Power-counting renormalizable, ghost-and-tachyon-free Poincaré gauge theories

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We present 48 further examples, in addition to the 10 identified in [1], of ghost-and-tachyon-free critical cases of parity-conserving Poincaré gauge theories of gravity (PGT$^+$) that are also power-counting renormalizable (PCR). This is achieved by extending the range of critical cases considered. Of the new PCR theories, seven have 2 massless degrees of freedom (d.o.f.) in propagating modes and a massive 0^- or 2^- mode, eight have only 2 massless d.o.f., and 33 have only massive mode(s). We also clarify the treatment of nonpropagating modes in determining whether a theory is PCR.

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In a recent paper [1], we presented a systematic method for identifying the ghost-and-tachyon-free critical cases of parity-preserving gauge theories of gravity and applied it to parity-preserving Poincaré gauge theory (PGT$^+$). The gravitational free-field Lagrangian for this theory may be written as [2]

$$\begin{aligned}\frac{\mathcal{L}_\mathrm{G}}{|b|} &= -\lambda\mathcal{R} + (r_4+r_5)\mathcal{R}^{AB}\mathcal{R}_{AB} + (r_4-r_5)\mathcal{R}^{AB}\mathcal{R}_{BA} + \left(\frac{r_1}{3}+\frac{r_2}{6}\right)\mathcal{R}^{ABCD}\mathcal{R}_{ABCD} + \left(\frac{2r_1}{3}-\frac{2r_2}{3}\right)\mathcal{R}^{ABCD}\mathcal{R}_{ACBD}\\ &\quad + \left(\frac{r_1}{3}+\frac{r_2}{6}-r_3\right)\mathcal{R}^{ABCD}\mathcal{R}_{CDAB} + \left(\frac{\lambda}{4}+\frac{t_1}{3}+\frac{t_2}{12}\right)\mathcal{T}^{ABC}\mathcal{T}_{ABC} + \left(-\frac{\lambda}{2}-\frac{t_1}{3}+\frac{t_2}{6}\right)\mathcal{T}^{ABC}\mathcal{T}_{BCA}\\ &\quad + \left(-\lambda-\frac{t_1}{3}+\frac{2t_3}{3}\right)\mathcal{T}_B{}^{AB}\mathcal{T}_{CA}{}^C, \end{aligned} \tag{1}$$

where the field strengths are

$$\mathcal{R}^{AB}{}_{\mu\nu} = 2(\partial_{[\mu}A^{AB}{}_{\nu]} + A^A{}_{E[\mu}A^{EB}{}_{\nu]}), \tag{2}$$

$$\mathcal{T}^A{}_{\mu\nu} = 2(\partial_{[\mu}b^A{}_{\nu]} + A^A{}_{E[\mu}b^E{}_{\nu]}), \tag{3}$$

in which $h_A{}^\mu$ is the translational gauge field, $b^A{}_\mu$ is its inverse, such that $b^A{}_\mu h_B{}^\mu = \delta^A_B$ and $b^A{}_\mu h_A{}^\nu = \delta^\nu_\mu$, and $A^{AB}{}_\mu = -A^{BA}{}_\mu$ is the gauge field corresponding to Lorentz transformations. Greek indices denote the coordinate frame, and Latin capital indices correspond to the local Lorentz frame. In our analysis, we linearized the gauge fields and decomposed the h field into its symmetric and antisymmetric parts $\mathfrak{s}$ and $\mathfrak{a}$, respectively, to obtain a quadratic Lagrangian, which we then decomposed into

$$\mathcal{L}_\mathrm{F} = \sum_{J,P,i,j} a(J^P)_{ij}\hat{\zeta}^\dagger\cdot\hat{P}(J^P)_{ij}\cdot\hat{\zeta} \tag{4}$$

using the spin projection operators (SPOs) $\hat{P}(J^P)_{ij}$ [3–5]. Please see Sec. II of [1] for a description of our notation. If any of the matrices $a(J^P)$ is singular, then the theory possesses gauge invariances. One may fix these gauges by deleting rows and columns of the a matrices such that they become nonsingular; the elements of the resulting matrices are usually denoted by $b_{ij}(J^P)$. The requirement that a theory is free from ghosts and tachyons places conditions on the b matrices; we traverse all critical cases to determine which (if any) satisfy these conditions.

In this way, in [1], we found 450 critical cases that are free from ghosts and tachyons, of which we identified 10 that are also power-counting renormalizable (PCR). The key quantity for determining whether a theory is PCR is the propagator,

$$\hat{D} = \sum_{J,P,i,j} b^{-1}_{ij}\hat{P}(J^P)_{ij}. \tag{5}$$

In particular, if the b matrices contain no elements linking any of the A, $\mathfrak{s}$, and $\mathfrak{a}$ fields, then it is straightforward to

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TABLE I. Parameter conditions for the PC renormalizable critical cases that are ghost and tachyon free and have both massless and massive propagating modes. The parameters listed in “Additional conditions” must be nonzero to prevent the theory becoming a different critical case.

No.	Critical condition	Additional conditions	No-ghost-and-tachyon condition
1	$r_1, \frac{r_3}{2} - r_4, t_1, t_3, \lambda = 0$	$r_2, r_3, 2r_3 + r_5, r_3 + 2r_5, t_2$	$t_2 > 0, r_2 < 0, r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$
2	$r_1, \frac{r_3}{2} - r_4, t_1, \lambda = 0$	$r_2, r_1 - r_3, 2r_3 + r_5, r_1 + r_3 + 2r_5, t_2, t_3$	$t_2 > 0, r_2 < 0, r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$
3	$r_1, r_3, r_4, t_1 + t_2, t_3, \lambda = 0$	$r_2, r_1 + r_5, 2r_1 + r_5, t_1, t_2$	$r_2 < 0, r_5 < 0, t_1 < 0$
4	$r_2, r_1 - r_3, r_4, t_1 + t_2, t_3, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5, t_1, t_2$	$t_1 > 0, r_1 + r_5 < 0, r_1 < 0$
5	$r_2, r_1 - r_3, r_4, t_2, t_1 + t_3, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5, t_1, t_3$	$r_5 > 0, 2r_1 + r_5 > 0, t_1 > 0, r_1 < 0$
6	$r_1, 2r_3 - r_4, t_1 + t_2, t_3, \lambda = 0$	$r_2, r_1 - r_3, r_1 - 2r_3 - r_5, 2r_3 + r_5, t_1, t_2$	$r_2 < 0, 2r_3 + r_5 < 0, t_1 < 0$
7	$r_2, 2r_1 - 2r_3 + r_4, t_1 + t_2, t_3, \lambda = 0$	$r_1, r_1 - r_3, r_1 - 2r_3 - r_5, 2r_3 + r_5, t_1, t_2$	$t_1 > 0, r_1 < 0, 2r_3 + r_5 < r_1$

obtain the propagators for these fields separately from $\hat{D}$. The original PCR criterion used in [2] requires the propagator of the A field to decay at least as quickly as k^{-2} at high energy, and those of the $\mathfrak{s}$ and $\mathfrak{a}$ fields to fall off at least as k^{-4}. By contrast, the alternative PCR criterion used in [1] also permits the presence of nonpropagating fields (for which the propagator decays no faster than $\sim k^0$), since these should completely decouple from the rest of the theory; we will compare these two criteria further below. Moreover, in [1], we restricted our PCR considerations to those cases for which the b matrices are diagonal, such that there are no mixing terms between the A, $\mathfrak{s}$, and $\mathfrak{a}$ fields in the gauge-fixed Lagrangian and so the physical interpretation is straightforward. Indeed, with this restriction, the high-energy behaviors of the propagators are equivalent to those of the corresponding diagonal elements in the b^{-1} matrices.

In this paper, we extend our search to include those cases for which the b matrices are block diagonal, with each block containing only one field. This clearly includes our previous study as a special case, but increases considerably the number of cases under consideration, while again ensuring that there are no mixing terms in the gauge-fixed Lagrangian.[1] It is worth noting that, even in this more general case, the behavior of the propagators at high energy goes as the highest power of the corresponding elements in the b^{-1} matrices. Moreover, in the PGT$^+$ cases we consider, any nondiagonal block of a b matrix that does not mix fields is always the only block in the matrix, contains only the A field, and has size 2×2. Moreover, these blocks occur only in the 1^- or 1^+ sector and have the following form:

$$b = \begin{pmatrix} rk^2 + (x + 4y) & -\sqrt{2}(x - 2y) \\ -\sqrt{2}(x - 2y) & 2(x + y) \end{pmatrix}, \tag{6}$$

where x, y, and r are real linear combinations of the parameters in the Lagrangian. Hence, the element with the highest power of k in b^{-1} is always a diagonal element. Note that when $x + y = 0$ and $r, x, y \neq 0$, the element with the highest power in b^{-1} goes as k^2, not k^{-2}, even though the highest power in b is also k^2. This is a similar case to that summarized in Eqs. (1.2)–(1.4) of [6]. Since there is no pole in the determinant $\det(b) = -18x^2$ in this case, there is no propagating mode in this sector.

Our main result is that, in addition to the 10 PCR cases found in [1], this new search yields a further 48 cases that are PCR. For completeness, we list all 58 cases (old and new) in Tables I–IV, in which the old cases are indicated with an asterisk followed by the old number of the case as given in [1]. Tables I and II summarize the seven cases with both massless and massive modes, all of which are new and have 2 massless degrees of freedom (d.o.f.) in propagating modes and a massive 0^- or 2^- mode. Tables III and IV summarize the 12 cases with only massless modes, of which eight are new and contain only 2 massless d.o.f. Finally, Tables V and VI summarize the 39 cases with only massive modes, of which 33 are new. For each set of tables, the first lists the various conditions for each critical case, and the second lists the “particle content” in terms of the diagonal elements in the b^{-1} matrix of each spin-parity sector in the sequence $\{0^-, 0^+, 1^-, 1^+, 2^-, 2^+\}$.

Since we adopt the PCR criterion in [1], which differs from the original criterion used in [2] by allowing the presence of nonpropagating fields, it is worth discussing further the status of such fields in the determination of whether a theory is PCR. We begin by noting that an important consequence of allowing the existence of nonpropagating fields is that whether some critical cases obey our PCR criterion may depend on the choice of gauge fixing. For example, in the spin-parity sector 0^+ in Case 8, the a matrix is

$$\begin{array}{cccc} & A & \mathfrak{s} & \mathfrak{s} \\ a(0^+) = & \left(\begin{array}{c} 2t_3 \\ 2i\sqrt{2}kt_3 \\ 0 \end{array}\right. & \begin{array}{c} -2i\sqrt{2}kt_3 \\ 4k^2t_3 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right) \end{array}, \tag{7}$$

[1] We note that this extension therefore does not include Einstein-Cartan theory.

TABLE II. Particle content of the PC renormalizable critical cases that are ghost and tachyon free and have both massless and massive propagating modes. All of these cases have 2 massless d.o.f. in propagating modes and also a massive mode. The column "b sectors" describes the diagonal elements in the b^{-1} matrix of each spin-parity sector in the sequence $\{0^-, 0^+, 1^-, 1^+, 2^-, 2^+\}$. Here and in Tables IV and VI it is notated as φ^n_v or φ^n_l, where φ is the field, $-n$ is the power of k in the element in the b^{-1} matrix when k goes to infinity, v means massive pole, and l means massless pole. If $n = \infty$, it represents that the diagonal element is zero. If $n \leq 0$, the field is not propagating. The "|" notation denotes the different form of the elements of the b^{-1} matrices in different choices of gauge fixing, and the "&" connects the diagonal elements in the same b^{-1} matrix. The superscript "N" represents that there is nonzero off-diagonal term in the b^{-1} matrix.

No.	Massless mode d.o.f.	Massive mode	b sectors
1	2	0^-	$\{A^2_{\rm v}, \times, A^2_{\rm l}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, \times, A^2_{\rm l}\}$
2	2	0^-	$\{A^2_{\rm v}, A^0 \vert \mathfrak{s}^2_{\rm l}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, \times, A^2_{\rm l}\}$
3	2	0^-	$\{A^2_{\rm v}, \times, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, (A^\infty\ \&\ A^{-2})^{\rm N} \vert (A^\infty\ \&\ \mathfrak{a}^0_{\rm l})^{\rm N}, A^0, A^0 \vert \mathfrak{s}^2_{\rm l}\}$
4	2	2^-	$\{A^0, \times, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, (A^\infty\ \&\ A^{-2})^{\rm N} \vert (A^\infty\ \&\ \mathfrak{a}^0_{\rm l})^{\rm N}, A^2_{\rm v}, A^0 \vert \mathfrak{s}^2_{\rm l}\}$
5	2	2^-	$\{\times, A_0 \vert \mathfrak{s}^2_{\rm l}, (A^\infty\ \&\ A^{-2})^{\rm N} \vert (A^\infty\ \&\ \mathfrak{s}^0_{\rm l})^{\rm N} \vert (A^\infty\ \&\ \mathfrak{a}^0_{\rm l})^{\rm N}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, A^2_{\rm v}, A^0 \vert \mathfrak{s}^2_{\rm l}\}$
6	2	0^-	$\{A^2_{\rm v}, A^2_{\rm l}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, (A^\infty\ \&\ A^{-2})^{\rm N} \vert (A^\infty\ \&\ \mathfrak{a}^0_{\rm l})^{\rm N}, A^0, A^0 \vert \mathfrak{s}^2_{\rm l}\}$
7	2	2^-	$\{A^0, A^2_{\rm l}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, (A^\infty\ \&\ A^{-2})^{\rm N} \vert (A^\infty\ \&\ \mathfrak{a}^0_{\rm l})^{\rm N}, A^2_{\rm v}, A^0 \vert \mathfrak{s}^2_{\rm l}\}$

TABLE III. Parameter conditions for the PC renormalizable critical cases that are ghost and tachyon free and have only massless propagating modes. The cases found previously in [1] are indicated with an asterisk followed by its original numbering.

No.	Critical condition	Additional condition	No-ghost-and-tachyon condition
8	$r_2, r_1 - r_3, r_4, t_1, t_2, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5, t_3$	$r_1(r_1 + r_5)(2r_1 + r_5) < 0$
*1 9	$r_2, r_1 - r_3, r_4, t_1, t_2, t_3, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5$	$r_1(r_1 + r_5)(2r_1 + r_5) < 0$
*3 10	$r_1, r_2, \frac{r_3}{2} - r_4, t_1, t_2, t_3, \lambda = 0$	$r_3, 2r_3 + r_5, r_3 + 2r_5$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$
*4 11	$r_1, \frac{r_3}{2} - r_4, t_1, t_2, t_3, \lambda = 0$	$r_2, r_3, 2r_3 + r_5, r_3 + 2r_5$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$
12	$r_1, r_2, \frac{r_3}{2} - r_4, t_1, t_3, \lambda = 0$	$r_3, 2r_3 + r_5, r_3 + 2r_5, t_2$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$
*2 13	$r_2, 2r_1 - 2r_3 + r_4, t_1, t_2, t_3, \lambda = 0$	$r_1, r_1 - r_3, r_1 - 2r_3 - r_5, 2r_3 + r_5$	$r_1(r_1 - 2r_3 - r_5)(2r_3 + r_5) > 0$
14	$r_1, r_2, \frac{r_3}{2} - r_4, t_1, t_2, \lambda = 0$	$2r_3 - r_4, 2r_3 + r_5, r_4 + r_5, t_3$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$
15	$r_1, r_2, \frac{r_3}{2} - r_4, t_1, \lambda = 0$	$r_3, 2r_3 + r_5, r_3 + 2r_5, t_2, t_3$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$
16	$r_1, \frac{r_3}{2} - r_4, t_1, t_2, \lambda = 0$	$r_2, r_3, 2r_3 + r_5, r_3 + 2r_5, t_3$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$
17	$r_1, r_2, r_3, r_4, t_1 + t_2, t_3, \lambda = 0$	$r_1 + r_5, 2r_1 + r_5, t_1, t_2$	$r_5 < 0, t_1 \neq 0$
18	$r_1, r_2, r_3, r_4, t_2, t_1 + t_3, \lambda = 0$	$r_1 + r_5, 2r_1 + r_5, t_1, t_3$	$r_5 > 0, t_1 \neq 0$
19	$r_1, r_2, 2r_3 - r_4, t_1 + t_2, t_3, \lambda = 0$	$r_1 - r_3, r_1 - 2r_3 - r_5, 2r_3 + r_5, t_1, t_2$	$r_3 < -\frac{r_5}{2}, t_1 \neq 0$

TABLE IV. Particle content of the PC renormalizable critical cases that are ghost and tachyon free and have only massless propagating modes. All of these cases have 2 massless d.o.f. of propagating mode. The cases found previously in [1] are indicated with an asterisk followed by its original numbering.

No.	Massless mode d.o.f.	b sectors
8	2	$\{\times, A^0 \vert \mathfrak{s}^2_{\rm l}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, A^2_{\rm l}, A^2_{\rm l}, \times\}$
*1 9	2	$\{\times, \times, A^2_{\rm l}, A^2_{\rm l}, A^2_{\rm l}, \times\}$
*3 10	2	$\{\times, \times, A^2_{\rm l}, A^2_{\rm l}, \times, A^2_{\rm l}\}$
*4 11	2	$\{A^2_{\rm l}, \times, A^2_{\rm l}, A^2_{\rm l}, \times, A^2_{\rm l}\}$
12	2	$\{A^0, \times, A^2_{\rm l}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, \times, A^2_{\rm l}\}$
*2 13	2	$\{\times, A^2_{\rm l}, A^2_{\rm l}, A^2_{\rm l}, A^2_{\rm l}, \times\}$
14	2	$\{\times, A^0 \vert \mathfrak{s}^2_{\rm l}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, A^2_{\rm l}, \times, A^2_{\rm l}\}$
15	2	$\{A^0, A^0 \vert \mathfrak{s}^2_{\rm l}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, \times, A^2_{\rm l}\}$
16	2	$\{A^2_{\rm l}, A^0 \vert \mathfrak{s}^2_{\rm l}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, A^2_{\rm l}, \times, A^2_{\rm l}\}$
17	2	$\{A^0, \times, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, (A^\infty\ \&\ A^{-2})^{\rm N} \vert (A^\infty\ \&\ \mathfrak{a}^0_{\rm l})^{\rm N}, A^0, A^0 \vert \mathfrak{s}^2_{\rm l}\}$
18	2	$\{\times, A^0 \vert \mathfrak{s}^2_{\rm l}, (A^\infty\ \&\ A^{-2})^{\rm N} \vert (A^\infty\ \&\ \mathfrak{s}^0_{\rm l})^{\rm N} \vert (A^\infty\ \&\ \mathfrak{a}^0_{\rm l})^{\rm N}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, A^0, A^0 \vert \mathfrak{s}^2_{\rm l}\}$
19	2	$\{A^0, A^2_{\rm l}, (A^2_{\rm l}\ \&\ A^0_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{s}^2_{\rm l})^{\rm N} \vert (A^2_{\rm l}\ \&\ \mathfrak{a}^2_{\rm l})^{\rm N}, (A^\infty\ \&\ A^{-2})^{\rm N} \vert (A^\infty\ \&\ \mathfrak{a}^0_{\rm l})^{\rm N}, A^0, A^0 \vert \mathfrak{s}^2_{\rm l}\}$

TABLE V. Parameter conditions for the PC renormalizable critical cases that are ghost and tachyon free and have only massive propagating modes. The cases found previously in [1] are indicated with an asterisk followed by its original numbering.

No.	Critical condition	Additional conditions	No-ghost-and-tachyon condition
20	$r_1, r_3, r_4, r_5, \lambda = 0$	$r_2, t_1, t_2, t_1+t_2, t_3, t_1+t_3$	$t_2 > 0, r_2 < 0$
21	$r_1, r_3, r_4, r_5, t_1+t_2, \lambda = 0$	$r_2, t_1, t_2, t_3, t_1+t_3$	$r_2 < 0, t_1 < 0$
22	$r_1, r_3, r_4, r_5, t_1+t_3, \lambda = 0$	$r_2, t_1, t_2, t_1+t_2, t_3$	$t_2 > 0, r_2 < 0$
23	$r_1, r_3, r_4, r_5, t_1+t_2, t_1+t_3, \lambda = 0$	r_2, t_1, t_2, t_3	$r_2 < 0, t_1 < 0$
24	$r_1, r_3, r_4, t_1, \lambda = 0$	$r_2, r_1+r_5, 2r_1+r_5, t_2, t_3$	$t_2 > 0, r_2 < 0$
*5 25	$r_1, r_3, r_4, r_5, t_1, \lambda = 0$	r_2, t_2, t_3	$t_2 > 0, r_2 < 0$
*6 26	$r_1, r_3, r_4, r_5, t_1, t_3, \lambda = 0$	r_2, t_2	$t_2 > 0, r_2 < 0$
27	$r_1, \frac{r_3}{2}-r_4, \frac{r_3}{2}+r_5, t_1, t_3, \lambda = 0$	r_2, r_3, t_2	$t_2 > 0, r_2 < 0$
28	$r_1, r_3, r_4, t_1, t_3, \lambda = 0$	r_2, r_5, t_2	$t_2 > 0, r_2 < 0$
29	$r_1-r_3, r_4, 2r_1+r_5, t_1, \lambda = 0$	$r_1, r_2, r_1+r_5, t_2, t_3$	$t_2 > 0, r_2 < 0$
*7 30	$r_1-r_3, r_4, 2r_1+r_5, t_1, t_3, \lambda = 0$	r_1, r_2, t_2	$t_2 > 0, r_2 < 0$
*8 31	$r_1, 2r_3-r_4, 2r_3+r_5, t_1, t_3, \lambda = 0$	r_2, r_3, t_2	$t_2 > 0, r_2 < 0$
32	$r_1, r_3, r_4, r_5, t_3, \lambda = 0$	r_2, t_1, t_2, t_1+t_2	$t_2 > 0, r_2 < 0$
33	$r_1, r_3, r_4, r_5, t_1+t_2, t_3, \lambda = 0$	r_2, t_1, t_2	$r_2 < 0, t_1 < 0$
34	$r_1, 2r_3-r_4, t_1, t_3, \lambda = 0$	$r_2, r_3, 2r_3+r_5, t_2$	$t_2 > 0, r_2 < 0$
*9 35	$r_1, \frac{r_3}{2}-r_4, 2r_3+r_5, t_1, t_3, \lambda = 0$	r_2, r_3, t_2	$t_2 > 0, r_2 < 0$
*10 36	$2r_1-2r_3+r_4, 2r_3+r_5, t_1, t_3, \lambda = 0$	r_1, r_2, r_1-r_3, t_2	$t_2 > 0, r_2 < 0$
37	$r_1, \frac{r_3}{2}-r_4, 2r_3+r_5, t_1, \lambda = 0$	$r_2, 2r_3-r_4, t_2, t_3$	$t_2 > 0, r_2 < 0$
38	$r_1, 2r_3-r_4, 2r_3+r_5, t_3, \lambda = 0$	$r_2, r_1-r_3, t_1, t_2, t_1+t_2$	$t_2 > 0, r_2 < 0$
39	$r_1, 2r_3-r_4, 2r_3+r_5, t_1+t_2, t_3, \lambda = 0$	r_2, r_1-r_3, t_1, t_2	$r_2 < 0, t_1 < 0$
40	$r_1, r_4+r_5, t_1, t_3, \lambda = 0$	$r_2, r_3-2r_4, 2r_3-r_4, t_2$	$t_2 > 0, r_2 < 0$
41	$r_1, \frac{r_3}{2}-r_4, \frac{r_3}{2}+r_5, t_1, \lambda = 0$	$r_2, 2r_3-r_4, t_2, t_3$	$t_2 > 0, r_2 < 0$
42	$r_1, r_3, r_4, t_1+t_2, \lambda = 0$	$r_2, r_1+r_5, 2r_1+r_5, t_1, t_2, t_3, t_1+t_3$	$t_3 > 0, r_2 < 0, r_5 < 0, t_1 < 0, t_1+t_3 < 0$
43	$r_1, r_3, r_4, t_1+t_3, \lambda = 0$	$r_2, r_1+r_5, 2r_1+r_5, t_1, t_2, t_1+t_2, t_3$	$r_5 > 0, t_2 > 0, t_1+t_2 > 0, r_2 < 0, t_1 < 0$
44	$r_2, r_1-r_3, r_4, t_1+t_2, \lambda = 0$	$r_1, r_1+r_5, 2r_1+r_5, t_1, t_2, t_3, t_1+t_3$	$t_1 > 0, r_1 < 0, r_1+r_5 < 0, t_3(t_1+t_3) > 0$
45	$r_2, r_1-r_3, r_4, t_1+t_3, \lambda = 0$	$r_1, r_1+r_5, 2r_1+r_5, t_1, t_2, t_1+t_2, t_3$	$r_5 > 0, 2r_1+r_5 > 0, t_1 > 0, t_1+t_2 > 0, r_1 < 0, t_2 < 0$
46	$r_1-r_3, r_4, 2r_1+r_5, t_1+t_3, \lambda = 0$	$r_1, r_2, r_1+r_5, t_1, t_2, t_1+t_2, t_3$	$t_1 > 0, t_2 > 0, r_1 < 0, r_2 < 0$
47	$r_1, r_2, r_3, r_4, t_1+t_2, \lambda = 0$	$r_1+r_5, 2r_1+r_5, t_1, t_2, t_3, t_1+t_3$	$r_5 < 0, t_1 t_3(t_1+t_3) > 0$
48	$r_1, r_2, r_3, r_4, t_1+t_3, \lambda = 0$	$r_1+r_5, 2r_1+r_5, t_1, t_2, t_1+t_2, t_3$	$r_5 > 0, t_1 t_2(t_1+t_2) < 0$
49	$r_1, r_3, r_4, t_1+t_2, t_1+t_3, \lambda = 0$	$r_2, r_1+r_5, 2r_1+r_5, t_1, t_2, t_3$	$r_2 < 0, t_1 < 0$
50	$r_2, r_1-r_3, r_4, r_1+r_5, t_1+t_2, \lambda = 0$	$r_1, 2r_1+r_5, t_1, t_2, t_3, t_1+t_3$	$t_1 > 0, r_1 < 0$
51	$r_2, r_1-r_3, r_4, 2r_1+r_5, t_1+t_3, \lambda = 0$	$r_1, r_1+r_5, t_1, t_2, t_1+t_2, t_3$	$t_1 > 0, r_1 < 0$
52	$r_2, r_1-r_3, r_4, t_1+t_2, t_1+t_3, \lambda = 0$	$r_1, r_1+r_5, 2r_1+r_5, t_1, t_2, t_3$	$t_1 > 0, r_1 < 0$
53	$r_2, r_1-r_3, r_4, r_1+r_5, t_1+t_2, t_1+t_3, \lambda = 0$	$r_1, 2r_1+r_5, t_1, t_2, t_3$	$t_1 > 0, r_1 < 0$
54	$r_2, r_1-r_3, r_4, 2r_1+r_5, t_1+t_2, t_1+t_3, \lambda = 0$	$r_1, r_1+r_5, t_1, t_2, t_3$	$t_1 > 0, r_1 < 0$
55	$r_2, r_1-r_3, r_4, r_1+r_5, t_1+t_2, t_3, \lambda = 0$	r_1, t_1, t_2	$t_1 > 0, r_1 < 0$
56	$r_2, r_1-r_3, r_4, 2r_1+r_5, t_2, t_1+t_3, \lambda = 0$	r_1, t_1, t_3	$t_1 > 0, r_1 < 0$
57	$r_1-r_3, r_4, 2r_1+r_5, t_2, t_1+t_3, \lambda = 0$	r_1, r_2, t_1, t_3	$t_1 > 0, r_1 < 0$
58	$r_2, 2r_1-2r_3+r_4, r_1-2r_3-r_5, t_1+t_2, t_3, \lambda = 0$	r_1, r_1-r_3, t_1, t_2	$t_1 > 0, r_1 < 0$

which is singular, indicating the presence of gauge invariances. One may render this matrix nonsingular by deleting rows and columns in two different ways, corresponding to two different gauge fixings, which in this case correspond simply to keeping either the first or the second column and row. If one chooses to keep only the second row and column, then this sector contains only an $\mathfrak{s}$ field, with a propagator that goes as $\sim k^{-2}$ at high energy, which thus violates both our alternative PCR criterion and the original one. Conversely, if one chooses to retain only the first column and row, then the 0^+ spin-parity sector contains only a nonpropagating A field, which we contend is harmless and thus satisfies our alternative PCR criterion, while violating the original one. The conclusions regarding PCR are therefore gauge dependent.

Overall, we take the view that a theory is PCR if one can find a gauge in which it satisfies our PCR criterion, irrespective of the existence of other gauge choices in which the PCR criterion is violated. The rationale for this view is that a theory should describe the same physics independently of which gauge one adopts. Thus, if one uses a particular gauge to make a physical prediction, then one should, in principle, be able to draw the same physical conclusion in any other gauge, although most often not in such a transparent manner.

We therefore consider the 0^+ sector of Case 8 to satisfy our PCR criterion, whereas it violates the original one in [2]. Moreover, although the total propagator for a field is the sum of the propagators across all sectors, it cannot satisfy either PCR condition if that same condition is

TABLE VI. Particle content of the PC renormalizable critical cases that are ghost and tachyon free and have only massive propagating modes. The cases found previously in [1] are indicated with an asterisk followed by its original numbering. Note that there are typos of the b sectors of Cases 30 and 31 (old numbers 7 and 8) in [1].

No.	Massive mode	b sectors
20	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^0 \,\&\, A^0)^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, (A^0 \,\&\, A^0)^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
21	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^0 \,\&\, A^0)^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, (A^\infty \,\&\, A^0)^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
22	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^0)^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, (A^0 \,\&\, A^0)^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
23	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^0)^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, (A^\infty \,\&\, A^0)^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
24	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^2_{\mathrm{l}} \,\&\, A^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, (A^2_{\mathrm{l}} \,\&\, A^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, \times, \times\}$
*5 25	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, \times, \times\}$
*6 26	0^-	$\{A^2_{\mathrm{v}}, \times, \times, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, \times, \times\}$
27	0^-	$\{A^2_{\mathrm{v}}, \times, \times, (A^2_{\mathrm{l}} \,\&\, A^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\}$
28	0^-	$\{A^2_{\mathrm{v}}, \times, A^2_{\mathrm{l}}, (A^2_{\mathrm{l}} \,\&\, A^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, \times, \times\}$
29	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^2_{\mathrm{l}} \,\&\, A^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, A^2_{\mathrm{l}}, \times\}$
*7 30	0^-	$\{A^2_{\mathrm{v}}, \times, A^2_{\mathrm{l}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, A^2_{\mathrm{l}}, \times\}$
*8 31	0^-	$\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, \times, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, \times, \times\}$
32	0^-	$\{A^2_{\mathrm{v}}, \times, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, (A^0 \,\&\, A^0)^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
33	0^-	$\{A^2_{\mathrm{v}}, \times, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert A^2_{\mathrm{l}}, (A^\infty \,\&\, A^0)^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
34	0^-	$\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, A^2_{\mathrm{l}}, (A^2_{\mathrm{l}} \,\&\, A^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, \times, \times\}$
*9 35	0^-	$\{A^2_{\mathrm{v}}, \times, A^2_{\mathrm{l}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, \times, A^2_{\mathrm{l}}\}$
*10 36	0^-	$\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, A^2_{\mathrm{l}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, A^2_{\mathrm{l}}, \times\}$
37	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^2_{\mathrm{l}} \,\&\, A^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, \times, A^2_{\mathrm{l}}\}$
38	0^-	$\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, (A^0 \,\&\, A^0)^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
39	0^-	$\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, (A^\infty \,\&\, A^0)^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
40	0^-	$\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, \times, (A^2_{\mathrm{l}} \,\&\, A^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\}$
41	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, (A^2_{\mathrm{l}} \,\&\, A^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^2_{\mathrm{l}} \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\}$
42	$0^-, 1^-$	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^2_{\mathrm{v}} \,\&\, A^0_{\mathrm{v}})^{\mathrm{N}}\vert(A^2_{\mathrm{v}} \,\&\, \mathfrak{s}^2_{\mathrm{vl}})^{\mathrm{N}}\vert(A^2_{\mathrm{v}} \,\&\, \mathfrak{a}^2_{\mathrm{vl}})^{\mathrm{N}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
43	$0^-, 1^+$	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, (A^2_{\mathrm{v}} \,\&\, A^0_{\mathrm{v}})^{\mathrm{N}}\vert(A^2_{\mathrm{v}} \,\&\, \mathfrak{a}^2_{\mathrm{vl}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
44	$1^-, 2^-$	$\{A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^2_{\mathrm{v}} \,\&\, A^0_{\mathrm{v}})^{\mathrm{N}}\vert(A^2_{\mathrm{v}} \,\&\, \mathfrak{s}^2_{\mathrm{vl}})^{\mathrm{N}}\vert(A^2_{\mathrm{v}} \,\&\, \mathfrak{a}^2_{\mathrm{vl}})^{\mathrm{N}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
45	$1^+, 2^-$	$\{A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, (A^2_{\mathrm{v}} \,\&\, A^0_{\mathrm{v}})^{\mathrm{N}}\vert(A^2_{\mathrm{v}} \,\&\, \mathfrak{a}^2_{\mathrm{vl}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
46	$0^-, 2^-$	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, (A^0 \,\&\, A^0)^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
47	1^-	$\{A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^2_{\mathrm{v}} \,\&\, A^0_{\mathrm{v}})^{\mathrm{N}}\vert(A^2_{\mathrm{v}} \,\&\, \mathfrak{s}^2_{\mathrm{vl}})^{\mathrm{N}}\vert(A^2_{\mathrm{v}} \,\&\, \mathfrak{a}^2_{\mathrm{vl}})^{\mathrm{N}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
48	1^+	$\{A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, (A^2_{\mathrm{v}} \,\&\, A^0_{\mathrm{v}})^{\mathrm{N}}\vert(A^2_{\mathrm{v}} \,\&\, \mathfrak{a}^2_{\mathrm{vl}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
49	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
50	2^-	$\{A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^0 \,\&\, A^0)^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
51	2^-	$\{A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, (A^0 \,\&\, A^0)^{\mathrm{N}}\vert(A^0 \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
52	2^-	$\{A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
53	2^-	$\{A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^0)^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
54	2^-	$\{A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, (A^\infty \,\&\, A^0)^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
55	2^-	$\{A^0, \times, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
56	2^-	$\{\times, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
57	2^-	$\{A^2_{\mathrm{l}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{s}^0_{\mathrm{l}})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$
58	2^-	$\{A^0, A^2_{\mathrm{l}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, (A^\infty \,\&\, A^{-2})^{\mathrm{N}}\vert(A^\infty \,\&\, \mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\}$

violated by the propagator in any sector individually. This occurs since the high-energy asymptotic behavior is determined by the term(s) with the highest power, unless they cancel out, but the SPO decomposition guarantees that such cancellations cannot happen if $k^2 \neq 0$, which is the case we are considering here. Thus, Case 8 as a whole violates the original PCR criterion in [2] because of the nature of the 0^+ sector, whereas one finds that it satisfies our alternative PCR criterion, and is hence listed in Tables III and IV.

We now explain why this does not, in fact, lead to a contradiction. If one chooses to keep only the first column and row in (7), the resulting b^{-1} matrix is clearly

$$b^{-1}(0^+) = \left(\frac{1}{2t_3}\right), \tag{8}$$

so the field in this sector is not propagating, and the corresponding propagator is $\sim k^0$ at high energy. The key point, however, is that there is no dynamical term in the Lagrangian for the field corresponding to (8). Thus, one can integrate out this nonpropagating field in the path integral, which is equivalent to substituting for it in the Lagrangian using its classical equation of motion obtained by varying the nonpropagating field. This is most transparently achieved by first introducing polarization basis vectors to decompose the fields and the SPOs, as discussed in Appendix A. One then expands the fields in terms of these basis vectors,

$$|A\rangle = \sum_{J,P,i,m} P\bar{A}_{i,J^P,m}|i, J^P, m\rangle, \tag{9}$$

from which one obtains the relation

$$\hat{P}_{ji}(J^P)|A\rangle = \bar{A}_{i,J^P,m}|j, J^P, m\rangle. \tag{10}$$

The Lagrangian corresponding to the 0^+ sector then becomes

$$\mathcal{L}(0^+) = t_3\bar{A}^2_{1,0^+,0}, \tag{11}$$

and the equation of motion is simply $\bar{A}_{1,0^+,0} = 0$, so one can simply ignore this sector. One might alternatively use the Lagrangian containing the source current here, so that the equation of motion becomes $2t_3\bar{A}_{1,0^+,0} = \bar{j}_{1,0^+,0}$, where $\bar{j}_{1,0^+,0}$ is appropriate expansion of the source current in the polarization. Since we are considering only free-field theories, however, the source currents can themselves be due only to the gauge fields and thus at least quadratic. Hence, these source currents can only affect the fields to the next order, so we can neglect them in the linearized Lagrangian.

The 1^- sector of Case 8 can also contain nonpropagating fields. The a matrix for this sector is

$$\begin{array}{cc} & \begin{array}{cccc} A & A & \mathfrak{s} & \mathfrak{a} \end{array} \\ a(1^-) = 2 & \begin{pmatrix} 3k^2(r_1+r_5)+2t_3 & \sqrt{2}t_3 & -i\sqrt{2}kt_3 & i\sqrt{2}kt_3 \\ \sqrt{2}t_3 & t_3 & -ikt_3 & ikt_3 \\ i\sqrt{2}kt_3 & ikt_3 & k^2t_3 & -k^2t_3 \\ -i\sqrt{2}kt_3 & -ikt_3 & -k^2t_3 & k^2t_3 \end{pmatrix}, \end{array}$$

which is singular as a result of gauge invariances. One may render the matrix nonsingular and thereby fix the gauge by, for example, choosing the first two rows and columns to form the corresponding b matrix, in which case the sector contains a propagating A particle and a nonpropagating A particle with some mixing term. The resulting determinant is

$$\det[b(1^-)] = \frac{4}{3}(r_1 + r_5)t_3k^2, \tag{12}$$

so there can only be massless modes in this sector. Using the expansion (10) to reconstruct the Lagrangian corresponding to the 1^- sector, one obtains

$$\mathcal{L}(1^-) = -\sum_{m=-1}^{1}\{\bar{A}_{1,1^-,m}[-3(r_1 + r_5)\partial^2 + 2t_3]\bar{A}_{1,1^-,m} + 2\sqrt{2}t_3\bar{A}_{1,1^-,m}\bar{A}_{2,1^-,m} + t_3\bar{A}^2_{2,1^-,m}\}. \tag{13}$$

Hence, it is clear that there is a propagating $\bar{A}_{1,1^-,m}$ field that is mixed with a $\bar{A}_{2,1^-,m}$ field without a dynamical term. One can thus integrate out the latter field using its classical equation of motion,

$$\bar{A}_{2,1^-,m} = -\sqrt{2}\bar{A}_{1,1^-,m}, \tag{14}$$

and the Lagrangian becomes

$$\mathcal{L}(1^-) = -\sum_{m=-1}^{1}\{\bar{A}_{1,1^-,m}[-3(r_1 + r_5)\partial^2]\bar{A}_{1,1^-,m}\}. \tag{15}$$

This is consistent with there being no massive mode in this sector. Furthermore, one finds that the effect of integrating out the nonpropagating fields in the 0^+ and 1^- sectors in Case 8 is the same as setting t_3 to zero, and all the b matrices become exactly the same as those of Case 9. Hence, at least in the free-field case we are considering, in which the gauge fields do not couple to external matter fields, Case 8 and 9 are actually describing the same theory. Moreover, since Case 9 may be shown to satisfy Sezgin's original PCR criterion in [2], there is thus no contradiction in Case 8 satisfying our alternative PCR criterion. Indeed, the alternative criterion allows us to identify Case 8 as PCR, which would be missed using the original PCR criterion.

For all Cases 1–58, one may similarly check whether, after integrating out the nonpropagating fields, the remaining fields are consistent with the particle contents that their determinants of b matrices indicate. Because all the b matrices containing nonpropagating terms in these cases are in the form of (6), one can perform this check by examining only all the "special cases" of the form (6) (including the critical cases and those with the parameters making any of the elements zero). We find that all of them are consistent.

Moreover, as one might expect, one may show that similar equivalences as Case 8 and 9 exist between other cases. For example, one may further demonstrate in the manner outlined above that: Case 2 is equivalent to Case 1; Cases 12, 14, and 15 are equivalent to Case 10; Case 16 is equivalent to Case 11; Case 25 is equivalent to Case 26; Case 29 is equivalent to Case 30; Case 37 is equivalent to Case 35; and Case 41 is equivalent to Case 27. Unfortunately, it is not so straightforward to establish the equivalences among the other cases. For the critical cases we do not list in this paper, we anticipate that there will similarly be some groups of equivalent cases in the above sense, provided they do not couple to external matter fields, so that one may simplify the "tree" of critical cases. We leave this analysis for future work. Nonetheless, we do find that after integrating out all the nonpropagating fields in Cases 1–58, all the resulting theories satisfy the original PCR condition. Hence, allowing for nonpropagating fields does not violate this criterion in practice.

In conclusion, we have found 48 further critical cases of PGT^+ that are both PCR and free of ghosts and tachyons. This is achieved by extending the range of critical cases considered beyond those investigated in [1], which previously identified 10 such theories. In future work, we plan to investigate all these theories further, but especially those that possess massless propagating particles, by considering their phenomenology in the context both of cosmological and compact object solutions. Note that while a theory may pass our PCR criterion, this is no guarantee that the theory is renormalizable, and this would take independent investigation and the inclusion of interactions. Indeed, it is shown in [7,8] that linearizing a theory can change its structure qualitatively, so that the degrees of freedom and gauge invariances may differ. One must therefore perform a full nonlinear analysis to determine whether this is the case for the theories considered here. We have also clarified the role played by nonpropagating modes in determining whether a theory is PCR. We illustrate this issue further in Appendix B, where we demonstrate the methods used in this paper in the more familiar and much simpler cases of the Proca and Stueckelberg theories for a massive spin-1 particle.

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APPENDIX A: POLARIZATION BASIS VECTORS

Assuming $k^A = (k^0, 0, 0, k^3)$, we define the polarization basis vectors for four-vectors as

$$\epsilon^A_{(1^-,1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}, \qquad \epsilon^A_{(1^-,-1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\i\\0\end{pmatrix},$$

$$\epsilon^A_{(1^-,0)} = \frac{1}{k}\begin{pmatrix}k^3\\0\\0\\k^0\end{pmatrix}, \qquad \epsilon^A_{(0^+,0)} = \frac{1}{k}\begin{pmatrix}k^0\\0\\0\\k^3\end{pmatrix}. \tag{A1}$$

The basis vectors satisfy the orthonormal and completeness conditions,

$$\epsilon^{*A}_{(J_1^{P_1},m_1)}\epsilon_{A,(J_2^{P_2},m_2)} = P_1\delta_{J_1,J_2}\delta_{P_1,P_2}\delta_{m_1,m_2}, \tag{A2}$$

$$\sum_{J,P,m} P\epsilon^A_{(J^P,m)}\epsilon^*_{B,(J^P,m)} = \delta^A_B. \tag{A3}$$

For the higher rank tensors, we can apply the addition rules for angular momentum. For example, a (2,0)-tensor f^{AB} can be decomposed as

$$\begin{aligned} f^{AB} &\in (\mathbf{0}^+ \oplus \mathbf{1}^-) \otimes (\mathbf{0}^+ \oplus \mathbf{1}^-) \\ &= (\mathbf{0}^+ \otimes \mathbf{0}^+) \oplus (\mathbf{0}^+ \otimes \mathbf{1}^-) \oplus (\mathbf{1}^- \otimes \mathbf{0}^+) \oplus (\mathbf{1}^- \otimes \mathbf{1}^-) \\ &= \mathbf{0}^+ \oplus \mathbf{1}^- \oplus \mathbf{1}^- \oplus (\mathbf{0}^+ \oplus \mathbf{1}^+ \oplus \mathbf{2}^+). \end{aligned} \tag{A4}$$

The polarization basis is obtained using Clebsch-Gordan coefficients.[2] For example, some basis elements $\epsilon^{AB}_{(J_1^{P_1},J_2^{P_2},J'^{P'},m_{J'})}$ for $J_1^{P_1} \otimes J_2^{P_2}$ are

$$\begin{aligned} \epsilon^{AB}_{(1^-,1^-,2^+,+2)} &= \epsilon^A_{(1^-,1)} \otimes \epsilon^B_{(1^-,1)}, \\ \epsilon^{AB}_{(1^-,1^-,2^+,+1)} &= \frac{1}{\sqrt{2}}(\epsilon^A_{(1^-,1)} \otimes \epsilon^B_{(1^-,0)} + \epsilon^A_{(1^-,0)} \otimes \epsilon^B_{(1^-,1)}). \end{aligned} \tag{A5}$$

Moreover, one can decompose any (2,0) tensor into $f^{AB} = \mathfrak{s}^{AB} + \mathfrak{a}^{AB}$, where $\mathfrak{s}$ is symmetric and $\mathfrak{a}$ is antisymmetric. One observes from the Clebsch-Gordan coefficients table that the $\mathbf{2}^+$ and $\mathbf{0}^+$ sectors are symmetric in A and B, whereas the $\mathbf{1}^+$ sector is antisymmetric. One may thus make a linear combination of the two $\mathbf{1}^-$ sectors to obtain a symmetric sector and an antisymmetric sector,

$$\epsilon^{AB}_{(\mathtt{sym},1^-,m)} \equiv \frac{1}{\sqrt{2}}(\epsilon^{AB}_{(0^+,1^-,1^-,m)} + \epsilon^{AB}_{(1^-,0^+,1^-,m)}), \tag{A6}$$

$$\epsilon^{AB}_{(\mathtt{ant},1^-,m)} \equiv \frac{1}{\sqrt{2}}(\epsilon^{AB}_{(0^+,1^-,1^-,m)} - \epsilon^{AB}_{(1^-,0^+,1^-,m)}). \tag{A7}$$

Hence, we can conclude that the symmetric part of (A4) is $\mathbf{2}^+ \oplus \mathbf{1}^- \oplus \mathbf{0}^+ \oplus \mathbf{0}^+$, which has $5+3+1+1=10$ degrees of freedom, and the antisymmetric part is $\mathbf{1}^+ \oplus \mathbf{1}^-$, which has $3+3=6$ degrees of freedom, all as expected.

One can similarly decompose the A^{ABC} fields, which are antisymmetric on A and B, into

$$\begin{aligned} A^{ABC} &\in (\mathbf{1}^+ \oplus \mathbf{1}^-) \otimes (\mathbf{0}^+ \oplus \mathbf{1}^-) \\ &= \mathbf{1}^+ \oplus (\mathbf{0}^- \oplus \mathbf{1}^- \oplus \mathbf{2}^-) \oplus \mathbf{1}^- \oplus (\mathbf{0}^+ \oplus \mathbf{1}^+ \oplus \mathbf{2}^+). \\ &= \mathbf{0}^- \oplus \mathbf{0}^+ \oplus 2(\mathbf{1}^-) \oplus 2(\mathbf{1}^+) \oplus \mathbf{2}^- \oplus \mathbf{2}^+, \end{aligned}$$

[2] We adopt the notation of the Particle Data Group, which can be found at http://pdg.lbl.gov/2008/reviews/clebrpp.pdf.

for which the basis is straightforwardly constructed following an analogous approach to that illustrated above.

The bases for higher rank tensors satisfy similar orthonormality and completeness conditions to (A2) and (A3),

$$\epsilon^{*\alpha}_{(i_1,J_1^{P_1},m_1)}\epsilon_{\alpha,(i_2,J_2^{P_2},m_2)} = P_1\delta_{i_1,i_2}\delta_{J_1,J_2}\delta_{P_1,P_2}\delta_{m_1,m_2}, \quad \text{(A8)}$$

$$\sum_{i,j,P,m}(P\epsilon^{\alpha}_{(i,J^P,m)}\epsilon^{*}_{\beta,(i,J^P,m)}) = \mathbb{I}^{\alpha}_{\beta}, \quad \text{(A9)}$$

where i is the label of the basis in the spin sector J^P, as there might be more than one basis in a sector. The α and β indices are shorthand for some generic indices, such as $\alpha = A_1A_2\ldots A_n$.

We can write the basis vectors together with its corresponding column vector $\mathbf{e}_a$ indicating the field (see (10) in [1]) in bra-ket notation $|i, J^P, m\rangle$, and the SPOs in [1] are related with those polarization basis vectors by

$$\hat{P}_{ij}(J^P) = \sum_m |i, J^P, m\rangle\,\langle j, J^P, m|. \quad \text{(A10)}$$

Note that the bras and kets here do not denote a quantum state, but are used merely to denote the field decomposition in a straightforward manner. We are taking inspiration from [9,10] in this section.

APPENDIX B: PROCA AND STUECKELBERG THEORIES

In this Appendix, we illustrate the methods used in this paper in the context of the more familiar and much simpler Proca and Stueckelberg theories.

Proca theory contains a massive vector field B_μ and has the free-field Lagrangian,

$$\mathcal{L}_{\mathrm{Pr}} = -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)(\partial^\mu B^\nu - \partial^\nu B^\mu) + \frac{1}{2}m^2 B_\mu B^\mu, \quad \text{(B1)}$$

with $m > 0$, which has no gauge freedoms. The corresponding SPOs are

$$\mathsf{P}(0^+) = \begin{array}{c} \\ B^*_\rho \end{array}\begin{array}{c} B_\mu \\ (\Omega_{\mu\rho}) \end{array}, \qquad \mathsf{P}(1^-) = \begin{array}{c} \\ B^*_\rho \end{array}\begin{array}{c} B_\mu \\ (\Theta_{\mu\rho}) \end{array}, \quad \text{(B2)}$$

where $\Omega^{\mu\rho} = k^\mu k^\rho/k^2$ and $\Theta^{\mu\rho} = \eta^{\mu\rho} - k^\mu k^\rho/k^2$. The a matrices of the theory are

$$a(0^+) = \begin{array}{c} \\ B^*_\mu \end{array}\begin{array}{c} B_\mu \\ (m^2) \end{array}, \qquad a(1^-) = \begin{array}{c} \\ B^*_\mu \end{array}\begin{array}{c} B_\mu \\ (-k^2 + m^2) \end{array}, \quad \text{(B3)}$$

which are identical to the b matrices because there are no gauge invariances and source constraints. Therefore, the 0^+ sector is nonpropagating and the 1^- sector corresponds to a k^{-2} propagator. Thus, Proca theory satisfies the alternative PCR condition in [1], and hence we classify it as PCR.

Conversely, Proca theory clearly violates Sezgin's original PCR condition in [2]. Indeed, Proca theory is generally considered to be non-PCR in the literature, because the propagator is

$$D(k)_{\mu\nu} = \frac{\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}}{k^2 - m^2}, \quad \text{(B4)}$$

so some components of it become $\sim k^0$ when $k^2 \to \infty$ and the offending term $k_\mu k_\nu$ cannot be eliminated by the renormalization procedure [11]. Using the polarization basis method mentioned in the main text, however, we can integrate out the nonpropagating 0^+ part. The free Lagrangian then becomes $\mathcal{L}_{\mathrm{Pr}}$ with the condition $\partial^\mu B_\mu = 0$, and the resulting propagator goes as k^{-2}, so the theory is PCR.

One may gain some insight into this apparent contradiction by noting that Proca theory may be considered as a gauge-fixed version of a gauge theory, namely the Stueckelberg theory, for which the Lagrangian is [12–14]

$$\mathcal{L}_{\mathrm{St}} = -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)(\partial^\mu B^\nu - \partial^\nu B^\mu) + \frac{1}{2}m^2 B_\mu B^\mu + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + m\phi\partial_\mu B^\mu \quad \text{(B5)}$$

and which possesses the gauge invariance,

$$B'_\mu = B_\mu + \partial_\mu\Lambda, \qquad \phi' = \phi + m\Lambda. \quad \text{(B6)}$$

The nonzero a matrices are

$$a(0^+) = \begin{array}{c} \\ \phi^* \\ B^*_\mu \end{array}\begin{array}{cc} \phi & B_\mu \\ \multicolumn{2}{c}{\begin{pmatrix} k^2 & -ikm \\ ikm & m^2 \end{pmatrix}} \end{array}, \quad \text{(B7)}$$

$$a(1^-) = \begin{array}{c} \\ B^*_\mu \end{array}\begin{array}{c} B_\mu \\ (-k^2 + m^2) \end{array}, \quad \text{(B8)}$$

and the corresponding SPOs are

$$\mathsf{P}(0^+) = \begin{array}{c} \\ \phi^* \\ B^*_\rho \end{array}\begin{array}{cc} \phi & B_\mu \\ \multicolumn{2}{c}{\begin{pmatrix} 1 & \tilde{k}_\mu \\ \tilde{k}_\rho & \Omega_{\mu\rho} \end{pmatrix}} \end{array}, \qquad \mathsf{P}(1^-) = \begin{array}{c} \\ B^*_\rho \end{array}\begin{array}{c} B_\mu \\ (\Theta_{\mu\rho}) \end{array}, \quad \text{(B9)}$$

where $\tilde{k}_\mu = k_\mu/\sqrt{k^2}$. As might be expected, the matrix $a(0^+)$ is singular, with rank one, and so we can choose to keep either the ϕ column/row or the B column/row. If we choose to keep B, then one recovers Proca's theory. If we instead choose to keep ϕ, then the b^{-1} matrices all go as $\sim k^{-2}$ in the high-energy limit and the theory thus satisfies the original PCR condition. Hence, Stueckelberg theory is PCR, and so Proca theory must also be PCR, since the two theories are physically equivalent. Thus, our alternative PCR criterion succeeds in identifying Proca theory as being PCR, whereas the theory violates the original PCR criterion.

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