 0, \quad (43)$$

where x is any timelike vector.

Lemma 6. When the conditions (43) of Lemma 5 are verified, the energy density ρ , the pressure p , and the unit velocity u of the fluid are given by

$$\rho = \frac{1}{4}(3q + t), \quad p = \frac{1}{4}(q - t), \quad u = \frac{Q(x)}{\sqrt{qQ(x, x)}}. \quad (44)$$

The extra conditions on the Ricci tensor for the fluid to be in LTE were presented in [1]. They are obtained in terms of the Ricci invariants (41) and (42) by writing their expression (8) with the values of ρ , p , and u given by Lemma 6 (see also [24]).

As we are concerned here by current generic gases, we must impose the energy conditions corresponding to a gas, $0 \leq \pi < 1$, $\pi = p/\rho$, as well as the LTE condition, which states (see Lemma 2) $d\chi \wedge d\pi = 0$, $\chi \neq \pi$. Moreover, the function $\chi(\pi)$ has to fulfill the compressibility conditions H_1 and H_2 for ideal gases given by Theorem 7.

When the perfect energy tensor is nonisoenergetic, $\dot{\rho} \neq 0$, the square of the speed of sound is given by the indicatrix function, $\chi = \chi \equiv \dot{p}/\dot{\rho}$. Then, taking into account Theorem 8 and the expressions in Lemma 5, we obtain the following.

Theorem 9. Let us consider, in terms of the above scalar and tensor functions (41), (42), and (44) of the Ricci tensor R , the following:

$$\pi \equiv \frac{p}{\rho}, \quad \chi \equiv \frac{Q(dp, d\rho)}{Q(d\rho, d\rho)}, \quad \chi' \equiv \frac{(d\chi, y)}{(d\pi, y)}, \quad (45)$$

where y is any transversal vector to $d\pi$: $(d\pi, y) \neq 0$.

A metric is a nonisoenergetic perfect fluid solution describing a generic ideal gas in LTE that fulfills the compressibility conditions H_1 and H_2 if, and only if, the Ricci tensor R satisfies

$$Q^2 + qQ = 0, \quad Q(x, x) > 0, \quad -t < q \leq t, \quad (46)$$

$$Q(d\rho) \neq 0, \quad d\chi \wedge d\pi = 0, \quad \chi \neq \pi, \quad (47)$$

$$\frac{\pi}{2\pi + 1} < \chi < 1, \quad (1 + \pi)(\chi - \pi)\chi' + 2\chi(1 - \chi) > 0, \quad (48)$$

where x is any timelike vector.

On the other hand, in the isoenergetic case, $\dot{\rho} = 0$, Theorem 8 states that the evolution is necessarily isobaric, $\dot{p} = 0$. Moreover, any function $\chi(\pi)$ fulfilling (48) is compatible with this evolution. Consequently, we have the following:

Theorem 10. A metric is an isoenergetic perfect fluid solution describing a generic ideal gas in LTE that fulfills the compressibility conditions H_1 and H_2 if, and only if, the Ricci tensor R satisfies (46) and

$$Q(dx) = 0, \quad Q(dp) = 0, \quad (49)$$

where x is any timelike vector, and Q , N , w , t , ρ , and p are given in (41), (42), and (44).

VI. CONSTRAINTS ON THE IDEAL GAS STEPHANI UNIVERSES

The *Stephani universes* were obtained as the conformally flat perfect fluid solutions to Einstein equations with nonzero expansion [25]. Later, they were achieved [26] as the conformally flat class of irrotational and shear-free perfect fluid spacetimes with nonzero expansion. They can also be characterized as the spacetimes verifying a weak cosmological principle without any hypothesis on the energy tensor [27,28].

In [28] Bona and Coll also showed that the necessary and sufficient condition for a Stephani universe to represent the evolution of a fluid in local thermal equilibrium is that it admits a three-dimensional isometry group on two-dimensional orbits. This result was later recovered in [29] and spherically symmetric Stephani universes that may be interpreted either as a classical monoatomic ideal gas or as a matter-radiation mixture were considered in [30].

In [5] we studied the Stephani universes that can be interpreted as a generic ideal gas in local thermal equilibrium. Now we summarize the main results in the following statements:

Thermodynamic Stephani universes [28]. The metric of the thermodynamic Stephani universes may be written as

$$ds^2 = -\alpha^2 dt^2 + \Omega^2 \delta_{ij} dx^i dx^j; \quad (50)$$

$$\alpha \equiv R \partial_R \ln L, \quad \Omega \equiv \frac{w}{2z} L, \quad (51)$$

$$L \equiv \frac{R(t)}{1 + b(t)w}, \quad w \equiv \frac{2z}{1 + \frac{\epsilon}{4} r^2}, \quad (52)$$

with $R(t)$ and $b(t)$ being two arbitrary functions of time. Its symmetry group is spherical, plane, or pseudospherical depending on ϵ to be 1, 0, or -1 and the Friedmann-Robertson-Walker limit occurs when $b = \text{constant}$. Furthermore, the energy density, the pressure, the expansion, and the three-space curvature are given by

$$\rho = \frac{3}{R^2} (\dot{R}^2 + \epsilon - 4b^2), \quad p = -\rho - \frac{R}{3} \frac{\partial_R \rho}{\alpha}, \quad (53)$$

$$\theta = \frac{3\dot{R}}{R} \neq 0, \quad \kappa = \frac{1}{R^2} (\epsilon - 4b^2). \quad (54)$$

Ideal gas Stephani universes [5]. A thermodynamic Stephani universe, (50), (51), and (52), represents a generic ideal gas if, and only if, the metric function $b(R)$ and the energy density $\rho(R)$ satisfy the equations:

$$Ra' = -a(c_1 a^2 + c_2 a + 1), \quad b''R = -c_1 a^2 b', \quad (55)$$

$$a = a(R) \equiv -\frac{R \partial_R \rho}{3\rho}, \quad (56)$$

where the principal constants c_1 and c_2 are arbitrary. Then, in terms of $b(R)$ and $\rho(R)$, the expansion factor $R(t)$ satisfies the generalized Friedmann equation:

$$\rho(R) = \frac{3}{R^2} [\dot{R}^2 + \epsilon - 4b^2(R)]. \quad (57)$$

Ideal gas Stephani models [5]. The indicatrix function $\chi = \chi(\pi)$ of the ideal gas Stephani universes takes the expression:

$$\chi(\pi) = \beta \pi^2 + \gamma \pi + \delta, \quad \delta \equiv \gamma - \beta - 2/3, \quad (58)$$

where the constants β and γ determine the fundamental constants as

$$c_1 = 3\beta, \quad c_2 = 3(\gamma - 2\beta - 1). \quad (59)$$

Then, depending on the roots of the equation $\chi(\pi) = \pi$, we find five classes C_n of ideal gas Stephani universes:

CLASS 1: $c_1 = c_2 = 0$.

CLASS 2: $c_1 = c_2 \neq 0$.

CLASS 3: $\Delta \equiv c_2 - 4c_1 = 0$, $c_1 \neq 0$.

CLASS 4: $\Delta \equiv c_2 - 4c_1 > 0$, $c_1 \neq 0$.

CLASS 5: $\Delta \equiv c_2 - 4c_1 < 0$.

Moreover, for every class, we can obtain the thermodynamic quantities (30) and (31) by using the generating functions (32) and (33), and we obtain five different thermodynamics.

On the other hand, the study of equations (55) and (56) leads us to distinguish the singular models, [$a'(R) = 0$], compatible with classes C2, C3, and C4, and regular models [$a'(R) \neq 0$], compatible with the five classes C_n [5].

Now we will study the compatibility of the above ideal gas Stephani models with the compressibility conditions studied in Sec. IV. It is worth remarking that a *full* analysis of the physical behavior of the solutions implies, not only a survey of the Plebański energy conditions [20],

$-p \leq \rho < p$, i.e., $-1 < \pi \leq 1$, but also the knowledge of the solution to the generalized Friedmann equation (57), which allows us to obtain the coordinate dependence of ρ and p by (53). But this is not our goal here.

Here we want to analyze when an ideal gas Stephani model fulfills the compressibility conditions H_1 and H_2 [i.e., verifies the corresponding expressions (39)], provided that it fulfills the energy conditions for a gas:

$$0 \leq \pi \leq 1. \quad (60)$$

We must look for the values of the principal constants c_1 and c_2 (or, equivalently, of the constants β and γ) for which the indicatrix function (58) satisfies the expressions (39) of the compressibility conditions H_1 and H_2 for a significant range of $\pi \in [0, 1]$. We shall consider here separately the models with a physically reasonable behavior at low or high temperature.

A. Low temperature: Models with $\chi(0) = 0$

For an ideal gas, good performance at low temperatures means a vanishing speed of sound when pressure vanishes, that is, $\chi(0) = 0$. Then, from (58) and (59), we have

$$\chi(\pi) = (\gamma - 2/3)\pi^2 + \gamma\pi, \quad (61)$$

$$c_1 = 3\gamma - 2 \neq 0, \quad c_2 = 1 - 3\gamma, \quad \Delta = 9(\gamma - 1)^2. \quad (62)$$

Now we study the values of the parameter γ for which the indicatrix function (61) fulfills the expressions (39) of the compressibility conditions H_1 and H_2 in a neighborhood of $\pi = 0$. Note that $\chi'(0) = \gamma$, and $\chi_*(0) = 0$, $\chi'_*(0) = 1$ for $\chi_*(\pi) = \pi/(2\pi + 1)$. Then, if we impose the first inequality in (39), $\chi_*(\pi) < \chi(\pi)$, we have, necessarily, $\gamma > 1$. On the other hand, a straightforward calculation shows that the second inequality in (39), $\chi(\pi) < 1$, holds in the interval $\pi \in [0, \pi_m]$, where

$$0 < \pi_m = \frac{2}{\gamma + \sqrt{\gamma^2 + 4(\gamma - 2/3)}} < 1. \quad (63)$$

Finally, under the already imposed constraints, the last inequality in (39) is satisfied. Note that, for $\gamma > 1$, one has $c_1 \neq 0$ and $\Delta > 0$. Consequently, the ideal gas thermodynamics belong to class C4. Thus, we have the following:

Proposition 1. The indicatrix function $\chi = \chi(\pi)$ of the ideal gas Stephani universes takes the expression (61) when it verifies $\chi(0) = 0$. When $\gamma > 1$ the associated thermodynamics belong to class C4 and fulfill the relativistic compressibility conditions H_1 and H_2 for $\pi \in [0, \pi_m]$, where π_m is given in (63).

Taking into account the results in [5] we can determine the generating functions (32) and (33) by using the

indicatrix function (61). And, from them, we can obtain the associated ideal gas thermodynamics (30) and (31). These thermodynamics of class C4 are compatible with both singular and regular models. The explicit expressions of these models can be found in [5], where the ideal gas Stephani models which approximate a classical ideal gas at first order in the temperature are studied. It is worth remarking that those models satisfy $\chi(0) = 0$ and the parameter γ coincides with the adiabatic index, which is subject to $1 < \gamma < 2$. Nevertheless, the compressibility conditions also hold for $\gamma \geq 2$ in $\pi \in [0, \pi_m]$.

B. High temperature: Models with $\chi(1/3) = 1/3$

For a relativistic Synge gas [13,31], the limit $\Theta \rightarrow \infty$ leads to a radiation fluid, $\rho = 3p$, and a speed of sound $c_s = 1/\sqrt{3}$. Consequently, in order to have a good behavior at high temperatures we will impose to the indicatrix function χ the constraint $\chi(1/3) = 1/3$. Then, from (58) and (59), and introducing the parameter $\lambda = 7 - 8\gamma$, we have

$$\chi(\pi) = \frac{1}{16} \left[3(1 - \lambda)\pi^2 + 2(7 - \lambda)\pi + \left(\lambda + \frac{1}{3} \right) \right], \quad (64)$$

$$c_1 = \frac{9}{16}(1 - \lambda), \quad c_2 = \frac{3}{4}(\lambda - 2), \quad \Delta = \frac{9}{16}\lambda^2. \quad (65)$$

We can see that, for any value of the parameter λ , the indicatrix function (64) fulfills the expressions (39) of the compressibility conditions H_1 and H_2 in $\pi = 1/3$, and, therefore, in its neighborhood. Moreover, depending on the values of λ , the ideal gas thermodynamics belong to classes C2, C3, or C4. Thus, we have the following:

Proposition 2. The indicatrix function $\chi = \chi(\pi)$ of the ideal gas Stephani universes takes the expression (64) when it verifies $\chi(1/3) = 1/3$. For any value of the parameter λ the associated thermodynamics fulfill the relativistic compressibility conditions H_1 and H_2 in a neighborhood of $\pi = 1/3$, and they belong to class C2 when $\lambda = 1$, to class C3 when $\lambda = 0$, and to class C4 otherwise.

Taking into account the results in [5] and by using the indicatrix function (64) we can determine the generating functions (32) and (33) and, from them, we can obtain the associated ideal gas thermodynamics (30) and (31). These thermodynamics of class C2, C3, and C4 are compatible with both singular and regular models. Here, we do not give the explicit expressions of all these models. We restrict ourselves to the cases which also are near to the Synge gas at the following order in temperature.

The derivative of the indicatrix function $\chi = \chi(\pi)$ of a Synge gas satisfies [13,31] $\chi'(1/3) = 1/2$. From (64) we have

$$\chi'(\pi) = \frac{1}{8} [3(1 - \lambda)\pi + (7 - \lambda)], \quad \chi''(\pi) = \frac{3}{8}(1 - \lambda). \quad (66)$$

If we impose $\chi'(1/3)$ to be in the neighborhood $[1/4, 3/4]$ of $1/2$, then the parameter λ satisfies $1 < \lambda < 3$, and $c_1 \neq 0$ and $\Delta > 0$. Consequently the thermodynamics belong to class C4.

Before determining the domain where the indicatrix function (64) fulfills the expressions (39) of the compressibility conditions H_1 and H_2 , we will obtain the generating functions (32) and (33) by using the indicatrix function (64). A straightforward calculation leads to

$$e(\pi) = e_0 \frac{|3(\lambda - 1)\pi + 3\lambda + 1|^{3+\frac{1}{\lambda}}}{|1 - 3\pi|^{\frac{1}{2}}(\pi + 1)^3}, \quad (67)$$

$$f(\pi) = f_0 \left| \frac{3(\lambda - 1)\pi + 3\lambda + 1}{1 - 3\pi} \right|^{\frac{1}{\lambda}}. \quad (68)$$

Note that both generating functions diverge in $\pi = 1/3$, and from (30) the temperature also diverges as occurs in the relativistic Synge gas, where $\pi \leq 1/3$. If we look for a model which approximates a Synge gas we must restrict ourselves to $\pi \in [0, 1/3]$. Now we shall see when the indicatrix function (64) fulfills the expressions (39) of the compressibility conditions H_1 and H_2 in this domain.

We have $\chi'(\pi) > 0$ in $[0, 1/3]$, and $\chi(0) = (\lambda + 1/3)/16 > 0$, $\chi(1/3) = 1/3$. Thus $0 < \chi(\pi) \leq 1/3 < 1$. The indicatrix function depends on the parameter λ and we can consider it as a function of two variables $\chi(\pi, \lambda)$. We have $\frac{\partial \chi}{\partial \lambda} > 0$ in the considered domain and, consequently, $\chi(\pi, \lambda) \geq \chi(\pi, 1)$. We can easily check that $\chi(\pi, 1) \geq \chi_*(\pi) = \frac{\pi}{2\pi+1}$. Therefore, the first compressibility condition in (39) holds.

On the other hand, if $\pi \in [0, 1/3]$, we have $\chi(\pi) \leq \chi(\pi, 1) \leq \pi$, $\chi'(\pi) > 0$, and $0 < \chi(\pi) < 1$, and consequently the second compressibility condition in (39) also holds.

Proposition 3. The indicatrix function $\chi = \chi(\pi)$ of the ideal gas Stephani universes takes the expression (64) when it is subject to $\chi(1/3) = 1/3$. If, additionally, the parameter $\lambda \in [1, 3]$ (or equivalently, $\chi'(1/3) \in [1/4, 3/4]$, i.e., it is closed to a Synge gas), then the associated thermodynamics fulfill the relativistic compressibility conditions H_1 and H_2 in $\pi = [0, 1/3]$, and they belong to class C4.

The ideal gas models of class C4 in proposition above demonstrate good physical behavior if the hydrodynamic variable $\pi = p/\rho$ ranges in $[0, 1/3]$, and they approach a Synge gas at high temperature, that is, in a neighborhood of $1/3$. Nevertheless, at low temperatures they differ from the Synge gas because the speed of sound does not approach zero when the temperature reaches zero.

From the generating functions, (67) and (68), we can obtain the associated ideal gas thermodynamics (30) and (31). These thermodynamics of class C4 are compatible with both singulars and regular models. The explicit expressions of these models can be easily obtained from

the results in [5]. By way of example, here we make explicit the generalized Friedmann equations for one of the singular models compatible with the case $\lambda = 2$ (the best fit for a Synge gas):

Generating functions:

$$e(\pi) = \frac{e_0(3\pi + 7)^{\frac{7}{2}}}{(1 - 3\pi)^{\frac{1}{2}}(\pi + 1)^3}, \quad f(\pi) = f_0 \left(\frac{3\pi + 7}{1 - 3\pi} \right)^2. \quad (69)$$

Generalized Friedmann equations:

$$\rho(R) = \frac{3}{R^2} [\dot{R}^2 + \varepsilon - 4b^2(R)], \quad (70)$$

where

$$\rho(R) = \rho_0 \left(\frac{R_0}{R} \right)^4, \quad b(R) = b_1 + b_2 \left(\frac{R}{R_0} \right)^2. \quad (71)$$

VII. ENDING COMMENTS

As we have seen in Sec. II, a thermodynamic perfect fluid in LTE may be considered as a conservative and deterministic hydrodynamic fluid endowed with subsidiary thermodynamic quantities. We know that causal and energy conditions directly constrain hydrodynamic variables and, consequently, they restrict the physically admissible energy tensors and, then, the gravitational fields that may be related to them. But whether or not the compressibility conditions, involving thermodynamic variables, also restrict the physically admissible energy tensors, remains an open problem, which we have solved in Sec. III (Theorems 2, 3, 4, 5, and 6). We have seen that, at most, only H_1 generically restricts them, and we have shown the form of the corresponding constraints. These results correspond to the conceptual point of view referred to in the Introduction.

An important task in relativity is the study of the physical meaning of the formal solutions $T = \{u, \rho, p\}$ to the conservative perfect fluid equations $\nabla \cdot T = 0$, whether or not they are coupled to the gravitational field by Einstein equations. At present, a wide family of such solutions is known without specific physical meanings. Our hydrodynamical approach (see also [2]) provides a tool to analyze the physical reality for this family of solutions and to solve the inverse problem for obtaining the specific thermodynamic interpretation. We have shown that the compressibility conditions H_1 impose constraints on the hydrodynamic quantities $\{u, \rho, p\}$, while the compressibility condition H_2 only imposes constraints on the thermodynamic subsidiary quantities.

We have applied these results to the particular case of generic ideal gases. In this case, because of the very definition of generic ideal gases, where hydrodynamic

and thermodynamic quantities are involved, not only H_1 but also H_2 constrain the hydrodynamic quantities (Theorems 7 and 8).

Our results can be useful in the determination of the metrics of a given family of perfect fluid solutions that represent the evolution of a specific set of fluids. The physical requirements studied here for generic ideal gases have been imposed on the ideal gas Stephani universes already obtained elsewhere [5] and physically realistic models have been selected (Sec. VI). A similar study has been performed elsewhere for the Szekeres-Szafron solutions of class II [8].

A Rainich-like theory of a physical medium is the *complete geometrization* of the physical relations characterizing it, that is to say, the formulation of these relations (equations and inequalities) in terms of concomitants of the sole space-time metric g . We have here formulated the Rainich approach to generic ideal gases subject to the compressibility conditions H_1 and H_2 (Theorems 9 and 10).

It is worth remarking that our approach offers an IDEAL (Intrinsic, Deductive, Explicit, and ALgorithmic) characterization of the perfect energy tensors that describe the evolution of a physically realistic perfect fluid. In [2] we

outlined the interest and the applications of this kind of approach.

Here we want to emphasize the relevance of our results in numerical relativity. On the one hand, the IDEAL approach to a solution to the Einstein equation or to the relativistic hydrodynamic equations may be the starting point in obtaining a fully algorithmic characterization of the initial data for this solution. For example, this study has been performed for the Schwarzschild and Kerr vacuum solutions [32–34] by considering our IDEAL labeling of these geometries [35,36]. On the other hand, physical constraints play a central role in designing numerical algorithms for solving the relativistic hydrodynamic equations [37–41] (see also the recent paper [42] and references therein).

These applications correspond to the practical point of view referred to in the Introduction.

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