Extended analysis for the evolution of the cosmological history in Einstein-aether scalar field theory

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We consider an Einstein-aether scalar field cosmological model where the aether and the scalar field are interacting. The model of our consideration consists of the two different interacting models proposed in the literature by Kanno *et al.* and Donnelly *et al.* We perform an extended analysis for the cosmological evolution as it is provided by the field equations by using methods from dynamical systems; specifically, we determine the stationary points and we perform a stability analysis of those exact solutions.

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I. INTRODUCTION

Gravitational theories where the Lorentz symmetry is violated have drawn the attention of cosmologists over the last years [1–6]. Hořava-Lifshitz theory is a theory of quantum gravity which provides Einstein's general relativity (GR) as a limit. Hořava-Lifshitz is a renormalization theory with consistent ultraviolet behavior exhibiting an anisotropic Lifshitz scaling between time and space [7]. Hořava-Lifshitz theory has various applications in gravitational theories from cosmological studies to compact stars [8–15].

There are various problems in Hořava-Lifshitz gravity of major significance which have yet to be addressed. For example, it has not been explained in detail how the Lorentz invariance is restored in the low-energy problem; indeed various proposals have been made on that problem based on the coexistence of Hořava-Lifshitz gravity with a Lorentz-invariant matter sector with controlled quantum corrections [16,17]. In addition the complete renormalization of Hořava-Lifshitz gravity has not been proved yet [18,19]. The renormalization of the projectable Hořava-Lifshitz gravity was recently proved in Ref. [20]; however while projectable Hořava-Lifshitz theory has similar physics properties as Einstein's GR, the latter theory is not fully recovered by the projectable Hořava-Lifshitz gravity [21]. For an extended discussion we refer the reader to Refs. [22–25] and references therein.

In the classical limit Hořava-Lifshitz gravity is related to the Einstein-aether gravitational theory. There is a one way equivalence, which means that every solution of Einstein-aether theory is also a solution of Hořava-Lifshitz gravity, while the inverse is not true [26,27]. The equivalence of the two theories is not generally true for other physical properties and results which follow from the direct form of the field equations, such as the parametrized post-Newtonian constraints [24,28].

The kinematic quantities of a timelike vector field, known as the aether field, are introduced in the Einstein-Hilbert action integral, where the selection of the aether field defines the preferred frame. Important characteristics of the Einstein-aether theory are that it preserves locality and covariance, while it contains Einstein's GR [29–31].

Similarly to the Hořava-Lifshitz theory, Einstein-aether gravity has many cosmological applications. Specifically it can describe various cosmological phases such as the early-time and late-time acceleration phases of the Universe [32–38]. Other applications of Einstein-aether theory in gravitational physics can be found in Refs. [39–49] and references therein.

In Ref. [29], Donnelly and Jacobson introduced a scalar field into Einstein-aether gravity such that the scalar field and the aether field are coupled and interact. In the model of Donnelly and Jacobson the interaction term between the scalar field and the aether field is introduced by the potential term. On the other hand, Kanno and Soda in Ref. [50] considered a scalar-aether interaction theory in which the interaction is introduced in the coefficient terms of the aether field.

There are various studies in the literature of Einstein-aether gravity with a scalar field. Static spherically symmetric solutions were studied in Refs. [43,44]. Anisotropic cosmological Einstein-aether scalar field models were studied in Ref. [51–53]. Inflationary solutions for this theory were presented for the first time in Ref. [32], while an analysis of the evolution of the dynamics of Einstein-aether scalar field theory was presented in Refs. [54,55]. The analysis presented in Refs. [54,55] was based on the Einstein-aether model proposed by Donnelly and Jacobson [29]. In Ref. [54] the authors performed a complete analysis for the given scalar-field interaction potential which was

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found in Ref. [32] and provided inflationary solutions. The scalar-field interaction potential of Ref. [32] is a power series in terms of exponential functions for the scalar field and the expansion rate of the underlying Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime.

In this work we extend the analysis of Ref. [55], by considering a more generic form of the scalar-field interaction model for the Einstein-aether cosmology. Because of the form of the interaction that we assume our analysis is also valid the two different Einstein-aether scalar field theories presented by Donnelly *et al.* [29] and Kanno *et al.* [50]. The scope of this analysis is to understand the change in the dynamics and the effects on the cosmological history of the new interaction terms, as well as to compare the two different Einstein-aether scalar field cosmological models when possible. The dynamics of the field equations and the evolution of the cosmological history are studied by determining the stationary/critical points of the field equations and determining their stability. Such an analysis has been widely used in the literature in various cosmological models [56–67]. The plan of the paper is as follows.

In Sec. II we briefly discuss the Einstein-aether scalar field gravitational model and we present the cosmological field equations for the model of our study. In Sec. III we write the field equations by using dimensionless variables and the H normalization. In addition we define the four different possible families of stationary points. The main results of this work are presented in Sec. IV where we derive the stationary points for the four possible families of points and determine the stability conditions. Finally, in Sec. V we discuss our results by comparing them with those of the analysis in Ref. [55] and we draw our conclusions.

II. EINSTEIN-AETHER COSMOLOGY

Einstein-aether theory is a Lorentz-violating gravitational theory which consists of GR coupled at second derivative order to a dynamical timelike unitary vector field, the aether field, u^μ. The vector field u^μ can be thought as the four-velocity of the preferred frame.

The action integral of the Einstein-aether theory is defined as [31]

$$S_{AE} = \int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}(K^{\alpha\beta\mu\nu}u_{\mu;\alpha}u_{\nu;\beta} + \lambda(u^c u_c + 1)) + S_m. \tag{1}$$

The first term on the rhs of the action integral is the Einstein-Hilbert Lagrangian where R is the Riccis calar of the underlying geometric space with metric $g^{\mu\nu}$; the second term on the rhs of Eq. (1) is introduced by the aether theory, where u^μ is the aether field, λ is a Lagrange multiplier and the tensor $K^{\alpha\beta\mu\nu}$ is defined as

$$K^{\alpha\beta\mu\nu} \equiv c_1 g^{\alpha\beta}g^{\mu\nu} + c_2 g^{\alpha\mu}g^{\beta\nu} + c_3 g^{\alpha\nu}g^{\beta\mu} + c_4 g^{\mu\nu}u^\alpha u^\beta. \tag{2}$$

The parameters c_1, c_2, c_3 and c_4 are dimensionless constants and define the coupling between the aether field and gravity. Finally, the third term on the rhs of Eq. (1) describes the matter source.

An equivalent way of writing the action integral (1) is by using the kinematic quantities θ, σ, ω and α for the aether field, u^μ. Hence, the action (1) is written as [26]

$$S_{EA} = \int \sqrt{-g}dx^4\left(R + \frac{c_\theta}{3}\theta^2 + c_\sigma\sigma^2 + c_\omega\omega^2 + c_\alpha\alpha^2\right) + S_m, \tag{3}$$

where the parameters c_θ, c_σ, c_ω, c_a are functions of c_1, c_2, c_3 and c_4. As far as the values of the free parameters of the theory, i.e., c_1, c_2, c_3 and c_4 are concerned, they have been constrained before in the literature. Observational data from binary pulsar systems was applied in Ref. [68], while recently the gravitational-wave event GW170817 was applied [69] to test the Einstein-aether theory and constrain the free parameters. In addition, in Ref. [70] cosmological constraints were applied to constrain the Einstein-aether theory.

In this work, we assume that the action integral of the matter source S_m describes a scalar field minimally coupled to gravity but coupled to the aether field, that is [29]

$$S_m = \int \left(\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\theta,\sigma,\omega,\alpha,\phi)\right), \tag{4}$$

where the interaction between the aether field and the scalar field is described by the potential $V(\theta,\sigma,\omega,\alpha,\phi)$.

According to the cosmological principle the Universe is considered to be homogeneous and isotropic which means that it is described by the FLRW spacetime. In addition we consider the spatial curvature to be zero, from which it follows that the line element which describes the Universe on large scales is

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2). \tag{5}$$

As far as the aether field is concerned, we choose $u^\mu = \delta^\mu_t$, and calculate $\sigma = 0$, $\omega = 0$ and $\alpha = 0$. Consequently, the action integral (4) is simplified as follows:

$$S_{EA} = \int \sqrt{-g}dx^4\left(R + \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\theta,\phi)\right), \tag{6}$$

where the term $\frac{c_\theta}{3}\theta^2$ has been absorbed into the potential function $V(\theta,\phi)$. In addition, we assume that the scalar field inherits the symmetries of the spacetime, that is, $\phi = \phi(t)$.

The gravitational field equations for the latter action integral and the line element (5) are [29]

$$\frac{1}{3}\theta^2 = \frac{1}{2}\dot{\phi}^2 + V - \theta V_\theta, \tag{7}$$

$$\frac{2}{3}\dot{\theta} = -\dot{\phi}^2 - \dot{\theta}V_{\theta\theta} - \dot{\phi}V_{\theta\phi}, \tag{8}$$

$$\ddot{\phi} + \theta\dot{\phi} + V_\phi = 0. \tag{9}$$

Recall that we have assumed $k = \frac{8\pi G}{c^2} = c = 1$; however, in the following section we work with dimensionless variables and thus the physical constants play no role in our analysis.

We observe that in the limit $V(\theta,\phi) = V(\phi)$ or $V(\theta,\phi) = V(\phi) + \kappa\theta^2$, the field equations of general relativity are recovered, while in the second case the constant κ changes the gravitational constant k.

A singular universe $a(t) = a_0 t^B$ is recovered when the scalar field potential $V(\theta,\phi)$ is of the form [32]

$$V(\phi,\theta) = V_0 e^{-\lambda\theta} + \sum_{r=0}^{n} V_r \theta^r e^{\frac{r-2}{2}\lambda\phi}, \tag{10}$$

in which V_0, V_r and λ are constants; specifically V_r is the coupling constant of the scalar field with the aether field. For the scalar field the exact solution is $\phi(t) = \ln t^{\frac{2}{\lambda}}$ and the expansion rate $\theta(t) = 3Bt^{-1}$ where $B = B(V_0, V_r, \lambda)$. In Ref. [54] the latter model was studied in detail, and the general cosmological evolution was studied by determining the stationary points and their stability.

In Ref. [55] the cosmological viability of Eqs. (7)–(9) were studied for a potential of the form $V(\theta,\phi) = U(\phi) + Y(\phi)\theta$ where $U(\phi)$ and $Y(\phi)$ were arbitrary. In this potential $Y(\phi)$ is the coupling function between the scalar field and the aether field. For this generic potential form exact solutions also determined, from which we found that except for the scaling solution $a(t) = a_0 t^p$ and the de Sitter universe $a(t) = a_0 e^{H_0 t}$, we can construct other kinds of solutions such is the ΛCDM universe with $a(t) = a_0 \sinh^{\frac{2}{3}}(\sqrt{\frac{2}{3}}\Lambda t)$.

In this work we extend the analysis of Ref. [55] by assuming the potential form to be

$$V(\theta,\phi) = U(\phi) + Y(\phi)\theta + \frac{1}{3}(W^2(\phi) - 1)\theta^2. \tag{11}$$

By inserting the potential (11) into Eqs. (7)–(9) we find

$$\frac{1}{3}W^2(\phi)\theta^2 = \frac{1}{2}\dot{\phi}^2 + U(\phi), \tag{12}$$

$$\frac{2}{3}W^2(\phi)\dot{\theta} = -\dot{\phi}^2 - Y(\phi)\dot{\phi} + \frac{4}{3}WW_\phi\theta\dot{\phi}, \tag{13}$$

$$\ddot{\phi} + \theta\dot{\phi} + U_\phi + Y_\phi\theta + \frac{1}{3}(W^2(\phi))_\phi\theta^2 = 0. \tag{14}$$

The modified Friedmann equations, namely Eqs. (12) and (13), can be written in an equivalent tensor form

$$W^2(\phi)G_{ab} = T_{ab}, \tag{15}$$

where G_{ab} is the Einstein tensor, and T_{ab} is the energy-momentum tensor which describes the effective fluid source and is written as

$$T_{ab} = \rho_\phi u_a u_b + p_\phi h_{ab}, \tag{16}$$

in which $h_{ab} = g_{ab} + u_a u_b$ is the projective tensor and ρ_ϕ and p_ϕ are the effective energy density and pressure components defined as

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + U(\phi), \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - U(\phi) + \dot{\phi}Y_{,\phi} + \frac{4}{3}WW_{,\phi}\theta\dot{\phi}. \tag{17}$$

At this point, it is important to mention that while we consider the scalar-aether model proposed in Ref. [29], for the function form (11) of the unknown potential, the field equations of our model for $Y(\phi) = 0$, reduce to the model of Kanno and Soda [50]. Hence, from the following analysis we are able to compare the dynamical evolution of the two different theories.

From Eq. (15), we see that the term provides the effects of a variable gravitational "constant" k, that is $k_{\mathrm{eff}} = (W^2(\phi))^{-1}$, a similar behavior as in the scalar-tensor theories. While the scalar field is minimally coupled to gravity it interacts with the aether field, which itself is coupled to gravity.

However, while scalar-tensor theories admit a minisuperspace description this is not true for this specific model. The energy density of the effective fluid is that of the minimally coupled scalar field, while the pressure p_ϕ differs from the other terms due to the coupling components of the scalar field with the aether field.

Finally, because of the $k_{\mathrm{eff}} = (W^2(\phi))^{-1}$ term we expect a different physical evolution of the system than in the model studied in Ref. [55] where the potential was considered to be $V(\theta,\phi) = U(\phi) + Y(\phi)\theta$.

III. DIMENSIONLESS VARIABLES

In order to study the general evolution of the field equations (12)–(14) we work with the dimensionless variables defined as [56,57]

$$x = \sqrt{\frac{3}{2}\frac{\dot{\phi}^2}{W^2\theta^2}}, \qquad y = \sqrt{\frac{3U}{W^2\theta^2}},$$

$$\lambda = \frac{U_{,\phi}}{U}, \qquad \xi = \sqrt{2}\frac{Y_{,\phi}}{\sqrt{U}}, \qquad \zeta = 2\frac{W_{,\phi}}{W}. \tag{18}$$

In the new variables, the field equations are written as the following algebraic-differential system:

$$\frac{dx}{d\tau}=\frac{1}{6}(x^2-1)(3x+2\sqrt{6}\zeta)-\frac{1}{6}y^2(3x+\sqrt{6}\lambda)+\frac{1}{2}(x^2-1)y\xi, \tag{19}$$

$$\frac{dy}{d\tau}=y^2\left((1-y^2)+\frac{1}{3}x(3(x+y\xi)+\sqrt{6})(\lambda+\sqrt{6}\zeta)\right), \tag{20}$$

$$\frac{d\lambda}{d\tau}=\sqrt{\frac{2}{3}}x\lambda(\zeta+\lambda(\Gamma^{(\lambda)}(\lambda)-1)), \tag{21}$$

$$\frac{d\xi}{d\tau}=\frac{\sqrt{3}}{6}x\xi(2\xi\Gamma^{(\xi)}(\xi)-\sqrt{2}\lambda), \tag{22}$$

$$\frac{d\zeta}{d\tau}=\frac{\sqrt{6}}{3}x\Gamma^{(\zeta)}(\zeta), \tag{23}$$

with the algebraic constraint equation

$$1-x^2-y^2=0. \tag{24}$$

The new independent variable τ is defined as $\frac{d\tau}{dt}=\theta$, that is $\tau=\frac{1}{3}\ln a$ and it describes the number of e-folds while the functions $\Gamma^{(\lambda)}(\lambda),\Gamma^{(\xi)}(\xi)$ and $\Gamma^{(\zeta)}(\zeta)$ are defined as

$$\Gamma^{(\lambda)}(\lambda)=\frac{U_{\phi\phi}U}{U_\phi^2},\qquad \Gamma^{(\xi)}(\xi)=\frac{Y_{\phi\phi}\sqrt{U}}{Y_\phi^2}\quad \text{and}$$
$$\Gamma^{(\zeta)}=\zeta^2\frac{W_{\phi\phi}W}{W_\phi^2}. \tag{25}$$

In the new coordinates, the equation-of-state parameter for the total fluid w_{tot} is written as

$$w_{\mathrm{tot}}=x^2-y^2+xy\xi+\frac{4\sqrt{6}}{3}x\zeta. \tag{26}$$

One can conclude that Eqs. (19)–(23) have more degrees of freedom than the field equations in the original variables of $\{\theta,\phi\}$. However that is not true since Eqs. (19)–(23) are not independent. Specifically the variables λ, ξ, ζ are not independent and in general one can always locally write $\phi=\phi(\lambda)$, such that $\xi=\xi(\lambda)$ and $\zeta=\zeta(\lambda)$. In that case, the independent equations of the dynamical system are Eqs. (19), (20) and (21). In addition when $\zeta=0$, that is $W(\phi)=\mathrm{const}$, we see that the latter dynamical system reduces to the one of Ref. [55] as expected.

Before we continue with the rest of our analysis we present the different families of stationary points. When the variables λ, ξ and ζ are constants, that is, $U(\phi)=U_0e^{\lambda\phi}$, $Y(\phi)=Y_0-\frac{\sqrt{2}}{4}\xi e^{-\frac{\lambda}{2}\phi}$ and $W(\phi)=W_0e^{\frac{\zeta}{2}\phi}$, then the rhs of Eqs. (21), (22) and (23) are identical to zero, and the dynamical system is reduced to the two equations (19) and (20). We say that the stationary points of that system belong to Family A. The stationary points which form Family B are those of the dynamical system (19), (20) and (22) where $\lambda=\mathrm{const}$ such that $\phi=\phi(\xi)$ and $\zeta=\zeta(\xi)$.

The third family of points, namely Family C, consists of the stationary points of the dynamical system (19), (20) and (23) in which $\lambda=\mathrm{const}$ and $\xi=\mathrm{const}$, such that $\phi=\phi(\zeta)$. However, for $U(\phi)\neq U_0e^{\lambda\phi}$, such that λ is a varying function, and $\phi=\phi(\lambda)$, we end with the dynamical system (19), (20) and (21) whose stationary points form Family D.

Therefore, we conclude that the points of Family A are defined on the two-dimensional space $A=(A_x,A_y)$, while the points of Families B, C and D are defined in the three-dimensional spaces $B=(B_x,B_y,B_\xi)$, $C=(C_x,C_y,C_\zeta)$ and $D=(D_x,D_y,D_\lambda)$ respectively. However, from the constraint equation (24) all the points in the plane x-y are on the border of the unitary circle, which means that each dynamical system can be reduced by one dimension.

IV. COSMOLOGICAL EVOLUTION

In this section we present the stationary points and their stability for the dynamical systems that we defined above, and we discuss the physical quantities of the exact solutions at the stationary points.

A. Family A

The two-dimensional dynamical system (19)–(20) admits the following four stationary points (A_x,A_y) which satisfy the constraint equation (24):

$$A_1^{\pm}=(\pm1,0),$$
$$A_2^{\pm}=\left(\frac{-2\sqrt{6}(2\zeta+\lambda)\pm\sqrt{3\xi^4-6((2\zeta+\lambda)^2-6)\xi^2}}{3(4+\xi^2)},\right.$$
$$\left.1-(A_{2(x)}^{\pm})^2\right). \tag{27}$$

We observe that there are two families of points, $A_1^{\pm}$ and $A_2^{\pm}$ which include mirror points on the unitary circle.

The points $A_1^{\pm}$ describe universes where only the kinetic part of the scalar field contributes to the energy density of the effective fluid. The total equation-of-state parameter is calculated as

$$w_{\mathrm{tot}}(A_1^{\pm})=1\pm\frac{4\sqrt{6}}{3}\zeta, \tag{28}$$

from which we infer that the coupling term $\theta^2W(\phi)$ also contributes to the pressure term and modifies the equation-of-state parameter from that of a stiff fluid as in the case of general relativity. From Eq. (28) we observe that now w_{tot} can take values lower than -1. If we constrain

$|w_{\rm tot}(A_1^{\pm})| \leq 1$, then we find that $\zeta(A_1^+) \in [-\frac{1}{2}\sqrt{\frac{3}{2}}, 0]$, for point A_1^+ and $\zeta(A_1^-) \in [0, \frac{1}{2}\sqrt{\frac{3}{2}}]$, for point A_1^-.

In order to study the stability of the stationary point we replace $x = \cos\omega$ and $y = \sin\omega$ from which we find the equation

$$\frac{d\omega}{d\tau} = \frac{2\zeta + \lambda}{\sqrt{6}}\cos\omega + \cos(2\omega) + \frac{\xi}{2}\sin(2\omega), \quad (29)$$

where the points $A_1^{\pm}$ correspond to $\omega_1^+ = 2\pi N$ and $\omega_1^- = \pi + 2\pi N$, where N is an integer number. Hence, the linearized equation $\omega = \omega_1^{\pm} + \delta\omega$ around the stationary points is

$$\frac{d(\delta\omega)}{d\tau} = \left(1 \pm \frac{2\zeta + \lambda}{\sqrt{6}}\right)\delta\omega, \quad (30)$$

from which it follows that the points $A_1^{\pm}$ are stable when $(1 \pm \frac{2\zeta+\lambda}{\sqrt{6}}) < 0$, that is $\zeta(A_1^+) < -\frac{6+\sqrt{6}\lambda}{2\sqrt{6}}$ for A_1^+ and $\zeta(A_1^-) < \frac{6-\sqrt{6}\lambda}{2\sqrt{6}}$ for A_1^-. Now if we assume that the points describe accelerated solutions, that is, $w_{\rm tot}(A_1^{\pm}) < -\frac{1}{3}$ and that they are attractors we find for point A_1^+, $\{\lambda < -2\sqrt{\frac{2}{3}}, -\frac{1}{2}\sqrt{\frac{3}{2}} \leq \zeta \leq -\frac{\sqrt{6}}{6}\} \cup \{-2\sqrt{\frac{2}{3}} < \lambda < -\sqrt{\frac{3}{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}} \leq \zeta < -\frac{6+\sqrt{6}\lambda}{2\sqrt{6}}\} \cup \{\lambda = -2\sqrt{\frac{2}{3}}, -\frac{1}{2}\sqrt{\frac{3}{2}} \leq \zeta < -\frac{\sqrt{6}}{6}\}$, while for point A_1^- we find $\{\sqrt{\frac{3}{2}} \leq \lambda \leq 2\sqrt{\frac{2}{3}}, \frac{6-\sqrt{6}\lambda}{2\sqrt{6}} < \zeta \leq \frac{1}{2}\sqrt{\frac{3}{2}}\} \cup \{\lambda > 2\sqrt{\frac{2}{3}}, \frac{\sqrt{6}}{6} < \zeta \leq \frac{1}{2}\sqrt{\frac{3}{2}}\}$. The latter regions are plotted in Fig. 1.

The points $A_2^{\pm}$ depend on the three constants of the problem. Points are physically accepted when $\xi^2(\xi^2 - 2((2\zeta + \lambda)^2 - 6)) \geq 0$, that is when $\xi^2 \geq 2((2\zeta + \lambda^2) - 6)$, or when $\xi = 0$. The equation-of-state parameter at the points is calculated to be

$$w_{\rm tot}(A_2^{\pm}) = -1 - (2\zeta - \lambda)\frac{4(2\zeta + \lambda) \pm \sqrt{2}\sqrt{6\xi^4 - 4((2\zeta + \lambda)^2 - 6)}}{3(4 + \xi^2)}. \quad (31)$$

In order to conclude about the stability of the stationary points we reduce the dynamical system to one equation with dependent variable the $\omega(\tau)$. Hence, the linearized system around the stationary points $\omega_2^{\pm}$, is

$$\frac{d(\delta\omega)}{d\tau} = \frac{\sqrt{3(4 + \xi^2) - 2(2\zeta + \lambda)^2}(\sqrt{2}(2\zeta + \lambda) \mp 2\sqrt{3(4 + \xi^2) - 2(2\zeta + \lambda)^2})}{6(4 + \xi^2)}\delta\omega, \quad (32)$$

from which it follows that the point A_2^+ is stable when $\{\xi < 0, -\frac{\sqrt{6(4+\xi^2)}}{2} < Z < -\sqrt{6}\} \cup \{\xi > 0, \sqrt{6} < Z < \sqrt{\frac{3}{2}(4 + \xi^2)}\}$ in which $Z = 2\zeta + \lambda$. On the other hand, point A_2^- is an attractor when $\{\xi > 0, -\sqrt{6} < Z < \frac{\sqrt{6(4+\xi^2)}}{2}\} \cup \{\xi \leq 0, -\sqrt{\frac{3}{2}(4 + \xi^2)} < Z < \sqrt{6}\}$.

In Fig. 2, we present the region in the three-dimensional space of the free parameters $\{\lambda, \xi, \zeta\}$ in which the points $A_2^{\pm}$ are attractors, and when the solution at the point is stable and describes an accelerated universe.

B. Family B

From the rhs of Eqs. (19), (20), and (22) we find that the stationary points $B = (B_x, B_y, B_\xi)$ which belong to family B are

$$B_1^{\pm} = (\pm 1, 0, 0), \quad (33)$$

$$B_2^{\pm} = (\pm 1, 0, \xi_0), \qquad \sqrt{2}\Gamma^{(\xi)}(\xi_0)\xi_0 = \lambda, \quad (34)$$

$$B_3^{\pm} = \left(-\frac{2\sqrt{2}(2(\xi_0) + \lambda) \pm \sqrt{3\xi_0^4 - 2((2\zeta(\xi_0) + \lambda)^2 - 6)\xi_0^2}}{\sqrt{3}(4 + \xi_0^2)}, 1 - (B_{4x}^{\pm})^2, \xi_0\right), \qquad \sqrt{2}\Gamma^{(\xi)}(\xi_0)\xi_0 = \lambda, \quad (35)$$

$$B_4^{\pm} = \left(0, 1, \pm\sqrt{\frac{2}{3}}(2\zeta(\xi_0) + \lambda)\right), \quad (36)$$

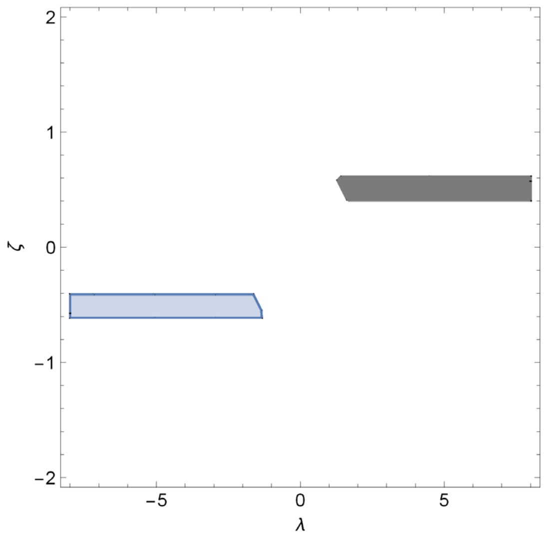

FIG. 1. Region plot in the space of variables $\{\lambda, \zeta\}$ where the exact solutions at points $A_1^{\pm}$ describe stable accelerated universes. The left region corresponds to point A_1^+, while the right region corresponds to point A_1^-.

$$B_5^{\pm} = \left(-\frac{2\zeta + \lambda}{\sqrt{6}}, \pm\frac{\sqrt{6 - (2\zeta(\xi_0) + \lambda^2)}}{\sqrt{6}}, 0\right). \quad (37)$$

The points $B_1^{\pm}$, $B_2^{\pm}$ describe the same physical solution as the points $A_1^{\pm}$ where the equation of state for the effective fluid is $w_{\mathrm{tot}}(B_1^{\pm}, B_2^{\pm}) = 1 \pm \frac{4\sqrt{6}}{3}\zeta$.

At the points $B_1^{\pm}$ there is no contribution to the evolution of the field equation from the term $Y(\phi)\theta$ since $\xi(B_2^{\pm}) = 0$. That is not true for the points $B_2^{\pm}$ where in general $\xi(B_2^{\pm}) \neq 0$ but because $y(B_2^{\pm}) = 0$ the contribution of the term $Y(\phi)\theta$ is neglected. In addition it is important to note that the points $B_2^{\pm}$ exist if and only if there exists a real solution of the algebraic equation $\sqrt{2}\Gamma^{(\xi)}(\xi_0)\xi_0 = \lambda$.

In addition the points $B_3^{\pm}$ describe the same physical solution as that of the points $A_2^{\pm}$ respectively, while $w_{\mathrm{tot}}(B_3^{\pm}) = w_{\mathrm{tot}}(A_2^{\pm})$.

The two new sets of points, namely $B_4^{\pm}$ and $B_5^{\pm}$ are of special interest since they provide additional phases of the cosmological evolution. The points $B_4^{\pm}$ describe de Sitter solutions since $w_{\mathrm{tot}}(B_3^{\pm}) = -1$. That is, the effective fluid source of the stationary points mimics the cosmological constant. On the other hand, the stationary points $B_5^{\pm}$ provide scaling solutions which can be seen as generalized versions of the scaling solution for the minimally coupled scalar field in general relativity. Indeed the limit of general relativity is recovered in the limit where $\zeta \to 0$.

1. Stability analysis

We proceed by studying the stability of the stationary points. To do that we prefer to reduce the dynamical by one dimension by applying the change of variables $x = \cos\omega$, $y = \sin\omega$, where the system (19), (20), and (22) is reduced to the following set of equations:

$$\frac{d\omega}{d\tau} = \frac{2\zeta + \lambda}{\sqrt{6}}\sin\omega + \frac{1}{2}\sin(2\omega) + \frac{\xi}{2}\sin^2\omega, \quad (38)$$

$$\frac{d\xi}{d\tau} = \frac{\sqrt{3}}{6}\xi\cos\omega(2\xi\Gamma^{(\xi)}(\xi) - \sqrt{2}\lambda). \quad (39)$$

For the points $B_1^{\pm}$ the eigenvalues of the linearized system are found to be

$$e_1(B_1^{\pm}) = \mp\frac{\lambda}{\sqrt{6}}, \qquad e_2(B_1^{\pm}) = 1 \pm \frac{\sqrt{6}}{6}(2\zeta + \lambda), \quad (40)$$

from which we can infer that B_1^+ is an attractor when $\lambda > 0$ and $(2\zeta + \lambda) < -\sqrt{6}$, while B_1^- is an attractor when $\lambda < 0$ and $(2\zeta + \lambda) > \sqrt{6}$.

For the stationary points $B_2^{\pm}$ the eigenvalues are found to be

$$e_1(B_2^{\pm}) = 1 \pm \frac{\sqrt{6}}{6}(2\zeta + \lambda),$$
$$e_2(B_2^{\pm}) = \pm\frac{1}{\sqrt{3}}\left(\frac{\sqrt{2}}{2}\lambda + \xi_0\Gamma^{(\xi)}_{,\xi}(\xi_0)\right). \quad (41)$$

Hence, at the point B_2^+ the solution is stable, when $(2\zeta + \lambda) < -\sqrt{6}$ and $\frac{\sqrt{2}}{2}\lambda + \xi_0\Gamma^{(\xi)}_{,\xi}(\xi_0) < 0$. Recall that $\sqrt{2}\Gamma^{(\xi)}(\xi_0)\xi_0 = \lambda$. In addition, point B_2^- is a stable point when $(2\zeta + \lambda) > \sqrt{6}$ and $\frac{\sqrt{2}}{2}\lambda + \xi_0\Gamma^{(\xi)}_{,\xi}(\xi_0) > 0$.

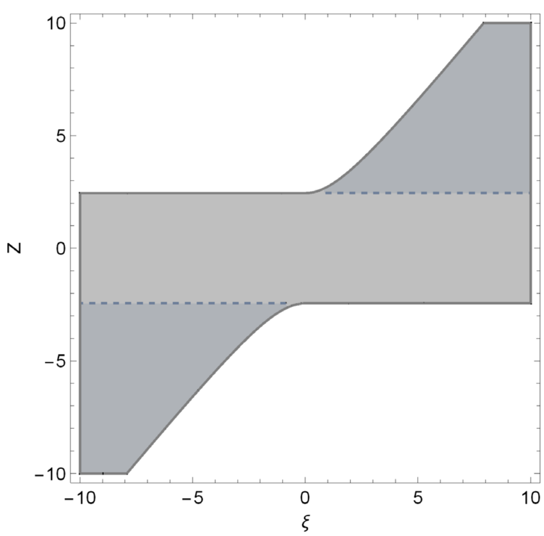

FIG. 2. Region plot in the space of variables $\{\xi, (2\zeta + \lambda)\}$ where the exact solutions at points $A_2^{\pm}$ are stable. The blue area corresponds to the values where point A_2^+ is stable, while the gray area is for point A_2^-.

The eigenvalues of the linearized system at point $B_3^{\pm}$ are derived to be

$$e_1(B_3^{\pm}) = -\frac{2\sqrt{2}(2\zeta+\lambda) \mp \sqrt{3\xi_0^4 - 2((2\xi_0+\lambda)^2-6)\xi_0^2}}{6(4+\xi_0^2)}(\sqrt{2}\lambda + 2\xi_0^2\Gamma_{,\xi}^{(\xi)}(\xi_0)), \tag{42}$$

$$e_2(B_3^{\pm}) = -1 + \frac{(2\zeta+\lambda)(4(2\zeta+\lambda) \mp \sqrt{2}\xi\sqrt{3\xi_0^4 - 2((2\xi_0+\lambda)^2-6)\xi_0^2})}{6(4+\xi_0^2)}; \tag{43}$$

however in order to infer about the stability, the parameter $\Gamma_{,\xi}^{(\xi)}(\xi_0)$ should be determined. Indeed for $\sqrt{2}\lambda + 2\xi_0^2\Gamma_{,\xi}^{(\xi)}(\xi_0) > 0$ the solution at point B_3^+ when $\xi_0 = 0$ is stable in the region $\{-\sqrt{6} < 2\zeta+\lambda < 0, \lambda > 0\}$ while when $\xi_0 \neq 0$ it is stable in the regions $\{-\sqrt{6} < 2\zeta+\lambda < 0\}$, $\{(2\zeta+\lambda) > 0, \sqrt{6}(2\zeta+\lambda) < 3\xi_0\}$ and $\{(2\zeta+\lambda) > 0, \sqrt{6}(2\zeta+\lambda) < -3\xi_0\}$. On the other hand, when $(\sqrt{2}\lambda + 2\xi_0^2\Gamma_{,\xi}^{(\xi)}(\xi_0)) < 0$ the solution at point B_3^+ when $\xi_0 = 0$ is stable in the region $\{0 < 2\zeta+\lambda < \sqrt{6}, \lambda < 0\}$ while when $\xi_0 \neq 0$ it is stable in the regions $\{(2\zeta+\lambda) < \sqrt{6}, \sqrt{6}(2\zeta+\lambda) > 3\xi_0, \xi_0 < 0\}$, $\{(2\zeta+\lambda) < \sqrt{6}, \sqrt{6}(2\zeta+\lambda) < -3\xi_0, \xi_0 > 0\}$, $\{2\zeta+\lambda > \sqrt{6}, \sqrt{6}(2\zeta+\lambda) > 3\xi_0, \sqrt{6}(\sqrt{(2\zeta+\lambda)^2-6} < 3\xi_0)\}$, and $\{2\zeta+\lambda > \sqrt{6}, \sqrt{6}(2\zeta+\lambda) > -3\xi_0, \sqrt{6}(\sqrt{(2\zeta+\lambda)^2-6} < -3\xi_0)\}$.

Similarly, when $\sqrt{2}\lambda + 2\xi_0^2\Gamma_{,\xi}^{(\xi)}(\xi_0) > 0$ the solution at point B_3^- when $\xi_0 = 0$ is an attractor in the region $\{0 < 2\zeta+\lambda < \sqrt{6}, \lambda > 0\}$ while when $\xi_0 \neq 0$ it is an attractor in the regions $\{2\zeta+\lambda < \sqrt{6}, \lambda < 0\}$, $\{2\zeta+\lambda > 0\}$, $\{\lambda < 0, 2\zeta+\lambda < 0\}$ and $\{\sqrt{6}(2\zeta+\lambda) < 3\xi_0, \sqrt{6}(2\zeta+\lambda) < -3\xi_0\}$. In addition when $\sqrt{2}\lambda + 2\xi_0^2\Gamma_{,\xi}^{(\xi)}(\xi_0) < 0$ the solution at point B_3^- $\xi_0 = 0$ is stable in the region $\{2\zeta+\lambda > -\sqrt{6}\}$ while when $\xi_0 \neq 0$ it is stable in the regions $\{\sqrt{6}(2\zeta+\lambda) > -3\xi, \sqrt{6}|(2\zeta+\lambda)| > 3|\xi|\}$, $\{\sqrt{6}|2\zeta+\lambda| > -3\xi, \sqrt{6}\sqrt{(2\zeta+\lambda)^2-6} < -3\xi\}$ and $\{\sqrt{6}|2\zeta+\lambda| > 3\xi, \sqrt{6}\sqrt{(2\zeta+\lambda)^2-6} < 3\xi\}$.

For the stationary points $B_4^{\pm}$ the eigenvalues are derived to be

$$e_1(B_4^{\pm}) = -\frac{(3 + \sqrt{9 \mp 2\Gamma_2(\xi_0)(2\zeta(\xi_0)+\lambda)(3\sqrt{2}+4\sqrt{3}\zeta_{,\xi}(\xi_0)) + 2\lambda(2\zeta(\xi_0)+\lambda)(3+2\sqrt{6})\zeta_{,\xi}(\zeta_0)})}{6}, \tag{44}$$

$$e_2(B_4^{\pm}) = -\frac{(3 - \sqrt{9 \mp 2\Gamma_2(\xi_0)(2\zeta(\xi_0)+\lambda)(3\sqrt{2}+4\sqrt{3}\zeta_{,\xi}(\xi_0)) + 2\lambda(2\zeta(\xi_0)+\lambda)(3+2\sqrt{6})\zeta_{,\xi}(\zeta_0)})}{6}. \tag{45}$$

With these eigenvalues and for $\zeta(\xi_0) = \text{const}$, i.e., $\zeta(\xi_0) = \zeta_0$ we find that the points $B_4^{\pm}$ are spiral attractors when $9 + (2\zeta_0+\lambda)(6\lambda - 4\sqrt{3}(2\zeta+\lambda)\Gamma_{,\xi}^{(\xi)}(\xi_0)) \leq 0$, while point B_4^- is also stable when $\{\lambda = 0, \zeta_0 \neq 0$ and $\Gamma_{,\xi}^{(\xi)}(\xi_0) < 0\}$ or $(2\zeta+\lambda)\Gamma_{,\xi}^{(\xi)}(\xi_0) + \sqrt{3}\lambda < 0\colon \{\lambda < 0, 2\zeta_0+\lambda < 0\}$ or the region $\{\lambda < 0, 0 < 2\zeta_0+\lambda, 8\zeta_0 + \frac{3}{\lambda} + 4\lambda \leq 0\}$ or $\{\lambda > 0, 8\zeta_0 + \frac{3}{\lambda} + 4\lambda > 0, 2\zeta_0+\lambda < 0\}$ or $\{\lambda > 0, 2\zeta_0+\lambda > 0\}$ or $(2\zeta_0+\lambda)\Gamma_{,\xi}^{(\xi)}(\xi_0) + \sqrt{3}\lambda > 0\colon \{3 + 4\lambda(3\zeta_0+\lambda) < 0\}$.

For the set of points $B_5^{\pm}$ we find the eigenvalues

$$e_1(B_5^{\pm}) = \frac{\lambda}{6}(2\zeta+\lambda), \qquad e_2(B_5^{\pm})\frac{1}{6}((2\zeta+\lambda)^2 - 6), \tag{46}$$

from which we conclude that the points $B_5^{\pm}$ are attractors for $\{\zeta < -\sqrt{15}, -2\sqrt{15} < 2\zeta+\lambda < 0\}$, $\{-\sqrt{15} < \zeta < 0, 2\zeta < 2\zeta+\lambda < 0\}$, $\{0 < \zeta < \sqrt{15}, 0 < 2\zeta+\lambda < 2\zeta\}$ and $\{\zeta > \sqrt{15}, 0 < 2\zeta+\lambda < 2\sqrt{15}\}$.

In Fig. 3 we present the phase-space diagram of the two-dimensional system in the variables $\{\omega, \xi\}$ for different values of the free parameters and for constant $\Gamma^{(\xi)}(\xi)$, i.e., $Y(\phi) = Y_0 \ln(Y_1 - Y_0 e^{-\frac{\lambda\phi}{2}})$.

C. Family C

The third system of our consideration consists of the differential equations (19), (20) and (23). This dynamical system admits the following stationary points:

$$C_1^{\pm} = (\pm 1, 0, \zeta_0), \qquad \Gamma^{(\zeta)}(\zeta_0) = 0, \tag{47}$$

$$C_2^{\pm} = \left(-\frac{2\sqrt{2}\lambda \pm \sqrt{\xi^2(3(4+\xi^2) - 2\lambda^2)}}{\sqrt{3}(4+\xi^2)}, 1 - (C_{2(x)}^{\pm})^2, \zeta_0\right), \qquad \Gamma^{(\zeta)}(\zeta_0) = 0, \tag{48}$$

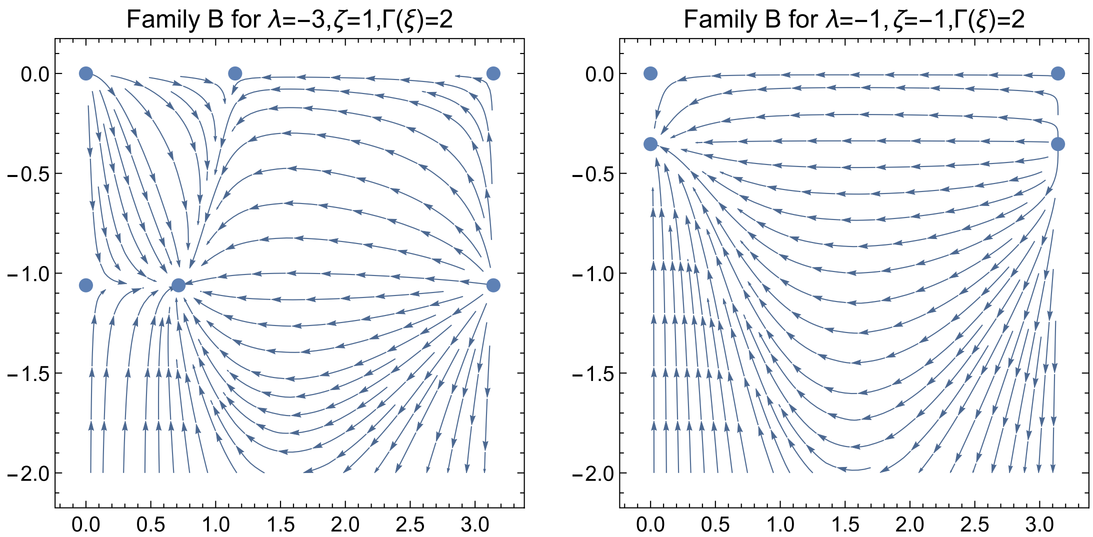

FIG. 3. Phase-space diagrams in the two-dimensional space $\{\omega,\xi\}$ for the dynamical system of Family B. The left plot is for $\{\lambda,\zeta,\Gamma^{(\xi)}\}=(-3,1,2)$ while the right plot is for $(-1,-1,2)$. The points in the plots are the critical points in the specific region of the variables.

$$C_3^{\pm}=\left(0,\pm 1,\frac{\sqrt{6}\xi-2\lambda}{4}\right), \tag{49}$$

defined in the space $C=(C_x,C_y,C_\zeta)$.

The physical properties of the solutions at points $C_1^{\pm}, C_2^{\pm}$ and $C_3^{\pm}$ are described by those of points $A_1^{\pm}, A_2^{\pm}$ and $B_4^{\pm}$ respectively, where $C_2^{\pm}$ should be seen as the special case of $A_2^{\pm}$ with $\zeta=0$. That is, points $C_1^{\pm}, C_2^{\pm}$ describe scaling solutions while points $C_3^{\pm}$ describe de Sitter universes.

1. Stability analysis

In order to study the stability of the stationary points we prefer to use the variables $\{\omega,\zeta\}$.

The eigenvalues of points $C_1^{\pm}$ are calculated as

$$e_1(C_1^{\pm})=1\pm\left(\sqrt{\frac{2}{3}}\zeta_0+\frac{\lambda}{\sqrt{6}}\right),$$
$$e_2(C_1^{\pm})=\pm\sqrt{\frac{2}{3}}\Gamma^{(\zeta)}_{,\zeta}(0), \tag{50}$$

from which we infer that point C_1^{+} is an attractor when $\{\lambda<-\sqrt{6},\Gamma^{(\zeta)}_{,\zeta}(\zeta_0)<0\}$, while C_2^{-} is an attractor when $\{\lambda>\sqrt{6},\Gamma^{(\zeta)}_{,\zeta}(\zeta_0)>0\}$.

As far as the linearized systems around the points $C_2^{\pm}$ are concerned the eigenvalues are found to be

$$e_1(C_2^{\pm})=-\frac{8\lambda\Gamma^{(\zeta)\prime}(\zeta_0)+8\zeta_0\lambda-4\lambda^2+6\xi^2+2\xi Y(\xi,\lambda)\Gamma^{(\zeta)\prime}(\zeta_0)\pm 2\zeta_0\xi Y(\xi,\lambda)\mp\lambda\xi Y(\xi,\lambda)+24+\Delta^2}{12(\xi^2+4)}, \tag{51}$$

$$e_2(C_2^{\pm})=-\frac{8\lambda\Gamma^{(\zeta)\prime}(\zeta_0)+8\zeta_0\lambda-4\lambda^2+6\xi^2+2\xi Y(\xi,\lambda)\Gamma^{(\zeta)\prime}(\zeta_0)+2\zeta_0\xi Y(\xi,\lambda)-\lambda\xi Y(\xi,\lambda)+24-\Delta^2}{12(\xi^2+4)}, \tag{52}$$

where

$$\begin{aligned}\Delta^2(\zeta_0,\xi,\lambda)=&\left(8\zeta_0\lambda-4\lambda^2+6\xi^2+2\Gamma^{(\zeta)}_{,\zeta}(\zeta_0)(4\lambda\pm\xi Y)\pm 2\zeta_0\xi Y\mp\lambda\xi Y+24\right)^2\\&+2\lambda^3(\xi^2-4)-4\lambda^2(\zeta_0(\xi^2-4)\pm\xi Y(\zeta_0,\xi,\lambda))\\&-16\Gamma^{(\zeta)}_{,\zeta}(\zeta_0)\lambda(-3\xi^4\pm 8\zeta_0\xi Y+48)+3\xi(\xi^2+4)(2\zeta_0\xi\pm Y),\end{aligned} \tag{53}$$

and $Y(\xi,\lambda)=\sqrt{6(4+\xi^2)-4\lambda^2}$. The stability conditions for that specific point will be determined in a specific application later.

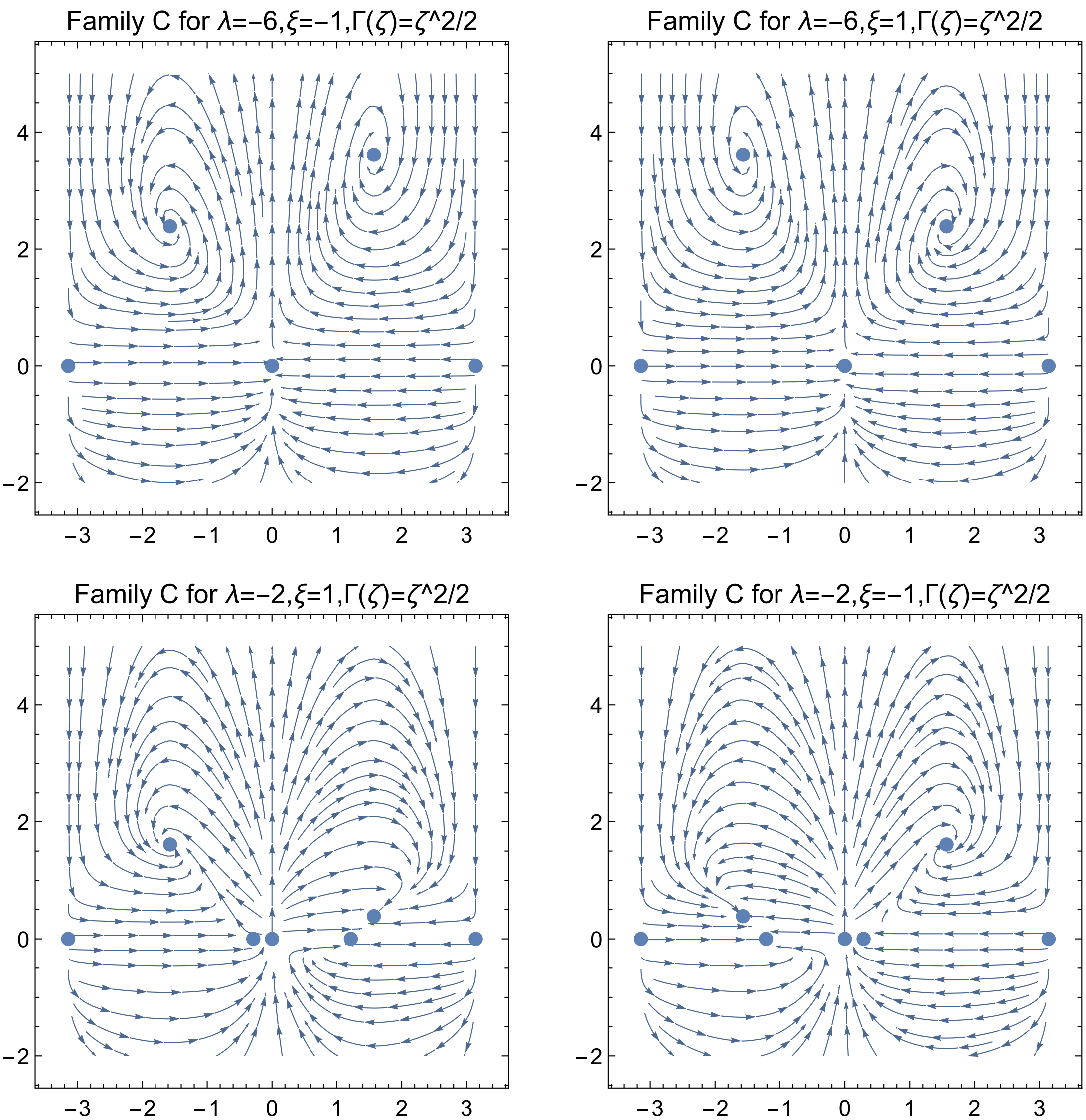

FIG. 4. Phase-space diagrams in the two-dimensional space $\{\omega, \zeta\}$ for the dynamical system of Family C. Plots are for different values of the free parameters as presented in the labels.

Finally, the eigenvalues of the linearized system at the points $C_3^{\pm}$ are

$$e_1(C_3^{\pm}) = -\frac{1}{6}(3 + \sqrt{9 - 24\Gamma^{(\zeta)}(\zeta_0)}), \qquad e_2(C_3^{\pm}) = -\frac{1}{6}(3 - \sqrt{9 - 24\Gamma^{(\zeta)}(\zeta_0)}), \tag{54}$$

where $\zeta_0 = \frac{\sqrt{6}\xi(\zeta_0) - 2\lambda}{4}$. Hence, the points $C_3^{\pm}$ are stable when $0 < \Gamma^{(\zeta)}(\zeta_0) < \frac{3}{8}$.

The phase-space diagram of the two-dimensional system in the variables $\{\omega, \zeta\}$ is presented in Fig. 4 for various values of the free parameters $\{\lambda, \xi\}$ and for $\Gamma^{(\zeta)}(\zeta) = \Gamma_0 \zeta^2$, that is, $W(\phi) = W_0 x^{-\Gamma_0}$.

D. Family D

For the fourth system of our consideration, in which $\lambda \neq \text{const}$, and from the system of equations (19), (20) and (21) we find the stationary points $D = (D_x, D_y, D_\lambda)$ as follows:

$$D_1^{\pm} = (1, 0, 0), \tag{55}$$

$$D_2^{\pm} = (1, 0, \lambda_0), \qquad \lambda_0(1 - \Gamma^{(\lambda)}(\lambda_0)) = \zeta(\lambda_0), \tag{56}$$

$$D_3^{\pm} = \left(\frac{-2\sqrt{2}(2\zeta + \lambda_0) \pm \sqrt{3(4+\xi^2)\xi^2 - 2((2\zeta+\lambda_0)\xi)^2}}{\sqrt{3}(4+\xi^2)}, 1 - (D_{3(x)}^{\pm})^2, \lambda_0 \right), \qquad \lambda_0(1 - \Gamma^{(\lambda)}(\lambda_0)) = \zeta(\lambda_0) \tag{57}$$

$$D_4^{\pm} = (0, \pm 1, \lambda_0), \qquad \lambda_0 = -\left(2\zeta + \sqrt{\frac{3}{2}}\xi\right), \tag{58}$$

$$D_5^{\pm} = \left(\frac{-4\sqrt{2}\zeta \pm \sqrt{3(4+\xi^2)\xi^2 - 8(\zeta\xi)^2}}{\sqrt{3}(4+\xi^2)}, 1 - (D_{5(x)}^{\pm})^2, 0 \right). \tag{59}$$

We observe that there are five sets of stationary points with physical properties as described by points $B_1^{\pm}, B_2^{\pm}, B_3^{\pm}, B_4^{\pm}$ and $B_5^{\pm}$ respectively. We proceed by studying the stability of the stationary points.

1. Stability analysis

As in the previous families of stationary points we study the stability of the stationary points for the two-dimensional system in the variables $\{\omega, \lambda\}$.

For the points $D_1^{\pm}$ the eigenvalues are calculated as

$$e_1(D_1^{\pm}) = \pm\sqrt{\frac{2}{3}}\zeta(0), \qquad e_2(D_1^{\pm}) = 1 \pm \frac{\sqrt{6}}{3}\zeta(0), \tag{60}$$

from which we infer that point D_1^{+} is an attractor when $\zeta(0) < -\frac{3}{2}$, while D_1^{-} is an attractor when $\zeta(0) > \frac{3}{2}$.

The eigenvalues of the linearized system at points $D_2^{\pm}$ are

$$e_1(D_2^{\pm}) = 1 \pm \frac{\sqrt{6}}{6}(2\zeta(\lambda_0) + \lambda_0),$$
$$e_2(D_2^{\pm}) = -\sqrt{\frac{2}{3}}(\zeta(\lambda_0) \mp \lambda_0(\lambda_0\Gamma_{,\lambda}^{(\lambda)}(\lambda_0) + \zeta_{\lambda}(\lambda_0))).$$

Hence, point D_2^{+} is an attractor when $(2\zeta(\lambda_0) + \lambda_0) < -\sqrt{6}$ and $\zeta(\lambda_0) > \lambda_0(\lambda_0\Gamma_{,\lambda}^{(\lambda)}(\lambda_0) + \zeta_{\lambda}(\lambda_0))$, while point D_2^{-} is an attractor when $(2\zeta(\lambda_0) + \lambda_0) > \sqrt{6}$ and $\zeta(\lambda_0) > -\lambda_0(\lambda_0\Gamma_{,\lambda}^{(\lambda)}(\lambda_0) + \zeta_{\lambda}(\lambda_0))$.

As far as the points $D_3^{\pm}$ are concerned, the eigenvalues are

$$e_1(D_3^{\pm}) = \frac{-2(3\xi^2 - 2(\lambda_0^2 - 6) + 8\zeta(\lambda_0 + \zeta)) \mp (\lambda_0 + 2\zeta)\sqrt{2(3\xi^2 - 2(\lambda_0^2 - 6) + 8\zeta(\lambda_0 + \zeta))}}{6(4+\xi^2)}, \tag{61}$$

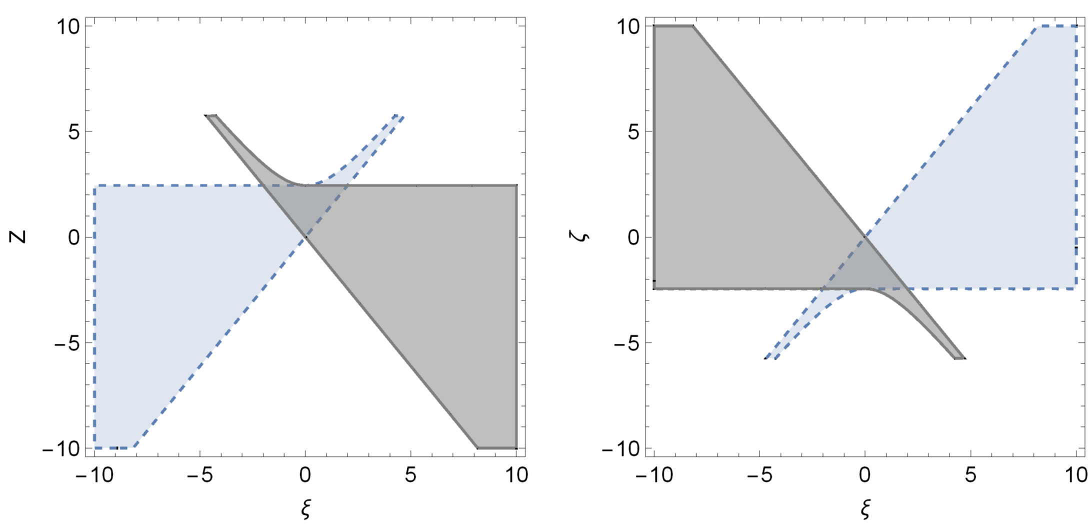

FIG. 5. Region plot in the $\{\xi, Z\}$ plane where the exact solutions at points $D_3^{\pm}$ are stable. The blue area corresponds to the values where point D_3^{+} is stable, while the gray area is for point D_3^{-}. The left panel is for $\Gamma_{\lambda}^{(\lambda)}(\lambda_0) > 0$ while the right panel is for $\Gamma_{\lambda}^{(\lambda)}(\lambda_0) < 0$.

$$e_2(D_3^{\pm}) = -\frac{2(2(\lambda_0+2\zeta)^2-3\xi^2)(\lambda_0(\lambda_0\Gamma^{(\lambda)}_{,\lambda}(\lambda_0)+\zeta_\lambda(\lambda_0))-\zeta)}{3(4\lambda_0+8\zeta\pm\xi\sqrt{2(3\xi^2-2(\lambda_0^2-6)+8\zeta(\lambda_0+\zeta))})}, \tag{62}$$

in which $\xi=\xi(\lambda_0)$ and $\zeta=\zeta(\lambda_0)$.

In order to simplify the stability conditions, we need to specify the unknown functions $\xi(\lambda),\zeta(\lambda)$ and $\Gamma^{(\lambda)}(\lambda)$. In the specific case where ξ,ζ are constants, it follows that D_3^+ is an attractor when $\Gamma^{(\lambda)}_{\lambda}(\lambda_0)>0$ in the regions $\{\lambda_0^2\Gamma^{(\lambda)}_{\lambda}(\lambda_0)>\zeta, Z<\sqrt{6},\xi<-2,|\xi|<\frac{2Z}{\sqrt{6}}\}$, $\{0<\xi<\frac{2Z}{\sqrt{6}},\sqrt{6(4+\xi^2)}>2Z\}$, and $\{-2<\xi<0,\xi<-\frac{2Z}{\sqrt{6}}\}$, while when $\Gamma^{(\lambda)}_{\lambda}(\lambda_0)<0$ it is an attractor in the regions $\{\frac{2Z}{\sqrt{6}}<\xi<0, \sqrt{6(4+\xi^2)}>-2Z\}$, $\{-\frac{2Z}{\sqrt{6}}<\xi<2\}$, $\{Z>-\sqrt{6}, 0<\xi<2,\xi<-\frac{2Z}{\sqrt{6}}\}$, and $\{\xi_0>2,\xi_0>\frac{2Z}{\sqrt{6}}\}$ where $Z=2\zeta+\lambda_0$. In addition, point D_3^- is an attractor when $\Gamma^{(\lambda)}_{\lambda}(\lambda_0)>0$ in the regions $\{Z<\sqrt{6},0<\xi<\frac{2Z}{\sqrt{6}}\}$, $\{Z<\sqrt{6}, 2<\xi<-\frac{2Z}{\sqrt{6}}\}$, $\{\xi<-\frac{2Z}{\sqrt{6}},2Z<\sqrt{6(4+\xi^2)}<0\}$, and $\{\frac{2Z}{\sqrt{6}}<\xi<2,\}$, while when $\Gamma^{(\lambda)}_{\lambda}(\lambda_0)<0$ it is an attractor in the regions $\{\xi_0<-\frac{2Z}{\sqrt{6}},\}$, $\{-2<\xi_0<\frac{2Z}{\sqrt{6}}\}$, $\{Z>-\sqrt{6},\xi<-2\}$, $\{\xi>0,\sqrt{6(4+\xi^2)}>-2Z\}$, and $\{Z>-\sqrt{6},\frac{2Z}{\sqrt{6}}<\xi<0\}$. The latter regions are plotted in Fig. 5.

For the points $D_4^{\pm}$ we find

$$e_1(D_4^{\pm}) = \frac{1}{6}(-3-\sqrt{3}\sqrt{\Delta(D_4^{\pm})}),$$
$$e_2(D_4^{\pm}) = \frac{1}{6}(-3+\sqrt{3}\sqrt{\Delta(D_4^{\pm})}), \tag{63}$$

with

$$\begin{aligned}\Delta(D_4^{\pm}) = {} & 3+4\lambda_0^2-4\lambda_0(\lambda_0\Gamma^{(\lambda)}(\lambda_0)+\zeta(\lambda_0)) \\ & -8\lambda\zeta_\lambda(\lambda_0)(\lambda_0(\Gamma^{(\lambda)}(\lambda_0)-1)+\zeta(\lambda_0)) \\ & \mp 2\sqrt{6}\lambda(\lambda(\Gamma^{(\lambda)}(\lambda_0)-1)+\zeta(\lambda_0))\xi_\lambda(\lambda_0).\end{aligned} \tag{64}$$

We cannot extract additional conditions for the stability of points $\Delta(D_4^{\pm})$ without considering special forms of the unknown functions.

The eigenvalues at points $D_5^{\pm}$ are

$$e_1(D_5^{\pm}) = -\frac{8\zeta\mp\sqrt{2}\sqrt{3(4+\xi^2)\xi^2-8(\zeta\xi)^2}}{3(4+\xi^2)}\zeta, \tag{65}$$

$$e_2(D_5^{\pm}) = \frac{(3(4+\xi^2)-8(\zeta)^2)\pm\sqrt{2}\zeta\sqrt{3(4+\xi^2)\xi^2-8(\zeta\xi)^2}}{3(4+\xi^2)}. \tag{66}$$

Therefore, point D_5^+ is an attractor when $\{\xi=0,0<\zeta<\frac{\sqrt{6}}{2}\}$; $\zeta>-\frac{\sqrt{6}}{2}$: $\{-2<\xi<0,\zeta<\frac{\sqrt{6}}{4}\xi\}$ or $\{\xi>0,\zeta<0\}$; $\zeta<\frac{\sqrt{6(4+\xi^2)}}{4}$: $\{\zeta>0,\xi<0\}$ or $\{0<\xi<\frac{4}{\sqrt{6}}\zeta\}$. On the other hand, point D_5^- is an attractor when $\xi>0$: $\{0<\zeta<\frac{3}{2}\}$; $\{4\zeta+\sqrt{6}(4+\xi^2)>0,4\zeta+\sqrt{6}\xi<0\}$ or $\xi<0$: $\{4\zeta+\sqrt{6(4+\xi^2)}>0,\zeta<0\}$ or $\{4\zeta+\sqrt{6}\xi>0,2\zeta<\sqrt{6}\}$. Recall that in the latter, ξ and ζ correspond to $\xi(0)$ and $\zeta(0)$. In Fig. 6 we present the regions in the $\{\xi(0),\zeta(0)\}$ plane where the points $D_5^{\pm}$ are attractors.

The phase-space diagram of the two-dimensional system in the variables $\{\omega,\lambda\}$ is presented in Fig. 7 for various functional forms of the free functions $\{\zeta(\lambda),\xi(\lambda),\Gamma^{(\lambda)}(\lambda)\}$.

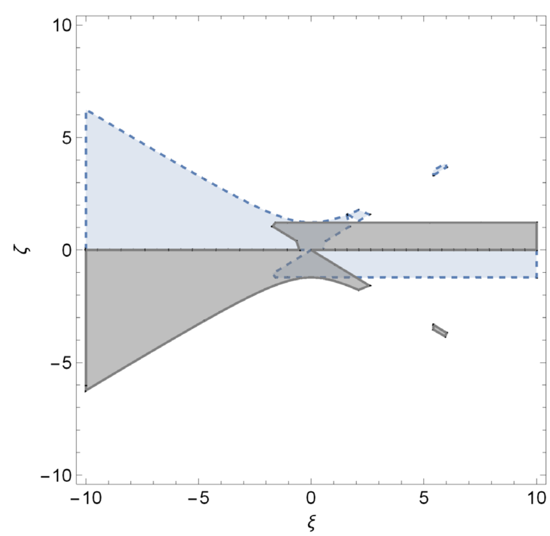

FIG. 6. Region plot in the $\{\xi,\zeta\}$ plane here the exact solutions at points $D_5^{\pm}$ are stable. The blue area corresponds to the values where point D_5^+ is stable, while the gray area is for point D_5^-.

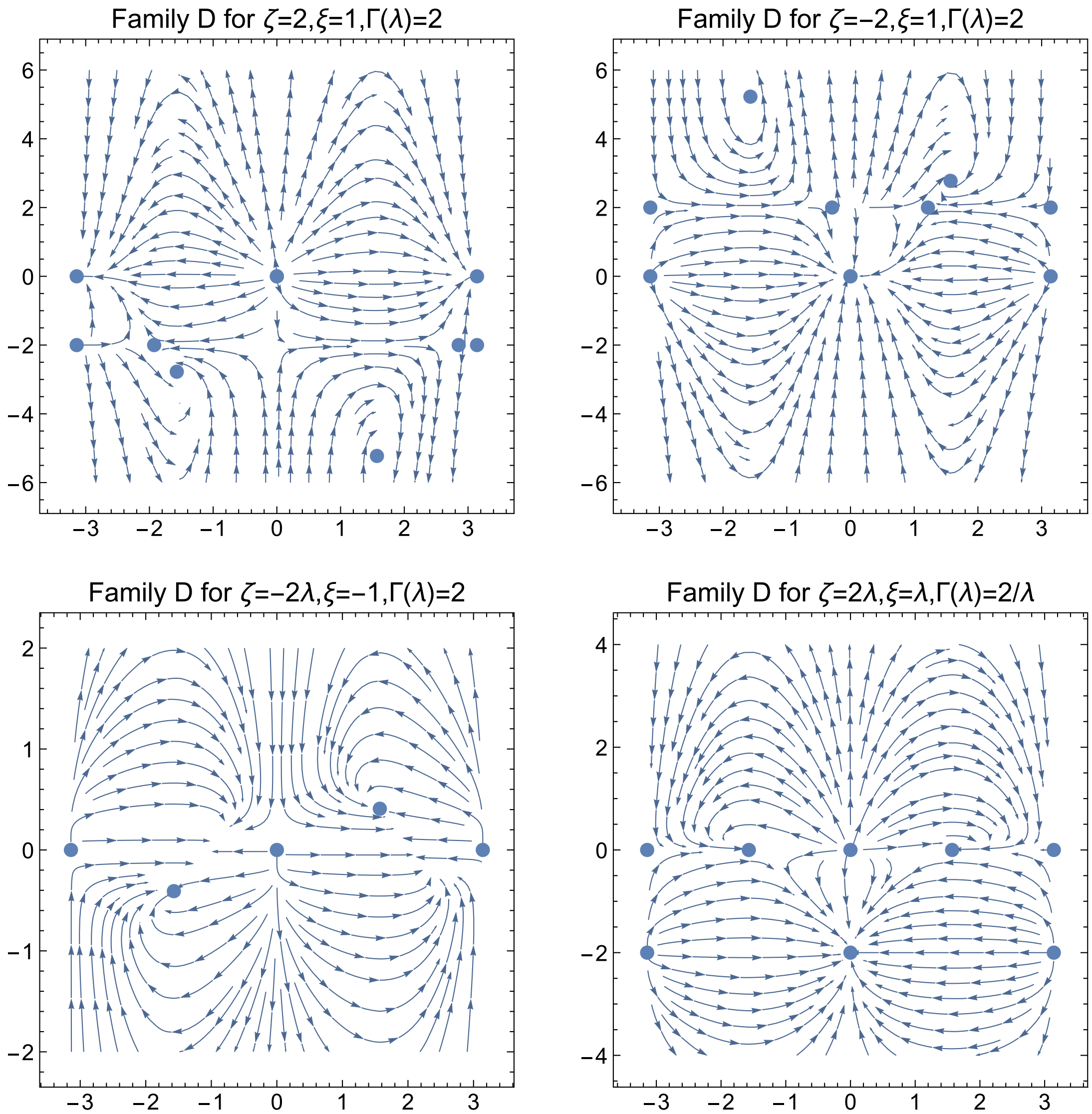

FIG. 7. Phase-space diagrams in the two-dimensional space $\{\omega,\lambda\}$ for the dynamical system of Family D. Plots are for different forms of the free functions $\{\zeta(\lambda),\xi(\lambda),\Gamma^{(\lambda)}(\lambda)\}$.

V. CONCLUSIONS

We preformed an extended analysis of the dynamics of the Einstein-aether cosmology with a scalar field coupled to the aether field by generalizing the analysis presented in Ref. [55]. Such an analysis is important in order to understand the viability of the Einstein-aether scalar field cosmology, as well as to understand the contribution of new interaction terms, between the scalar field and the aether field, in the gravitational field equations. In order to study the dynamics of the cosmological evolution we studied the field equations in dimensionless form by using the θ normalization, and we determined the stationary points. Each stationary point describes a specific phase in the cosmological history of the model.

We assumed that the scalar field and the aether field contribute to the gravitational integral a potential term of the form $V(\theta,\phi)=U(\phi)+Y(\phi)\theta+\frac{1}{3}(W^2(\phi)-1)\theta^2$, where $U(\phi), Y(\phi)$ and $W(\phi)$ are arbitrary functions. When $W^2(\phi)=1$, or in general when $W(\phi)=\text{const}$, the results of Ref. [55] are recovered. In addition when $W(\phi)=\text{const}$, $U(\phi)=V_0e^{-\lambda\phi}$ and $Y(\phi)=Y_0e^{-\frac{1}{2}\lambda\phi}$ the results of Ref. [54] are recovered. Indeed when the function $W(\phi)=\text{const}$, in Ref. [55] it was found that there are three families of stationary points, while in our consideration for the arbitrary function $W(\phi)$ there are four families of stationary points.

By writing the field equations with the use of the Einstein tensor, we observe that the contribution of $W(\phi)$ is similar with the coupling function of the scalar field with gravity in Scalar-tensor theories. While in our model the scalar field is only coupled with the aether field, however there is an undirected coupling with the gravity.

In particular $W(\phi)$ can be used to define an effective varying gravitational "constant" $k_{\rm eff} = W^{-2}(\phi)$.

The first family of stationary points in Ref. [55], namely Family $\bar{\rm A}$, consists of two sets of stationary points which describe scaling solutions. The stationary points of Family $\bar{\rm B}$ are four pairs of stationary points, while the third family of stationary points, namely Family $\bar{\rm C}$, are again four pairs of stationary points. It is important to mention that in Ref. [55] it was assumed that the parameter y is always positive.

In the model of this work, the models of Families A, B and D can be seen as the generalized versions of Families $\bar{\rm A}$, $\bar{\rm B}$ and $\bar{\rm C}$ respectively. On the other hand, Family C describes new stationary points provided by our model and specifically by the nonconstant function $W(\phi)$.

Family A consists of two pairs of stationary points which describe scaling solutions of the points of Family $\bar{\rm A}$. The stationary points of Families B and C consist of five pairs of points, with four pairs describing scaling solutions and only one pair describing a de Sitter universe. However all the points have their equivalents in Families $\bar{\rm B}$ and $\bar{\rm C}$ by using the presentation of Ref. [55]. Because the dimension of the system is different from the case where $W(\phi) = {\rm const}$ the stability conditions and the physical variables are modified; however when $W(\phi) = {\rm const}$ we end up with the same results as Ref. [55]. Family C for the model of our consideration admits six stationary points in three pairs. Two pairs describe scaling universes while the third pair of points describe de Sitter universes. Recall that the de Sitter solution is supported by cosmological observations to be the attractor of the late-time cosmic acceleration phase of the Universe.

From our analysis we found that the introduction of the new potential term in the field equations modifies the dynamics. However, while one may expect the stationary points to be different we found that there is a one-to-one correspondence between all the stationary points for $W(\phi) = {\rm const}$ and the case where $W(\phi)$ is an arbitrary function. The only new stationary points are those of Family C.

Consequently, when $V(\theta,\phi) = U(\phi) + Y(\phi)\theta$ or $V(\theta,\phi) = U(\phi) + Y(\phi)\theta + \bar{W}^2(\phi)\theta^2$, the cosmological history has a similar evolution. From the results of this work we can conclude that the model $V(\theta,\phi) = U(\phi) + Y(\phi)\theta + \bar{W}^2(\phi)\theta^2$ can describe the basic cosmological history, a similar result as that expected for the general model $V(\theta,\phi)$, since more degrees of freedom are introduced. Of course the latter conclusion follows from the evolution of the solution trajectories of the field equations. Recall that when $Y(\phi) = 0$, that is, $V(\theta,\phi) = U(\phi) + \bar{W}^2(\phi)\theta^2$ our model also describes the one considered by Kanno *et al.* [50].

For the model with $Y(\phi) = 0$ the field equations reduces to that with $\xi = 0$. Therefore, only the stationary points of families A, C and D exist with the additional constraint $\xi = 0$. Consequently, we can conclude that the introduction of the function $Y(\phi)$ enriches the evolution of the cosmological history.

Let us now discuss the physical interpretation of the critical points. Points $A_1^\pm$ and $A_2^\pm$ describe scaling solutions in general; however for specific values of the free parameters these solutions can also describe de Sitter spacetimes. Consequently, for specific ranges of the free parameters the points of Family A can describe an unstable scaling solution which describes the inflationary era, as well as a future de Sitter attractor. The situation is similar for the other families of critical points. Families B, C and D can admit more than two sets of critical points, but that does not mean that all those solutions can play a role in the cosmological evolution, since the cosmological evolution described by the field equations depends on the initial conditions, as demonstrated by the phase-space diagrams presented in Figs. 3, 4 and 7. Recall that Einstein-aether theory has been tested as a dark energy alternative in Ref. [34].

Nevertheless, if one would like to describe the complete cosmological history then radiation and dust fluids should be introduced in the field equations in order to describe the radiation- and matter-dominated epochs. By performing a similar analysis in a scenario with more matter sources in the cosmological model, we except to find critical points where the radiation fluid or the dust fluid contribute or dominate in the cosmological evolution, in order to describe the radiation and matter eras. Further, the existence of new critical points where all the fluid sources contribute are expected to exist, similarly to the quintessence and the scalar tensor theories; for more details we refer the reader to the Appendix. On the other hand, we can require the scaling solutions that we found before to describe the additional eras of the cosmological evolution; for example Brans-Dicke theory provides an ideal gas solution and $f(R)$ theory provides a radiation epoch [57,71,72].

In addition, we remark that there exist other exact solutions for the field equations (12), (13) and (14) except for the scaling and de Sitter solutions. As it was discussed in Ref. [55] because there are a greater number of unknown functions than equations of motion one can construct various analytical solutions which can describe well-studied cosmological solutions. For instance, if we assume $W(\phi) = W_0\phi(t), \phi(t) = \phi_0 t$ and $\theta(t) = \theta_1 \coth(\theta_0 t)$ in order to describe the Λ cosmology, it follows necessarily that $U(\phi) = \frac{1}{6}(2W_0^2\phi^2(\theta_1)^2\coth^2(\theta_0\phi) - 3)$ and $Y(\phi) = -\frac{2}{3}W_0^2\phi\theta_1\coth(\theta_0\phi)$; however for different functional forms of $\phi(t)$, the Λ cosmology can be recovered but for different functions $U(\phi)$ and $Y(\phi)$. The main difference between the various classical solutions that can be found is the attractor of the solution: the scaling solution $\theta(t) \simeq t^{-1}$ or the de Sitter solution $\theta(t) \simeq \theta_1$.

Additional analyses should include cosmological observations, and the effects of the interaction term at the perturbation level should also be studied. However, such analyses are beyond the purpose of this work.

From the above results we see that maybe it is not necessary to introduce more nonlinear interactions between the scalar field and the aether field, at least in the context of the cosmological solutions.

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APPENDIX: STATIONARY POINTS IN THE PRESENCE OF MATTER

In this appendix, we consider the existence of a dust fluid in the cosmological model, which does not interact with the aether or the scalar fields. The only field equation that is modified is Eq. (12) which becomes

$$\frac{1}{3}W^2(\phi)\theta^2 = \frac{1}{2}\dot{\phi}^2 + U(\phi) + \rho_m \tag{A1}$$

where ρ_m is the energy density of the fluid.

Because the second-order differential equations (13) and (14) remain the same, when we introduce the pressureless fluid, we infer that the field equations in the dimensionless variables (19)–(23) are the same. However, the constraint equation (24) reads

$$\Omega_m = 1 - x^2 - y^2 \tag{A2}$$

where now the new variable $\Omega_m = \frac{3\rho_m}{W^2(\phi)\theta^2}$, describes the energy density of the dust fluid source constrained in the range $0 \leq \Omega_m \leq 1$.

There are two main differences with the results presented in Sec. IV. First, the critical points are not necessarily points on the unitary circle in the two-dimensional space $\{x, y\}$, but they are located in the unitary disk with its center at the point (0,0). Moreover, because of the introduction of the extra variable Ω_m, the dimension of the dynamical system has been increased by one. We present the additional points for the field equations where Ω_m is different from zero. However, we do not perform a complete analysis, that is, we do not calculate the new stability conditions.

In Family A, the additional critical points are

$$A_1^m = \left(-2\sqrt{\frac{2}{3}}\zeta, 0\right), \qquad A_{2(\pm)}^m = \left(-\frac{1}{\lambda}\sqrt{\frac{3}{2}}, \frac{-\sqrt{3}\xi \pm \sqrt{3(4+\xi^2) - 16\zeta\lambda}}{2\sqrt{2}\lambda}\right)$$

from which the energy density of the dust fluid is calculated as

$$\Omega_m(A_1^m) = 1 - \frac{8}{3}\zeta^2, \tag{A3}$$

and

$$\Omega_m(A_{2(\pm)}^m) = 1 - \frac{3}{\lambda^2} + \frac{2\zeta}{\lambda} - \frac{3\xi^2}{4\lambda^2} \pm \frac{\sqrt{3}\xi}{4\lambda^2}\sqrt{3(4+\xi^2) - 16\zeta\lambda}. \tag{A4}$$

Point A_1^m is physically accepted when $|\zeta| \leq \frac{1}{2}\sqrt{\frac{3}{2}}$, while for the points $A_{2(\pm)}^m$ it follows that they are physically accepted when $3(4+\xi^2) - 16\zeta\lambda \geq 0$ and $0 \leq \Omega_m(A_{2(\pm)}^m) \leq 1$. Hence, for point $A_{2(+)}^m$ we find the constraints $\xi \leq 0$: $\{\xi = \frac{3(4+\xi^2)}{16\lambda}, \pm 4\lambda + \sqrt{6(4+\xi^2)} = 0\}$ or $\{\frac{3(4+\xi^2)}{16\lambda} \leq \zeta \leq \frac{6-2\lambda^2-|\xi|\sqrt{6-\frac{9}{\lambda^2}}\lambda}{4\lambda}, 4\lambda + \sqrt{6(4+\xi^2)} < 0\}$ or $\{\frac{6-2\lambda^2+|\xi|\sqrt{6-\frac{9}{\lambda^2}}\lambda}{4\lambda} \leq \zeta \leq \frac{3(4+\xi^2)}{16\lambda}, -4\lambda + \sqrt{6(4+\xi^2)} < 0\}$, while when $\xi > 0$: $\{4\zeta \pm \sqrt{6} = 0, \sqrt{6} + 2\lambda = 0\}$ or $\{\zeta > \frac{3(4+\xi^2)}{16\lambda}, 2(2\zeta+\lambda) \leq \frac{6+\sqrt{6-\frac{9}{\lambda^2}}\lambda\xi}{\lambda}, 4\lambda + \sqrt{6(4+\xi^2)} \leq 0\}$ or $\{\zeta \leq \frac{3(4+\xi^2)}{16\lambda}, -4\lambda + \sqrt{6(4+\xi^2)} \leq 0, 4\zeta + 2\lambda + \sqrt{6 - \frac{9}{\lambda^2}} \geq \frac{6}{\lambda^2}\}$ or $\{2(2\zeta+\lambda) \leq \frac{6+\sqrt{6-\frac{9}{\lambda^2}}\lambda\xi}{\lambda}, 4\zeta + 2\lambda + \sqrt{6-\frac{9}{\lambda^2}} \geq \frac{6}{\lambda^2}\}$ with $\{\lambda < \pm\frac{\sqrt{6}}{2}, \mp 4\lambda + \sqrt{6(4+\xi^2)} > 0\}$.

As far as the physical properties of the exact solutions at those new critical points are concerned, we observe that both the fluid sources contribute to the cosmological solution. However, in general for these points we find that the parameter for the equation of state for the effective fluid of the scalar and aether fields is different from zero at these points, which indicates that they are not tracking solutions.

In Family B we find the critical points

$$B_1^m = (A_1^m, 0), B_{2(\pm)}^m = (A_{2(\pm)}^m, \xi_0),$$
$$B_3^m = (A_1^m, \xi_0), B_{4(\pm)}^m = (A_{2(\pm)}^m, 0),$$

where $\sqrt{2}\Gamma_2(\xi_0)\xi_0 = \lambda$. We easily observe that the sets of points $\{B_1^m, B_3^m\}$ and $\{B_{2(\pm)}^m, B_{4(\pm)}^m\}$ have similar physical properties as points A_1^m and $A_{2(\pm)}^m$ respectively.

For Family C the additional stationary points are

$$C_1^m = (1, 0, 0) \quad \text{and} \quad C_{2(\pm)}^m = (A_{2(\pm)}^m, 0),$$

where C_1^m describes a universe dominated by the pressureless fluid, while $C_{2(\pm)}^m$ have the same physical properties as points $A_{2(\pm)}^m$. However, for points $C_{2(\pm)}^m$ because $\zeta = 0$ we find that the exact solutions at $C_{2(\pm)}^m$ describe tracking solutions, that is, the effective fluid of the scalar and aether fields behaves like the dust fluid.

Finally for Family D the new critical points are found to be

$$D_1^m = (A_1^m, 0), \qquad D_{2(\pm)}^m = (A_{2(\pm)}^m, \xi_0),$$
$$D_3^m = (A_1^m, \xi_0), \qquad D_{4(\pm)}^m = (A_{2(\pm)}^m, 0),$$

with a one-to-one physical correspondence with points B_1^m–$B_{4(\pm)}^m$.

In the generic scenario that the additional matter source has a pressure term of the form $p_m = (\gamma - 1)\rho_m$ where the limit $\gamma = 1$ corresponds to the dust fluid source, the field equations (19)–(23) in the dimensionless variables are modified as

$$\frac{dx}{d\tau} = \frac{1}{6}(x^2 - 1)(3x + 2\sqrt{6}\zeta) - \frac{1}{6}y^2(3x + \sqrt{6}\lambda) + \frac{1}{2}(x^2 - 1)y\xi + \frac{(\gamma - 1)}{2}x\Omega_m, \tag{A5}$$

$$\frac{dy}{d\tau} = y^2((1 - y^2) + \frac{1}{3}x(3(x + y\xi) + \sqrt{6})(\lambda + \sqrt{6}\zeta)) + y^2(\gamma - 1)\Omega_m \tag{A6}$$

$$\frac{d\lambda}{d\tau} = \sqrt{\frac{2}{3}}x\lambda(\zeta + \lambda(\Gamma^{(\lambda)}(\lambda) - 1)), \tag{A7}$$

$$\frac{d\xi}{d\tau} = \frac{\sqrt{3}}{6}x\xi(2\xi\Gamma^{(\xi)}(\xi) - \sqrt{2}\lambda), \tag{A8}$$

$$\frac{d\zeta}{d\tau} = \frac{\sqrt{6}}{3}x\Gamma^{(\zeta)}(\zeta), \tag{A9}$$

from which we find the same families of stationary points which now depend on the equation-of-state parameter for the ideal gas γ.

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