Initial conditions of inflation in a Bianchi I Universe

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We investigate the initial conditions of inflation in a Bianchi I universe that is homogeneous but not isotropic. We use the Eisenhart lift to describe such a theory geometrically as geodesics on a field-space manifold. We construct the phase-space manifold of the theory by considering the tangent bundle of the field space and equipping it with a natural metric. We find that the total volume of this manifold is finite for a wide class of inflationary models. We therefore take the initial conditions to be uniformly distributed over it in accordance with Laplace's principle of indifference. This results in a normalizable, reparametrization invariant measure on the set of initial conditions of inflation in a Bianchi I universe. We find that this measure favors an initial state in which the inflaton field is at or near its minimum, with a mild preference for some initial anisotropy. Since inflation requires an initial field value with a large displacement from its minimum, we therefore conclude that the theory of inflation requires finely tuned initial conditions.

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I. INTRODUCTION

Inflation [1–3] is the leading theory describing the early Universe. In particular, inflation is often invoked as the solution to the classic cosmological puzzles of the horizon and flatness problems. Both of these are problems of fine-tuning of the initial state of the Universe. There is nothing preventing a completely flat and homogeneous universe arising in the Hot Big Bang model, but it requires incredibly specific initial conditions. It is therefore imperative that inflation require less finely tuned initial conditions if it is to solve these problems satisfactorily.

There has been much debate in the literature as to whether the initial conditions required for inflation are finely tuned [4–24]. However, there is still no consensus with some authors claiming that inflation happens generically [25–28], while others argue precisely the opposite [29–32].

The main reason for the differences in conclusion is the infinite size of the space of allowed initial conditions. This infinity must be regulated in order to obtain a finite number for the likelihood of inflation and, as shown by [33,34], the result one obtains can depend strongly on the choice of regulator.

In [35] (hereafter F18), we constructed a measure of initial conditions that is finite for a large class of inflationary models and thus has no need for regularization. We achieved this by using the Eisenhart lift [36,37] to describe inflation as the geodesic motion on a field-space manifold. We then took as our measure of initial conditions the diffeomorphism invariant volume element of the tangent bundle of that manifold. We found the total volume of the tangent bundle to be finite provided the inflationary potential diverges faster than φ^2 as $|\varphi| \to \infty$ and is nonzero everywhere. We were thus able to normalize our measure without the use of a regulator.

In this paper, we extend the applicability of F18, whose results were calculated for a flat, homogeneous, and isotropic universe. We still assume the Universe to be flat and homogeneous, but shall relax the assumption of isotropy and thus consider a Bianchi I universe [38]. Although we do not observe anisotropy in the Universe today, there is no reason to assume it was not present in the early Universe. Indeed, smoothing out initial anisotropy is one of the key achievements of inflation [39–41]. The aim of this paper is therefore to see whether allowing for such initial anisotropy changes the results of F18.

II. THE EISENHART LIFT

We begin by briefly reviewing the Eisenhart lift [36,37] and showing how it can be used to describe scalar field theories geometrically. Consider a theory with N degrees of freedom, labeled by φ^i (collectively $\boldsymbol{\varphi}$) and with a Lagrangian $\mathcal{L}$. Let us split the Lagrangian into two parts,

$$\mathcal{L} = \sqrt{|g|}[\mathcal{L}_1(\boldsymbol{\varphi}) + \mathcal{L}_2(\boldsymbol{\varphi})], \tag{1}$$

where $g_{\mu\nu}$ with determinant g is the metric of spacetime. Our results will not depend on the nature of this splitting, but the most useful case will be when $\mathcal{L}_1$ contains the kinetic terms and $\mathcal{L}_2$ contains the potential and interaction terms.

We now add to our theory a vector field B^μ and consider a new Lagrangian,

$$\mathcal{L}' = \sqrt{|g|}\left[\mathcal{L}_1 - \frac{1}{2}\frac{M^4}{\mathcal{L}_2}\nabla_\mu B^\mu \nabla_\nu B^\nu\right], \tag{2}$$

where M is an arbitrary mass scale, introduced to keep dimensions consistent. Note that we can always set $M = M_{\mathrm{Pl}}$ through an appropriate redefinition of B^μ.

Varying (2) with respect to B^μ gives the equation of motion,

$$\partial_\mu\left(\frac{M^2\nabla_\nu B^\nu}{\mathcal{L}_2}\right) = 0, \tag{3}$$

which implies

$$A = \frac{M^2\nabla_\nu B^\nu}{\mathcal{L}_2} \tag{4}$$

is a constant of motion.

Varying (2) with respect to φ^i yields the other equation of motion,

$$\nabla_\mu\left(\frac{\partial\mathcal{L}_1}{\partial(\partial_\mu\varphi^i)}\right) - \frac{\partial\mathcal{L}_1}{\partial\varphi^i} + \nabla_\mu\left(\frac{A^2}{2}\frac{\partial\mathcal{L}_2}{\partial(\partial_\mu\varphi^i)}\right) - \frac{A^2}{2}\frac{\partial\mathcal{L}_2}{\partial\varphi^i} = 0. \tag{5}$$

As we have just seen, A is a constant. If we choose that constant to be $A = \pm\sqrt{2}$, we see that (5) reduces to

$$\nabla_\mu\left(\frac{\partial(\mathcal{L}_1+\mathcal{L}_2)}{\partial(\partial_\mu\varphi^i)}\right) - \frac{\partial(\mathcal{L}_1+\mathcal{L}_2)}{\partial\varphi^i} = 0. \tag{6}$$

These are exactly the equations of motion for Lagrangian (1). Thus, the theory described by Lagrangian (1) and the theory described by Lagrangian (2) yield exactly the same classical predictions.

We can use this result to describe any homogeneous scalar field theory in a geometric way, as was shown in [35,37]. Such a theory will have a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}k_{ij}(\boldsymbol{\varphi})\dot{\varphi}^i\dot{\varphi}^j - V(\varphi). \tag{7}$$

We now wish to apply the Eisenhart lift to this theory. Since working with homogeneous fields is equivalent to working in one dimension, the field B^μ has only one component B^0, which we can treat as another scalar field. For consistency with previous work, we shall relabel this field $B^0 \equiv \chi$.

Taking $\mathcal{L}_1 = \frac{1}{2}k_{ij}\dot{\varphi}^i\dot{\varphi}^j$ and $\mathcal{L}_2 = -V(\varphi)$, we arrive at the following equivalent Lagrangian:

$$\mathcal{L}' = \frac{1}{2}k_{ij}\dot{\varphi}^i\dot{\varphi}^j + \frac{1}{2}\frac{M^4}{V}\dot{\chi}^2 = \frac{1}{2}G_{AB}\dot{\phi}^A\dot{\phi}^B. \tag{8}$$

Here $\phi^A \equiv \{\varphi^i, \chi\}$, the indices A and B run from 1 to $N+1$ and

$$G_{AB} \equiv \begin{pmatrix} k_{ij} & 0 \\ 0 & \frac{M^4}{V} \end{pmatrix}. \tag{9}$$

The Lagrangian (8) describes a system that follows the geodesics of the $N+1$ dimensional field-space manifold with metric (9). We can therefore describe any theory of the form (7) in a purely geometric manner using this manifold.

III. INFLATION IN A BIANCHI I UNIVERSE

We shall study the theory of a single minimally coupled scalar field in a universe described by Einstein gravity. We therefore take the Lagrangian of the theory to be

$$\mathcal{L} = \sqrt{|g|}\left[-\frac{1}{2}R + \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi)\right], \tag{10}$$

where R is the Ricci scalar.

As discussed in the Introduction, we shall restrict our attention to spacetimes of Bianchi I type. The line element in such a spacetime is given by

$$ds^2 = dt^2 - a_x^2(t)dx^2 - a_y^2(t)dy^2 - a_z^2(t)dz^2. \tag{11}$$

Furthermore, we shall take the inflaton field to be homogeneous. With these restrictions, the Lagrangian (10) becomes

$$\begin{aligned}\mathcal{L} = &-a_x\dot{a}_y\dot{a}_z - a_y\dot{a}_x\dot{a}_z - a_z\dot{a}_x\dot{a}_y \\ &+ \frac{1}{2}a_xa_ya_z\dot{\varphi}^2 - a_xa_ya_zV(\varphi).\end{aligned} \tag{12}$$

This Lagrangian is of the form (7). We can therefore introduce a new scalar field χ and use the Eisenhart lift to construct an equivalent Lagrangian,

$$\mathcal{L}' = \frac{1}{2}G_{AB}\dot{\phi}^A\dot{\phi}^B, \tag{13}$$

where $\phi^A = \{a_x, a_y, a_z, \varphi, \chi\}$ and

$$G_{AB} = \begin{pmatrix} 0 & -a_z & -a_y & 0 & 0 \\ -a_z & 0 & -a_x & 0 & 0 \\ -a_y & -a_x & 0 & 0 & 0 \\ 0 & 0 & 0 & a_xa_ya_z & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a_xa_ya_zV(\varphi)} \end{pmatrix}. \tag{14}$$

Notice that we have chosen to set $M = 1$ in Planck units.

There is a one-to-one correspondence between trajectories of inflation in a Bianchi I universe and geodesics of the five-dimensional manifold with coordinates ϕ^A and metric G_{AB}.

IV. A MEASURE ON INITIAL CONDITIONS

Following the procedure from F18, we can use the field-space manifold constructed in the previous section to define a measure on the set of initial conditions for inflation. We start by constructing the phase-space manifold for the system described by (13). This is a ten-dimensional space with coordinates $\Phi^\alpha = \{\boldsymbol{\phi}, \dot{\boldsymbol{\phi}}\}$. As discussed in F18, the natural metric for the phase-space manifold is the Sasaki metric [42],

$$\mathcal{G}_{\alpha\beta} = \begin{pmatrix} G_{AB} + G_{CD}\Gamma^C_{AE}\Gamma^D_{BF}\dot{\phi}^E\dot{\phi}^F & G_{CB}\Gamma^C_{AD}\dot{\phi}^D \\ G_{AC}\Gamma^C_{DB}\dot{\phi}^D & G_{AB} \end{pmatrix}, \quad (15)$$

where $\Gamma^A_{BC} = \frac{1}{2}G^{AD}(G_{BD,C} + G_{DC,B} - G_{BC,D})$ is the Christoffel symbol for the field-space manifold.

The invariant volume element of the phase-space manifold,

$$d\Omega = \sqrt{\det \mathcal{G}} d^{10}\Phi = \det G d^{10}\Phi, \quad (16)$$

provides a natural measure on the initial conditions in this model. As shown in F18, the measure (16) is equivalent to the Liouville measure [25,30,43].

The Lagrangian (13) has five symmetries, which leave the equations of motion invariant. These symmetries are shifts of χ,

$$\chi \to \chi + c, \quad (17)$$

three spatial dilations,

$$a_i \to c a_i, \qquad \dot{a}_i \to c\dot{a}_i, \qquad \chi \to c\chi, \qquad \dot{\chi} \to c\dot{\chi}, \quad (18)$$

for $i \in \{x, y, z\}$ and time dilation,

$$\dot{a}_i \to c\dot{a}_i \quad \forall\ i, \qquad \dot{\chi} \to c\dot{\chi}, \qquad \dot{\varphi} \to c\dot{\varphi}. \quad (19)$$

In (17)–(19), c represents an arbitrary constant.

Because of these symmetries, there are redundancies in our description of the initial conditions. Any two sets of initial conditions related by one or more of the transformations (17)–(19) are physically indistinguishable and, in fact, represent the same Universe. We will therefore integrate out these symmetries to construct a measure of physically distinct initial conditions, as we did in F18.

In order to achieve this, we need to change variables to isolate the redundant degrees of freedom. To this end, we define the variables

$$H_i \equiv \frac{\dot{a}_i}{a_i}, \qquad H_\chi \equiv \frac{\dot{\chi}}{a_x a_y a_z}, \qquad \tilde{\chi} \equiv \frac{\chi}{a_x a_y a_z}. \quad (20)$$

Now, only $\tilde{\chi}$ is affected by the transformation (17) and only a_i is affected by the transformation (18). Thus, these symmetries have been isolated.

We further define

$$H_1 \equiv \frac{1}{3}(H_x + H_y + H_z), \quad (21)$$

$$H_2 \equiv \frac{1}{6}(2H_x - H_y - H_z), \quad (22)$$

$$H_3 \equiv \frac{1}{\sqrt{12}}(H_y - H_z), \quad (23)$$

which simplify the algebra by diagonalizing part of the phase-space metric (15). Note that the isotropic case now corresponds to $H_1 = H$, $H_2 = H_3 = 0$.

Finally, we isolate the symmetry (19) by defining

$$H_1 = \frac{1}{\sqrt{6}}\rho\cos\alpha\cos\gamma, \qquad H_2 = \frac{1}{\sqrt{6}}\rho\cos\alpha\sin\gamma\cos\delta,$$
$$H_3 = \frac{1}{\sqrt{6}}\rho\cos\alpha\sin\gamma\sin\delta,$$
$$H_\chi = \rho\sqrt{V}\sin\alpha\sin\beta, \qquad \dot{\varphi} = \rho\sin\alpha\cos\beta, \quad (24)$$

so that only ρ is affected by time dilations. Note that the angles defined above cover the ranges

$$\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \qquad \beta \in [0, 2\pi],$$
$$\gamma \in \left[0, \frac{\pi}{2}\right], \qquad \delta \in [0, 2\pi]. \quad (25)$$

Using the above definitions, initial conditions of inflation in this model are fully described by the initial values of the variables,

$$\Phi^\alpha = \{a_x, a_y, a_z, \varphi, \tilde{\chi}, \rho, \alpha, \beta, \gamma, \delta\}. \quad (26)$$

Of these, $\tilde{\chi}$, a_x, a_y, a_z, and ρ correspond to redundancies of description since their initial values can be arbitrarily changed by the symmetry transformations (17)–(19). We will therefore integrate out these degrees of freedom.

This means that the physically distinct sets of initial conditions can be parametrized by φ, α, β, γ, and δ. Of these, φ, α, and β were used in F18 and control the initial inflaton field value, the initial expansion rate, and the initial inflaton field velocity, respectively. In addition to those, we now have γ, which controls the total degree of initial anisotropy and δ, which controls the direction of that anisotropy.

There is one additional consideration we must take into account. Varying the action (10) with respect to g_{00} (also known as the lapse) yields the Hamiltonian constraint, which can be expressed in the variables (26) as

$$\mathcal{H} = \frac{1}{2} a_x a_y a_z \rho^2 [-\cos^2\alpha \cos(2\gamma) + \sin^2\alpha] = 0, \tag{27}$$

where $\mathcal{H}$ is the Hamiltonian of the theory (which is equal to the Lagrangian for a kinetic-only theory such as (13). This is an algebraic constraint on the variables (26) and so describes a nine-dimensional submanifold of the phase space, which we call the Hamiltonian hypersurface. Only configurations that lie on the Hamiltonian hypersurface are physically allowed initial conditions for inflation, and thus we should use a measure based on this submanifold, not the full phase space.

Following F18, we take the metric on the Hamiltonian hypersurface to be that induced on it by virtue of being embedded in the phase-space manifold. Explicitly, we choose a set of coordinates $\tilde{\Phi}^a$ (collectively $\tilde{\mathbf{\Phi}}$) on the Hamiltonian hypersurface and encode the embedding through $\Phi^\alpha = F^\alpha(\tilde{\mathbf{\Phi}})$. The induced metric is then given by

$$\tilde{\mathcal{G}}_{ab} = \frac{\partial F^\alpha}{\partial \tilde{\Phi}^a} \frac{\partial F^\beta}{\partial \tilde{\Phi}^b} \mathcal{G}_{\alpha\beta}. \tag{28}$$

We therefore take as our measure of initial conditions the invariant volume element on the Hamiltonian hypersurface,

$$d\tilde{\Omega} = \sqrt{\det \tilde{\mathcal{G}}} d^9 \tilde{\Phi}. \tag{29}$$

We choose to use (27) to eliminate the variable α and thus describe the Hamiltonian hypersurface using the coordinates

$$\tilde{\Phi}^a = \{a_x, a_y, a_z, \varphi, \tilde{\chi}, \rho, \beta, \gamma, \delta\}. \tag{30}$$

The embedding is then described by

$$F^\alpha = \{a_x, a_y, a_z, \varphi, \tilde{\chi}, \rho, \arctan(\sqrt{\cos(2\gamma)}), \beta, \gamma, \delta\}. \tag{31}$$

Note, however, that F^α only exists if

$$\gamma \leq \frac{\pi}{4}. \tag{32}$$

Therefore, as well as fixing the value of α, the Hamiltonian constraint also restricts the allowed degree of anisotropy.

With this in mind, we can proceed to calculate the induced metric on the Hamiltonian hypersurface using (28). However, as in F18, we find this metric to be singular with $\tilde{\mathcal{G}}_{6a} = \tilde{\mathcal{G}}_{a6} = 0$ for all values of the index a. Here 6 refers to the ρ coordinate. We must therefore introduce a regularization technique in order to obtain a sensible measure.

We will use the following, parametrization independent, regularization method which was also used in F18. We consider a submanifold very close to the Hamiltonian hypersurface where $\mathcal{H} = \epsilon$ and calculate the volume element on this hypersurface before taking the limit $\epsilon \to 0$.

Let us denote the induced metric on the surface $\mathcal{H} = \epsilon$ by $\tilde{\mathcal{G}}_{ab}(\epsilon)$. Then the invariant volume element on this surface is

$$d\tilde{\Omega}(\epsilon) = \sqrt{\det \tilde{\mathcal{G}}(\epsilon)} d^9 \tilde{\Phi} \approx \sqrt{\epsilon} \sqrt{\mathrm{adj}[\tilde{\mathcal{G}}(0)]^{\alpha\beta} \frac{d\tilde{\mathcal{G}}_{\alpha\beta}}{d\epsilon}\bigg|_{\epsilon=0}} d^9 \tilde{\Phi}, \tag{33}$$

where $\mathrm{adj}[\tilde{\mathcal{G}}]^{\alpha\beta}$ is the adjugate of $\tilde{\mathcal{G}}^{\alpha\beta}$ and we have used Jacobi's identity [44] to evaluate the derivative of the determinant. We see that the volume element is proportional to $\sqrt{\epsilon}$ and is thus singular when $\epsilon \to 0$ as expected. However, this overall factor will drop out when the measure is properly normalized and we can safely ignore it. We can therefore take the limit in a sensible fashion at which point the approximation in (33) becomes exact.

We perform this calculation using the variables (30) and find

$$\lim_{\epsilon \to 0} d\tilde{\Omega}(\epsilon) = \sqrt{\frac{\epsilon}{2} a_x^3 a_y^3 a_z^3 \rho^2} \frac{\sin(\gamma)}{\cos^3(\gamma)} \frac{1}{\sqrt{V(\varphi)}} d^9 \tilde{\Phi}. \tag{34}$$

As explained earlier, the initial values of $\tilde{\chi}$, a_x, a_y, a_z, and ρ are redundant degrees of freedom and so we integrate them out to obtain a measure on the physically distinguishable initial conditions. We therefore obtain the main result of this paper,

$$dP = \frac{1}{\mathcal{N}} \frac{\sin(\gamma)}{\cos^3(\gamma)} \frac{1}{\sqrt{V(\varphi)}} d\varphi d\beta d\gamma d\delta, \tag{35}$$

where dP is the measure on the initial conditions and

$$\begin{aligned} \mathcal{N} &= \int_0^{2\pi} d\beta \int_0^{\frac{\pi}{4}} d\gamma \int_0^{2\pi} d\delta \int_{-\infty}^{\infty} d\varphi \left[\frac{\sin(\gamma)}{\cos^3(\gamma)} \frac{1}{\sqrt{V(\varphi)}}\right] \\ &= 2\pi^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{V(\varphi)}} d\varphi \end{aligned} \tag{36}$$

is a normalization constant.

Notice that, just as in F18, the normalization constant $\mathcal{N}$ is finite provided the inflationary potential diverges quicker than φ^2 as $|\varphi| \to \infty$ and is nonzero everywhere. Therefore, the measure (35) is well defined and requires no regularization for this class of inflationary theories.

V. CONCLUSIONS

We have used the Eisenhart lift to describe inflation in a Bianchi I universe as the geodesic motion on a five-dimensional field-space manifold with metric (14). The tangent bundle of that manifold, equipped with a natural metric, provides a phase-space manifold that describes all possible sets of initial conditions for inflation in a Bianchi I universe.

We have shown that, once the Hamiltonian constraint has been taken into account and all redundant degrees of freedom have been integrated out, the total volume of this phase-space manifold is finite provided the inflationary potential is strictly positive and diverges quicker than φ^2 as $|\varphi| \to \infty$. This is the same class of theories found in F18. We can therefore employ Laplace's principle of indifference [45] to argue that the initial conditions should be uniformly distributed over this manifold. This results in the well-defined, normalized measure (35).

The measure (35) factorizes into two parts,

$$dP = dP_{\mathrm{FRW}} dP_{\mathrm{anis}}, \tag{37}$$

where

$$dP_{\mathrm{FRW}} = \frac{\pi}{\mathcal{N}} \frac{1}{\sqrt{V(\varphi)}} d\varphi d\beta \tag{38}$$

is the measure for the initial conditions in an FRW universe and

$$dP_{\mathrm{anis}} = \frac{1}{\pi} \frac{\sin(\gamma)}{\cos^3(\gamma)} d\gamma d\delta \tag{39}$$

is the measure for the anisotropies. The normalization constants are chosen so that the measures (38) and (39) are individually normalized. This separation allows us to analyze independently the initial inflaton field configuration and the initial spacetime geometry.

The measure (38) is identical to the one found in F18, and thus many of the same conclusions will still hold. In particular, provided $\mathcal{N}$ is finite, we find that the region of phase space in which the inflaton field is displaced from its minimum takes up a significantly smaller fraction of the measure than the region of phase space with the inflaton at or near its minimum. Thus, under this measure, initial conditions that lead to significant inflation are indeed finely tuned. A full analysis of this measure, including quantitative results, can be found in F18.

The measure on the initial spacetime geometry, which in our case is restricted to an initial anisotropy parametrized by γ and δ, is given by (39). We see that this measure is independent of the inflationary potential and is uniform over δ. The measure can therefore be considered as a distribution on the degree of anisotropy present in the initial state of the Universe, which is independent of the inflationary model. This distribution is shown in Fig. 1.

As we can see in Fig. 1, the measure slightly favors anisotropic universes over isotropic ones. However, if inflation does occur, it will dilute any initial anisotropy by an exponential amount. Therefore, anisotropy will only be observable today if it was initially exponentially large.

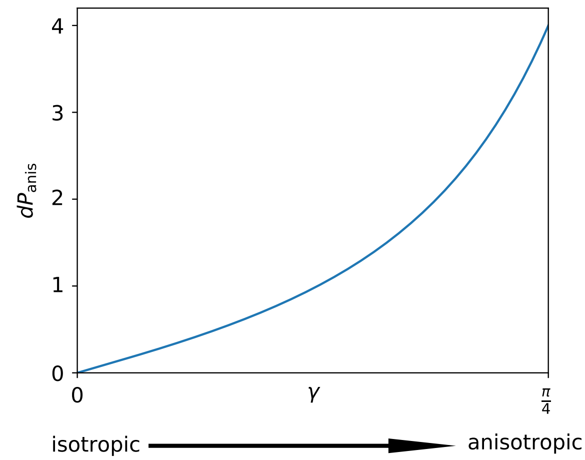

FIG. 1. Measure on the initial anisotropy of the Universe as given by (39).

Such initial conditions represent a tiny fraction of the measure (39), despite the mild enhancement of anisotropic universes. We therefore see that the inclusion of anisotropy has a negligible impact on the results of F18.

While we have relaxed the assumption of isotropy in this paper, we have still considered a patch of the Universe that is flat and homogeneous. Since inflation only requires a small patch of the Universe to be homogeneous in order to begin [46–49], this assumption is far weaker than the one made in the Hot Big Bang model. We defer investigation of the effects of inhomogeneities on the measure (35) for future work, but note that allowing for such inhomogeneities can only make inflation less likely [22–24]. Thus, the results of this paper (and indeed F18) should be thought of as upper bounds on the likelihood of inflation.

Since we are only focusing on one patch, there will be others, and one might argue that even if very few patches undergo inflation, those that do will grow to be exponentially larger than those that do not [19] leading to a Universe whose volume is dominated by patches that underwent inflation. However, such volume weighting arguments are inherently problematic. Since most models of inflation lead to an infinite universe, both the inflationary and noninflationary patches occupy an infinite volume [50–52]. Thus, it is ambiguous which occupies a greater fraction of the volume of the Universe.

Furthermore, these arguments inevitably lead to the so-called *youngness paradox* [51,53]. In most models, inflation is eternal and new universes are being born all the time. So, it is difficult to understand why we live in such an old universe. There are $e^{10^{37}}$ times as many universes that were born even one second later than ours.

We note that, as in F18, the measure (35) is finite only if the inflationary potential is strictly positive. Restricting our attention to such potentials can be justified by considering the cosmological constant [54]. If a cosmological constant is the true cause of the current era of accelerated expansion, then the minimum of the inflaton potential that we are currently in must necessarily be greater than zero. Thus, excluding a deeper minimum that would lead to instability, the potential must be greater than zero everywhere.

However, recent work on the string theory swampland conjectures has placed doubt on the possibility of a truly eternal cosmological constant (see [55] and references therein). This has led some to embrace alternatives to the cosmological constant such as quintessence [56–58]. If dark energy was indeed described by quintessence, then the inflationary potential could pass through zero leading to a divergent phase-space volume. We note, however, that no evidence for quintessence has so far been found and the swampland conjectures remain unproven. The simplest explanation of late time acceleration therefore remains the cosmological constant, which leads to a strictly positive inflationary potential and a finite measure.

Relaxing the assumption of isotropy does not solve the fine-tuning issues observed in F18. In the manifold of all possible initial conditions for a single scalar field in a Bianchi I universe, the set that allows $N > 60$ e-foldings of inflation represents only a tiny fraction. It is therefore far from clear that inflation truly solves the fine-tuning puzzles that it was designed for.

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