Wormhole solutions in modified Brans-Dicke theory

Eleftherios Papantonopoulos[*] and Christoforos Vlachos[†]

Physics Division, National Technical University of Athens, 15780 Zografou Campus, Athens, Greece

We consider a modified Brans-Dicke theory in which, instead of the usual Brans-Dicke parameter, a new dimensionful parameter appears that modifies the kinetic term of the scalar field coupled to gravity. Solving the coupled Einstein-Klein-Gordon equations we find new spherically symmetric solutions. Depending on the choices of the parameters these solutions reduce to the Schwarzschild solution of general relativity, and they give new wormhole solutions that depend on the new parameter.

I. INTRODUCTION

Scalar fields play an important role in the general relativity (GR) on short and large distances. On short distances they dress the local black hole solutions with scalar hair and provide wormhole solutions, while on large distances they describe the early inflationary universe and also its late-times cosmic evolution. In an attempt to provide a viable theory of gravity and to cure certain inconsistencies of GR, scalar-tensor theories were introduced. As is well known, Brans-Dicke (BD) theory [1] is one of the first scalar-tensor gravity theories that modifies GR in a viable way and respects Mach's principle and weak equivalence principle (WEP). In this theory there is an effective Newtonian gravitational constant G that is the inverse of the scalar field, $G \sim \frac{1}{\phi}$. It is characterized by a new dimensionless coupling constant ω, large values of which mean a significant contribution from the tensor part, while the scalar field contribution is important for small values. GR is recovered in the limit $\omega \to \infty$.

It is interesting to note that BD theory appears in supergravity models such as in string theory at low energies or in the Kaluza-Klein theories after a dimensional reduction process [2]. These theories yield the correct Newtonian weak-field limit, but care should be taken when one studies these theories and compares their predictions with GR. In general, scalar fields, depending on their coupling to gravity, mediate fifth forces. In the case of BD solar system measurements of post-Newtonian corrections require that ω is larger than a few thousands [3]. Therefore in these theories scalar fields should accommodate a mechanism to suppress the scalar interaction on small scales. There are various screening mechanisms to suppress scalar interactions on small scales. One of the basic screening mechanisms is the Vainshtein mechanism [4] which was

developed for the massive gravity (for an extensive review on the Vainshtein mechanism in massive gravity, see [5]).

On large scales, ω gets substantially lower values in a model dependent way [6], from cosmological observations. On the other side, the gravitational coupling may depend on the scale [7], having different values at local and at cosmological scales. In this case, ω can be smaller at cosmological scales giving deviations from GR, while agreement with local tests is preserved. Special solutions on BD cosmology have been given in [8] which are generalizations of the dust solution first given by the Brans-Dicke theory. In [9] special radiation solutions for spatially curved space were found and in [10] vacuum solutions were given. In [11] the general spatially flat cosmological solution was obtained in parametric form for any barotropic perfect fluid. In [12] the general stiff and radiation solutions for all kinds of spatial curvature are found. Similar solutions were found in [13]. In [14] exact solutions were found in the presence of the cosmological constant. Other works with exact solutions are given in [15]. In [16] it was argued that the initial singularity in the BD theory can be resolved. The BD theory is used to solve some problems of the inflationary scenario [17]. Also a solution to the "graceful exit" problem of inflation [18] was first obtained in BD without fine-tuning.

On small scales after the introduction of the theory, Brans found four families of static spherically symmetric solutions [19]. For a long time many authors claimed that these solutions can describe nontrivial black holes different from those of GR. However, it was proven by Hawking [20] that all those spherically symmetric black hole solutions are the same as in GR. This result was further extended to scalar-tensor theories [21] and to compact objects in the presence of a cosmological constant [22]. In [23] it was claimed that black hole solutions in the BD theory were found violating the WEC. However, in [24] it was shown that their solutions describe either wormholes or naked singularities. It was further proved in [25] that the static

---
[*]lpapa@central.ntua.gr
[†]cvlach@mail.ntua.gr

064025-1

spherically symmetric solutions of BD theory describe either wormholes or naked singularities. A more extensive study was performed in [26] where it was shown that the static and spherically symmetric BD solutions of scalar-tensor gravity, analyzed in both the Jordan and the Einstein conformal frames, describe wormholes, naked singularities, or the Schwarzschild solution. Thus, they do not describe black hole solutions besides those in GR.

If an electromagnetic field is introduced, then in four dimensions the local solutions reduce to the Reissner-Nordström black hole solution with a constant scalar field, as it was proven in [20] using the WEC. However, in higher dimensions the vacuum Brans-Dicke-Maxwell theory has black hole solutions [27]. This is a consequence of the presence of the electromagnetic field in the scalar field equation, and in this way it can be considered as a source of a nontrivial scalar field.

Recently a modification of BD gravity theory was proposed [28] in which the scalar field, in addition to its coupling to the metric, is also coupled to matter [29]. This coupling, except for the BD parameter ω, introduces another parameter in the coupling of the scalar field to gravity. This coupling is a new scale in the theory and modifies the matter content of the BD theory; this scale also appears in the vacuum equations of the modified BD theory. In [30] the cosmological implications of an extended BD theory presented in [29] was discussed. The new mass scale introduced in the theory modifies the Friedmann equations with field-dependent corrected kinetic terms. In the general solutions of a radiation universe it was found that there are branches with complete removal of the initial singularity, while at the same time a transient accelerating period can occur within deceleration. Entropy production is also possible in the early universe. In the dust era, late-times acceleration has been found numerically in agreement with the correct behavior of the density parameters and the dark energy equation of state, while the gravitational constant has only a slight variation over a large redshift interval in agreement with observational bounds.

Motivated by the cosmological results of introducing another coupling of the scalar field to gravity in the BD theory, in this work we will investigate what are the effects of the new coupling parameter on small scales. As we have already discussed, in BD theory all the static spherically symmetric solutions of the theory, except the Schwarzschild solution, describe either wormholes or naked singularities [31]. To evade this problem one has to introduce a potential, and in this case nontrivial black hole solutions can be obtained [23,32]. Therefore, it is interesting to see whether the introduction of the new coupling parameter, which does not introduce new extra matter in the BD theory in the form of a potential and cannot be absorbed in the redefinition of ω, leads to new spherically symmetric black hole or wormhole solutions.

For this, after reviewing in Sec. II the BD vacuum solutions, we solve the Einstein and scalar equations of this modified BD gravity theory in Sec. III. In this section we perform a detailed investigation of the solutions. By varying this new coupling we find two branches of solutions that describe naked singularities and generate new wormhole geometries whose behavior depends on the coupling of the scalar field to matter. We did not find new black hole solutions, and therefore our solutions are in accordance with the two previously stated theorems [20,25]. As is well known, the wormhole solutions violate the WECs. For this reason we also study the WEC and find that they are violated. Finally, in Sec. IV are our conclusions.

II. BRANS-DICKE VACUUM SOLUTIONS

In this section we will review the local solutions of the BD theory mainly following the work of [27]. In the $D(\geq 4)$ dimensions, the action of the vacuum BD theory is given by

$$I = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left(\phi R - \frac{\omega}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right). \quad (2.1)$$

In this Jordan frame version of BD theory, test particles have constant rest mass and move along the geodesics. That is, matter fields are coupled to gravity only via the metric. Varying (2.1) yields the equations of motion,

$$\phi G_{\mu\nu} \equiv \phi \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right)$$
$$= \frac{\omega}{\phi} \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right]$$
$$+ \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi, \quad (2.2)$$

$$\nabla^2 \phi = 0, \quad (2.3)$$

where $d = D - 3$. Solving (2.2) and (2.3) we consider the following conformal transformation:

$$g_{\mu\nu} = \Omega^2 \bar{g}_{\mu\nu}, \quad (2.4)$$

with

$$\Omega^{-(d+1)} = \phi \quad (2.5)$$

and

$$\bar{\phi} = \sqrt{2a} \int^\phi \frac{d\phi}{\phi} = \sqrt{2a} \ln \phi, \qquad a = \frac{d+2}{d+1} + \omega. \quad (2.6)$$

The BD theory (2.1) can be transformed into the Einstein frame version of BD theory with a minimally coupled scalar field ($\bar{\phi}$)

$$\bar{I} = \frac{1}{16\pi} \int d^D x \sqrt{-\bar{g}} \left[\bar{R} - \frac{1}{2} (\bar{\nabla} \bar{\phi})^2 \right], \qquad (2.7)$$

where $\bar{R}$ and $\bar{\nabla}$ are the scalar curvature and covariant differentiation in the new metric $\bar{g}_{\mu\nu}$, respectively. We note the following:

(i) Relation (2.6) implies $a > 0$ ($\omega > -\frac{d+2}{d+1}$), and one has $\bar{\phi} = 0$ at spacelike infinity.

(ii) The action under the conformal transformation gets a simpler form of a minimally coupled scalar field.

(iii) The Brans-Dicke theory (2.1) is equivalent to the theory (2.7) up to a conformal transformation. However, note that in the Einstein frame, a test particle will take the variable rest mass with spacetime and is no longer going to move along the geodesics. This physical inequivalence can be understood from the conformal transformation of the metric (2.4) and (2.5). The conformal transformation depends on the scalar field ϕ which parametrizes the matter of the theory. Therefore the physical behavior of the theory can be understood only if the coupling to matter is specified.

Hence, one can argue that the two theories are equivalent from a mathematical point of view but not from a physical one.

Varying the action (2.7) we can obtain the equations of motion that are connected with the equations of motion of (2.2) and (2.3) through the relation

$$z(g_{\mu\nu}, \phi) = \left(e^{-\frac{2}{(d+1)\sqrt{2a}} \bar{\phi}} \bar{g}_{\mu\nu}, e^{\frac{1}{\sqrt{2a}} \bar{\phi}} \right). \qquad (2.8)$$

Introducing isotropic coordinates [33]

$$d\bar{s}^2 = -e^f dt^2 + e^{-h} (d\rho^2 + \rho^2 d\Omega_{d+1}^2), \qquad (2.9)$$

in the D-dimensional vacuum BD theory, using (2.8), we can obtain its solution,

$$ds^2 = \Omega^2 d\bar{s}^2 = \left(\frac{\rho^d + \rho_o^d}{\rho^d - \rho_o^d} \right)^{\frac{2}{d+1} \left[\frac{(d+1)(1-\gamma^2)}{ad} \right]^{1/2}} d\bar{s}^2, \qquad (2.10)$$

$$\phi = \left(\frac{\rho^d - \rho_o^d}{\rho^d + \rho_o^d} \right)^{\left[\frac{(d+1)(1-\gamma^2)}{ad} \right]^{1/2}}, \qquad (2.11)$$

where γ is a constant and $d\bar{s}^2$ is given by Eq. (2.9). It is easy to show that the solution (2.10) is asymptotically flat and the point $\rho = \rho_o$ corresponds to a naked singularity still. This can be found from calculating the scalar curvature of the solution (2.10) through the relation,

$$R = \Omega^{-2} \bar{R} - 2(d+2) \Omega^{-3} \bar{g}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \Omega$$
$$- (d+2)(d-1) \Omega^{-4} \bar{g}^{\mu\nu} \bar{\nabla}_\mu \Omega \bar{\nabla}_\nu \Omega, \qquad (2.12)$$

and show that it diverges at zero areal radius. Again we observe the following:

(i) When $\gamma = 1$, the solution (2.10) is reduced to the D-dimensional Schwarzschild solution with the constant scalar field ($\phi = 1$). In that case, the BD theory degenerates into the Einstein theory of gravitation.

(ii) The scalar ϕ in the BD theory belongs to the region $\phi \in (0, 1]$. From the action (2.7) we can see that the equations of motion remain unchanged under the transformation: $\bar{\phi} \to -\bar{\phi}$. Thus, we can obtain another solution of the vacuum BD theory,

$$ds^2 = \left(\frac{\rho^d - \rho_o^d}{\rho^d + \rho_o^d} \right)^{\frac{2}{d+1} \left[\frac{(d+1)(1-\gamma^2)}{ad} \right]^{1/2}} d\bar{s}^2, \qquad (2.13)$$

$$\phi = \left(\frac{\rho^d + \rho_o^d}{\rho^d - \rho_o^d} \right)^{\left[\frac{(d+1)(1-\gamma^2)}{ad} \right]^{1/2}}, \qquad (2.14)$$

where $d\bar{s}^2$ is still given by Eq. (2.9). In this case, the scalar field ϕ takes values in the region $[1, \infty)$. But the spacetime is still an asymptotically flat region and the point $\rho = \rho_o$ is a curvature singularity unless $\gamma = 1$. When $\gamma = 1$, the scalar field is a constant and the solution (2.13) is the D-dimensional Schwarzschild solution. We note that $\phi = 0$ corresponds to infinite gravitational coupling while $\phi = \infty$ corresponds to zero gravitational coupling, so ϕ should be allowed to diverge or vanish only at singularities.

The metric (2.10) is actually a Brans Class 1 solution since if we substitute $D = 4$, then $d = 1$ and $\alpha = \frac{3+2\omega}{2}$, which is related with our λ parameter by $\frac{1}{\alpha} = \frac{1}{\omega+3/2} = \lambda$ (see Sec. III). We substitute all these in the line element, together with the form of $d\bar{s}^2$, and we obtain

$$ds^2 = -\left(\frac{1 - \rho_o/\rho}{1 + \rho_o/\rho} \right)^{\alpha(\lambda,\gamma)+2\gamma} dt^2$$
$$+ \left(1 - \frac{\rho_o^2}{\rho^2} \right)^2 \left(\frac{1 + \rho_o/\rho}{1 - \rho_o/\rho} \right)^{\alpha(\lambda,\gamma)+2\gamma} [d\rho^2 + \rho^2 d\Omega^2], \qquad (2.15)$$

which is exactly what we find in (3.66). The requirement that the scalar field $\bar{\phi}$, in the Einstein frame, is real implies $\alpha > 0$ ($\omega > -\frac{3}{2}$ in $D = 4$); hence, only Brans Class I solution can be obtained in the Jordan frame. However, there are three other classes of Brans solutions that correspond to $\omega < -\frac{3}{2}$ (for a detailed discussion on the Brans solutions see the review in [26]).

Therefore, again the black hole solution of the vacuum BD theory is the Schwarzschild solution with a constant scalar field.

III. MODIFIED BRANS-DICKE THEORY

We will consider a modified BD theory presented in [28] and described by the following equations:

$$G^\mu{}_\nu = \frac{8\pi}{\phi}(T^\mu{}_\nu + \mathcal{T}^\mu{}_\nu), \qquad (3.1)$$

$$
\begin{aligned}
T^\mu{}_\nu =& \frac{\phi}{2\lambda(\nu + 8\pi\phi^2)^2}\{2[(1+\lambda)\nu + 4\pi(2-3\lambda)\phi^2]\phi^{;\mu}\phi_{;\nu} \\
&- [(1+2\lambda)\nu + 4\pi(2-3\lambda)\phi^2]\delta^\mu{}_\nu\phi^{;\rho}\phi_{;\rho}\} \\
&+ \frac{\phi^2}{\nu + 8\pi\phi^2}(\phi^{;\mu}{}_{;\nu} - \delta^\mu{}_\nu\Box\phi),
\end{aligned}
\qquad (3.2)
$$

$$\Box\phi = 4\pi\lambda\mathcal{T}, \qquad (3.3)$$

$$\mathcal{T}^\mu{}_{\nu;\mu} = \frac{\nu}{\phi(\nu + 8\pi\phi^2)}\mathcal{T}^\mu{}_\nu\phi_{;\mu}, \qquad (3.4)$$

where $\mathcal{T}^\mu{}_\nu$ is the energy momentum tensor of matter (e.g., the standard matter radiation in the case of cosmology). We can see that matter couples directly to the scalar field ϕ. Since in the action of BD gravity the scalar field is nonminimally coupled to the curvature, the same mechanism could also lead to a coupling between the scalar and matter fields, as happens here. Various studies have analyzed the exchange of energy from ordered motion by entropy generation due to bulk viscosity. Of course, the parameters λ and ν should be such that the equivalence principle is not violated at the ranges that it has been tested. However, the new parameter ν violates the exact conservation of the matter energy-momentum tensor $\mathcal{T}^\mu{}_\nu$ in (3.4). Interesting cosmological results were obtained in [29] as we have already discussed. For $\nu = 0$ the system (3.1)–(3.4) reduces to the BD equations of motion (with unit velocity of light)

$$G^\mu{}_\nu = \frac{8\pi}{\phi}(T^\mu{}_\nu + \mathcal{T}^\mu{}_\nu), \qquad (3.5)$$

$$T^\mu{}_\nu = \frac{2-3\lambda}{16\pi\lambda\phi}\left(\phi^{;\mu}\phi_{;\nu} - \frac{1}{2}\delta^\mu{}_\nu\phi^{;\rho}\phi_{;\rho}\right) + \frac{1}{8\pi}(\phi^{;\mu}{}_{;\nu} - \delta^\mu{}_\nu\Box\phi), \qquad (3.6)$$

$$\Box\phi = 4\pi\lambda\mathcal{T}, \qquad (3.7)$$

$$\mathcal{T}^\mu{}_{\nu;\mu} = 0, \qquad (3.8)$$

which is described by the action

$$
\begin{aligned}
S_{\text{BD}} =& \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \frac{\omega_{\text{BD}}}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right) \\
&+ \int d^4x\sqrt{-g}L_m,
\end{aligned}
\qquad (3.9)
$$

where $L_m(g_{\kappa\lambda}, \Psi)$ is the matter Lagrangian depending on some extra fields Ψ. The parameter $\lambda \neq 0$ is related to the standard BD parameter $\omega_{\text{BD}} = \frac{2-3\lambda}{2\lambda}$.

As we discussed in the Introduction of this work, we are interested to see what is the effect of the new coupling ν on local spherically symmetric solutions of the vacuum BD theory without extra matter. Therefore we will study the modified BD theory with $\mathcal{T}^\mu{}_\nu = 0$. Although the extra matter vanishes, it leaves an impact on the vacuum equation (3.1) through the parameter ν, and this is the novel difference compared to the vacuum BD equation (3.5). This vacuum theory arises from the action [29]

$$
\begin{aligned}
S =& \frac{\eta}{2(8\pi)^{3/2}}\int d^4x\sqrt{-g}\left[\sqrt{|\nu + 8\pi\phi^2|}R\right. \\
&\left. - \frac{8\pi}{\lambda}\frac{\nu + 4\pi(2-3\lambda)\phi^2}{|\nu + 8\pi\phi^2|^{3/2}}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right],
\end{aligned}
\qquad (3.10)
$$

where $\eta = \text{sgn}(\phi)$. It is clear in the above action that the kinetic term of the scalar field is modified compared to the original Brans-Dicke theory, and it results in a modified energy momentum tensor for the scalar field.

In this work, we will consider the modified BD theory given by the action (3.10) and study spherically symmetric solutions of this theory. It is useful for the following analysis of spherically symmetric solutions to transform the action (3.10) to its canonical form. Following the discussion in Sec. II, consider the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}, \qquad (3.11)$$

where

$$\Omega = \left(\frac{|\nu + 8\pi\phi^2|}{8\pi}\right)^{\frac{1}{4}}, \qquad (3.12)$$

together with a field redefinition from the field $\phi(x)$ to the new field $\sigma(x)$ defined by

$$\frac{d\phi}{d\sigma} = \sqrt{\frac{|\lambda|}{16\pi}}\sqrt{|\nu + 8\pi\phi^2|}. \qquad (3.13)$$

The action (3.10) takes the form

$$S = \frac{\eta}{16\pi}\int d^4x\sqrt{-\tilde{g}}\left(\tilde{R} - \frac{1}{2}\epsilon\epsilon_\lambda\tilde{g}^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu}\right), \qquad (3.14)$$

where $\epsilon = \text{sgn}(\nu + 8\pi\phi^2)$ and $\epsilon_\lambda = \text{sgn}(\lambda)$. The Lagrangian (3.14) refers to the Einstein frame where the gravitational coupling is a true constant and the field σ behaves as a usual scalar field. In this work we are looking for black hole and wormhole solutions. The action (3.14) describes in the Einstein frame the kinetic energy of a scalar field coupled to gravity. We will look first for spherically

symmetric black hole solutions in this frame. For this reason we will assume $\epsilon\epsilon_\lambda > 0$ for the scalar field to have a physically propagating mode. This condition is a necessary condition in order not to have instabilities in our theory. However, as we will see in the Jordan frame, we will find wormhole solutions in which the energy conditions are violated [34] and ghostlike instabilities appear. Therefore, this condition does not guarantee the stability of the theory under perturbations. This behavior is not a surprise since stable wormholes can be supported only in higher-derivative theories of the beyond-Horndeski type [35–37], a class in which our theory (3.10) clearly does not belong.

We assume throughout that $\epsilon\epsilon_\lambda = 1$. For $\epsilon > 0$, the integration of Eq. (3.13) gives

$$\sigma = \sqrt{\frac{2}{|\lambda|}} \ln\left|4\pi\phi + \sqrt{2\pi}\sqrt{\nu + 8\pi\phi^2}\right|, \quad (3.15)$$

where an additive integration constant σ_0 has been absorbed into σ. Inversely,

$$\phi = \frac{s}{8\pi}\left(e^{\sqrt{\frac{|\lambda|}{2}}\sigma} - 2\pi\nu e^{-\sqrt{\frac{|\lambda|}{2}}\sigma}\right), \quad (3.16)$$

where $\quad s = \text{sgn}(4\pi\phi + \sqrt{2\pi}\sqrt{\nu + 8\pi\phi^2}) = \text{sgn}(e^{\sqrt{\frac{|\lambda|}{2}}\sigma} + 2\pi\nu e^{-\sqrt{\frac{|\lambda|}{2}}\sigma})$. The conformal factor Ω in terms of the new field σ takes the form

$$\Omega = \frac{1}{\sqrt{8\pi}}\left|e^{\sqrt{\frac{|\lambda|}{2}}\sigma} + 2\pi\nu e^{-\sqrt{\frac{|\lambda|}{2}}\sigma}\right|^{\frac{1}{2}}. \quad (3.17)$$

For the physically more interesting case with $\phi > 0$, the absolute value in (3.17) disappears.

For $\epsilon < 0$, the integration of Eq. (3.13) gives

$$\sigma = \sqrt{\frac{2}{|\lambda|}} \arcsin\left(\sqrt{\frac{8\pi}{|\nu|}}\phi\right), \quad (3.18)$$

where again an additive integration constant σ_0 has been absorbed into σ and it is $-\frac{\pi}{2} < \sqrt{\frac{|\lambda|}{2}}\sigma < \frac{\pi}{2}$. Inversely,

$$\phi = \sqrt{\frac{|\nu|}{8\pi}} \sin\left(\sqrt{\frac{|\lambda|}{2}}\sigma\right). \quad (3.19)$$

The conformal factor Ω in terms of the new field σ takes the form

$$\Omega = \left(\frac{|\nu|}{8\pi}\right)^{\frac{1}{4}}\left[\cos\left(\sqrt{\frac{|\lambda|}{2}}\sigma\right)\right]^{\frac{1}{2}}. \quad (3.20)$$

After the solution of the fields $\tilde{g}_{\mu\nu}, \sigma$ governed by the action (3.14) has been derived, the solution for the initial

fields $g_{\mu\nu}, \phi$ is found through Eqs. (3.11), (3.16), and (3.19) as functions of σ. The action (3.14) defines Einstein gravity minimally coupled to a scalar field whose equations of motion are

$$\tilde{G}_{\mu\nu} = \frac{1}{2}\sigma_{,\mu}\sigma_{,\nu} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{g}^{\kappa\lambda}\sigma_{,\kappa}\sigma_{,\lambda}, \quad (3.21)$$

$$\tilde{\Box}\sigma = 0. \quad (3.22)$$

The solution of this system of a minimally coupled scalar field to gravity, assuming spherical symmetry, has been found in [33]. In the Einstein frame, we consider a static spherically symmetric line element in isotropic coordinates

$$d\tilde{s}^2 = -e^f dt^2 + e^{-h}[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)], \quad (3.23)$$

where f and h are functions of the radial coordinate ρ (we keep the symbol r for the radius in the standard coordinates). Because of the symmetry it is also $\sigma(\rho)$. The solution of the system (3.21) and (3.22) is the following [33]:

$$\sigma = 2\sqrt{1-\gamma^2}\ln\frac{\rho-\rho_o}{\rho+\rho_o}, \quad (3.24)$$

$$e^f = \left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{2\gamma}, \quad (3.25)$$

$$e^{-h} = \left(1 - \frac{\rho_o^2}{\rho^2}\right)^2\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{2\gamma}, \quad (3.26)$$

where $\rho_o > 0, \gamma$ are integration constants. The reason we restrict ρ_o in this range of values will become clear in the next sections, where the limit to the Schwarzschild solution will be considered.

Let us express the quantity Ω as a function of ρ through Eq. (3.24). For $\epsilon > 0$ it is

$$\Omega = \frac{1}{\sqrt{8\pi}}\left|\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}} + 2\pi\nu\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}}\right|^{\frac{1}{2}}, \quad (3.27)$$

while for $\epsilon < 0$,

$$\Omega = \left(\frac{|\nu|}{8\pi}\right)^{\frac{1}{4}}\left[\cos\left(\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right]^{\frac{1}{2}}. \quad (3.28)$$

Note that in the limit $\rho \to \rho_o$, it is $\Omega \to \infty$ for $\epsilon > 0$, while Ω remains finite for $\epsilon < 0$.

Going back to the original frame we have in the isotropic coordinates

$$ds^2 = -\Omega^{-2}e^f dt^2 + \Omega^{-2}e^{-h}[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \quad (3.29)$$

The metric in the standard coordinates is obtained setting

$$r = \rho\Omega^{-1}e^{-\frac{h}{2}}, \tag{3.30}$$

and then

$$ds^2 = -\Omega^{-2}e^f dt^2 + \frac{r^2}{\rho^2}\frac{dr^2}{(\frac{dr}{d\rho})^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3.31}$$

Let us now take a closer look at the behavior of this metric for both branches.

A. Spherically symmetric solutions in the Jordan frame

We consider the line element in the Jordan frame

$$ds^2 = -\Omega^{-2}e^f dt^2 + \Omega^{-2}e^{-h}(d\rho^2 + \rho^2 d\Omega^2), \tag{3.32}$$

in isotropic coordinates (t, ρ, θ, ϕ), where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of the unit 2-sphere. The exponentials are given by (3.25) and (3.26) and the conformal factors by (3.27) and (3.28) for $\epsilon > 0$ and $\epsilon < 0$, respectively. It must be $\rho > \rho_o > 0$ and $0 \le \gamma^2 \le 1$. The $\lambda \ne 0$ parameter is related to the standard BD parameter by $\lambda = \frac{1}{\omega+3/2}$. Hence in our model, λ and ν are parameters of the theory and ρ_o and γ are the parameters of this specific family of solutions.

B. Branch $\epsilon < 0$

As previously stated, we require $\epsilon < 0$ and $\epsilon_\lambda < 0$ to avoid ghost solutions of the field σ. That is,

$$\epsilon = \text{sgn}(\nu + 8\pi\phi^2) < 0,$$

$$\epsilon_\lambda = \text{sgn}(\lambda) < 0.$$

Substituting (3.24) into (3.19) and using the preceding inequalities we can find that when $\nu < 0$ and $\rho > \rho_o\frac{e^K+1}{e^K-1}$, where $K = \frac{\pi}{2\sqrt{2|\lambda|(1-\gamma^2)}}$, the relation $\epsilon < 0$ is always satisfied. Basically, this is the range of ρ such that $-\frac{\pi}{2} < \sqrt{\frac{|\lambda|}{2}}\sigma < \frac{\pi}{2}$. Therefore, in the general case, we will only consider such values of ρ in the following analysis. Moreover, the condition $\epsilon_\lambda < 0$ is the equivalent to $\omega < -3/2$.

1. Scalar field

The Brans-Dicke scalar field is given by the relation

$$\phi = \sqrt{\frac{|\nu|}{8\pi}}\sin\left(\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right). \tag{3.33}$$

Note that ϕ becomes constant when $\gamma = \pm 1$, and the theory reduces to GR. Moreover, the scalar field vanishes in the limits $\rho \to \infty$ or $\lambda \to 0$ (i.e., $\omega \to \infty$), which means the effective gravitational constant diverges.

2. Metric components

The metric components in isotropic coordinates are given by

$$g_{tt} = -\left(\frac{8\pi}{|\nu|}\right)^{\frac{1}{2}}\sec\left(\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{2\gamma},$$

$$g_{\rho\rho} = \left(\frac{8\pi}{|\nu|}\right)^{\frac{1}{2}}\sec\left(\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\left(1-\frac{\rho_o^2}{\rho^2}\right)^2$$
$$\times \left(\frac{1+\rho_o/\rho}{1-\rho_o/\rho}\right)^{2\gamma}.$$

The appearance of the secant function prevents g_{tt} and $g_{\rho\rho}$ from vanishing; however, it causes divergences at the points that satisfy the equation $\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o} = -\frac{\pi}{2}$, i.e., $\rho = \rho_o\frac{e^K+1}{e^K-1}$, if $\gamma \ne \pm 1$.

It is rather easy to check the asymptotic behavior of our metric. A straightforward calculation gives

$$\lim_{\rho\to\infty} g_{tt} = -\sqrt{\frac{8\pi}{|\nu|}}, \qquad \lim_{\rho\to\infty} g_{\rho\rho} = \sqrt{\frac{8\pi}{|\nu|}}. \tag{3.34}$$

The fact that ν is a parameter of a theory and not a dynamic variable, means that we can absorb the above factors in the line element by redefining dt and $d\rho$. Thus, in this case the line element (3.32) with Ω given by (3.28), describes an asymptotically flat spacetime. Moreover, note that when $\nu = 0$ the line element diverges and the solution does not produce the corresponding solutions of BD. This is not surprising since setting $\nu = 0$ automatically corresponds to $\epsilon = \text{sgn}(8\pi\phi^2) > 0$, and that is why only the branch $\epsilon > 0$ has a correct limit to Brans-Dicke.

Let us now consider two particular cases.

If $\gamma = 1$, then

$$g_{tt} = -\left(\frac{8\pi}{|\nu|}\right)^{\frac{1}{2}}\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^2, \tag{3.35}$$

$$g_{\rho\rho} = \left(\frac{8\pi}{|\nu|}\right)^{\frac{1}{2}}\left(1+\frac{\rho_o}{\rho}\right)^4. \tag{3.36}$$

Now the range $\rho_o < \rho < \rho_o\frac{e^K+1}{e^K-1}$ is physically meaningful. Note that $g_{tt} \overset{\rho\to\rho_o}{\to} 0$ while it remains negative for $\rho \ne \rho_o$. The BD scalar becomes constant for $\gamma = 1$, and as can be seen from the forms of g_{tt} and $g_{\rho\rho}$, the solution reduces to the standard Schwarzschild metric in isotropic coordinates, with mass $M = 2\rho_o$. In the case $\gamma = -1$ one can find that

$$g_{tt} = -\left(\frac{8\pi}{|\nu|}\right)^{\frac{1}{2}}\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^2 \overset{\rho\to\rho_o}{\to} -\infty \qquad (3.37)$$

and

$$g_{\rho\rho} = \left(\frac{8\pi}{|\nu|}\right)^{\frac{1}{2}}\left(1-\frac{\rho_o}{\rho}\right)^4 \overset{\rho\to\rho_o}{\to} 0. \qquad (3.38)$$

The scalar field is again constant; thus the solution corresponds to Schwarzschild spacetime with a negative mass $M = -2\rho_o$.

3. Areal radius and Ricci scalar

Now we wish to analyze the behavior of the areal radius since it can give us extra information about the geometry

and also help us to deduce which ranges of the spatial coordinate ρ are physically meaningful. Basically, what we call areal radius is just the radial coordinate of the spherical (Schwarschild) coordinates and that is why we are going to denote it by r. On the other hand, the study of scalar quantities helps us to detect real spacetime singularities since they do not depend on our choice of coordinates. They are invariants, and thus, if we manage to find a point where a scalar curvature diverges, we know that it will correspond to a true spacetime singularity.

The areal radius is read off the line element (3.29)

$$r = \rho\Omega^{-1}e^{-\frac{h}{2}},$$

and in this case it is

$$r = \rho\Omega^{-1}e^{\frac{h}{2}} = \Omega^{-1}\frac{1}{\rho}\frac{(\rho+\rho_o)^{\gamma+1}}{(\rho-\rho_o)^{\gamma-1}}$$
$$= \left(\frac{|\nu|}{8\pi}\right)^{-1/4}\sec^{\frac{1}{2}}\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\frac{1}{\rho}(\rho+\rho_o)^{1+\gamma}(\rho-\rho_o)^{1-\gamma}, \qquad (3.39)$$

while its derivative is given by

$$\frac{dr}{d\rho} = \sqrt[4]{\frac{2\pi}{|\nu|}}\frac{1}{p^2}\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\gamma}\left[\sqrt{2}(\rho^2+\rho_o^2-2\gamma\rho\rho_o)\cos\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right.$$
$$\left.+\sqrt{2}\alpha(\lambda,\gamma)\rho\rho_o\sin\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right]\sec^{\frac{3}{2}}\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right). \qquad (3.40)$$

Furthermore, the Ricci scalar is given by

$$\mathcal{R} = \sqrt{\frac{|\nu|}{2\pi}}(1-\gamma^2)\rho^4\rho_o^2\frac{(\rho-\rho_o)^{2(\gamma-2)}}{(\rho+\rho_o)^{2(\gamma+2)}}\sec\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)$$
$$\times\left\{4\cos^2\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)+3|\lambda|\left[\cos\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)-5\right]\right\}. \qquad (3.41)$$

Note that all of these quantities as well as the metric components are proportional to $\sec(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o})$ which forces them to diverge near the point $\rho = \rho_o\frac{e^K+1}{e^K-1}$, if $\gamma \neq \pm 1$. Here a few additional points should be stressed.

If $\gamma = 1$, then

$$r = \left(\frac{8\pi}{|\nu|}\right)^{1/4}\frac{(\rho+\rho_o)^2}{\rho}, \qquad (3.42)$$

$$\frac{dr}{d\rho} = \left(\frac{8\pi}{|\nu|}\right)^{1/4}\left(1-\frac{\rho_o^2}{\rho^2}\right), \qquad (3.43)$$

and if $\gamma = -1$,

$$r = \left(\frac{8\pi}{|\nu|}\right)^{1/4}\frac{(\rho-\rho_o)^2}{\rho}, \qquad (3.44)$$

$$\frac{dr}{d\rho} = \left(\frac{8\pi}{|\nu|}\right)^{1/4}\left(1-\frac{\rho_o^2}{\rho^2}\right), \qquad (3.45)$$

which means that for these two particular cases, the areal radius decreases for $0 < \rho < \rho_o$, has an absolute minimum at $\rho = \rho_o$ [whose value is $r = \left(\frac{8\pi}{|\nu|}\right)^{1/4}4\rho_o > 0$ if $\gamma = 1$, and $r = 0$ if $\gamma = -1$], and increases for $\rho > \rho_o$. Thus, for $\gamma = -1$ the range $0 < \rho < \rho_o$ is unphysical. Moreover, notice that $r \to +\infty$ in the limits $\rho \to 0$ and $\rho \to \infty$, for both cases $\gamma = \pm 1$. Therefore the region near $\rho \to 0$ corresponds to a second asymptotically flat region of spacetime.

For these two particular values of γ we know that the scalar field becomes trivial and the solution describes a Schwarzschild spacetime.

If $\gamma \neq \pm 1$, the areal radius approaches infinity if $\rho \to \infty$ or $\rho \to \rho_o \frac{e^K+1}{e^K-1}$ where $K = \frac{\pi\alpha(\lambda,\gamma)}{2}$ and has a point of minimum value that satisfies the equation

$$\frac{dr}{d\rho} = 0 \to -\frac{1}{\alpha(\lambda,\gamma)} \left(\frac{\rho_o}{\rho} + \frac{\rho}{\rho_o} - 2\gamma \right)$$

$$= \tan \left(\alpha(\lambda,\gamma) \ln \frac{\rho - \rho_o}{\rho + \rho_o} \right). \tag{3.46}$$

Since r goes to infinity if $\rho \to \infty$ or $\rho = \rho_o \frac{e^K+1}{e^K-1} \geq \rho_o$, the range $0 \leq \rho \leq \rho_o \frac{e^K+1}{e^K-1}$ is unphysical. By taking a closer look at (3.39) it is evident that $r(\rho)$ is always positive in the range $\rho > \rho_o \frac{e^K+1}{e^K-1}$. Such a behavior indicates a wormhole structure. To justify this last statement let us represent (3.32) in the Morris-Thorne form [38]

$$ds^2 = -e^{-2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.47}$$

Following [38,39], certain conditions have to be imposed in order for a line element to be considered as a wormhole spacetime, namely

(1) $\frac{b(r)}{r} \leq 1$ for every $[r_{\text{th}}, +\infty)$, where r_{th} is the radius of the throat. This condition ensures that the proper radial distance defined by $l(r) = \pm \int_{r_{\text{th}}}^{r} \frac{dr}{1 - \frac{b(r)}{r}}$ is finite everywhere in spacetime. Note that in the

coordinates (t, l, θ, ϕ) the line element (3.47) can be written as

$$ds^2 = -e^{2\Phi(l)}dt^2 + dl^2 + r^2(l)(d\theta^2 + \sin^2\theta d\phi).$$

In this case the throat radius would be given by $r_{\text{th}} = \min\{r(l)\}$.

(2) $\frac{b(r_{\text{th}})}{r_{\text{th}}} = 1$ at the throat. This relation comes from requiring the throat to be a stationary point of $r(l)$. Equivalently, one may arrive at this equation by demanding the embedded surface of the wormhole to be vertical at the throat.

(3) $b'(r) < \frac{b(r)}{r}$ which reduces to $b'(r_{\text{th}}) < 1$ at the throat. This is known as the flare-out condition since it guarantees r_{th} to be a minimum and not any other stationary point.

Confronting the two metrics we directly see that the radial coordinate is given by the already known relation (3.30). The redshift and shape functions are, respectively, given by

$$\Phi(r) = \frac{1}{2}(\ln e^f + \ln \Omega^{-2}), \tag{3.48}$$

$$b(r) = r \left[1 - \frac{1}{\Omega^{-2}e^{-h}} \left(\frac{dr}{d\rho} \right)^2 \right], \tag{3.49}$$

or in terms of the coordinates

$$\Phi(r) = \frac{1}{4}\ln\left(\frac{8\pi}{|\nu|}\right) + \frac{1}{2}\left\{ \ln\left(\frac{\rho(r)-\rho_o}{\rho(r)+\rho_o}\right)^{2\gamma} - \ln\left[\cos\left(\alpha(\lambda,\gamma) \ln \frac{\rho(r)-\rho_o}{\rho(r)+\rho_o} \right) \right] \right\}, \tag{3.50}$$

$$\frac{b(r)}{r} = 1 - \left\{ \frac{\rho(r)^2 + \rho_o^2 - 2\gamma\rho(r)\rho_o + \rho(r)\rho_o\alpha(\lambda,\gamma)\tan\left(\alpha(\lambda,\gamma)\ln\frac{\rho(r)-\rho_o}{\rho(r)+\rho_o}\right)}{\sqrt{2}(\rho(r)^2 - \rho_o^2)} \right\}^2. \tag{3.51}$$

All the aforementioned conditions are verified in the range $[r_{\text{th}}, +\infty)$. The throat condition $\frac{b(r_{\text{th}})}{r_{\text{th}}} = 1$ has the same roots with (3.46); i.e., it is satisfied when r reaches its minimum value. Substituting these roots back in r, one can determine the area of this spatially finite "bridge" which is given by $A(\rho) = 4\pi r(\rho)^2$ [39]. Moreover, since from (3.39) we observe that $r \propto |\nu|^{-1/4}$, then we conclude for the size of the bridge that

$$A(\rho) \propto |\nu|^{-1/2}. \tag{3.52}$$

The Ricci scalar is finite at the throat, and hence, the solution indeed describes a geometrically traversable wormhole. On the other hand, it diverges as one approaches the second asymptotic spatial infinity $\rho = \rho_o \frac{e^K+1}{e^K-1}$ $(r \to \infty)$

associated with our wormhole. Thus the region near $\rho = \rho_o \frac{e^K+1}{e^K-1}$ is asymptotically large in the sense that the proper area of a circle at radius ρ, $A(\rho) = 4\pi r(\rho)^2$, goes to infinity as one approaches that region but not asymptotically flat, making the wormhole asymmetric under the interchange of the two asymptotic regions. This feature is also exhibited in the Brans Class I solution for a specific range of its parameters (see [26,40] for more details). Also, notice that even though it is not evident from the form of (3.51), the parameter ν determines the behavior of $b(r)$ as it is hidden inside $\rho(r)$, the inverse of (3.39). This behavior is depicted in Fig. 1.

In [38] the authors proved that every wormhole, by definition, must violate the null energy condition (NEC) if one assumes that a perfect fluid generates the wormhole
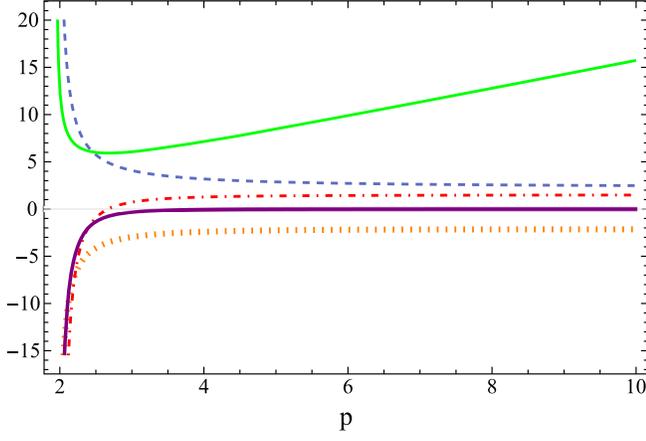
FIG. 1. Plot of g_{tt} (orange dashed line), $g_{\rho\rho}$ (blue dashed line), areal radius r (green solid line), $\frac{dr}{d\rho}$ (red dash-dotted line), and Ricci scalar (purple solid line) vs the isotropic radius with parameter values $\rho_o = 1, \gamma = 0.2, \lambda = -1, \nu = 3$. All quantities diverge at the point $\rho = \rho_o \frac{e^K+1}{e^K-1}$ that, as can be seen, corresponds to $r \to \infty$. The areal radius (green line) has a point of minimum value (vanishing of the red curve) at which $g_{tt} \neq 0$ and $g_{\rho\rho}, \mathcal{R}$ are finite. This point connects the two asymptotically flat regions at $\rho \to \infty$ and $\rho \to \rho_o \frac{e^K+1}{e^K-1}$. The scalar field remains finite for every $\rho > \rho_o \frac{e^K+1}{e^K-1}$ while it vanishes as $\rho \to \infty$.

FIG. 2. Plot of energy density ϱ (blue solid curve) and the derivative of $r(\rho)$ with respect to the isotropic radius ρ for $\rho_o = 1, \gamma = -0.2, \lambda = -2, \nu = 3$. The energy density becomes negative as one approaches the wormhole throat (vanishing of $\frac{dr}{d\rho}$) indicating the presence of "exotic" matter and the violation of the WEC.

$$\nu\left[1 - 4\pi + 4\pi^2\nu\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{2\alpha(\lambda,\gamma)}\right] > -\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{2\alpha(\lambda,\gamma)},$$
(3.55)

where we have denoted $\alpha(\lambda,\gamma) \equiv \sqrt{2|\lambda|(1-\gamma^2)}$ for brevity. It can be seen that $0 \leq \alpha(\lambda,\gamma) \leq \sqrt{2\lambda}$. Numerically it is found that the last relation holds for every $\rho > \rho_o$ and puts no further bounds on the values of ν. Furthermore, the condition $\epsilon_\lambda > 0$ corresponds to $\omega > -3/2$.

1. Scalar field

In this case the Brans-Dicke scalar field takes the form

$$\phi = \frac{s}{8\pi}\left|\left(\frac{1-\rho_o/\rho}{1+\rho_o/\rho}\right)^{\alpha(\lambda,\gamma)} - 2\pi\nu\left(\frac{1+\rho_o/\rho}{1-\rho_o/\rho}\right)^{\alpha(\lambda,\gamma)}\right|,$$
(3.56)

$$s = \mathrm{sgn}\left(4\pi\phi + \sqrt{2\pi(\nu + 8\pi\phi^2)}\right),$$
(3.57)

for which we observe the following:

  (i) The limit $\lambda \to 0$ (i.e., $\omega \to \infty$) corresponds to $\phi \to$ const.

  (ii) Additionally, for $\gamma = \pm 1$ the scalar field also becomes trivial.

  (iii) When $\gamma \neq \pm 1$ and $\nu \neq 0$, then ϕ diverges in the limit $\rho \to \rho_o$; thus, the effective gravitational constant vanishes.

  (iv) In the limit $\nu \to 0$ the scalar field yields

$$\phi = \frac{s}{8\pi}\left|\left(\frac{1-\rho_o/\rho}{1+\rho_o/\rho}\right)^{\alpha(\lambda,\gamma)}\right|,$$
(3.58)

$$s = \mathrm{sgn}(\phi).$$
(3.59)

spacetime. This is guaranteed by the flaring-out condition $b'(r) < \frac{b(r)}{r}$. By going in a proper reference frame, i.e., a frame of a static observer with respect to (r, θ, ϕ) coordinates, one can express the energy density ϱ and radial pressure p_r with respect to the isotropic radius by using the known relations [38]

$$\varrho = \frac{1}{r^2}\frac{db}{d\rho}\left(\frac{dr}{d\rho}\right)^{-1},$$
(3.53)

$$p_r = \frac{2}{r}\left(1 - \frac{b}{r}\right)\frac{d\Phi}{d\rho}\left(\frac{dr}{d\rho}\right)^{-1} - \frac{b^3}{r}.$$
(3.54)

Their explicit forms are given in Appendix and, as Fig. 2 indicates, for general values of the parameters (i.e., for $\gamma \neq \pm 1, \lambda \neq 0$, and $\nu \neq 0$) the energy density becomes negative close to the throat. Thus, the WEC is also violated.

C. Branch $\epsilon > 0$

We now require $\epsilon > 0$ and $\epsilon_\lambda > 0$ to avoid ghost solutions of the field σ. That is,

$$\epsilon = \mathrm{sgn}(\nu + 8\pi\phi^2) > 0,$$

$$\epsilon_\lambda = \mathrm{sgn}(\lambda) > 0.$$

Following the same procedure as before, we substitute (3.24) into (3.16) and solve the first inequality, i.e., $\nu > -8\pi\phi^2$. This leads to

2. Metric components

The metric components are given by

$$g_{tt} = -8\pi \left| \frac{(\rho^2 - \rho_o^2)^{\alpha(\lambda,\gamma)}}{(\rho - \rho_o)^{2\alpha(\lambda,\gamma)} + 2\pi\nu(\rho + \rho_o)^{2\alpha(\lambda,\gamma)}} \right|$$
$$\times \left(\frac{1 - \rho_o/\rho}{1 + \rho_o/\rho} \right)^{2\gamma}, \tag{3.60}$$

$$g_{\rho\rho} = 8\pi \left| \frac{(\rho^2 - \rho_o^2)^{\alpha(\lambda,\gamma)}}{(\rho - \rho_o)^{2\alpha(\lambda,\gamma)} + 2\pi\nu(\rho + \rho_o)^{2\alpha(\lambda,\gamma)}} \right|$$
$$\times \left(1 - \frac{\rho_o^2}{\rho^2} \right)^2 \left(\frac{1 + \rho_o/\rho}{1 - \rho_o/\rho} \right)^{2\gamma}, \tag{3.61}$$

and we calculate

$$\lim_{\rho \to \infty} g_{tt} = -\frac{8\pi}{|2\pi\nu + 1|}, \qquad \lim_{\rho \to \infty} g_{\rho\rho} = \frac{8\pi}{|2\pi\nu + 1|}. \tag{3.62}$$

Again, the asymptotic behavior of $g_{tt}, g_{\rho\rho}$ depends only on the parameter ν, which means spacetime becomes Minkowski in the large distance limit. We note the following.

In the limit $\lambda \to 0 (\omega \to \infty)$ the metric takes the form

$$g_{tt} = -\frac{8\pi}{|1 + 2\pi\nu|} \left(\frac{1 - \rho_o/\rho}{1 + \rho_o/\rho} \right)^{2\gamma},$$
$$g_{\rho\rho} = \frac{8\pi}{|1 + 2\pi\nu|} \left(1 - \frac{\rho_o^2}{\rho^2} \right)^2 \left(\frac{1 + \rho_o/\rho}{1 - \rho_o/\rho} \right)^{2\gamma}. \tag{3.63}$$

This is the solution (3.24)–(3.26) with the introduction of a new scale, supported by the new parameter ν.

As $\rho \to \rho_o$, $g_{\rho\rho}$ vanishes except when $\gamma = 1$ for which it blows up.

If $-1 \le \gamma \le -\sqrt{\frac{|\lambda|}{|\lambda|+2}}$, the temporal component diverges at ρ_o, whereas if $-\sqrt{\frac{|\lambda|}{|\lambda|+2}} \le \gamma \le 1$, it is $g_{tt} \overset{\rho \to \rho_o}{\to} 0$.

If $\gamma = 1$, then

$$g_{tt} = -\frac{8\pi}{|1 + 2\pi\nu|} \left(\frac{1 - \rho_o/\rho}{1 + \rho_o/\rho} \right)^2,$$
$$g_{\rho\rho} = \frac{8\pi}{|1 + 2\pi\nu|} \left(1 + \frac{\rho_o}{\rho} \right)^4, \tag{3.64}$$

while the Brans-Dicke scalar becomes constant and the solution reduces to the Schwarzschild solution with $M = 2\rho_o$. On the other hand, when $\gamma = -1$, one directly sees that

$$g_{tt} = -\frac{8\pi}{|1 + 2\pi\nu|} \left(\frac{1 + \rho_o/\rho}{1 - \rho_o/\rho} \right)^2,$$
$$g_{\rho\rho} = \frac{8\pi}{|1 + 2\pi\nu|} \left(1 - \frac{\rho_o}{\rho} \right)^4, \tag{3.65}$$

which corresponds again to the Schwarzschild solution with negative mass $M = -2\rho_o$. Therefore ρ_o can be considered as the mass parameter of the solution. Consequently, we will consider only positive values of ρ_o since, even in the case $\gamma = -1$, we can recover a positive mass by taking $-\rho_o$ to be the mass parameter instead of just ρ_o.

As we consider the limit to Brans-Dicke $\nu \to 0$, the metric reads

$$g_{tt} = -8\pi \left(\frac{1 - \rho_o/\rho}{1 + \rho_o/\rho} \right)^{2\gamma - \alpha(\lambda,\gamma)},$$
$$g_{\rho\rho} = 8\pi \left(1 - \frac{\rho_o^2}{\rho^2} \right)^2 \left(\frac{1 + \rho_o/\rho}{1 - \rho_o/\rho} \right)^{\alpha(\lambda,\gamma) + 2\gamma}, \tag{3.66}$$

and we notice that $2\gamma - \alpha(\lambda, \gamma) > 0$ implies $\sqrt{\frac{|\lambda|}{2+|\lambda|}} < \gamma \le 1 \to g_{tt} \overset{\rho \to \rho_o}{\to} 0$, whereas $2\gamma - \alpha(\lambda, \gamma) < 0$ gives $-1 \le \gamma < \sqrt{\frac{|\lambda|}{2+|\lambda|}} \to g_{tt} \overset{\rho \to \rho_o}{\to} \infty$. For the radial component one can calculate that $\alpha(\lambda, \gamma) + 2(\gamma - 1) > 0$ leads to $-1 \le \gamma < \frac{2-|\lambda|}{2+|\lambda|}$ and consequently $g_{\rho\rho} \overset{\rho \to \rho_o}{\to} 0$, while in the range $\frac{2-|\lambda|}{2+|\lambda|} < \gamma < 1$ we obtain $\alpha(\lambda, \gamma) + 2(\gamma - 1) > 0$ and it is $g_{\rho\rho} \overset{\rho \to \rho_o}{\to} \infty$.

More importantly, solution (3.66) can be identified as Brans Class I solution

$$ds_{(I)}^2 = -\left(\frac{1 - B/\rho}{1 + B/\rho} \right)^{2/\lambda_I} dt^2 + \left(1 + \frac{B}{\rho} \right)^4$$
$$\times \left(\frac{1 - B/\rho}{1 + B/\rho} \right)^{\frac{2(\lambda_I - C - 1)}{\lambda_I}} (d\rho^2 + \rho^2 d\Omega_{(2)}^2), \tag{3.67}$$

$$\phi_I = \phi_0 \left(\frac{1 - B/\rho}{1 + B/\rho} \right)^{C/\lambda_I}, \tag{3.68}$$

$$\lambda_I^2 = (C + 1)^2 - C \left(C - \frac{\omega C}{2} \right), \tag{3.69}$$

if we make the substitutions

$$2\gamma - \alpha(\lambda, \gamma) = \frac{2}{\lambda_I}, \quad 2\gamma + \alpha(\lambda, \gamma) = 2 \left(\frac{C+1}{\lambda_I} \right), \quad B = \rho_o. \tag{3.70}$$

The above relations yield

$$\gamma = \frac{C + 2}{2\lambda_I}, \qquad \alpha(\lambda, \gamma) = \frac{C}{\lambda_I}. \tag{3.71}$$

Thus, just as (B, C) [or (B, λ_I)] are the two independent parameters of the Brans Class I solution, in our case they are (ρ_o, γ). The Schwarzschild solution is obtained by setting $C = 0$, which forces $\lambda_I \pm 1$. These values correspond to $\gamma = \pm 1$ in our solution, which indeed yields the Schwarzschild solution as we saw earlier.

3. Areal radius and Ricci scalar

By substituting the corresponding Ω, the areal radius and its derivative take the form

$$r = \frac{\sqrt{8\pi}}{\rho} \left| \frac{(\rho - \rho_o)^{\alpha(\lambda,\gamma)-2(\gamma-1)} \cdot (\rho + \rho_o)^{\alpha(\lambda,\gamma)+2(\gamma+1)}}{(\rho - \rho_o)^{2\alpha(\lambda,\gamma)} + 2\pi\nu(\rho + \rho_o)^{2\alpha(\lambda,\gamma)}} \right|^{1/2},$$

$$(3.72)$$

$$\frac{dr}{d\rho} = \left(\frac{\rho + \rho_o}{\rho - \rho_o} \right)^{\gamma} \frac{1}{\rho^2 \Omega} \cdot \left\{ (\rho^2 + \rho_o^2 - 2\rho\rho_o\gamma) - \rho\rho_o\alpha(\lambda,\gamma) \right.$$
$$\left. \times \left[\frac{(\frac{\rho-\rho_o}{\rho+\rho_o})^{2\alpha(\lambda,\gamma)} - 2\pi\nu}{(\frac{\rho-\rho_o}{\rho+\rho_o})^{2\alpha(\lambda,\gamma)} + 2\pi\nu} \right] \right\},$$

$$(3.73)$$

while the Ricci scalar is given by

$$\mathcal{R} = (\gamma^2 - 1)\rho^4 \rho_o^2 \frac{(\rho - \rho_o)^{2(\gamma-2)}}{(\rho + \rho_o)^{2(\gamma+2)}}$$
$$\times \frac{(3|\lambda| - 2)[(\frac{\rho-\rho_o}{\rho+\rho_o})^{2\alpha(\lambda,\gamma)} + 4\pi^2\nu^2(\frac{\rho+\rho_o}{\rho-\rho_o})^{2\alpha(\lambda,\gamma)}] - 4\pi\nu(15|\lambda| + 2)}{2\pi|(\frac{\rho-\rho_o}{\rho+\rho_o})^{\alpha(\lambda,\gamma)} + 2\pi\nu(\frac{\rho+\rho_o}{\rho-\rho_o})^{\alpha(\lambda,\gamma)}|}.$$

$$(3.74)$$

One can observe the following.

In general, the factor $(\rho - \rho_o)^{\alpha(\lambda,\gamma)-2(\gamma-1)}$ compels $r(\rho)$ to vanish at the point $\rho = \rho_o$ since $\alpha(\lambda, \gamma) - 2(\gamma - 1) > 0$ for every $\gamma^2 < 1$. The Ricci scalar is singular at $\rho = \rho_o$, and since g_{tt} does not vanish anywhere at $\rho > \rho_o$, we conclude that this point corresponds to a naked singularity at the center of the spherical symmetry $\rho = \rho_o$ (or $r = 0$). Note that the function $r(\rho)$ is almost always increasing in the range $\rho > \rho_o$ and therefore $\rho \to \infty$ corresponds to the asymptotic region $r \to \infty$.

However, as we saw earlier, our solution contains Brans Class I as a special case, a family of solutions that is known to describe wormholes for some range of its parameters. With this in mind, it would be no surprise if our solution can describe wormhole spacetimes besides the naked singularities. A wormhole spacetime should contain two asymptotic regions and a throat that connects them. Hence, the areal radius should diverge at a second point except the one at $\rho \to \infty$ and additionally should have a minimum value that is greater than zero. The only possible case for which (3.72) produces a second asymptotic region is if we demand the denominator to vanish, i.e., $(\rho - \rho_o)^{2\alpha(\lambda,\gamma)} + 2\pi\nu(\rho + \rho_o)^{2\alpha(\lambda,\gamma)} = 0$. Solving this equation we can find the point ρ at which the second asymptotic region occurs. It is given by

$$\rho = \rho_o \frac{1 + N}{1 - N}, \qquad N = (-2\pi\nu)^{1/2\alpha}, \qquad (3.75)$$

and in order to be greater than ρ_o the following relation must hold:

$$0 < (-\nu)^{1/2\alpha} < (2\pi)^{-1/2\alpha}, \qquad (3.76)$$

which is satisfied irrespective of the values of λ and γ provided that ν lies in the range

$$-\frac{1}{2\pi} < \nu < 0. \qquad (3.77)$$

Again, to determine the redshift and shape functions we make use of (3.48) and (3.49). Their explicit forms are given in Appendix. In the following diagrams (Figs. 3 and 4) we present the behavior of the metric, areal radius, Ricci scalar, and energy density ϱ with respect to the isotropic radius. In Figs. 5 and 6 we compare the energy density of our model with the one obtained from the Brans Class 1 solution for two different values of the parameter ν. Both solutions violate the WEC near the wormhole throats.

If $\gamma = 1$, then

$$r = \left(\frac{8\pi}{1 + 2\pi\nu} \right)^{1/2} \frac{(\rho + \rho_o)^2}{\rho}, \qquad (3.78)$$
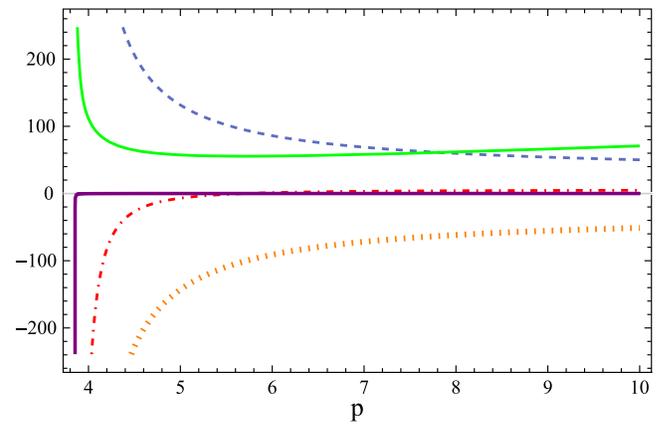
FIG. 3. Plot of g_{tt} (orange dashed line), $g_{\rho\rho}$ (blue dashed line), areal radius r (green solid line), $\frac{dr}{d\rho}$ (red dash-dotted line), and Ricci scalar (purple solid line) for parameter values $\rho_o = 1, \gamma = 0, \lambda = 2, \nu = -0.019$, i.e., satisfying (3.77). The areal radius diverges at two different points [one at $\rho \to \infty$ and one at the point given by (3.75)] corresponding to two different asymptotic regions. Between these two regions there is a point of minimum $r(\rho)$ where the throat is located.
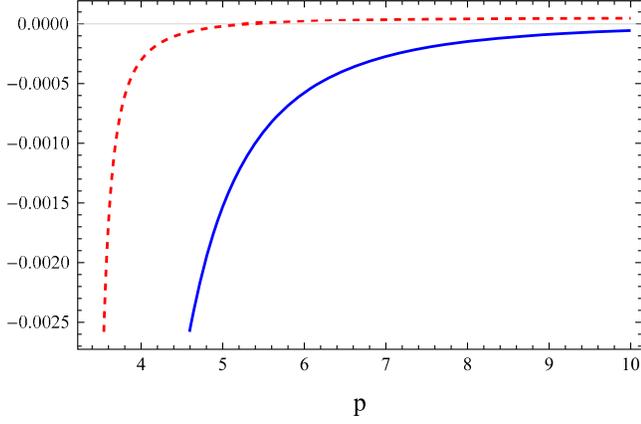
FIG. 4.   Plot of energy density ϱ (blue solid curve) and $\frac{dr}{d\rho}$ (red dash-dotted line) for $\rho_o = 1, \gamma = 0.5, \lambda = 2, \nu = -0.019$, i.e., satisfying (3.77). The energy density becomes negative as we approach the wormhole throat violating the WEC.

$$\frac{dr}{d\rho} = \left(\frac{8\pi}{1+2\pi\nu}\right)^{1/2}\left(1 - \frac{\rho_o^2}{\rho^2}\right). \qquad (3.79)$$

If $\gamma = -1$, we get

$$r = \left(\frac{8\pi}{1+2\pi\nu}\right)^{1/2}\frac{(\rho - \rho_o)^2}{\rho}, \qquad (3.80)$$

$$\frac{dr}{d\rho} = \left(\frac{8\pi}{1+2\pi\nu}\right)^{1/2}\left(1 - \frac{\rho_o^2}{\rho^2}\right). \qquad (3.81)$$

Hence, just as in case $\epsilon < 0$, if $\gamma = \pm 1$, then the areal radius is a decreasing function for $0 < \rho < \rho_o$, has an absolute minimum at $\rho = \rho_o$, and increases for $\rho > \rho_o$. If $\gamma = -1$, its
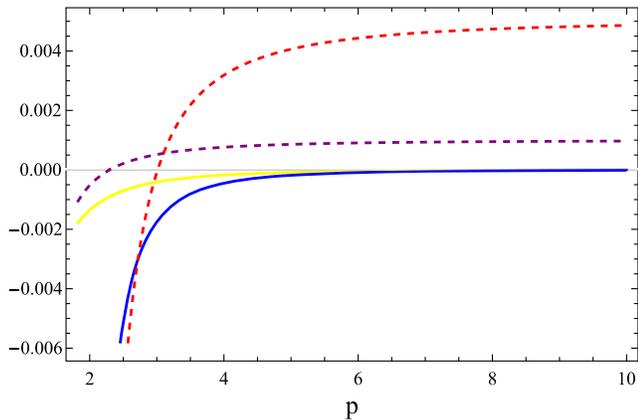
FIG. 5.   Plot of energy densities ϱ (blue solid curve), ϱ_{BD} (yellow solid curve), $\frac{dr}{d\rho}$ (red dash-dotted line), and $\frac{dr_{\mathrm{BD}}}{d\rho}$ (purple dash-dotted line) for $\rho_o = 1, \gamma = 0.5, \lambda = 2, \nu \simeq -2.19 \times 10^{-3}$, i.e., satisfying (3.77). The energy density becomes negative as we approach the wormhole throat for both theories violating the WEC.
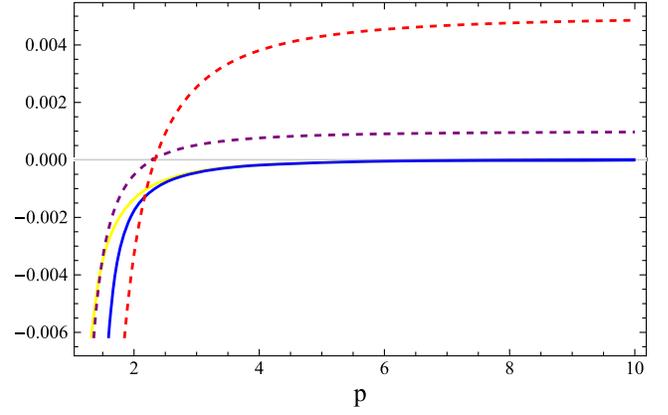
FIG. 6.   Plot of energy densities ϱ (blue solid curve), ϱ_{BD} (yellow solid curve), $\frac{dr}{d\rho}$ (red dash-dotted line), and $\frac{dr_{\mathrm{BD}}}{d\rho}$ (purple dash-dotted line) for $\rho_o = 1, \gamma = 0.5, \lambda = 2, \nu \simeq -5.8 \times 10^{-4}$, i.e., satisfying (3.77). The energy density becomes negative as we approach the wormhole throat for both theories violating the WEC.

minimum value is $r(\rho = \rho_o) = 0$, whereas if $\gamma = 1$, then $r(\rho = \rho_o) = \left(\frac{8\pi}{1+2\pi\nu}\right)^{1/2}4\rho_o > 0$.

In the limit $\nu \to 0$ the areal radius yields

$$r = \sqrt{8\pi}\frac{(1+\rho_o/\rho)^{\alpha(\lambda,\gamma)+2(\gamma+1)}}{(1-\rho_o/\rho)^{\alpha(\lambda,\gamma)+2(\gamma-1)}}\rho, \qquad (3.82)$$

and it is $\alpha(\lambda,\gamma) + 2(\gamma - 1) > 0 \to \frac{2-|\lambda|}{2+|\lambda|} < \gamma < 1$. If the γ parameter lies within this range, then the point $\rho = \rho_o$ corresponds to $r \to \infty$. Its derivative is given by

$$\frac{dr}{d\rho} = \frac{\sqrt{8\pi}}{\rho^2}\left|\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{2\gamma+\alpha(\lambda,\gamma)}\right|^{1/2}$$
$$\times (\rho^2 + \rho_o^2 - 2\rho\rho_o\gamma - \rho\rho_o\alpha(\lambda,\gamma)), \qquad (3.83)$$

and the point of minimum r corresponds to

$$\rho = \frac{\rho_o}{2}\left(2\gamma + \alpha(\lambda,\gamma) \pm \sqrt{-4 + (2\gamma + \alpha(\lambda,\gamma))^2}\right).$$

By the use of (3.71) we can express these roots in terms of the parameters of Brans Class I, and the result is given by

$$\rho = \frac{B(C+1)}{\lambda_I}\left(1 \pm \sqrt{1 - \left(\frac{\lambda_I}{C+1}\right)^2}\right)^2, \qquad (3.84)$$

which is in agreement with the previously obtained result in [26].

The behavior of the metric, areal radius, and Ricci scalar, for values of ν that do not satisfy (3.77), is depicted in the graph Fig. 7.

To summarize for the branch $\epsilon > 0$, if $\gamma = \pm 1$, the spherically symmetric solutions reduce to the Schwarzschild
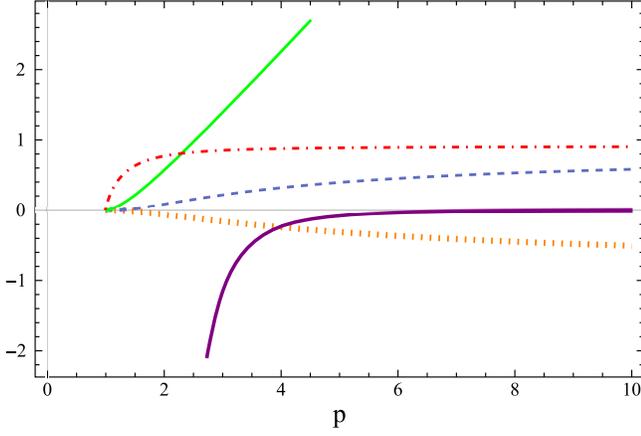
FIG. 7. Plot of g_{tt} (orange dashed line), $g_{\rho\rho}$ (blue dashed line), areal radius r (green solid line), $\frac{dr}{d\rho}$ (red dash-dotted line), and Ricci scalar (purple solid line) with respect to the isotropic radius for $\rho_o = 1, \gamma = 0.2, \lambda = 2, \nu = -5$. At ρ_o the areal radius as well as the metric components vanish, whereas the Ricci scalar diverges. Unless $\gamma = \pm 1$, the solution produces a singularity at $\rho = \rho_o$, which is not covered by an event horizon.

solution with a constant scalar field, while if $\gamma \neq 1$, then we have a naked singularity when the parameter ν does not satisfy the bound (3.77) and a wormhole solution otherwise.

IV. CONCLUSIONS

The Brans Classes I–IV of solutions of BD theory according to Hawkings theorem [20] always describe either wormholes or horizonless geometries containing naked singularities, and they never describe black holes. Spacetimes having naked singularities are unphysical in the sense that the initial value problem fails, leaving the theory void of predictability and wormholes as the only remaining Brans solutions. The Brans solutions are vacuum solutions and the BD scalar acts as the only form of effective matter. However, the BD scalar can violate all of the energy conditions; therefore it is no surprise that one can obtain wormholes as solutions of vacuum BD theory.

Wormholes are exotic objects that require the energy conditions to be violated. This can be understood because a wormhole requires light rays that enter at one mouth and emerge from the other to have cross-sectional areas initially decreasing and then increasing. This conversion can be

produced by gravitational repulsion that acts on the light rays passing near the throat, and this can happen provided that in this region a negative energy density resides, as it is effectively guaranteed by the BD scalar field ϕ in the case of $\gamma > 1$. Note that in GR the properties required for the functions $\Phi(r)$ and $b(r)$ cause such constraints on the matter stress tensor as to make necessary the occupancy of exotic matter, especially in the wormhole throat, where the absence of a horizon is required. In the BD theory the role of exotic matter is instead played, if $\gamma > 1$ (or $\omega < -2$), by the scalar field ϕ.

The aim of this work was to study spherically symmetric solutions and in particular wormhole solutions in a modified BD theory. The BD theory is modified by introducing a dimensionful parameter ν [28,29] in the kinetic coupling of the scalar field to gravity. This coupling gives an extra contribution to the matter content of the BD theory. Solving the coupled Einstein-Klein-Gordon equations we found new spherically symmetric solutions that depend on the new parameter ν. Demanding the absence of unphysical propagating modes, such as ghosts, we find two branches of solutions.

The spherically symmetric solutions of the first branch for any choice of parameters and independently of the value of the new parameter ν reduce to the Schwarzschild black hole or to a naked singularity with a constant scalar field and also give new wormhole solutions. The second branch of solutions, except the Schwarzschild solution for negative values of the parameter ν, gives wormholes whose throat size is inversely proportional to the coupling parameter ν and a nontrivial scalar field. Also we checked the WEC for both branches, and we found that they are violated in both cases.

APPENDIX: ENERGY DENSITY AND RADIAL PRESSURE

In this appendix we present the explicit form for the radial pressure and the energy density with respect to the isotropic radius for both branches.

$\epsilon < 0$:

$$\varrho = -\frac{\rho^4 \rho_o^2 \sqrt{|\nu|}\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{-2\gamma}}{2\sqrt{2\pi}(\rho-\rho_o)^4(\rho+\rho_o)^4} \sec\left(\alpha(\lambda,\gamma)\log\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right)$$

$$\times \left\{-2\gamma\alpha(\lambda,\gamma)\sin\left(2\alpha(\lambda,\gamma)\log\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right) + 4(\gamma^2-1)\cos^2\left(\alpha(\lambda,\gamma)\log\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right)\right.$$

$$\left.+ (\gamma^2-1)|\lambda|\left(\cos\left(2\alpha(\lambda,\gamma)\log\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right) - 9\right)\right\},$$

$$p_r = \frac{\rho^3 \rho_o \sqrt{|\nu|}(\frac{\rho+\rho_o}{\rho-\rho_o})^{-2\gamma}}{2\sqrt{\pi}(\rho-\rho_o)^4(\rho+\rho_o)^4} \left\{ -4\sqrt{1-\gamma^2}\sqrt{|\lambda|}(-\rho^2+\gamma\rho\rho_o-\rho_o^2)\tan\left(\alpha(\lambda,\gamma)\log\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right) \right.$$

$$\left. - 3\sqrt{2}(\gamma^2-1)\rho\rho_o|\lambda|\tan^2\left(\alpha(\lambda,\gamma)\log\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right) - 2\sqrt{2}(\gamma^2-1)\rho\rho_o \right\}$$

$$\times \cos\left(\alpha(\lambda,\gamma)\log\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right).$$

$\epsilon > 0$:

$$\varrho = \frac{\rho^4 \rho_o^2 (\frac{\rho+\rho_o}{\rho-\rho_o})^{-2\gamma}(\rho^2-\rho_o^2)^{-2\alpha(\lambda,\gamma)}}{4\pi(\rho-\rho_o)^4(\rho+\rho_o)^4\sqrt{(\rho^2-\rho_o^2)^{-2\alpha(\lambda,\gamma)}}(2\pi\nu(\rho+\rho_o)^{2\alpha(\lambda,\gamma)}+(\rho-\rho_o)^{2\alpha(\lambda,\gamma)})^2}$$

$$\times \{(\gamma^2-1)|\lambda|(4\pi^2\nu^2(\rho+\rho_o)^{4\alpha(\lambda,\gamma)}-36\pi\nu(\rho-\rho_o)^{2\alpha(\lambda,\gamma)}(\rho+\rho_o)^{2\alpha(\lambda,\gamma)}$$

$$+(\rho-\rho_o)^{4\alpha(\lambda,\gamma)})-2\gamma\alpha(\lambda,\gamma)((\rho-\rho_o)^{4\alpha(\lambda,\gamma)}-4\pi^2\nu^2(\rho+\rho_o)^{4\alpha(\lambda,\gamma)})$$

$$-2(\gamma^2-1)(2\pi\nu(\rho+\rho_o)^{2\alpha(\lambda,\gamma)}+(\rho-\rho_o)^{2\alpha(\lambda,\gamma)})^2\},$$

$$p_r = \{4\pi(\rho-\rho_o)^4(\rho+\rho_o)^4(\rho^2-\rho_o^2)^{-\alpha(\lambda,\gamma)}(2\pi\nu(\rho+\rho_o)^{2\alpha(\lambda,\gamma)}+(\rho-\rho_o)^{2\alpha(\lambda,\gamma)})\}^{-1}$$

$$\times \rho^3\rho_o\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{-2\gamma}(\rho^2-\rho_o^2)^{-2\alpha(\lambda,\gamma)}[-2\alpha(\lambda,\gamma)(-\rho^2+\gamma\rho\rho_o-\rho_o^2)$$

$$\times ((\rho-\rho_o)^{4\alpha(\lambda,\gamma)}-4\pi^2\nu^2(\rho+\rho_o)^{4\alpha(\lambda,\gamma)})$$

$$+2(\gamma^2-1)\rho\rho_o(2\pi\nu(\rho+\rho_o)^{2\sqrt{2}\sqrt{1-\gamma^2}\sqrt{|\lambda|}}+(\rho-\rho_o)^{2\alpha(\lambda,\gamma)})^2$$

$$+3(\gamma^2-1)\rho\rho_o|\lambda|((\rho-\rho_o)^{2\alpha(\lambda,\gamma)}-2\pi\nu(\rho+\rho_o)^{2\alpha(\lambda,\gamma)})^2],$$

where $\alpha(\lambda,\gamma) = \sqrt{2|\lambda|(1-\gamma^2)}$.

[1] C. Brans and R. H. Dicke, Mach's principle and a relativistic theory of gravitation, Phys. Rev. **124**, 925 (1961).

[2] P. G. O. Freund, Kaluza-Klein cosmologies, Nucl. Phys. **B209**, 146 (1982); T. Appelquist, A. Chodos, and P. G. O. Freund, *Modern Kaluza-Klein Theories* (Addison-Wesley, Reading, MA, 1987); E. S. Fradkin and A. A. Tseytlin, Effective field theory from quantized strings, Phys. Lett. **158B**, 316 (1985); E. S. Fradkin and A. A. Tseytlin, Quantum string theory effective action, Nucl. Phys. **B261**, 1 (1985); C. G. Callan, Jr., E. J. Martinec, M. J. Perry, and D. Friedan, Strings in background fields, Nucl. Phys. **B262**, 593 (1985); C. G. Callan, Jr., I. R. Klebanov, and M. J. Perry, String theory effective actions, Nucl. Phys. **B278**, 78 (1986).

[3] B. Bertotti, L. Iess, and P. Tortora, A test of general relativity using radio links with the Cassini spacecraft, Nature (London) **425**, 374 (2003); C. M. Will, The confrontation between general relativity and experiment, Living Rev. Relativity **9**, 3 (2006); A. De Felice, G. Mangano, P. D. Serpico, and M. Trodden, Relaxing nucleosynthesis constraints on Brans-Dicke theories, Phys. Rev. D **74**, 103005 (2006).

[4] A. I. Vainshtein, To the problem of nonvanishing gravitation mass, Phys. Lett. **39B**, 393–394 (1972).

[5] E. Babichev and C. Deffayet, An introduction to the Vainshtein mechanism, Classical Quantum Gravity **30**, 184001 (2013).

[6] V. Acquaviva, C. Baccigalupi, S. M. Leach, A. R. Liddle, and F. Perrotta, Structure formation constraints on the Jordan-Brans-Dicke theory, Phys. Rev. D **71**, 104025 (2005); J. C. Fabris, S. V. B. Goncalves, and R. de Sa Ribeiro, Late time accelerated Brans-Dicke pressureless solutions and the Supernovae Type IA data, Gravitation Cosmol. **12**, 49 (2006); A. Avilez and C. Skordis, Cosmological Constraints on Brans-Dicke Theory, Phys. Rev. Lett. **113**, 011101 (2014); Y. C. Li, F. Q. Wu, and X. Chen, Constraints on the Brans-Dicke gravity theory with the Planck data, Phys. Rev. D **88**, 084053 (2013); O. Hrycyna, M. Szydlowski, and M. Kamionka, Dynamics and cosmological constraints on Brans-Dicke cosmology, Phys. Rev. D **90**, 124040 (2014).

[7] P. D. Mannheim, Attractive and repulsive gravity, Found. Phys. **30**, 709 (2000); A. Barnaveli and M. Gogberashvili, Gravitational repulsion of spherical domain walls, Gen. Relativ. Gravit. **26**, 1117 (1994).

[8] H. Nariai, On the Green's function in an expanding Universe and its role in the problem of Mach's principle, Prog. Theor. Phys. **40**, 49 (1968); Gravitational instability in the Brans-Dicke cosmology, Prog. Theor. Phys. **42**, 544 (1969).

[9] R. E. Morganstern, Exact solutions to radiation-filled Brans-Dicke cosmologies, Phys. Rev. D **4**, 282 (1971); O. Obregon and P. Chauvet, Exact solution to radiation-filled Brans-Dicke closed space cosmology, Astrophys. Space Sci. **56**, 335 (1978).

[10] J. O' Hanlon and B. O. J. Tupper, Vacuum-field solutions in the Brans-Dicke theory, Nuovo Cimento B **7**, 305 (1972); C. Romero and A. Barros, Brans-dicke vacuum solutions and the cosmological constant: A qualitative analysis, Gen. Relativ. Gravit. **25**, 491 (1993); C. Romero and A. Barros, Brans-dicke cosmology and the cosmological constant: The spectrum of vacuum solutions, Astrophys. Space Sci. **192**, 263 (1992); J. M. Cervero and P. G. Estevez, Astrophys. Space Sci. **15** (1984).

[11] L. E. Gurevich, A. M. Finkelstein., and V. A. Ruban, On the problem of the initial state in the isotropic scalar-tensor cosmology of Brans-Dicke, Astrophys. Space Sci. **22**, 231 (1973); R. E. Morganstern, Exact solutions to Brans-Dicke cosmologies in flat Friedmann universes, Phys. Rev. D **4**, 946 (1971).

[12] D. Lorenz-Petzold, Exact-Brans-Dicke-Friedmann-Robertson-Walker-Bianchi-Kantowski-Sachs vacuum solutions, Astrophys. Space Sci. **98**, 101 (1984); Exact perfect fluid solutions in the Brans-Dicke-theory, Astrophys. Space Sci. **98**, 249 (1984); Exact non-flat Friedmann-Robertson-Walker solutions in Barker's theory of gravitation, Astrophys. Space Sci. **106**, 419 (1984); J. D. Barrow, Scalar-tensor cosmologies, Phys. Rev. D **47**, 5329 (1993); J. J. Levin and K. Freese, Curvature and flatness in a Brans-Dicke universe, Nucl. Phys. **B421**, 635 (1994); J. P. Mimoso and D. Wands, Massless fields in scalar-tensor cosmologies, Phys. Rev. D **51**, 477 (1995).

[13] V. A. Ruban and A. M. Finkelstein, Astrofiz. **12**, 371 (1976); N. Van den Bergh, Exact solutions for the early Friedmann-Robertson-Walker universe in general scalar-tensor theories, Gen. Relativ. Gravit. **15**, 441 (1983).

[14] K. Uehara and C. W. Kim, Brans-dicke Cosmology with the cosmological constant, Phys. Rev. D **26**, 2575 (1982); N. Riazi and E. Ahmadi-Azar, A class of exact cosmological solutions of Brans-Dicke theory with cosmological constant, Astrophys. Space Sci. **226**, 1 (1995); L. O. Pimentel, Exact cosmological solutions in the scalar-tensor theory with cosmological constant, Astrophys. Space Sci. **112**, 175 (1985); S. Ram and C. P. Singh, Early cosmological models with bulk viscosity in Brans-Dicke theory, Astrophys. Space Sci. **254**, 143 (1997); S. N. Pandey, Astrophys. Space Sci. **277**, 403 (2001).

[15] S. W. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972); O. Bertolami, Brans-Dicke cosmology with a scalar field dependent cosmological term, Fortschr. Phys. **34**, 829 (1986); J. D. Barrow, Scalar-tensor cosmologies, Phys. Rev. D **47**, 5329 (1993); V. B. Johri and K. Desikan, Cosmological models with constant deceleration parameter in Brans-Dicke theory, Gen. Relativ. Gravit. **26**, 1217 (1994); Possible solution to the horizon problem:

Modified aging in massless scalar theories of gravity, Phys. Rev. D **47**, 4282 (1993); S. Capozziello, R. De Ritis, C. Rubano, and P. Scudellaro, Int. J. Mod. Phys. D **05**, 85 (1996).

[16] C. Park and S. J. Sin, P-brane cosmology and phases of Brans-Dicke theory with matter, Phys. Rev. D **57**, 4620 (1998); S. Kalyana Rama, Can string theory avoid cosmological singularities?, Phys. Lett. B **408**, 91 (1997).

[17] C. Mathiazhagan and V. B. Johri, An inflationary Universe in Brans-Dicke theory: A hopeful sign of theoretical estimation of the gravitational constant, Classical Quantum Gravity **1**, L29 (1984).

[18] D. La and P. J. Steinhardt, Extended Inflationary Cosmology, Phys. Rev. Lett. **62**, 376 (1989); Extended Inflationary Cosmology, Phys. Rev. Lett. **62**, 1066 (1989).

[19] C. H. Brans, Mach's principle and a relativistic theory of gravitation. II, Phys. Rev. **125**, 2194 (1962).

[20] S. W. Hawking, Black holes in the Brans-Dicke theory of gravitation, Commun. Math. Phys. **25**, 167 (1972).

[21] T. P. Sotiriou and V. Faraoni, Black Holes in Scalar-Tensor Gravity, Phys. Rev. Lett. **108**, 081103 (2012).

[22] S. Bhattacharya, K. F. Dialektopoulos, A. E. Romano, and T. N. Tomaras, Brans-Dicke Theory with $\Lambda > 0$: Black Holes and Large Scale Structures, Phys. Rev. Lett. **115**, 181104 (2015).

[23] M. Campanelli and C. O. Lousto, Are black holes in Brans-Dicke theory precisely the same as a general relativity?, Int. J. Mod. Phys. D **02**, 451 (1993).

[24] L. Vanzo, S. Zerbini, and V. Faraoni, The Campanelli-Lousto and veiled spacetimes, Phys. Rev. D **86**, 084031 (2012).

[25] A. G. Agnese and M. La Camera, Wormholes in the Brans-Dicke theory of gravitation, Phys. Rev. D **51**, 2011 (1995).

[26] V. Faraoni, F. Hammad, and S. D. Belknap-Keet, Revisiting the Brans solutions of scalar-tensor gravity, Phys. Rev. D **94**, 104019 (2016).

[27] R. G. Cai and Y. S. Myung, Black holes in the Brans-Dicke-Maxwell theory, Phys. Rev. D **56**, 3466 (1997).

[28] G. Kofinas, The complete Brans-Dicke theories, Ann. Phys. (Amsterdam) **376**, 425 (2017).

[29] G. Kofinas and M. Tsoukalas, On the action of the complete BransDicke theory, Eur. Phys. J. C **76**, 686 (2016).

[30] G. Kofinas, E. Papantonopoulos, and E. N. Saridakis, Modified Brans Dicke cosmology with matter-scalar field interaction, Classical Quantum Gravity **33**, 155004 (2016).

[31] F. S. N. Lobo and M. A. Oliveira, General class of vacuum Brans-Dicke wormholes, Phys. Rev. D **81**, 067501 (2010).

[32] C. J. Gao and S. N. Zhang, Black holes in Brans-Dicke theory with a cosmological constant, arXiv:gr-qc/0604083.

[33] B. C. Xanthopoulos and T. Zannias, Einstein gravity coupled to a massless scalar field in arbitrary space-time dimensions, Phys. Rev. D **40**, 2564 (1989).

[34] V. A. Rubakov, The null energy condition and its violation, Usp. Fiz. Nauk **184**, 137 (2014) [Phys. Usp. **57**, 128 (2014)].

[35] O. A. Evseev and O. I. Melichev, No static spherically symmetric wormholes in Horndeski theory, Phys. Rev. D **97**, 124040 (2018).

[36] G. Franciolini, L. Hui, R. Penco, L. Santoni, and E. Trincherini, Stable wormholes in scalar-tensor theories, J. High Energy Phys. 01 (2019) 221.

[37] S. Mironov, V. Rubakov, and V. Volkova, More about stable wormholes in beyond Horndeski theory, Classical Quantum Gravity **36**, 135008 (2019).

[38] M. S. Morris and K. S. Thorne, Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, Am. J. Phys. **56**, 5 (1988).

[39] M. Visser, *Lorentzian Wormholes: From Einstein to Hawking* (AIP, Woodbury, 1995), p. 412.

[40] M. Visser and D. Hochberg, Generic wormhole throats, Ann. Isr. Phys. Soc. **13**, 249 (1997).