

Scalar-field dark energy nonminimally and kinetically coupled to dark matter

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 (Received 7 October 2019; accepted 11 February 2020; published 9 March 2020)

We provide a general framework for studying the dark energy cosmology in which a scalar field ϕ is nonminimally and kinetically coupled to cold dark matter (CDM). The scalar-graviton sector is described by the action of Horndeski theories with the speed of gravitational waves equivalent to that of light, whereas CDM is treated as a perfect fluid given by a Schutz-Sorkin action. We consider two interacting Lagrangians of the forms $f_1(\phi, X)\rho_c(n_c)$ and $f_2(n_c, \phi, X)J_c^\mu\partial_\mu\phi$, where $X = -\partial^\mu\phi\partial_\mu\phi/2$, ρ_c and n_c are the energy density and number density of CDM respectively, and J_c^μ is a vector field related to the CDM four velocity. We derive the scalar perturbation equations of motion without choosing any special gauges and identify conditions for the absence of ghosts and Laplacian instabilities on scales deep inside the sound horizon. Applying a quasistatic approximation in a gauge-invariant manner, we also obtain the effective gravitational couplings felt by CDM and baryons for the modes relevant to the linear growth of large-scale structures. In particular, the n_c dependence in the coupling f_2 gives rise to an interesting possibility for realizing the gravitational coupling with CDM weaker than the Newton gravitational constant G .

DOI: [10.1103/PhysRevD.101.063511](https://doi.org/10.1103/PhysRevD.101.063511)

I. INTRODUCTION

In spite of the numerous observational evidence for the existence of dark energy and dark matter, the origins of them have not been identified yet. Dark energy is responsible for today's cosmic acceleration [1–3], whereas dark matter is the main source for the growth of large-scale structures [4,5]. The standard cosmological paradigm is known as the Λ -cold dark matter (Λ CDM) model, in which dark energy and dark matter are attributed to the cosmological constant and weakly interacting nonrelativistic particles, respectively. The Λ CDM model is overall consistent with the current observational data, but there have been tensions for today's Hubble constant H_0 and the amplitude σ_8 of matter density contrast between the high- and low-redshift measurements [6–8].

The cosmological constant is not the only possibility for the origin of dark energy, but there are also other candidates like a time-varying scalar field ϕ . If we allow scalar nonminimal and derivative couplings to gravity, Horndeski theories [9] are the most general scalar-tensor theories with second-order field equations [10–12]. Meanwhile, the gravitational wave (GW) event GW170817 [13] constrains the speed of gravity c_t to be very close to that of light c , at the level $|c_t/c - 1| \lesssim 10^{-15}$. If we strictly demand that $c_t = c$ without allowing any tuning among functions, the Horndeski Lagrangian is restricted to be of the form $L_H = G_2(\phi, X) + G_3(\phi, X)\square\phi + G_4(\phi)R$, where $G_{2,3}$ are functions of ϕ and X , and G_4 is a function of ϕ

alone multiplied by the Ricci scalar R [14–20]. In this theoretical scheme, there are some dark energy models fitting to the observational data better than the Λ CDM [21,22]. Nevertheless, it is still a challenging issue to alleviate the tensions of H_0 and σ_8 between the high- and low-redshift measurements.

From the viewpoint of particle physics, it is natural to expect that dark energy interacts with dark matter [23]. Such a coupling can potentially alleviate the coincidence problem as to why the energy densities of two dark components are of the same orders today. For a quintessence scalar field ϕ whose continuity equation is sourced by the term $\beta\rho_c\dot{\phi}$, where β is coupling constant, ρ_c is the CDM density, and $\dot{\phi}$ is the time derivative of ϕ , there exists a scaling matter era characterized by a constant field density parameter $\Omega_\phi = 2\beta^2/3$ followed by the dark energy dominated epoch [24–27]. In this model, the likelihood analysis with cosmic microwave background (CMB) temperature anisotropies from *Planck* and other data give rise to the upper limit $\beta < 0.062$ (95% C.L.), with a mild peak of the marginalized posterior distribution around $\beta = 0.036$ [28].

There have been also several phenomenological approaches to the interacting dark energy scenario in which the couplings with CDM are added to the continuity equations by hand [29–44]. One of such examples is to introduce the interacting term $\xi H\rho_c$ or $\xi H\rho_{DE}$ to the right-hand side of the CDM continuity equation, where ξ is a

coupling constant, H is the Hubble expansion rate, and ρ_{DE} is the dark energy density. For the interaction $\xi H \rho_{\text{DE}}$, the recent joint data analysis based on *Planck*, the direct measurement of H_0 , galaxy clusterings, and cosmic shear measurements shows that the model with negative ξ (i.e., the energy transfer from CDM to dark energy) significantly reduces the tensions of H_0 and σ_8 between CMB and low-redshift measurements below the 68% C.L. level [44]. A similar conclusion was also reached in Refs. [45,46] for interacting models in which a subdominant fraction of dark matter decays after the recombination epoch.

At the covariant level, the phenomenological interactions mentioned above can render difficulties for defining physical quantities properly [47], e.g., the CDM velocity. This problem manifests itself for the dynamics of cosmological perturbations, leading to unphysical instabilities in the early Universe [36,37]. In contrast, the Lagrangian formulation of coupled dark energy and dark matter with their fully covariant energy-momentum tensors does not give rise to any ambiguities for physical quantities at the perturbation level. In this vein, Poursidou *et al.* [48] and Boehmer *et al.* [49,50] constructed interacting theories of the dark sector by using an action of the relativistic perfect fluid for CDM (see also Refs. [51–53]). The variational approach to the perfect fluid was originally advocated by Schutz and Sorkin [54] in 1977 and was further developed by Brown [55] in 1993. The corresponding action, which we call the Schutz-Sorkin action, contains an energy density ρ and a vector field J^μ associated with the fluid four velocity, together with Lagrange multipliers. Since both scalar and vector degrees of freedom can be consistently implemented in this framework, the Schutz-Sorkin action has been used to describe perfect fluids in the late-time cosmology in scalar-tensor and vector-tensor theories [56–67].

Neglecting the dependence of entropy density s , the possible interacting Lagrangian between the scalar field ϕ and the CDM density ρ_c is of the form $\mathcal{L}_{\text{int1}} = -\sqrt{-g}f_1(\phi)\rho_c(n_c)$, where g is the determinant of metric tensor $g_{\mu\nu}$, $f_1(\phi)$ is a function of ϕ , and ρ_c depends on the CDM number density n_c [49].¹ The Schutz-Sorkin Lagrangian contains the contribution $\mathcal{L}_M = -\sqrt{-g}\rho_c(n_c)$, so the total effective CDM density is given by $\hat{\rho}_c = (1 + f_1)\rho_c$. As we will see later, the energy transfer between CDM and the scalar field is particularly transparent by considering continuity equations for $\hat{\rho}_c$ and ρ_{DE} . Indeed, this interacting model accommodates the coupled quintessence scenario advocated in Refs. [23,25].

Since the Schutz-Sorkin action contains a vector field J_c^μ , we can also think of the scalar derivative coupling $J_c^\mu \partial_\mu \phi$. Indeed, the interacting Lagrangian of the form $\mathcal{L}_{\text{int2}} = f_2(n_c, \phi)J_c^\mu \partial_\mu \phi$, where f_2 is a function of n_c and ϕ , was proposed in Ref. [50]. This coupling opened up a new

window for the study of coupled dark energy. For instance, the effective gravitational interaction with matter density perturbations can be either weaker or stronger than the Newton gravitational constant on scales relevant to the linear growth of large-scale structures [59]. This is not the case for the coupled quintessence with the interacting Lagrangian $\mathcal{L}_{\text{int1}}$ mentioned above, in that the scalar-matter interaction is always attractive. Hence the signature of different couplings can be probed from the observations of galaxy clusterings and weak lensing. In particular, it will be of interest to study whether models with the weak gravitational interaction for the large-scale structure growth, as indicated in current observations [28,68–71], can be consistently constructed in this framework.

The couplings $f_1(\phi)$ and $f_2(n_c, \phi)$ can be generalized to include the dependence of field kinetic energy $X = -\partial^\mu \phi \partial_\mu \phi / 2$, such that the two interacting Lagrangians are given by $\mathcal{L}_{\text{int1}} = -\sqrt{-g}f_1(\phi, X)\rho_c(n_c)$ and $\mathcal{L}_{\text{int2}} = f_2(n_c, \phi, X)J_c^\mu \partial_\mu \phi$. The background cosmological dynamics with the first interaction was recently discussed for a canonical field with the potential $V(\phi)$ [72]. The interacting Lagrangians $\mathcal{L}_{\text{int1}}$ and $\mathcal{L}_{\text{int2}}$ may also be partially related to the theories in which CDM is conformally and disformally coupled to the metric $\bar{g}_{\mu\nu}$ different from the metric $g_{\mu\nu}$ felt by baryons [73–75]. So far, most of the past papers considered the canonical scalar or k-essence field for the dark energy sector, but we would like to extend the analysis to more general coupled Horndeski theories satisfying $c_t = c$. This allows the possibility for realizing the dark energy equation of state w_{DE} smaller than -1 [21,22].

In this paper, we provide a general framework of coupled Horndeski scalar dark energy with two interacting Lagrangians $\mathcal{L}_{\text{int1}}$ and $\mathcal{L}_{\text{int2}}$ containing the X dependence in f_1 and f_2 . We derive the scalar perturbation equations of motion on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background in a gauge-ready form and obtain the stability conditions in the small-scale limit. We also compute the effective gravitational couplings of CDM and baryons associated with the linear growth of large-scale structures. This general formulation will be useful to construct theoretically consistent models of coupled dark energy and to confront them with observations.

This paper is organized as follows. In Sec. II, we derive the background equations of motion on the flat FLRW spacetime and discuss how dark energy and CDM interact with each other. In Sec. III, we expand the total action up to second order in scalar perturbations and obtain the perturbation equations without fixing gauge conditions. In Sec. IV, we identify conditions for the absence of ghosts and Laplacian instabilities in the small-scale limit by choosing several different gauges and show that they are independent of the choice of gauges. In Sec. V, we apply the quasistatic approximation to the perturbations deep inside the sound horizon and derive the general expression

¹As in Refs. [49,65,66], this interacting Lagrangian can be written in the more general form $\mathcal{L}_{\text{int1}} = -\sqrt{-g}\rho_c(n_c, \phi)$.

of the effective gravitational couplings of CDM and baryons. Section VI is devoted to conclusions.

We use the natural unit where the speed of light c , the reduced Planck constant \hbar , and the Boltzmann constant k_B are equivalent to 1. The Newton gravitational constant G is related to the reduced Planck mass M_{pl} , as $G = 1/(8\pi M_{\text{pl}}^2)$. The Greek and Latin indices denote components in space-time and in a three-dimensional space-adapted basis, respectively.

II. LAGRANGIAN FORMULATION OF COUPLED DARK ENERGY

We consider the Lagrangian formulation of scalar-field dark energy coupled to CDM. The scalar-graviton sector is described by the action of Horndeski theories \mathcal{S}_H with the tensor propagation speed c_t equivalent to 1. Besides CDM, we also take baryons and radiations into account and assume that they do not have direct couplings to the scalar field ϕ . The noninteracting matter sector of perfect fluids is described by the Schutz-Sorkin action \mathcal{S}_M [54,55]. For the coupled action \mathcal{S}_{int} between CDM and ϕ , we consider the two types of interactions mentioned in the Introduction. Then, the total action is given by

$$\mathcal{S} = \mathcal{S}_H + \mathcal{S}_M + \mathcal{S}_{\text{int}}, \quad (2.1)$$

where

$$\mathcal{S}_H = \int d^4x \sqrt{-g} [G_4(\phi)R + G_2(\phi, X) + G_3(\phi, X)\square\phi], \quad (2.2)$$

$$\mathcal{S}_M = - \sum_{I=c,b,r} \int d^4x [\sqrt{-g}\rho_I(n_I) + J_I^\mu \partial_\mu \ell_I], \quad (2.3)$$

$$\begin{aligned} \mathcal{S}_{\text{int}} = & - \int d^4x \sqrt{-g} f_1(\phi, X) \rho_c(n_c) \\ & + \int d^4x f_2(n_c, \phi, X) J_c^\mu \partial_\mu \phi. \end{aligned} \quad (2.4)$$

Here, R is the Ricci scalar, G_4 is a function of ϕ alone, $G_{2,3}$ depend on both ϕ and $X = -\partial^\mu \phi \partial_\mu \phi/2$, and $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the d'Alembertian with the covariant derivative operator ∇_μ . The Schutz-Sorkin action (2.3) describes the perfect fluids of CDM, baryons, and radiations, labeled by c, b, r , respectively. The energy density ρ_I is a function of each fluid number density n_I , which can be expressed in terms of the vector field J_I^μ , as

$$n_I = \sqrt{\frac{J_I^\mu J_I^\nu g_{\mu\nu}}{g}}. \quad (2.5)$$

The fluid four-velocity $u_{I\mu}$ is defined by

$$u_{I\mu} \equiv \frac{J_{I\mu}}{n_I \sqrt{-g}}. \quad (2.6)$$

From Eq. (2.5), there is the relation $u_I^\mu u_{I\mu} = -1$. The scalar quantity ℓ_I in \mathcal{S}_M is a Lagrange multiplier, with the notation of the partial derivative $\partial_\mu \ell_I \equiv \partial \ell_I / \partial x^\mu$. Throughout the paper, we do not include the contribution of entropy density s_I per particle [54,55] to the matter action. The vector degrees of freedom are generally present in \mathcal{S}_M [60,63], but we do not take them into account as they are irrelevant to the cosmological dynamics in scalar-tensor theories.

Now, we are considering interacting theories in which CDM is coupled to the scalar field through the dependence of both ϕ and X . The scalar quantities constructed from the one derivative $\partial_\mu \phi$ correspond to $X = -\partial^\mu \phi \partial_\mu \phi/2$ and $Y = J_c^\mu \partial_\mu \phi$. The first interacting action in Eq. (2.4) possesses the ϕ and X dependent coupling $f_1(\phi, X)$ with the CDM density $\rho_c(n_c)$. The second interacting action in Eq. (2.4) contains the derivative coupling Y as the form $f_2(n_c, \phi, X)Y$, where f_2 is a general function of n_c , ϕ , and X .

The theories in Ref. [49] can be recovered by choosing the couplings $f_1 = f_1(\phi)$ and $f_2 = 0$ with $G_4 = M_{\text{pl}}^2/2$, $G_2 = X - V(\phi)$, and $G_3 = 0$, whereas the interactions in Ref. [50] correspond to $f_1 = 0$ and $f_2 = f_2(n_c, \phi)$. Our analysis encompasses these two coupled dark energy theories as special cases. We note that the analysis can be extended to more general couplings² containing the nonlinear terms in Y , e.g., $f_2(n_c, \phi, X, Y)$ [48,51–53]. The theories in which CDM is conformally and disformally coupled to the metric $\tilde{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\partial_\mu \phi \partial_\nu \phi$ [73–75] may be accommodated in this extended class since the term $J_c^\mu J_c^\nu \tilde{g}_{\mu\nu}$ in the definition of Eq. (2.5) generates a nonlinear term in Y . In this paper we focus on the interacting theories containing the linear term in Y , leaving more general couplings for a future work.

Variation of the action (2.1) with respect to ℓ_I leads to

$$\partial_\mu J_I^\mu = 0, \quad \text{for } I = c, b, r. \quad (2.7)$$

Varying the action (2.1) with respect to J_I^μ and using the relation $\partial n_I / \partial J_I^\mu = J_{I\mu} / (n_I g)$, it follows that

$$\partial_\mu \ell_c = u_{c\mu} \rho_{c,n_c} (1 + f_1) - \frac{1}{\sqrt{-g}} u_{c\mu} f_{2,n_c} J_c^\alpha \partial_\alpha \phi + f_2 \partial_\mu \phi, \quad (2.8)$$

$$\partial_\mu \ell_I = u_{I\mu} \rho_{I,n_I}, \quad \text{for } I = b, r, \quad (2.9)$$

where $\rho_{I,n_I} \equiv \partial \rho_I / \partial n_I$ and $f_{2,n_c} \equiv \partial f_2 / \partial n_c$.

²We thank Edmund Copeland for pointing out this issue after the initial submission of our paper on arXiv.

A. Background equations of motion

We derive the field equations of motion on the flat FLRW background given by the line element

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (2.10)$$

where $N(t)$ is the lapse, $a(t)$ is the scale factor, and t is the cosmic time. Since the fluid four-velocity in its rest frame is given by $u_I^\mu = (N^{-1}, 0, 0, 0)$, Eq. (2.6) gives the relation $J_I^0 = n_I \sqrt{-g} u_I^0$, i.e.,

$$J_I^0 = n_I a^3. \quad (2.11)$$

From Eq. (2.7), we obtain

$$\mathcal{N}_I \equiv J_I^0 = n_I a^3 = \text{constant}, \quad (2.12)$$

which means that the particle number \mathcal{N}_I of each matter species is conserved.

On the background (2.10), the action (2.1) reduces to

$$\begin{aligned} \mathcal{S} = & \int d^4x \left\{ \frac{a^3}{N} \left[N^2 G_2(\phi, X) - \left(\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} \right) G_3(\phi, X) + 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) G_4(\phi) \right] + \frac{\dot{N}}{N^2} a^3 \left[\dot{\phi} G_3(\phi, X) - 6 \frac{\dot{a}}{a} G_4(\phi) \right] \right\} \\ & - \int d^4x [N a^3 \{ [1 + f_1(\phi, X)] \rho_c + \rho_b + \rho_r \} + a^3 (n_c \dot{\epsilon}_c + n_b \dot{\epsilon}_b + n_r \dot{\epsilon}_r) - n_c a^3 f_2(n_c, \phi, X) \dot{\phi}], \end{aligned} \quad (2.13)$$

where a dot represents the derivative with respect to t . The field kinetic energy X is given by

$$X = \frac{\dot{\phi}^2}{2N^2}, \quad (2.14)$$

which contains the N dependence.

From Eq. (2.12), the number densities n_I satisfy the differential equations $\dot{n}_I + 3Hn_I = 0$ (with $I = c, b, r$), where $H \equiv \dot{a}/a$ is the Hubble expansion rate. The pressure of each matter component is given by

$$P_I(n_I) = n_I \rho_{I, n_I} - \rho_I. \quad (2.15)$$

In terms of $\rho_I(n_I)$ and $P_I(n_I)$, the conservations of particle numbers translate to the continuity equations,

$$\dot{\rho}_I + 3H(\rho_I + P_I) = 0. \quad (2.16)$$

In what follows, we will consider the perfect fluid satisfying the weak energy conditions, i.e.,

$$\rho_I > 0, \quad \rho_I + P_I > 0. \quad (2.17)$$

On the background (2.10), Eqs. (2.8) and (2.9) reduce, respectively, to

$$\dot{\epsilon}_c = -N \rho_{c, n_c} (1 + f_1) + (n_c f_{2, n_c} + f_2) \dot{\phi}, \quad (2.18)$$

$$\dot{\epsilon}_I = -N \rho_{I, n_I}, \quad \text{for } I = b, r. \quad (2.19)$$

Varying the action (2.1) with respect to N, a, ϕ , using Eqs. (2.18)–(2.19), and setting $N = 1$ at the end, we obtain the background equations,

$$\mathcal{C}_1 + \dot{\phi}^2 f_{1, X} \rho_c - n_c \dot{\phi}^3 f_{2, X} - (1 + f_1) \rho_c - \rho_b - \rho_r = 0, \quad (2.20)$$

$$\mathcal{C}_2 - n_c^2 f_{2, n_c} \dot{\phi} + (1 + f_1) P_c + P_b + P_r = 0, \quad (2.21)$$

$$\begin{aligned} \mathcal{C}_3 + [f_{1, \phi} - \dot{\phi}^2 f_{1, X\phi} - \ddot{\phi} (f_{1, X} + \dot{\phi}^2 f_{1, XX})] \rho_c + 3H \dot{\phi} f_{1, X} P_c \\ - n_c (3H n_c f_{2, n_c} - 3\dot{\phi} \ddot{\phi} f_{2, X} + 3H \dot{\phi}^2 n_c f_{2, X n_c} \\ - \dot{\phi}^3 f_{2, X\phi} - \dot{\phi}^3 \ddot{\phi} f_{2, XX}) = 0, \end{aligned} \quad (2.22)$$

where

$$\begin{aligned} \mathcal{C}_1 = & 6G_4 H^2 + G_2 - \dot{\phi}^2 G_{2, X} - \dot{\phi}^2 (G_{3, \phi} - 3H \dot{\phi} G_{3, X}) \\ & + 6H \dot{\phi} G_{4, \phi}, \end{aligned} \quad (2.23)$$

$$\mathcal{C}_2 = \mathcal{C}_1 + 2q_t \dot{H} - D_6 \ddot{\phi} + D_7 \dot{\phi}, \quad (2.24)$$

$$\mathcal{C}_3 = 2D_1 \ddot{\phi} + 3D_6 \dot{H} - D_5 + 3HD_7, \quad (2.25)$$

and

$$q_t = 2G_4, \quad (2.26)$$

$$\begin{aligned} D_1 = & \frac{1}{2} G_{2, X} + G_{3, \phi} + \frac{1}{2} \dot{\phi}^2 (G_{2, XX} + G_{3, X\phi}) \\ & - \frac{3}{2} H \dot{\phi} (2G_{3, X} + \dot{\phi}^2 G_{3, XX}), \end{aligned} \quad (2.27)$$

$$\begin{aligned} D_5 = & G_{2, \phi} - \dot{\phi}^2 (G_{2, X\phi} + G_{3, \phi\phi}) \\ & + 3H \dot{\phi} (\dot{\phi}^2 G_{3, X\phi} + 2G_{4, \phi\phi}) + 6H^2 G_{4, \phi}, \end{aligned} \quad (2.28)$$

$$D_6 = -\dot{\phi}^2 G_{3, X} - 2G_{4, \phi}, \quad (2.29)$$

$$D_7 = \dot{\phi} (G_{2, X} + 2G_{3, \phi} + 2G_{4, \phi\phi}) - H (3\dot{\phi}^2 G_{3, X} + 2G_{4, \phi}). \quad (2.30)$$

As we will see in Sec. II B, the quantity q_t is associated with the no-ghost condition of tensor perturbations.

The coefficients $D_{1,5,6,7}$ appear in the second-order action of scalar perturbations derived later in Sec. III.

As long as the condition

$$q_s \equiv 4D_1 q_t + 3D_6^2 - 2q_t(f_{1,X} + \dot{\phi}^2 f_{1,XX})\rho_c + 2q_t n_c \dot{\phi}(3f_{2,X} + \dot{\phi}^2 f_{2,XX}) \neq 0 \quad (2.31)$$

is satisfied, Eqs. (2.21) and (2.22) can be solved for \dot{H} and $\ddot{\phi}$. As we will show in Sec. IV, the absence of scalar ghosts requires that $q_s > 0$, under which the background system does not cross the singular point at which the corresponding determinant vanishes (i.e., $q_s = 0$).

We define the CDM density $\hat{\rho}_c$ and the pressure \hat{P}_c containing the effect of interactions with the scalar field, as

$$\hat{\rho}_c \equiv (1 + f_1)\rho_c, \quad (2.32)$$

$$\hat{P}_c \equiv (1 + f_1)P_c, \quad (2.33)$$

together with the CDM effective equation of state,

$$\hat{w}_c \equiv \frac{\hat{P}_c}{\hat{\rho}_c} = \frac{P_c}{\rho_c}. \quad (2.34)$$

The coupling f_1 modifies both the energy density and pressure of CDM. For the pressureless dust ($P_c = 0$), the effective pressure \hat{P}_c vanishes.

On using the continuity Eq. (2.16), the energy density $\hat{\rho}_c$ obeys

$$\dot{\hat{\rho}}_c + 3H(\hat{\rho}_c + \hat{P}_c) = \frac{f_{1,\phi} + \ddot{\phi}f_{1,X}}{1 + f_1} \hat{\rho}_c \dot{\phi}. \quad (2.35)$$

We can express Eqs. (2.20) and (2.21) in the following forms,

$$3M_{\text{pl}}^2 H^2 = \rho_{\text{DE}} + \hat{\rho}_c + \rho_b + \rho_r, \quad (2.36)$$

$$M_{\text{pl}}^2 (2\dot{H} + 3H^2) = -P_{\text{DE}} - \hat{P}_c - P_b - P_r, \quad (2.37)$$

where

$$\rho_{\text{DE}} = -G_2 + \dot{\phi}^2 G_{2,X} + \dot{\phi}^2 (G_{3,\phi} - 3H\dot{\phi}G_{3,X}) - 6H\dot{\phi}G_{4,\phi} + 3(M_{\text{pl}}^2 - 2G_4)H^2 - \dot{\phi}^2 f_{1,X}\rho_c + n_c \dot{\phi}^3 f_{2,X}, \quad (2.38)$$

$$P_{\text{DE}} = G_2 + \dot{\phi}^2 (G_{3,\phi} + \ddot{\phi}G_{3,X}) + 2G_{4,\phi}(\dot{\phi} + 2H\dot{\phi}) + 2\dot{\phi}^2 G_{4,\phi\phi} - (2\dot{H} + 3H^2)(M_{\text{pl}}^2 - 2G_4) - n_c^2 f_{2,n_c} \dot{\phi}. \quad (2.39)$$

Differentiating Eq. (2.38) with respect to t and exploiting Eq. (2.22), we obtain

$$\dot{\rho}_{\text{DE}} + 3H(\rho_{\text{DE}} + P_{\text{DE}}) = -\frac{f_{1,\phi} + \ddot{\phi}f_{1,X}}{1 + f_1} \hat{\rho}_c \dot{\phi}. \quad (2.40)$$

The sign on the right-hand side of Eq. (2.40) is opposite to that of Eq. (2.35), showing the energy exchange between the scalar field and CDM. For the couplings $f_1(\phi) = e^{Q\phi/M_{\text{pl}}} - 1$ and $f_2 = 0$, where Q is a dimensionless constant, the right-hand sides of Eqs. (2.35) and (2.40) reduce, respectively, to $Q\hat{\rho}_c \dot{\phi}/M_{\text{pl}}$ and $-Q\hat{\rho}_c \dot{\phi}/M_{\text{pl}}$. This corresponds to the coupled dark energy scenario originally proposed in Refs. [23,25].

The coupling $f_2(n_c, \phi, X)$ does not give rise to its contribution to the right-hand sides of Eqs. (2.35) and (2.40). This reflects the fact that the interaction induced by $f_2(n_c, \phi, X)$ corresponds to the momentum transfer [48,51,52]. By defining $\bar{P}_c = \hat{P}_c - n_c^2 f_{2,n_c} \dot{\phi}$ and $\bar{P}_{\text{DE}} = P_{\text{DE}} + n_c^2 f_{2,n_c} \dot{\phi}$, respectively, the right-hand sides of continuity equations for CDM and dark energy acquire the terms $-3Hn_c^2 f_{2,n_c} \dot{\phi}$ and $3Hn_c^2 f_{2,n_c} \dot{\phi}$, respectively. Here, we do not choose these definitions of effective pressures to show explicitly that the energy transfer solely comes from the coupling $f_1(\phi, X)$. With $\hat{\rho}_c$ and \hat{P}_c defined by Eqs. (2.32) and (2.33), the CDM effective equation of state (2.34) is also equivalent to the standard value P_c/ρ_c .

B. Tensor perturbations

We consider the propagation of tensor perturbations h_{ij} obeying the traceless and transverse conditions $h_i^i = 0$ and $\partial_i h_{ij} = 0$ on the flat FLRW background. The perturbed line element containing h_{ij} is given by

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j. \quad (2.41)$$

The nonvanishing components of h_{ij} can be chosen as $h_{11} = h_1(t, z)$, $h_{22} = -h_1(t, z)$, and $h_{12} = h_{21} = h_2(t, z)$, where the two independent modes h_1 and h_2 depend on t and z . We expand the total action (2.1) up to second order in h_1 , h_2 and use the background Eqs. (2.21) and (2.22). After the integration by parts, the resulting second-order tensor action yields

$$\mathcal{S}_t^{(2)} = \int dt d^3x \sum_{i=1}^2 \frac{a^3}{4} q_t \left[\dot{h}_i^2 - \frac{c_i^2}{a^2} (\partial h_i)^2 \right], \quad (2.42)$$

where q_t is defined by Eq. (2.26), and

$$c_t^2 = 1. \quad (2.43)$$

The tensor ghost is absent under the condition

$$q_t = 2G_4 > 0. \quad (2.44)$$

Since the propagation speed c_t of tensor perturbations is equivalent to that of light, the coupled dark energy theory

given by the action (2.1) is consistent with the observational bound of c_t constrained from the GW170817 event [13]. The values of q_t and c_t^2 are equivalent to those in the uncoupled theories ($f_1 = 0$ and $f_2 = 0$), so the interaction between the scalar field and CDM does not affect the propagation of gravitational waves.

III. SCALAR PERTURBATION EQUATIONS IN GAUGE-READY FORM

In this section, we first expand the action (2.1) up to second order in scalar perturbations without choosing any particular gauges. Then, we derive linear perturbation equations of motion in the form ready for fixing any gauge conditions, i.e., the gauge-ready form [64,76,77].

The general perturbed line element containing four scalar metric perturbations α, χ, ζ, E is given by [78–80]

$$ds^2 = -(1 + 2\alpha)dt^2 + 2\partial_i\chi dt dx^i + a^2(t)[(1 + 2\zeta)\delta_{ij} + 2\partial_i\partial_j E]dx^i dx^j, \quad (3.1)$$

where the perturbed quantities depend on t and x^i . The scalar field ϕ is also decomposed as

$$\phi = \bar{\phi}(t) + \delta\phi, \quad (3.2)$$

where $\bar{\phi}(t)$ is the background value and $\delta\phi$ is the perturbation. In the following, we omit the overbar from background quantities. The perturbation of field kinetic energy X , expanded up to second order, is given by

$$\delta X = \dot{\phi}(\delta\dot{\phi} - \dot{\phi}\alpha) + \frac{1}{2} \left[(\delta\dot{\phi} - 2\dot{\phi}\alpha)^2 - \frac{1}{a^2} (\partial\delta\phi + \dot{\phi}\partial\chi)^2 \right] + \mathcal{O}(\varepsilon^3), \quad (3.3)$$

where ε^n represents the n th order of perturbations.

For quantities in the Schutz-Sorkin action (2.3), we decompose the temporal and spatial components of J_I^μ (with $I = c, b, r$) into the background and perturbed parts, as

$$J_I^0 = \mathcal{N}_I + \delta J_I, \quad J_I^i = \frac{\delta^{ik}\partial_k\delta J_I}{a^2(t)}, \quad (3.4)$$

where \mathcal{N}_I is the conserved particle number, and $\delta J_I, \delta j_I$ correspond to scalar perturbations. For baryons and radiations, we recall that the Lagrange multipliers ℓ_I satisfy the relations (2.9). The velocity potentials v_I are defined by

$$u_{Ii} = -\partial_i v_I. \quad (3.5)$$

Since $J_{Ii} = J_I^0 g_{0i} + J_I^j g_{ij} = \mathcal{N}_I \partial_i \chi + \partial_i \delta J_I$ at linear order in perturbations, Eq. (2.6) gives

$$\partial_i \delta J_I = -\mathcal{N}_I (\partial_i \chi + \partial_i v_I). \quad (3.6)$$

From Eq. (2.8), the spatial derivative of ℓ_c associated with CDM perturbations yields

$$\partial_i \ell_c = -\{\rho_{c,n_c}(t)[1 + f_1(t)] - n_c(t)f_{2,n_c}(t)\dot{\phi}(t)\}\partial_i v_c + f_2(t)\partial_i \delta\phi, \quad (3.7)$$

whose integrated solution is

$$\ell_c = \mathcal{A}(t) - \{\rho_{c,n_c}(t)[1 + f_1(t)] - n_c(t)f_{2,n_c}(t)\dot{\phi}(t)\}v_c + f_2(t)\delta\phi. \quad (3.8)$$

The time-dependent function $\mathcal{A}(t)$ is determined by the condition that $\partial_0 \ell_c$ computed from Eq. (2.8) coincides with $\dot{\mathcal{A}}(t)$ at the background level. Then, it follows that

$$\begin{aligned} \ell_c = & - \int^t \{\rho_{c,n_c}(\tilde{t})[1 + f_1(\tilde{t})] \\ & - [n_c(\tilde{t})f_{2,n_c}(\tilde{t}) + f_2(\tilde{t})]\dot{\phi}(\tilde{t})\}d\tilde{t} \\ & - \{\rho_{c,n_c}(t)[1 + f_1(t)] - n_c(t)f_{2,n_c}(t)\dot{\phi}(t)\}v_c \\ & + f_2(t)\delta\phi. \end{aligned} \quad (3.9)$$

For baryons and radiations, Eq. (2.9) gives

$$\ell_I = - \int^t \rho_{I,n_I}(\tilde{t})d\tilde{t} - \rho_{I,n_I}(t)v_I, \quad \text{for } I = b, r. \quad (3.10)$$

We define the density perturbation of each matter fluid, as

$$\delta\rho_I \equiv \frac{\rho_{I,n_I}}{a^3} [\delta J_I - \mathcal{N}_I(3\zeta + \partial^2 E)]. \quad (3.11)$$

Then, the perturbation of n_I , which is expanded up to second order, is given by

$$\begin{aligned} \delta n_I = & \frac{\delta\rho_I}{\rho_{I,n_I}} - \frac{\mathcal{N}_I(\partial v_I)^2}{2a^5} - (3\zeta + \partial^2 E) \frac{\delta\rho_I}{\rho_{I,n_I}} \\ & - \frac{\mathcal{N}_I(\zeta + \partial^2 E)(3\zeta - \partial^2 E)}{2a^3} + \mathcal{O}(\varepsilon^3), \end{aligned} \quad (3.12)$$

whose first term on the right-hand side shows the consistency with the left-hand side.

The fluid density ρ_I , which depends on n_I , is expressed in the form

$$\begin{aligned} \rho_I(n_I) = & \rho_I + \rho_{I,n_I}\delta n_I + \frac{1}{2}\rho_{I,n_I n_I}\delta n_I^2 + \mathcal{O}(\varepsilon^3) \\ = & \rho_I + (\rho_I + P_I) \frac{\delta n_I}{n_I} + \frac{1}{2}(\rho_I + P_I)c_I^2 \left(\frac{\delta n_I}{n_I}\right)^2 \\ & + \mathcal{O}(\varepsilon^3), \end{aligned} \quad (3.13)$$

where c_I^2 is the matter sound speed squared defined by

$$c_I^2 = \frac{P_{I,n_I}}{\rho_{I,n_I}} = \frac{n_I \rho_{I,n_I n_I}}{\rho_{I,n_I}}. \quad (3.14)$$

In the interacting action (2.4), the coupling terms f_1 and f_2 are also expanded as

$$f_1(\phi, X) = f_1 + f_{1,\phi} \delta\phi + f_{1,X} \delta X + \frac{1}{2} f_{1,\phi\phi} \delta\phi^2 + \frac{1}{2} f_{1,XX} \delta X^2 + f_{1,X\phi} \delta X \delta\phi + \mathcal{O}(\varepsilon^3), \quad (3.15)$$

$$f_2(n_c, \phi, X) = f_2 + f_{2,n_c} \delta n_c + f_{2,\phi} \delta\phi + f_{2,X} \delta X + \frac{1}{2} f_{2,n_c n_c} \delta n_c^2 + \frac{1}{2} f_{2,\phi\phi} \delta\phi^2 + \frac{1}{2} f_{2,XX} \delta X^2 \\ + f_{2,n_c\phi} \delta n_c \delta\phi + f_{2,n_c X} \delta n_c \delta X + f_{2,X\phi} \delta X \delta\phi + \mathcal{O}(\varepsilon^3), \quad (3.16)$$

where δX and δn_c are given, respectively, by Eqs. (3.3) and (3.12).

A. Second-order action

We first expand the Horndeski action (2.2) up to quadratic order in scalar perturbations without using the background Eqs. (2.20)–(2.22). After the integration by parts, the second-order Horndeski action yields

$$S_H^{(2)} = \int dt d^3x (L_H^{\text{flat}} + L_H^\zeta + L_H^E), \quad (3.17)$$

where

$$L_H^{\text{flat}} = a^3 \left[D_1 \delta\dot{\phi}^2 + D_2 \frac{(\partial\delta\phi)^2}{a^2} + D_3 \delta\phi^2 + \left(D_4 \delta\dot{\phi} + D_5 \delta\phi + D_6 \frac{\partial^2 \delta\phi}{a^2} \right) \alpha - (D_6 \delta\dot{\phi} - D_7 \delta\phi) \frac{\partial^2 \chi}{a^2} \right. \\ \left. + (\dot{\phi} D_6 - 2H q_t) \alpha \frac{\partial^2 \chi}{a^2} + (\dot{\phi}^2 D_1 + 3H \dot{\phi} D_6 - 3H^2 q_t) \alpha^2 + \frac{1}{2} C_1 \left\{ \frac{(\partial\chi)^2}{a^2} - \alpha^2 \right\} \right], \quad (3.18)$$

$$L_H^\zeta = a^3 \left[\left\{ 3D_6 \delta\dot{\phi} - 3D_7 \delta\phi - 3(\dot{\phi} D_6 - 2H q_t) \alpha + 2q_t \frac{\partial^2 \chi}{a^2} \right\} \dot{\zeta} - 3q_t \dot{\zeta}^2 + q_t \frac{(\partial\zeta)^2}{a^2} \right. \\ \left. - (B_1 \delta\phi + 2q_t \alpha) \frac{\partial^2 \zeta}{a^2} + 3 \left(C_1 \alpha + \frac{1}{2} C_2 \zeta - C_3 \delta\phi \right) \zeta \right], \quad (3.19)$$

$$L_H^E = a^3 \left[2q_t \ddot{\zeta} + 2B_2 \dot{\zeta} - D_6 \delta\ddot{\phi} - B_3 \delta\dot{\phi} + B_4 \delta\phi + \frac{1}{a^3} \frac{d}{dt} \{ a^3 (\dot{\phi} D_6 - 2H q_t) \alpha \} \right. \\ \left. + C_1 \alpha + C_2 \left(\zeta - \frac{1}{2} \partial^2 E \right) - C_3 \delta\phi \right] \partial^2 E, \quad (3.20)$$

with the coefficients

$$D_2 = -\frac{1}{2} G_{2,X} - G_{3,\phi} + 2H \dot{\phi} G_{3,X} + \frac{1}{2} \dot{\phi}^2 G_{3,X\phi} + \frac{1}{2} (2G_{3,X} + G_{3,XX} \dot{\phi}^2) \ddot{\phi}, \\ D_3 = \frac{1}{2} G_{2,\phi\phi} - \frac{1}{2} (G_{2,X\phi\phi} + G_{3,\phi\phi\phi}) \dot{\phi}^2 + \frac{3}{2} (G_{3,X\phi\phi} \dot{\phi}^2 - G_{2,X\phi} - 2G_{3,\phi\phi}) H \dot{\phi} \\ + \frac{3}{2} (\dot{\phi}^2 G_{3,X\phi} + 2G_{4,\phi\phi}) \dot{H} + \frac{3}{2} (3\dot{\phi}^2 G_{3,X\phi} + 4G_{4,\phi\phi}) H^2 \\ - \left[\frac{1}{2} G_{2,X\phi} + G_{3,\phi\phi} - \frac{3}{2} (G_{3,XX\phi} \dot{\phi}^2 + 2G_{3,X\phi}) H \dot{\phi} + \frac{1}{2} (G_{2,XX\phi} + G_{3,X\phi\phi}) \dot{\phi}^2 \right] \ddot{\phi}, \\ D_4 = -(G_{2,X} + 2G_{3,\phi}) \dot{\phi} - (G_{2,XX} + G_{3,X\phi}) \dot{\phi}^3 + 3(3G_{3,X} \dot{\phi}^2 + G_{3,XX} \dot{\phi}^4 + 2G_{4,\phi}) H, \quad (3.21)$$

$$B_1 = \frac{2}{\dot{\phi}} \dot{q}_t, \quad B_2 = \dot{q}_t + 3Hq_t, \quad B_3 = \dot{D}_6 + 3HD_6 - D_7, \quad B_4 = \dot{D}_7 + 3HD_7. \quad (3.22)$$

We recall that the coefficients $\mathcal{C}_{1,2,3}$ and $D_{1,5,6,7}$ are given by Eqs. (2.23)–(2.25) and Eqs. (2.27)–(2.30), respectively. The coefficients $D_{2,3,4}$ can be expressed by using q_t , $D_{1,5,6,7}$ and their time derivatives, as

$$2\dot{\phi}^2 D_2 = -2H\dot{q}_t - \dot{\phi}(\dot{D}_6 + HD_6 + D_7), \quad (3.23)$$

$$2\dot{\phi} D_3 = \dot{D}_5 + 3HD_5 - 3H(\dot{D}_7 + 3HD_7) - 3H[f_{1,\phi} - f_{1,X}\dot{\phi}^2 - (f_{1,X} + f_{1,XX}\dot{\phi}^2)\dot{\phi}]\rho_c - 9H^2\dot{\phi}f_{1,X}P_c \\ + 3Hn_c(3Hn_c f_{2,n_c} - 3\dot{\phi}\ddot{\phi}f_{2,X} + 3H\dot{\phi}^2 n_c f_{2,n_c X} - \dot{\phi}^3 f_{2,X\phi} - \dot{\phi}^3 \ddot{\phi} f_{2,XX}), \quad (3.24)$$

$$D_4 = -2\dot{\phi}D_1 - 3HD_6, \quad (3.25)$$

where we used Eq. (2.22) for the derivation of Eq. (3.24). The second-order Horndeski action (3.17) is written in the gauge-ready form. If we choose the flat gauge in which both ζ and E vanish, we have that $L_H^\zeta = 0$ and $L_H^E = 0$. In this case, what is left in $\mathcal{S}_H^{(2)}$ is the Lagrangian L_H^{flat} alone. As we will see in Sec. IV A, we can also choose other gauges depending on the problem at hand.

We also expand the sum of actions $\mathcal{S}_M + \mathcal{S}_{\text{int}}$ up to quadratic order in scalar perturbations. After integrating out the fields δj_I by using the relation (3.6), the second-order action in the matter sector is given by the sum of $\mathcal{S}_M^{(2)}$ and $\mathcal{S}_{\text{int}}^{(2)}$. The former is expressed in the form

$$\mathcal{S}_M^{(2)} = \int dt d^3x (L_M^{\text{flat}} + L_M^\zeta + L_M^E), \quad (3.26)$$

where

$$L_M^{\text{flat}} = \sum_{I=c,b,r} a^3 \left[\left\{ (\rho_I + P_I) \frac{\partial^2 \chi}{a^2} - \dot{\delta}\rho_I - 3H(1 + c_I^2)\delta\rho_I \right\} v_I - \frac{1}{2}(\rho_I + P_I) \frac{(\partial v_I)^2}{a^2} - \frac{c_I^2}{2(\rho_I + P_I)} \delta\rho_I^2 \right. \\ \left. - \alpha\delta\rho_I - \frac{1}{2}\rho_I \left\{ \frac{(\partial\chi)^2}{a^2} - \alpha^2 \right\} \right], \quad (3.27)$$

$$L_M^\zeta = - \sum_{I=c,b,r} 3a^3 \left[(\rho_I + P_I) v_I \dot{\zeta} + \rho_I \alpha \zeta - \frac{1}{2} P_I \zeta^2 \right], \quad (3.28)$$

$$L_M^E = \sum_{I=c,b,r} a^3 \left[(\rho_I + P_I)(v_I - 3Hc_I^2 v_I) - \rho_I \alpha + P_I \left(\zeta - \frac{1}{2} \partial^2 E \right) \right] \partial^2 E. \quad (3.29)$$

The latter interacting second-order action is given by

$$\mathcal{S}_{\text{int}}^{(2)} = \int dt d^3x (L_{\text{int}1}^{\text{flat}} + L_{\text{int}1}^\zeta + L_{\text{int}1}^E + L_{\text{int}2}^{\text{flat}} + L_{\text{int}2}^\zeta + L_{\text{int}2}^E), \quad (3.30)$$

where

$$\begin{aligned}
L_{\text{int1}}^{\text{flat}} = a^3 \left[\left\{ (\rho_c + P_c) \frac{\partial^2 \chi}{a^2} - \dot{\delta} \rho_c - 3H(1 + c_c^2) \delta \rho_c \right\} f_1 v_c - \frac{1}{2} (\rho_c + P_c) f_1 \frac{(\partial v_c)^2}{a^2} - \frac{c_c^2}{2(\rho_c + P_c)} f_1 \delta \rho_c^2 \right. \\
- \left. \left\{ (f_1 - \dot{\phi}^2 f_{1,X}) \alpha + \dot{\phi} f_{1,X} \delta \dot{\phi} + f_{1,\phi} \delta \phi \right\} \delta \rho_c - \rho_c f_{1,X} \left\{ \dot{\phi} \delta \phi \frac{\partial^2 \chi}{a^2} - \frac{(\partial \delta \phi)^2}{2a^2} \right\} \right. \\
+ \frac{1}{2} \left\{ \rho_c (f_{1,X\phi} + \dot{\phi}^2 f_{1,XX\phi}) \ddot{\phi} - 3HP_c \dot{\phi} f_{1,X\phi} - \rho_c (f_{1,\phi\phi} - \dot{\phi}^2 f_{1,X\phi\phi}) \right\} \delta \phi^2 - \rho_c (f_{1,\phi} - \dot{\phi}^2 f_{1,X\phi}) \alpha \delta \phi \\
\left. - \frac{1}{2} \rho_c (f_{1,X} + \dot{\phi}^2 f_{1,XX}) (\dot{\phi} \alpha - \delta \dot{\phi})^2 - \frac{1}{2} \rho_c (f_1 - \dot{\phi}^2 f_{1,X}) \left\{ \frac{(\partial \chi)^2}{a^2} - \alpha^2 \right\} \right], \quad (3.31)
\end{aligned}$$

$$\begin{aligned}
L_{\text{int1}}^\zeta = -3a^3 \left[\left\{ (\rho_c + P_c) f_1 v_c - \rho_c \dot{\phi} f_{1,X} \delta \phi \right\} \dot{\zeta} + \rho_c (f_1 - \dot{\phi}^2 f_{1,X}) \alpha \zeta - \frac{1}{2} P_c f_1 \zeta^2 \right. \\
\left. + \left\{ \rho_c (f_{1,\phi} - \dot{\phi}^2 f_{1,X\phi}) - \rho_c \ddot{\phi} (f_{1,X} + \dot{\phi}^2 f_{1,XX}) + 3HP_c \dot{\phi} f_{1,X} \right\} \delta \phi \zeta \right], \quad (3.32)
\end{aligned}$$

$$\begin{aligned}
L_{\text{int1}}^E = -a^3 \left[\left\{ (\rho_c + P_c) f_1 v_c - \rho_c \dot{\phi} f_{1,X} \delta \phi \right\} \partial^2 \dot{E} + \rho_c (f_1 - \dot{\phi}^2 f_{1,X}) \alpha \partial^2 E - P_c f_1 \zeta \partial^2 E + \frac{1}{2} P_c f_1 (\partial^2 E)^2 \right. \\
\left. + \left\{ \rho_c (f_{1,\phi} - \dot{\phi}^2 f_{1,X\phi}) - \rho_c \ddot{\phi} (f_{1,X} + \dot{\phi}^2 f_{1,XX}) + 3HP_c \dot{\phi} f_{1,X} \right\} \delta \phi \partial^2 E \right], \quad (3.33)
\end{aligned}$$

and

$$\begin{aligned}
L_{\text{int2}}^{\text{flat}} = -a^3 n_c \left[\left\{ (\rho_c + P_c) \frac{\partial^2 \chi}{a^2} - \delta \dot{\rho}_c - 3H(1 + c_c^2) \delta \rho_c \right\} \frac{\dot{\phi} n_c f_{2,n_c}}{\rho_c + P_c} v_c - \dot{\phi} n_c f_{2,n_c} \frac{(\partial v_c)^2}{2a^2} - \frac{\dot{\phi} n_c (2f_{2,n_c} + n_c f_{2,n_c n_c})}{2(\rho_c + P_c)^2} \delta \rho_c^2 \right. \\
- \frac{1}{\rho_c + P_c} \left\{ n_c f_{2,n_c} \delta \dot{\phi} - \dot{\phi}^2 (f_{2,X} + n_c f_{2,n_c X}) (\dot{\phi} \alpha - \delta \dot{\phi}) + [n_c (3H f_{2,n_c} + \dot{\phi} f_{2,n_c \phi}) - f_{2,X} \dot{\phi} \ddot{\phi}] \delta \phi \right\} \delta \rho_c \\
- \dot{\phi} f_{2,X} \left\{ \dot{\phi} \delta \phi \frac{\partial^2 \chi}{a^2} - \frac{(\partial \delta \phi)^2}{2a^2} \right\} + \frac{1}{2} \left\{ \dot{\phi} \ddot{\phi} (3f_{2,X\phi} + \dot{\phi}^2 f_{2,XX\phi}) - 3H n_c (f_{2,n_c \phi} + \dot{\phi}^2 f_{2,n_c X\phi}) + \dot{\phi}^3 f_{2,X\phi\phi} \right\} \delta \phi^2 \\
\left. + \dot{\phi}^3 f_{2,X\phi} \alpha \delta \phi - \frac{1}{2} \dot{\phi} (3f_{2,X} + \dot{\phi}^2 f_{2,XX}) (\dot{\phi} \alpha - \delta \dot{\phi})^2 + \frac{1}{2} \dot{\phi}^3 f_{2,X} \left\{ \frac{(\partial \chi)^2}{a^2} - \alpha^2 \right\} \right], \quad (3.34)
\end{aligned}$$

$$\begin{aligned}
L_{\text{int2}}^\zeta = 3a^3 n_c \left[\left\{ \dot{\phi} (n_c f_{2,n_c} v_c - \dot{\phi} f_{2,X} \delta \phi) \right\} \dot{\zeta} - \dot{\phi}^3 f_{2,X} \alpha \zeta - \frac{1}{2} \dot{\phi} n_c f_{2,n_c} \zeta^2 \right. \\
\left. - \left\{ \dot{\phi} \ddot{\phi} (3f_{2,X} + \dot{\phi}^2 f_{2,XX}) - 3H n_c (f_{2,n_c} + \dot{\phi}^2 f_{2,n_c X}) + \dot{\phi}^3 f_{2,X\phi} \right\} \delta \phi \zeta \right], \quad (3.35)
\end{aligned}$$

$$\begin{aligned}
L_{\text{int2}}^E = a^3 n_c \left[\left\{ \dot{\phi} (n_c f_{2,n_c} v_c - \dot{\phi} f_{2,X} \delta \phi) \right\} \partial^2 \dot{E} - \dot{\phi} n_c f_{2,n_c} \zeta \partial^2 E - \dot{\phi}^3 f_{2,X} \alpha \partial^2 E + \frac{1}{2} \dot{\phi} n_c f_{2,n_c} (\partial^2 E)^2 \right. \\
\left. - \left\{ \dot{\phi} \ddot{\phi} (3f_{2,X} + \dot{\phi}^2 f_{2,XX}) - 3H n_c (f_{2,n_c} + \dot{\phi}^2 f_{2,n_c X}) + \dot{\phi}^3 f_{2,X\phi} \right\} \delta \phi \partial^2 E \right]. \quad (3.36)
\end{aligned}$$

Now, we are ready for computing the total second-order action $\mathcal{S}_s^{(2)} = \mathcal{S}_H^{(2)} + \mathcal{S}_M^{(2)} + \mathcal{S}_{\text{int}}^{(2)}$. On using the background Eqs. (2.20)–(2.22), the terms containing \mathcal{C}_1 , \mathcal{C}_2 , and \mathcal{C}_3 in Eqs. (3.18)–(3.20) cancel the sum of last contributions to Eqs. (3.27), (3.31), (3.34), last two contributions to Eqs. (3.28)–(3.29), and the terms except for $\dot{\zeta}$ and $\partial^2 \dot{E}$ in Eqs. (3.32)–(3.33) and (3.35)–(3.36). Then, the resulting full second-order action is given by

$$\mathcal{S}_s^{(2)} = \int dt d^3x (L_0 + L_{\text{int}}), \quad (3.37)$$

where

$$\begin{aligned}
L_0 = & a^3 \left\{ D_1 \delta \dot{\phi}^2 + D_2 \frac{(\partial \delta \phi)^2}{a^2} + D_3 \delta \phi^2 + \left(D_4 \delta \dot{\phi} + D_5 \delta \phi + D_6 \frac{\partial^2 \delta \phi}{a^2} \right) \alpha - (D_6 \delta \dot{\phi} - D_7 \delta \phi) \frac{\partial^2 \chi}{a^2} \right. \\
& + (\dot{\phi} D_6 - 2H q_t) \alpha \frac{\partial^2 \chi}{a^2} + (\dot{\phi}^2 D_1 + 3H \dot{\phi} D_6 - 3H^2 q_t) \alpha^2 \\
& + \sum_{l=c,b,r} \left\{ (\rho_l + P_l) v_l \frac{\partial^2 \chi}{a^2} - v_l \delta \dot{\rho}_l - 3H(1 + c_l^2) v_l \delta \rho_l - \frac{\rho_l + P_l}{2a^2} (\partial v_l)^2 - \frac{c_l^2}{2(\rho_l + P_l)} \delta \rho_l^2 - \alpha \delta \rho_l \right\} \\
& + \left\{ 3D_6 \delta \dot{\phi} - 3D_7 \delta \phi - 3(\dot{\phi} D_6 - 2H q_t) \alpha - \sum_{l=c,b,r} 3(\rho_l + P_l) v_l + 2q_t \frac{\partial^2 \chi}{a^2} \right\} \dot{\xi} - 3q_t \dot{\xi}^2 \\
& - (B_1 \delta \phi + 2q_t \alpha) \frac{\partial^2 \zeta}{a^2} + q_t \frac{(\partial \zeta)^2}{a^2} + \left[2q_t \ddot{\xi} + 2B_2 \dot{\xi} - D_6 \delta \ddot{\phi} - B_3 \delta \dot{\phi} + B_4 \delta \phi + \frac{1}{a^3} \frac{d}{dt} \{ a^3 (\dot{\phi} D_6 - 2H q_t) \alpha \} \right. \\
& \left. + \sum_{l=c,b,r} (\rho_l + P_l) (\dot{v}_l - 3H c_l^2 v_l) \right] \partial^2 E \Big\}, \tag{3.38}
\end{aligned}$$

and

$$\begin{aligned}
L_{\text{int}} = & a^3 \left[\left(f_1 - \frac{\dot{\phi} n_c^2 f_{2,n_c}}{\rho_c + P_c} \right) \left\{ (\rho_c + P_c) \frac{\partial^2 \chi}{a^2} - \dot{\delta} \rho_c - 3H(1 + c_c^2) \delta \rho_c \right\} v_c - \{ (\rho_c + P_c) f_1 - \dot{\phi} n_c^2 f_{2,n_c} \} \frac{(\partial v_c)^2}{2a^2} \right. \\
& - \left\{ \left(f_1 - \dot{\phi}^2 f_{1,X} + \frac{\dot{\phi}^3 n_c \{ f_{2,X} + n_c f_{2,n_c X} \}}{\rho_c + P_c} \right) \alpha + \left(\dot{\phi} f_{1,X} - \frac{n_c \{ n_c f_{2,n_c} + \dot{\phi}^2 (f_{2,X} + n_c f_{2,n_c X}) \}}{\rho_c + P_c} \right) \delta \dot{\phi} \right. \\
& + \left. \left(f_{1,\phi} - \frac{n_c \{ n_c (3H f_{2,n_c} + \dot{\phi} f_{2,n_c \phi}) - \dot{\phi} \ddot{\phi} f_{2,X} \}}{\rho_c + P_c} \right) \delta \phi \right\} \delta \rho_c - (f_{1,X} \rho_c - \dot{\phi} n_c f_{2,X}) \left\{ \dot{\phi} \delta \phi \frac{\partial^2 \chi}{a^2} - \frac{(\partial \delta \phi)^2}{2a^2} \right\} \\
& + \frac{1}{2} \{ \rho_c (f_{1,X\phi} + \dot{\phi}^2 f_{1,XX\phi}) \ddot{\phi} - n_c (3f_{2,X\phi} + \dot{\phi}^2 f_{2,XX\phi}) \dot{\phi} \ddot{\phi} - \rho_c (f_{1,\phi\phi} - \dot{\phi}^2 f_{1,X\phi\phi}) - 3HP_c \dot{\phi} f_{1,X\phi} \\
& + n_c (3H n_c \{ f_{2,n_c\phi} + \dot{\phi}^2 f_{2,n_c X\phi} \} - \dot{\phi}^3 f_{2,X\phi\phi}) \} \delta \phi^2 - \frac{1}{2(\rho_c + P_c)} \left\{ f_1 c_c^2 - \frac{\dot{\phi} n_c^2 (2f_{2,n_c} + n_c f_{2,n_c n_c})}{\rho_c + P_c} \right\} \delta \rho_c^2 \\
& - \{ \rho_c (f_{1,\phi} - \dot{\phi}^2 f_{1,X\phi}) + \dot{\phi}^3 n_c f_{2,X\phi} \} \alpha \delta \phi - \frac{1}{2} \{ \rho_c (f_{1,X} + \dot{\phi}^2 f_{1,XX}) - \dot{\phi} n_c (3f_{2,X} + \dot{\phi}^2 f_{2,XX}) \} (\dot{\phi} \alpha - \delta \dot{\phi})^2 \\
& \left. - \{ [(\rho_c + P_c) f_1 - \dot{\phi} n_c^2 f_{2,n_c}] v_c - \dot{\phi} (\rho_c f_{1,X} - \dot{\phi} n_c f_{2,X}) \delta \phi \} (3\dot{\xi} + \partial^2 \dot{E}) \right]. \tag{3.39}
\end{aligned}$$

The Lagrangian L_{int} characterizes the interaction between CDM and the scalar field arising from nonvanishing couplings f_1 and f_2 .

B. Perturbation equations

We derive all the linear perturbation equations of motion in Fourier space with the comoving wave number k . Varying the action (3.37) with respect to nondynamical fields α , χ , v_l , and E , it follows that

$$\begin{aligned}
& [D_4 + \dot{\phi} (f_{1,X} + \dot{\phi}^2 f_{1,XX}) \rho_c - n_c \dot{\phi}^2 (3f_{2,X} + \dot{\phi}^2 f_{2,XX})] \delta \dot{\phi} - 3(\dot{\phi} D_6 - 2H q_t) \dot{\xi} \\
& + [D_5 - (f_{1,\phi} - \dot{\phi}^2 f_{1,X\phi}) \rho_c - \dot{\phi}^3 n_c f_{2,X\phi}] \delta \phi + [2\dot{\phi}^2 D_1 + 6H \dot{\phi} D_6 - 6H^2 q_t - \dot{\phi}^2 (f_{1,X} + \dot{\phi}^2 f_{1,XX}) \rho_c \\
& + \dot{\phi}^3 n_c (3f_{2,X} + \dot{\phi}^2 f_{2,XX})] \alpha + \frac{k^2}{a^2} [2q_t \dot{\xi} - (\dot{\phi} D_6 - 2H q_t) (\chi - a^2 \dot{E}) - D_6 \delta \phi] - \sum_{l=c,b,r} \delta \rho_l \\
& - \left[f_1 - \dot{\phi}^2 f_{1,X} + \frac{\dot{\phi}^3 n_c (f_{2,X} + n_c f_{2,n_c X})}{\rho_c + P_c} \right] \delta \rho_c = 0, \tag{3.40}
\end{aligned}$$

$$D_6\delta\dot{\phi} - 2q_t\dot{\zeta} - (D_7 - \dot{\phi}f_{1,X}\rho_c + \dot{\phi}^2n_c f_{2,X})\delta\phi - (\dot{\phi}D_6 - 2Hq_t)\alpha - \sum_{I=c,b,r}(\rho_I + P_I)v_I - [f_1(\rho_c + P_c) - \dot{\phi}n_c^2 f_{2,n_c}]v_c = 0, \quad (3.41)$$

$$\delta\rho_I + 3H(1 + c_I^2)\delta\rho_I + 3(\rho_I + P_I)\dot{\zeta} + \frac{k^2}{a^2}(\rho_I + P_I)(v_I + \chi - a^2\dot{E}) = 0, \quad \text{for } I = c, b, r, \quad (3.42)$$

$$2q_t\dot{\zeta} + 2B_2\dot{\zeta} - D_6\delta\ddot{\phi} - (B_3 + \dot{\phi}f_{1,X}\rho_c - \dot{\phi}^2n_c f_{2,X})\delta\dot{\phi} + [B_4 - \{\dot{\phi}^2f_{1,X\phi} + \ddot{\phi}(f_{1,X} + \dot{\phi}^2f_{1,XX})\}\rho_c + 3HP_c\dot{\phi}f_{1,X} + \dot{\phi}n_c\{\ddot{\phi}(2f_{2,X} + \dot{\phi}^2f_{2,XX}) + \dot{\phi}^2f_{2,X\phi}\} - 3H\dot{\phi}^2n_c^2 f_{2,n_cX}]\delta\phi + \frac{1}{a^3}\frac{d}{dt}\{a^3(\dot{\phi}D_6 - 2Hq_t)\alpha\} + \sum_{I=c,b,r}(\rho_I + P_I)(\dot{v}_I - 3Hc_I^2v_I) + [f_1(\rho_c + P_c) - \dot{\phi}n_c^2 f_{2,n_c}]\dot{v}_c - [\{3Hc_c^2f_1 - \dot{\phi}(f_{1,\phi} + \ddot{\phi}f_{1,X})\}(\rho_c + P_c) + n_c^2\{\dot{\phi}(f_{2,n_c} - 3H\dot{\phi}) + \dot{\phi}^2(f_{2,n_c\phi} + \ddot{\phi}f_{2,n_cX})\}]v_c = 0. \quad (3.43)$$

These equations can be used to eliminate α , χ , v_I , and E from the action (3.37). In Sec. IV B, we will derive the stability conditions of dynamical perturbations in the small-scale limit after the elimination of nondynamical perturbations.

Variations of the action (3.37) with respect to dynamical fields $\delta\phi$, $\delta\rho_I$, and ζ lead to

$$\begin{aligned} & \dot{Z} + 3HZ + 3(D_7 - \dot{\phi}f_{1,X}\rho_c + \dot{\phi}^2n_c f_{2,X})\dot{\zeta} + M_\phi^2\delta\phi - [\{\dot{\phi}(f_{1,X\phi} + \dot{\phi}^2f_{1,XX\phi}) - f_{1,\phi\phi} + \dot{\phi}^2f_{1,X\phi\phi}\}\rho_c \\ & - 3HP_c\dot{\phi}f_{1,X\phi} - \dot{\phi}n_c\{\ddot{\phi}(3f_{2,X\phi} + \dot{\phi}^2f_{2,XX\phi}) + \dot{\phi}^2f_{2,X\phi\phi}\} + 3Hn_c^2(f_{2,n_c\phi} + \dot{\phi}^2f_{2,n_cX\phi})]\delta\phi \\ & + \left[f_{1,\phi} - \frac{n_c}{\rho_c + P_c}(3Hn_c f_{2,n_c} + \dot{\phi}n_c f_{2,n_c\phi} - \dot{\phi}\ddot{\phi}f_{2,X}) \right] \delta\rho_c - [D_5 - (f_{1,\phi} - \dot{\phi}^2f_{1,X\phi})\rho_c - \dot{\phi}^3n_c f_{2,X\phi}]\alpha \\ & - \frac{k^2}{a^2}[2D_2\delta\phi - D_6\alpha - D_7\chi + B_1\zeta - a^2B_4E + (f_{1,X}\rho_c - \dot{\phi}n_c f_{2,X})\{\delta\phi + \dot{\phi}(\chi - a^2\dot{E})\}] = 0, \end{aligned} \quad (3.44)$$

$$\begin{aligned} & \left(1 + f_1 - \frac{\dot{\phi}n_c^2 f_{2,n_c}}{\rho_c + P_c}\right)\dot{v}_c - \left[3Hc_c^2(1 + f_1) - \dot{\phi}(f_{1,\phi} + \ddot{\phi}f_{1,X}) + \frac{n_c^2}{\rho_c + P_c}\{\dot{\phi}f_{2,n_c} - 3H\dot{\phi}(f_{2,n_c} + n_c f_{2,n_c n_c})\right. \\ & \left. + \dot{\phi}^2(f_{2,n_c\phi} + \ddot{\phi}f_{2,n_cX})\right]v_c - \left[1 + f_1 - \dot{\phi}^2f_{1,X} + \frac{\dot{\phi}^3n_c(f_{2,X} + n_c f_{2,n_cX})}{\rho_c + P_c}\right]\alpha \\ & - \left[\dot{\phi}f_{1,X} - \frac{n_c(n_c f_{2,n_c} + \dot{\phi}^2f_{2,X} + \dot{\phi}^2n_c f_{2,n_cX})}{\rho_c + P_c}\right]\delta\dot{\phi} - \left[f_{1,\phi} - \frac{n_c(3Hn_c f_{2,n_c} + \dot{\phi}n_c f_{2,n_c\phi} - \dot{\phi}\ddot{\phi}f_{2,X})}{\rho_c + P_c}\right]\delta\phi \\ & - \frac{1}{\rho_c + P_c}\left[(1 + f_1)c_c^2 - \frac{\dot{\phi}n_c^2(2f_{2,n_c} + n_c f_{2,n_c n_c})}{\rho_c + P_c}\right]\delta\rho_c = 0, \end{aligned} \quad (3.45)$$

$$\dot{v}_I - 3Hc_I^2v_I - \frac{c_I^2}{\rho_I + P_I}\delta\rho_I - \alpha = 0, \quad \text{for } I = b, r, \quad (3.46)$$

$$\begin{aligned} & \dot{W} + 3HW + \sum_{I=c,b,r}(\rho_I + P_I)(\dot{v}_I - 3Hc_I^2v_I) + [(\rho_c + P_c)f_1 - \dot{\phi}n_c^2 f_{2,n_c}]\dot{v}_c + \frac{k^2}{3a^2}(2q_t\alpha + 2q_t\zeta + B_1\delta\phi) \\ & - [n_c^2\{\dot{\phi}f_{2,n_c} - 3H\dot{\phi}(f_{2,n_c} + n_c f_{2,n_c n_c}) + \dot{\phi}^2(f_{2,n_c\phi} + \ddot{\phi}f_{2,n_cX})\}] \\ & + (\rho_c + P_c)\{3Hc_c^2f_1 - \dot{\phi}(f_{1,\phi} + \ddot{\phi}f_{1,X})\}v_c = 0, \end{aligned} \quad (3.47)$$

where

$$M_\phi^2 \equiv -2D_3, \quad (3.48)$$

$$\begin{aligned} \mathcal{Z} \equiv & [2D_1 - (f_{1,X} + \dot{\phi}^2 f_{1,XX})\rho_c + \dot{\phi} n_c (3f_{2,X} + \dot{\phi}^2 f_{2,XX})] \dot{\delta\phi} + 3D_6 \dot{\zeta} + [D_4 + \dot{\phi} (f_{1,X} + \dot{\phi}^2 f_{1,XX})\rho_c \\ & - \dot{\phi}^2 n_c (3f_{2,X} + \dot{\phi}^2 f_{2,XX})] \alpha - \left[\dot{\phi} f_{1,X} - \frac{n_c (n_c f_{2,n_c} + \dot{\phi}^2 f_{2,X} + \dot{\phi}^2 n_c f_{2,n_c X})}{\rho_c + P_c} \right] \delta\rho_c \\ & + \frac{k^2}{a^2} [D_6 \chi - a^2 (D_6 \dot{E} + D_7 E)], \end{aligned} \quad (3.49)$$

$$\mathcal{W} \equiv 2q_t \dot{\zeta} - D_6 \dot{\delta\phi} + (D_7 - \dot{\phi} f_{1,X} \rho_c + \dot{\phi}^2 n_c f_{2,X}) \delta\phi + (\dot{\phi} D_6 - 2H q_t) \alpha + \frac{2k^2}{3a^2} q_t (\chi - a^2 \dot{E}). \quad (3.50)$$

The coefficient $-2D_3$ contains the term $-G_{2,\phi\phi}$. For a canonical scalar with the potential $V(\phi)$, i.e., $G_2 = X - V(\phi)$, the term $-G_{2,\phi\phi}$ reduces to $V_{,\phi\phi}$, which corresponds to the field mass squared. In Ref. [50], the scalar perturbation equations were derived in coupled quintessence with the interacting Lagrangian $\mathcal{L}_{\text{int}} = q J_c^\mu \partial_\mu \phi / n_c$, where q is a constant. They can be recovered by setting $f_1 = 0$, $f_2 = q/n_c$, $G_2 = X - V$, $G_3 = 0$, $G_4 = M_{\text{pl}}^2/2$, $\rho_b = 0$, and $\rho_r = 0$ in Eqs. (3.40)–(3.42), (3.44), (3.45), and (3.47).

On using Eqs. (3.43) and (3.47), the time derivatives $\ddot{\zeta}$ and $\dot{\delta\phi}$ are eliminated to give

$$\begin{aligned} q_t [\alpha + \dot{\chi} + \zeta + H\chi - a^2 (\ddot{E} + 3H\dot{E})] \\ + \dot{q}_t \left(\chi - a^2 \dot{E} + \frac{\delta\phi}{\dot{\phi}} \right) = 0, \end{aligned} \quad (3.51)$$

where we used the relation $B_1 = 2\dot{q}_t/\dot{\phi}$. The above perturbation equations of motion can be applied to any choices of gauges.

IV. SMALL-SCALE STABILITY CONDITIONS

By using the second-order action of scalar perturbations derived in Sec. III, we identify conditions for the absence of ghosts and Laplacian instabilities in the small-scale limit. Before doing so, we introduce commonly used gauge-invariant variables and discuss several different choices of gauges.

A. Gauge-invariant variables and gauge fixings

Let us consider the infinitesimal gauge transformation of time and spatial coordinates of the forms $\tilde{t} = t + \xi^0$ and $\tilde{x}^i = x^i + \delta^{ij} \partial_j \xi$, where ξ^0 and ξ are scalar quantities. Then, the four scalar metric perturbations in the line element (3.1) transform as [78–80]

$$\begin{aligned} \tilde{\alpha} &= \alpha - \dot{\xi}^0, & \tilde{\chi} &= \chi + \xi^0 - a^2 \dot{\xi}, \\ \tilde{\zeta} &= \zeta - H \xi^0, & \tilde{E} &= E - \xi. \end{aligned} \quad (4.1)$$

The transformations of $\delta\phi$, $\delta\rho_I$, and v_I are given, respectively, by

$$\begin{aligned} \tilde{\delta\phi} &= \delta\phi - \dot{\phi} \xi^0, & \tilde{\delta\rho}_I &= \delta\rho_I - \dot{\rho}_I \xi^0, & \tilde{v}_I &= v_I - \xi^0. \end{aligned} \quad (4.2)$$

Then, the following perturbed quantities are invariant under the gauge transformation,

$$\delta\phi_{\text{I}} = \delta\phi - \frac{\dot{\phi}}{H} \zeta, \quad \delta\rho_{\text{I}} = \delta\rho_I - \frac{\dot{\rho}_I}{H} \zeta, \quad v_{\text{I}} = v_I - \frac{\zeta}{H}, \quad (4.3)$$

$$\mathcal{R} = \zeta - \frac{H}{\dot{\phi}} \delta\phi, \quad \delta\rho_{\text{Iu}} = \delta\rho_I - \frac{\dot{\rho}_I}{\dot{\phi}} \delta\phi, \quad v_{\text{Iu}} = v_I - \frac{\delta\rho_I}{\dot{\rho}_I}, \quad (4.4)$$

$$\begin{aligned} \delta\phi_{\text{N}} &= \delta\phi + \dot{\phi} (\chi - a^2 \dot{E}), & \delta\rho_{\text{IN}} &= \delta\rho_I + \dot{\rho}_I (\chi - a^2 \dot{E}), \\ v_{\text{IN}} &= v_I + \chi - a^2 \dot{E}. \end{aligned} \quad (4.5)$$

The gauge-invariant gravitational potentials first introduced by Bardeen [78] are given by

$$\Psi = \alpha + \frac{d}{dt} (\chi - a^2 \dot{E}), \quad \Phi = \zeta + H (\chi - a^2 \dot{E}). \quad (4.6)$$

We recall that the background CDM density containing the effect of interactions is given by $\hat{\rho}_c = (1 + f_1)\rho_c$. We also introduce the corresponding gauge-invariant CDM density perturbation, as

$$\hat{\delta\rho}_{c\text{N}} = (1 + f_1) \delta\rho_{c\text{N}} + [f_{1,\phi} \delta\phi_{\text{N}} + f_{1,X} \dot{\phi} (\delta\phi_{\text{N}} - \dot{\phi} \Psi)] \rho_c. \quad (4.7)$$

The gauge choice corresponds to fixing the infinitesimal scalars ξ^0 and ξ . The latter can be fixed by choosing

$$E = 0. \quad (4.8)$$

There are several different gauge choices for the fixing of ξ^0 . The representative examples are

$$(i) \quad \zeta = 0 \quad (\text{flat gauge}), \quad (4.9)$$

$$(ii) \quad \delta\phi = 0 \quad (\text{unitary gauge}), \quad (4.10)$$

$$(iii) \quad \chi = 0 \quad (\text{Newtonian gauge}). \quad (4.11)$$

In the flat gauge, the dynamical scalar perturbations are given by $\delta\phi_f = \delta\phi$ and $\delta\rho_{If} = \delta\rho_I$, while, in the unitary gauge, they correspond to $\mathcal{R} = \zeta$ and $\delta\rho_{Iu} = \delta\rho_I$. We note that the comoving curvature perturbation \mathcal{R} [81,82] is related to the Mukhanov-Sasaki variable $\delta\phi_f$ [83,84], as $\mathcal{R} = -(H/\dot{\phi})\delta\phi_f$. In the Newtonian gauge, the perturbations in Eqs. (4.5) and (4.6) reduce, respectively, to $\delta\phi_N = \delta\phi$, $\delta\rho_{IN} = \delta\rho_I$, $v_{IN} = v_I$, $\Psi = \alpha$, and $\Phi = \zeta$. For this gauge choice, the gauge-invariant dynamical scalar perturbations are $\mathcal{R} = \Phi - (H/\dot{\phi})\delta\phi_N$ (or $\delta\phi_f = \delta\phi_N - (\dot{\phi}/H)\Phi$) and $\delta\rho_{IN}$.

B. Stability conditions in the small-scale limit

In order to obtain conditions for the absence of ghosts and Laplacian instabilities, we choose two different gauges and show that the small-scale stability conditions are independent of the choice of gauges.

1. Flat gauge

Let us begin with the flat gauge characterized by $\zeta = 0$ and $E = 0$. We first solve Eqs. (3.40)–(3.42) for non-dynamical perturbations $\alpha, \chi, v_c, v_b, v_r$ and substitute them into Eq. (3.37). After the integration by parts, the resulting second-order action is expressed in the form

$$\mathcal{S}_s^{(2)} = \int dt d^3x a^3 \left(\dot{\vec{\chi}}^t \mathbf{K} \dot{\vec{\chi}} - \frac{k^2}{a^2} \vec{\chi}^t \mathbf{G} \vec{\chi} - \vec{\chi}^t \mathbf{M} \vec{\chi} - \vec{\chi}^t \mathbf{B} \dot{\vec{\chi}} \right), \quad (4.12)$$

where $\mathbf{K}, \mathbf{G}, \mathbf{M}, \mathbf{B}$ are 4×4 matrices, and

$$\vec{\chi}^t = (\delta\phi_f, \delta\rho_{cf}/k, \delta\rho_{bf}/k, \delta\rho_{rf}/k). \quad (4.13)$$

The leading-order terms in the components of matrices \mathbf{M} and \mathbf{B} are of order k^0 . Taking the small-scale limit ($k \rightarrow \infty$), the nonvanishing matrix components of \mathbf{K} and \mathbf{G} are given by

$$\begin{aligned} K_{11}^{(f)} &= \frac{H^2 q_t q_s}{(2Hq_t - \dot{\phi}D_6)^2}, & K_{22}^{(f)} &= \frac{a^2}{2(\rho_c + P_c)} \left(1 + f_1 - \frac{\dot{\phi}n_c^2 f_{2,n_c}}{\rho_c + P_c} \right), \\ K_{33}^{(f)} &= \frac{a^2}{2(\rho_b + P_b)}, & K_{44}^{(f)} &= \frac{a^2}{2(\rho_r + P_r)}, \end{aligned} \quad (4.14)$$

$$\begin{aligned} G_{11}^{(f)} &= -D_2 + \frac{D_6 D_7 - [\sum_{l=c,b,r}(\rho_l + P_l) + f_1(\rho_c + P_c) - \dot{\phi}n_c^2 f_{2,n_c}] \mathcal{G}_1}{2Hq_t - \dot{\phi}D_6} + \dot{\mathcal{G}}_1 + H\mathcal{G}_1 \\ &\quad - \frac{(f_{1,X}\rho_c - \dot{\phi}n_c f_{2,X})(2Hq_t + \dot{\phi}D_6)}{2(2Hq_t - \dot{\phi}D_6)}, \\ G_{22}^{(f)} &= \frac{a^2}{2(\rho_c + P_c)} \left[(1 + f_1)c_c^2 - \frac{\dot{\phi}n_c^2(2f_{2,n_c} + n_c f_{2,n_c n_c})}{\rho_c + P_c} \right], & G_{33}^{(f)} &= \frac{a^2 c_b^2}{2(\rho_b + P_b)}, & G_{44}^{(f)} &= \frac{a^2 c_r^2}{2(\rho_r + P_r)}, \end{aligned} \quad (4.15)$$

where

$$\mathcal{G}_1 = \frac{D_6^2}{2(2Hq_t - \dot{\phi}D_6)}, \quad (4.16)$$

and we used the relation (3.25). We recall that q_t and q_s are defined, respectively, by Eqs. (2.26) and (2.31). The scalar ghost associated with the field perturbation $\delta\phi_f$ is absent for $K_{11}^{(f)} > 0$. Provided that there is no ghost in the tensor sector ($q_t > 0$), the condition $K_{11}^{(f)} > 0$ translates to

$$q_s > 0. \quad (4.17)$$

Since q_s contains the X derivatives of f_1 and f_2 , the X dependence in f_1 and f_2 affects the no-ghost condition of $\delta\phi_f$.

For the CDM perturbation $\delta\rho_{cf}$, the ghost is absent if $K_{22}^{(f)} > 0$, i.e.,

$$q_c \equiv 1 + f_1 - \frac{\dot{\phi}n_c^2 f_{2,n_c}}{\rho_c + P_c} > 0. \quad (4.18)$$

Hence the coupling f_1 and the n_c dependence in f_2 lead to the modification to q_c . For baryons and radiations, the positivities of $K_{33}^{(f)}$ and $K_{44}^{(f)}$ are ensured under the weak energy conditions (2.17).

Since all the off-diagonal components of \mathbf{K} and \mathbf{G} vanish in the small-scale limit, the propagation speed squares of four dynamical perturbations in Eq. (4.13) are simply given by $G_{ii}^{(f)}/K_{ii}^{(f)}$, with $i = 1, 2, 3, 4$. For baryons and radiations, $G_{33}^{(f)}/K_{33}^{(f)}$ and $G_{44}^{(f)}/K_{44}^{(f)}$ coincide with c_b^2 and c_r^2 , respectively. In contrast, the CDM propagation speed squared yields

$$\hat{c}_c^2 = \frac{G_{22}^{(f)}}{K_{22}^{(f)}} = c_c^2 - \frac{[(2 - c_c^2)f_{2,n_c} + n_c f_{2,n_c n_c}] \dot{\phi} n_c^2}{(1 + f_1)(\rho_c + P_c) - \dot{\phi} n_c^2 f_{2,n_c}}. \quad (4.19)$$

The existence of nonvanishing coupling $f_1(\phi, X)$ does not modify the standard value $\hat{c}_c^2 = c_c^2$, but the n_c dependence in f_2 leads to the deviation of \hat{c}_c^2 from c_c^2 .

The propagation speed squared associated with the field perturbation $\delta\phi_{\mathcal{F}}$ is given by

$$c_s^2 = \frac{G_{11}^{(f)}}{K_{11}^{(f)}}. \quad (4.20)$$

To avoid the Laplacian instabilities of $\delta\rho_{cf}$ and $\delta\phi_{\mathcal{F}}$, we require that

$$\hat{c}_c^2 > 0, \quad \text{and} \quad c_s^2 > 0. \quad (4.21)$$

Under the no-ghost conditions $q_s > 0$ and $q_c > 0$, $G_{11}^{(f)}$ and $G_{22}^{(f)}$ must be positive for the consistency with (4.21).

2. Unitary gauge

We also derive stability conditions of scalar perturbations by choosing the unitary gauge characterized by $\delta\phi = 0$ and $E = 0$. After eliminating the nondynamical perturbations $\alpha, \chi, v_c, v_b, v_r$ on account of Eqs. (3.40)–(3.42), the second-order scalar action (3.37) reduces to the form (4.12), with the dynamical perturbations given by

$$\vec{\mathcal{X}}^i = (\mathcal{R}, \delta\rho_{cu}/k, \delta\rho_{bu}/k, \delta\rho_{ru}/k). \quad (4.22)$$

In the small-scale limit, the nonvanishing components of \mathbf{K} and \mathbf{G} are diagonal terms, with the same matrix components for $\delta\rho_{cu}/k, \delta\rho_{bu}/k, \delta\rho_{ru}/k$ as those obtained in the flat gauge. Then, the stability conditions for CDM are satisfied for $q_c > 0$ and $\hat{c}_c^2 > 0$.

The nonvanishing matrix components associated with the curvature perturbation \mathcal{R} are

$$K_{11}^{(u)} = \frac{\dot{\phi}^2 q_t q_s}{(2Hq_t - \dot{\phi} D_6)^2}, \quad (4.23)$$

$$\begin{aligned} G_{11}^{(u)} = & -q_t \\ & - \frac{[\sum_{I=c,b,r}(\rho_I + P_I) + f_1(\rho_c + P_c) - \dot{\phi} n_c^2 f_{2,n_c}] \mathcal{G}_2}{2Hq_t - \dot{\phi} D_6} \\ & + \dot{\mathcal{G}}_2 + H\mathcal{G}_2, \end{aligned} \quad (4.24)$$

where

$$\mathcal{G}_2 = \frac{2q_t^2}{2Hq_t - \dot{\phi} D_6}. \quad (4.25)$$

Provided that the tensor ghost is absent ($q_t > 0$), the no-ghost condition $K_{11}^{(u)} > 0$ is satisfied for $q_s > 0$. We note that $K_{11}^{(u)}$ is related to $K_{11}^{(f)}$ given in Eq. (4.14), as

$$\frac{K_{11}^{(u)}}{K_{11}^{(f)}} = \frac{\dot{\phi}^2}{H^2}. \quad (4.26)$$

The propagation speed squared associated with the perturbation \mathcal{R} is

$$c_s^2 = \frac{G_{11}^{(u)}}{K_{11}^{(u)}}. \quad (4.27)$$

On using Eqs. (2.20), (2.21), and (3.23), we find that there is the following relation

$$\frac{G_{11}^{(u)}}{G_{11}^{(f)}} = \frac{\dot{\phi}^2}{H^2}, \quad (4.28)$$

where $G_{11}^{(f)}$ is given in Eq. (4.15). From Eqs. (4.26) and (4.28), the propagation speed squared (4.27) in the unitary gauge is identical to the corresponding value (4.20) in the flat gauge.

The matrix component $G_{11}^{(u)}$ in Eq. (4.24) contains the time derivative of \mathcal{G}_2 , which generates the terms \dot{q}_t, \dot{D}_6 , and $\dot{\phi}$ in c_s^2 . After eliminating these time derivatives by using Eqs. (2.20), (2.21), (3.23), and the relation $B_1 = 2\dot{q}_t/\dot{\phi}$, we can express Eq. (4.27) in the compact form,

$$c_s^2 = - \frac{D_6^2 + 2B_1 D_6 + 2q_t(2D_2 + f_{1,X}\rho_c - \dot{\phi} n_c f_{2,X})}{q_s}, \quad (4.29)$$

where q_s contains the derivatives $f_{1,X}, f_{2,X}, f_{1,XX}$, and $f_{2,XX}$. This means that the derivative couplings $f_1(X)$ and $f_2(X)$ nontrivially modify the propagation speed of perturbation associated with the scalar field.

The above discussion shows that, in the small-scale limit, the conditions for the absence of ghosts and Laplacian instabilities for CDM and ϕ are independent of the choice of gauges. Indeed, the analysis in the Newtonian gauge also leads to the same stability conditions as those derived above (as performed for the uncoupled case in Ref. [64]).

V. EFFECTIVE GRAVITATIONAL COUPLINGS FOR NONRELATIVISTIC MATTER

We proceed to the derivation of the effective gravitational couplings felt by CDM and baryons for perturbations deep

inside the sound horizon. Since we are interested in the cosmological dynamics in the late Universe, we ignore the contribution of radiations to the background and perturbation equations of motion. For CDM and baryons, we consider nonrelativistic matter satisfying

$$P_I = 0, \quad c_I^2 = 0, \quad \text{for } I = c, b. \quad (5.1)$$

We caution that, even under the conditions (5.1), the n_c dependence in f_2 gives rise to the nonvanishing sound speed squared \hat{c}_s^2 for CDM while the effective pressure \hat{P}_c vanishes, see Eqs. (2.33) and (4.19). From Eq. (4.5), we define the gauge-invariant matter density contrast,

$$\delta_{IN} \equiv \frac{\delta\rho_{IN}}{\rho_I} = \frac{\delta\rho_I}{\rho_I} - 3H(\chi - a^2\dot{E}), \quad \text{for } I = c, b, \quad (5.2)$$

where we used the continuity Eq. (2.16). From Eq. (3.42), the matter density contrast obeys

$$\dot{\delta}_{IN} + \frac{k^2}{a^2}v_{IN} + 3\dot{\Phi} = 0, \quad (5.3)$$

where v_{IN} and Φ are gauge-invariant quantities defined in Eqs. (4.5) and (4.6).

From Eqs. (3.45) and (3.46), the velocity potentials of CDM and baryons satisfy

$$\dot{v}_{cN} + a_1 H v_{cN} - (1 + a_2)\Psi + \frac{H}{\dot{\phi}} \left[a_2 \frac{\dot{\delta}\phi_N}{H} - (3\hat{c}_c^2 + a_1 + a_2\epsilon_\phi)\delta\phi_N \right] - \hat{c}_c^2\delta_{cN} = 0, \quad (5.4)$$

$$\dot{v}_{bN} - \Psi = 0, \quad (5.5)$$

respectively, where

$$a_1 = \frac{\dot{q}_c}{Hq_c}, \quad a_2 = -\frac{\dot{\phi}[\rho_c\dot{\phi}f_{1,X} - \dot{\phi}^2 n_c(f_{2,X} + n_c f_{2,n_c X}) - n_c^2 f_{2,n_c}]}{(1+f_1)\rho_c - \dot{\phi}n_c^2 f_{2,n_c}}, \quad \epsilon_\phi = \frac{\ddot{\phi}}{H\dot{\phi}}. \quad (5.6)$$

Taking the time derivatives of Eq. (5.3) for $I = c, b$ and using Eqs. (5.4) and (5.5), it follows that

$$\begin{aligned} \ddot{\delta}_{cN} + (2 + a_1)H\dot{\delta}_{cN} + \hat{c}_c^2 \frac{k^2}{a^2}\delta_{cN} + (1 + a_2)\frac{k^2}{a^2}\Psi - \frac{k^2 H}{a^2} \frac{\dot{\phi}}{\dot{\phi}} \left[a_2 \frac{\dot{\delta}\phi_N}{H} - (3\hat{c}_c^2 + a_1 + \epsilon_\phi a_2)\delta\phi_N \right] \\ = -3\ddot{\Phi} - 3(2 + a_1)H\dot{\Phi}, \end{aligned} \quad (5.7)$$

$$\ddot{\delta}_{bN} + 2H\dot{\delta}_{bN} + \frac{k^2}{a^2}\Psi = -3\ddot{\Phi} - 6H\dot{\Phi}. \quad (5.8)$$

The scalar perturbation equations derived in Sec. III B can be expressed in terms of the gauge-invariant gravitational potentials Ψ and Φ as well as the other gauge-invariant perturbations given in Eq. (4.5). First of all, Eq. (3.51) translates to

$$\Psi + \Phi = -\frac{H\alpha_M}{\dot{\phi}}\delta\phi_N, \quad (5.9)$$

where

$$\alpha_M = \frac{\dot{q}_t}{Hq_t}. \quad (5.10)$$

A. Quasistatic approximation for the modes deep inside the sound horizon

In what follows, we employ the so-called quasistatic approximation for the modes deep inside the sound horizon [85–87]. Under this approximation scheme, the dominant contributions to the scalar perturbation equations of motion are the terms containing δ_{cN} , δ_{cN} , δ_{bN} , and k^2/a^2 . We also take into account the field mass squared M_ϕ^2 appearing in Eq. (3.44) to accommodate the case in which the field is heavy at high redshifts as in $f(R)$ models of late-time cosmic acceleration [88–90]. The tachyonic instability can be avoided for $M_\phi^2 \geq 0$. Applying the quasistatic approximation to Eqs. (3.40) and (3.44), it follows that

$$\frac{k^2}{a^2}(2q_t\Phi - D_6\delta\phi_N) - (1 + a_2)q_c\rho_c\delta_{cN} - \rho_b\delta_{bN} \simeq 0, \quad (5.11)$$

$$M_\phi^2 \delta\phi_N + \frac{k^2}{a^2} \left[\frac{1}{2q_t} \left(q_s c_s^2 + D_6^2 + \frac{4Hq_t D_6 \alpha_M}{\dot{\phi}} \right) \delta\phi_N + D_6 \Psi - \frac{2H\alpha_M q_t}{\dot{\phi}} \Phi \right] + \frac{Hq_c \rho_c}{\dot{\phi}} \left[a_2 \frac{\dot{\delta}_{cN}}{H} + \{3\hat{c}_c^2 + a_1 + a_2(a_1 + \epsilon_{a_2})\} \delta_{cN} \right] \simeq 0, \quad (5.12)$$

where we used Eq. (4.29), and

$$\epsilon_{a_2} = \frac{\dot{a}_2}{Ha_2}. \quad (5.13)$$

The difference from the uncoupled Horndeski theories is that there exists the time derivative $\dot{\delta}_{cN}$ in Eq. (5.12). We solve Eqs. (5.9), (5.11), and (5.12) for Ψ , Φ , and $\delta\phi_N$ to express them in terms of δ_{cN} , $\dot{\delta}_{cN}$, and δ_{bN} . In doing so, we introduce the following dimensionless variables³:

$$\Delta_1 = \alpha_B - \alpha_M, \quad \Delta_2 = \frac{\dot{\phi}^2 q_s c_s^2}{4H^2 q_t^2} \left(1 + \frac{2q_t a^2 M_\phi^2}{q_s c_s^2 k^2} \right), \quad \Delta_3 = (1 + a_2)\Delta_1 + 3\hat{c}_c^2 + a_1 + a_2(a_1 + \epsilon_{a_2}),$$

$$\alpha_B = -\frac{D_6 \dot{\phi}}{2Hq_t}. \quad (5.14)$$

Then, Ψ , Φ , and $\delta\phi_N$ are expressed in the forms

$$\Psi = -\frac{a^2}{2q_t \Delta_2 k^2} \left[\{\Delta_1 \Delta_3 + (1 + a_2)\Delta_2\} q_c \rho_c \delta_{cN} + (\Delta_1^2 + \Delta_2) \rho_b \delta_{bN} + a_2 q_c \Delta_1 \rho_c \frac{\dot{\delta}_{cN}}{H} \right], \quad (5.15)$$

$$\Phi = \frac{a^2}{2q_t \Delta_2 k^2} \left[\{\alpha_B \Delta_3 + (1 + a_2)\Delta_2\} q_c \rho_c \delta_{cN} + (\alpha_B \Delta_1 + \Delta_2) \rho_b \delta_{bN} + a_2 q_c \alpha_B \rho_c \frac{\dot{\delta}_{cN}}{H} \right], \quad (5.16)$$

$$\delta\phi_N = -\frac{a^2 \dot{\phi}}{2Hq_t \Delta_2 k^2} \left(\Delta_3 q_c \rho_c \delta_{cN} + \Delta_1 \rho_b \delta_{bN} + a_2 q_c \rho_c \frac{\dot{\delta}_{cN}}{H} \right). \quad (5.17)$$

The gravitational slip parameter is given by

$$\eta \equiv -\frac{\Phi}{\Psi} = \frac{[\alpha_B(\Delta_3 + a_2 \epsilon_{\delta_c}) + (1 + a_2)\Delta_2] q_c \Omega_c \delta_{cN} + (\alpha_B \Delta_1 + \Delta_2) \Omega_b \delta_{bN}}{[\Delta_1(\Delta_3 + a_2 \epsilon_{\delta_c}) + (1 + a_2)\Delta_2] q_c \Omega_c \delta_{cN} + (\Delta_1^2 + \Delta_2) \Omega_b \delta_{bN}}, \quad (5.18)$$

where

$$\Omega_c = \frac{\rho_c}{3M_{\text{pl}}^2 H^2}, \quad \Omega_b = \frac{\rho_b}{3M_{\text{pl}}^2 H^2}, \quad \epsilon_{\delta_c} = \frac{\dot{\delta}_c}{H\delta_c}. \quad (5.19)$$

We substitute Eqs. (5.15), (5.17), and the time derivative of Eq. (5.17) into Eqs. (5.7) and (5.8). In doing so, the terms on the right-hand sides of Eqs. (5.7) and (5.8) are neglected relative to those on their left-hand sides. We also introduce the following dimensionless variables:

$$Q_t = \frac{q_t}{M_{\text{pl}}^2}, \quad \epsilon_H = \frac{\dot{H}}{H^2}, \quad \epsilon_{\Delta_i} = \frac{\dot{\Delta}_i}{H\Delta_i} \text{ (for } i = 1, 2, 3), \quad (5.20)$$

and

$$b_1 = \frac{3a_2^2 q_c \Omega_c}{2Q_t \Delta_2}, \quad b_2 = -b_1(1 + 2\epsilon_H + \alpha_M + \epsilon_{\Delta_2} - 2a_1 - 2\epsilon_{a_2}), \quad b_3 = \frac{3a_2 \Delta_1 \Omega_b}{4Q_t \Delta_2}, \quad (5.21)$$

³The definition of Δ_2 in Eq. (6.14) of Ref. [64] contains the term q_t in the denominator, but the definition of Δ_2 in Eq. (5.14) of this paper is different from the former in that q_t is replaced by q_t^2 to make Δ_2 dimensionless.

where $b_1 > 0$ under the absence of ghosts and Laplacian/tachyonic instabilities. Then, it follows that

$$\ddot{\delta}_{cN} + \frac{2 + a_1 + b_2}{1 + b_1} H \dot{\delta}_{cN} + \frac{2b_3}{1 + b_1} H \dot{\delta}_{bN} + \frac{\hat{c}_c^2}{1 + b_1} \frac{k^2}{a^2} \delta_{cN} - \frac{3H^2}{2G} (G_{cc} \Omega_c \delta_{cN} + G_{cb} \Omega_b \delta_{bN}) \simeq 0, \quad (5.22)$$

$$\ddot{\delta}_{bN} + 2H \dot{\delta}_{bN} - 2H q_c b_3 \frac{\Omega_c}{\Omega_b} \dot{\delta}_{cN} - \frac{3H^2}{2G} (G_{bc} \Omega_c \delta_{cN} + G_{bb} \Omega_b \delta_{bN}) \simeq 0, \quad (5.23)$$

where $G = 1/(8\pi M_{\text{pl}}^2)$, and the effective gravitational couplings are given by

$$G_{cc} = \frac{2q_c}{2Q_I \Delta_2 + 3a_2^2 q_c \Omega_c} [(1 + a_2)^2 \Delta_2 + \Delta_3 \{ \Delta_1 + 3\hat{c}_c^2 + a_1 + a_2(1 + \epsilon_H + \alpha_M + \Delta_1 - a_1 + \epsilon_{\Delta_2} - \epsilon_{\Delta_3}) \}] G, \quad (5.24)$$

$$G_{cb} = \frac{2}{2Q_I \Delta_2 + 3a_2^2 q_c \Omega_c} [(1 + a_2)(\Delta_1^2 + \Delta_2) + \Delta_1 \{ 3\hat{c}_c^2 + a_1 + a_2(1 + \alpha_M + \epsilon_H - \epsilon_{\Delta_1} + \epsilon_{\Delta_2}) \}] G, \quad (5.25)$$

$$G_{bc} = \frac{q_c [\Delta_1 \Delta_3 + (1 + a_2) \Delta_2]}{Q_I \Delta_2} G, \quad (5.26)$$

$$G_{bb} = \frac{\Delta_1^2 + \Delta_2}{Q_I \Delta_2} G. \quad (5.27)$$

The gravitational couplings G_{cc} and G_{cb} , which are multiplied by $\Omega_c \delta_{cN}$ and $\Omega_b \delta_{bN}$ respectively, affect the growth of CDM density contrast δ_{cN} . Meanwhile, the baryon density contrast δ_{bN} has the gravitational couplings G_{bb} and G_{bc} multiplied by $\Omega_b \delta_{bN}$ and $\Omega_c \delta_{cN}$, respectively. The baryon perturbation is affected by the evolution of the CDM perturbation, and vice versa.

In the absence of matter couplings ($f_1 = 0 = f_2$), the quantities a_1, a_2, b_1, b_2, b_3 vanish. In this case we have $q_c = 1, \hat{c}_c^2 = 0$ and $\Delta_3 = \Delta_1$, so that G_{cc}, G_{cb}, G_{bc} reduce to the same form as G_{bb} . We note that G_{bb} is equivalent to the effective gravitational coupling G_{eff} derived for uncoupled Horndeski theories [64,87]. When the matter couplings f_1 and f_2 are present, G_{cc}, G_{cb}, G_{bc} are generally different from G_{bb} .

Since we are considering the case in which $P_c = 0$ and $c_c^2 = 0$, the effective CDM sound speed squared is

$$\hat{c}_c^2 = -\frac{(2f_{2,n_c} + n_c f_{2,n_c n_c}) \dot{\phi} n_c^2}{(1 + f_1) \rho_c - \dot{\phi} n_c^2 f_{2,n_c}}. \quad (5.28)$$

Apart from the specific coupling satisfying the condition $2f_{2,n_c} + n_c f_{2,n_c n_c} = 0$, the n_c dependence in f_2 leads to \hat{c}_c^2 different from 0. For $\hat{c}_c^2 > 0$, the growth of δ_{cN} is prevented by the positive pressure induced by the coupling $f_2(n_c)$, whereas, for $\hat{c}_c^2 < 0$, there is the additional enhancement of δ_{cN} besides the growth induced by gravitational instabilities. For the consistency with the observed galaxy power spectrum in the linear regime of perturbations, the effective sound speed squared needs to be much smaller than 1, say,

$|\hat{c}_c^2| \lesssim 10^{-5}$ in unified perfect fluid models of dark energy and dark matter [91]. We note that the ϕ and X dependence in f_1 and f_2 does not modify the value of \hat{c}_c^2 .

The above results have been obtained for the gauge-invariant density contrasts (5.2) for CDM and baryons. There is also the gauge-invariant CDM density perturbation (4.7) containing the effect of coupling f_1 . Dividing $\hat{\delta}_{\rho_{cN}}$ by the background total CDM density $\hat{\rho}_c = (1 + f_1) \rho_c$, the corresponding density contrast is

$$\hat{\delta}_{cN} \equiv \frac{\hat{\delta}_{\rho_{cN}}}{\hat{\rho}_c} = \delta_{cN} + \frac{f_{1,\phi} \delta \phi_N + f_{1,X} \dot{\phi} (\delta \dot{\phi}_N - \dot{\phi} \Psi)}{1 + f_1}. \quad (5.29)$$

Under the quasistatic approximation, the terms containing $\delta \phi_N, \delta \dot{\phi}_N$, and Ψ in Eq. (5.29) are suppressed relative to δ_{cN} . Indeed, this property can be confirmed by the solutions (5.15) and (5.17) in the small-scale limit. In this case, we have $\hat{\delta}_{cN} \simeq \delta_{cN}$ and hence the effective gravitational couplings for $\hat{\delta}_{cN}$ are approximately the same as those derived above for the perturbations deep inside the sound horizon.

B. Concrete theories

On using the general formulas (5.24)–(5.27), we compute the effective gravitational couplings for specific theories and the choice of couplings. We consider three different cases: (i) $f_1 = f_1(\phi), f_2 = 0$, (ii) $f_1 = f_1(\phi), f_2 = f_2(n_c, \phi)$, and (iii) $f_1 = 0, f_2 = f_2(n_c, X)$.

1. Theories with $f_1=f_1(\phi)$ and $f_2=0$

In this case, we introduce the dimensionless coupling

$$Q = \frac{M_{\text{pl}} f_{1,\phi}}{1 + f_1}. \quad (5.30)$$

Then, the right-hand sides of Eqs. (2.35) and (2.40) reduce to $Q\hat{\rho}_c\dot{\phi}/M_{\text{pl}}$ and $-Q\hat{\rho}_c\dot{\phi}/M_{\text{pl}}$, respectively, showing that the quantity Q characterizes the strength of nonminimal coupling $f_1(\phi)$. The quantities a_1 , a_2 , q_c , \hat{c}_c^2 , and Δ_3 are given, respectively, by

$$\begin{aligned} a_1 &= Qx, & a_2 &= 0, & q_c &= 1 + f_1, \\ \hat{c}_c^2 &= 0, & \Delta_3 &= \Delta_1 + Qx, \end{aligned} \quad (5.31)$$

where

$$x \equiv \frac{\dot{\phi}}{M_{\text{pl}}H}. \quad (5.32)$$

Since the coupling f_2 is absent, \hat{c}_c^2 vanishes. From Eqs. (5.24)–(5.27), it follows that

$$\begin{aligned} G_{cc} &= \frac{G}{Q_t}(1 + f_1) \left[1 + \frac{(Qx + \Delta_1)^2}{\Delta_2} \right], \\ G_{cb} &= \frac{G}{Q_t} \left[1 + \frac{\Delta_1(Qx + \Delta_1)}{\Delta_2} \right], \\ G_{bc} &= \frac{G}{Q_t}(1 + f_1) \left[1 + \frac{\Delta_1(Qx + \Delta_1)}{\Delta_2} \right], \\ G_{bb} &= \frac{G}{Q_t} \left(1 + \frac{\Delta_1^2}{\Delta_2} \right). \end{aligned} \quad (5.33)$$

As long as the stability conditions of tensor and scalar perturbations are satisfied, it follows that $Q_t > 0$, $q_c = 1 + f_1 > 0$, and $\Delta_2 > 0$. Hence both G_{cc} and G_{bb} are positive, with the positive scalar-matter interactions characterized by $(Qx + \Delta_1)^2/\Delta_2 > 0$ and $\Delta_1^2/\Delta_2 > 0$. According to our knowledge, the gravitational couplings (5.33) have not been explicitly derived in coupled Horndeski theories given by the action (2.2).

In the following, we specify the scalar-graviton sector to the minimally coupled k-essence given by the action,

$$\mathcal{S}_H = \int d^4x \sqrt{-g} \left[\frac{M_{\text{pl}}^2}{2} R + G_2(\phi, X) \right]. \quad (5.34)$$

We also consider a light-mass scalar field satisfying

$$\frac{q_t a^2 M_\phi^2}{q_s c_s^2 k^2} \ll 1, \quad (5.35)$$

in Δ_2 of Eq. (5.14). We also adopt the condition (5.35) for the theories discussed later in Secs. V B 2 and V B 3. Then, we have

$$\begin{aligned} Q_t &= 1, & q_s c_s^2 &= 2M_{\text{pl}}^2 G_{2,X}, & \alpha_M &= 0, \\ \alpha_B &= 0, & \Delta_1 &= 0, & \Delta_2 &= \frac{x^2}{2} G_{2,X}. \end{aligned} \quad (5.36)$$

From Eq. (5.33), we obtain

$$\begin{aligned} G_{cc} &= G(1 + f_1) \left(1 + \frac{2Q^2}{G_{2,X}} \right), & G_{cb} &= G, \\ G_{bc} &= G(1 + f_1), & G_{bb} &= G. \end{aligned} \quad (5.37)$$

The coupling term $2Q^2/G_{2,X}$ in G_{cc} coincides with that derived in Refs. [86,92]. For a canonical scalar field given by the Lagrangian $G_2 = X - V(\phi)$, the scalar-matter interaction in G_{cc} reduces to the well-known form $2Q^2$ [28,93], which enhances the growth rate of CDM perturbations.

2. Theories with $f_1=f_1(\phi)$ and $f_2=f_2(n_c, \phi)$

Besides the nonminimal coupling $f_1 = f_1(\phi)$, we consider the case in which the second interaction of the form $f_2 = f_2(n_c, \phi)$ is present. For concreteness, we focus on the couplings,

$$f_1 = f_1(\phi), \quad f_2 = c_p n_c^p + F_2(\phi), \quad (5.38)$$

where f_1, F_2 are functions of ϕ , and c_p, p are constants. If

$$p = -1, \quad (5.39)$$

then the CDM sound speed squared (5.28) yields

$$\hat{c}_c^2 = 0. \quad (5.40)$$

For $p \neq -1$, the gravitational instability of CDM perturbations is strongly modified by the additional pressure arising from the nonvanishing \hat{c}_c^2 , unless $|\hat{c}_c^2|$ is very much smaller than 1. Hence we focus on the case $p = -1$ in the following discussion. Together with the coupling Q defined in Eq. (5.30), we introduce the following dimensionless quantities,

$$q = \frac{M_{\text{pl}} Q_{,\phi}}{Q}, \quad \Omega_{f_2} = \frac{c_p \dot{\phi}}{3M_{\text{pl}}^2 H^2}, \quad (5.41)$$

and consider a light scalar field whose mass M_ϕ is in the range (5.35).

For the scalar-graviton sector, we adopt a canonical scalar field given by the action

$$\mathcal{S}_H = \int d^4x \sqrt{-g} \left[\frac{M_{\text{pl}}^2}{2} R + X - V(\phi) \right], \quad (5.42)$$

where $V(\phi)$ is the field potential. Then, it follows that

$$a_1 = \frac{(1+f_1)\Omega_c Qx + (3+\epsilon_\phi)\Omega_{f_2}}{(1+f_1)\Omega_c + \Omega_{f_2}}, \quad a_2 = -\frac{\Omega_{f_2}}{(1+f_1)\Omega_c + \Omega_{f_2}}, \quad q_c = 1 + f_1 + \frac{\Omega_{f_2}}{\Omega_c}, \quad (5.43)$$

and

$$\begin{aligned} \alpha_M = 0, \quad \alpha_B = 0, \quad \Delta_1 = 0, \quad \Delta_2 = \frac{x^2}{2}, \quad \Delta_3 = \frac{(1+f_1)\Omega_c Qx}{(1+f_1)\Omega_c + \Omega_{f_2}}, \\ \epsilon_{\Delta_1} = 0, \quad \epsilon_{\Delta_2} = 2(\epsilon_\phi - \epsilon_H), \quad \epsilon_{\Delta_3} = \frac{(1+f_1)\Omega_c(\epsilon_\phi - \epsilon_H + qx) + \Omega_{f_2}[(Q+q)x - 3 - \epsilon_H]}{(1+f_1)\Omega_c + \Omega_{f_2}}. \end{aligned} \quad (5.44)$$

The quantity x , which is defined by Eq. (5.32), is expressed as $x = \epsilon\sqrt{2\Delta_2}$, where $\epsilon = +1$ for $\dot{\phi} > 0$ and $\epsilon = -1$ for $\dot{\phi} < 0$. The effective gravitational couplings (5.24)–(5.27) reduce, respectively, to

$$\begin{aligned} G_{cc} &= G(1+f_1) \left[1 + 2Q^2 - \Omega_{f_2} \frac{2\Delta_2(1-2Qq) - 4\epsilon Q\sqrt{2\Delta_2} + 3\Omega_{f_2}(1+2Q^2)}{2\Omega_c\Delta_2(1+f_1) + \Omega_{f_2}(2\Delta_2 + 3\Omega_{f_2})} \right], \\ G_{cb} &= G \frac{2\Omega_c(1+f_1)\Delta_2}{2[\Omega_c(1+f_1) + \Omega_{f_2}]\Delta_2 + 3\Omega_{f_2}^2}, \quad G_{bc} = G(1+f_1), \quad G_{bb} = G. \end{aligned} \quad (5.45)$$

Compared to the values given in Eq. (5.37), the n_c dependence in f_2 modifies both G_{cc} and G_{cb} . We note that there is no modification to the gravitational couplings from the term $F_2(\phi)$ in f_2 .

If we consider the theories with [50]

$$f_1 = 0, \quad (5.46)$$

i.e., $Q = 0$, Eq. (5.45) gives

$$\begin{aligned} G_{cc} = G_{cb} &= G \frac{2\Omega_c\Delta_2}{2(\Omega_c + \Omega_{f_2})\Delta_2 + 3\Omega_{f_2}^2}, \\ G_{bc} = G_{bb} &= G, \end{aligned} \quad (5.47)$$

where G_{cc} matches with that derived in Eq. [59]. Since $\Delta_2 > 0$ and $\Omega_c > 0$, the gravitational couplings with CDM can be smaller than G (i.e., $G_{cc} = G_{cb} < G$) for

$$\Omega_{f_2} > 0, \quad \text{i.e.,} \quad c_p \dot{\phi} > 0. \quad (5.48)$$

This is an explicit example in which the n_c dependence in f_2 allows the possibility for realizing weak gravity on scales relevant to the linear growth of large-scale structures. We note that, under the absence of ghosts and Laplacian instabilities, the effective gravitational coupling in uncoupled Horndeski theories is destined to be larger than G [19,64]. Hence, the coupled model with weak gravitational interactions can be observationally distinguished from uncoupled Horndeski theories. It is of interest to explore further whether the coupled dark energy model with the coupling $f_2 \propto n_c^{-1}$ can be in better fit to the observational data in comparison to the Λ CDM model. In the presence of nonvanishing $f_1(\phi)$, the gravitational

couplings G_{cc} and G_{cb} in Eq. (5.45) differ from those in Eq. (5.47), so the observational signatures are different.

3. Theories with $f_1 = 0$ and $f_2 = f_2(n_c, X)$

Finally, we study the effect of X -dependence in f_2 on the effective gravitational couplings. For concreteness, let us consider the functions

$$f_1 = 0, \quad f_2 = \beta n_c^{-1} X^s, \quad (5.49)$$

where β and s are constants. For this choice of f_2 , the CDM sound speed squared \hat{c}_c^2 vanishes. On using the definition of fluid four-velocity (2.6), the interacting action (2.4) can be written as

$$\mathcal{S}_{\text{int}} = \int d^4x \sqrt{-g} \beta X^s u_c^\mu \partial_\mu \phi. \quad (5.50)$$

On the flat FLRW background (2.10) with $N = 1$, the term $u_c^\mu \partial_\mu \phi$ is equivalent to $\dot{\phi}$. Then, the above interacting Lagrangian is proportional to $\dot{\phi}^{2s+1}$. For the scalar-graviton sector, we adopt the minimally coupled k-essence described by the action (5.34). Then, Eqs. (2.38) and (2.39) reduce, respectively, to

$$\rho_{\text{DE}} = -G_2 + \dot{\phi}^2 G_{2,X} + 2s\bar{\beta}\dot{\phi}^{2s+1}, \quad (5.51)$$

$$P_{\text{DE}} = G_2 + \bar{\beta}\dot{\phi}^{2s+1}, \quad (5.52)$$

where $\bar{\beta} = 2^{-s}\beta$. For the power,

$$s = \frac{1}{2}, \quad (5.53)$$

the last terms in Eqs. (5.51) and (5.52) are proportional to $\dot{\phi}^2$, i.e., the same time dependence as the standard kinetic term X in G_2 . If $s \neq 1/2$, the last terms in Eqs. (5.51) and (5.52) can dominate over X either in the asymptotic past or future. In the following, we focus on the model with the power (5.53). Then, it follows that

$$\begin{aligned} a_1 &= \frac{\bar{\beta}x^2(3 + 2\epsilon_\phi)}{3\Omega_c + \bar{\beta}x^2}, & a_2 &= -\frac{\bar{\beta}x^2}{3\Omega_c + \bar{\beta}x^2}, \\ q_c &= 1 + \frac{\bar{\beta}x^2}{3\Omega_c}, & \hat{c}_c^2 &= 0, \end{aligned} \quad (5.54)$$

where x is defined by Eq. (5.32), and

$$\begin{aligned} Q_t &= 1, & q_s c_s^2 &= 2M_{\text{pl}}^2(G_{2,X} + \bar{\beta}), & \alpha_M &= 0, & \alpha_B &= 0, \\ \Delta_1 &= 0, & \Delta_2 &= \frac{x^2}{2}(G_{2,X} + \bar{\beta}), & \Delta_3 &= 0. \end{aligned} \quad (5.55)$$

Substituting Eqs. (5.54) and (5.55) into (5.24)–(5.27), we obtain

$$G_{cc} = G_{cb} = G \left(1 + \frac{G_{2,X} + 2\bar{\beta} \frac{\bar{\beta}x^2}{3\Omega_c}}{G_{2,X} + \bar{\beta}} \right)^{-1}, \quad (5.56)$$

$$G_{bc} = G_{bb} = 1. \quad (5.57)$$

From Eq. (5.54), the ghost does not appear from the CDM sector for $\bar{\beta} > 0$. Moreover, the condition $G_{2,X} + \bar{\beta} > 0$ is required for the positivity of $q_s c_s^2$ in Eq. (5.55). Then, the second term in the parentheses of Eq. (5.56) is positive, so that $G_{cc} = G_{cb} < G$. This is the explicit example in which the X dependence in f_2 leads to the realization of weak gravity on cosmological scales.

The interacting action (5.50) with the power (5.53) can be further generalized to the theories containing the non-linear dependence of $u_c^\mu \partial_\mu \phi$, e.g.,

$$\mathcal{S}_{\text{int}} = \int d^4x \sqrt{-g} \beta X^s (u_c^\mu \partial_\mu \phi)^{2-2s}. \quad (5.58)$$

By setting $s = 0$, the interacting action (5.58) recovers the theory studied in Ref. [52] as a special case. The detailed analysis of cosmological dynamics for the extended coupling (5.58) is given in Ref. [94] (which was submitted one month after the initial submission of this paper).

VI. CONCLUSIONS

We studied the interacting dark energy scenario in which a scalar field ϕ is coupled to the CDM perfect fluid given by the Schutz-Sorkin action (2.3). The scalar-graviton sector is described by the Horndeski action (2.2) with the tensor propagation speed squared equivalent to that of light. We considered the new interacting action (2.4) containing the

X dependence in the couplings f_1 and f_2 . Our analysis is sufficiently general in that it accommodates a wide variety of nonminimal and derivative couplings studied in the literature [23,25,49,50,59,72]. Moreover, unlike most of past related papers, we did not restrict the dark energy field to quintessence or k-essence.

In Sec. II, we derived the background equations of motion on the flat FLRW background in the forms (2.20)–(2.22). As long as the quantity q_s defined by Eq. (2.31) does not vanish, the dynamical system can be solved for \dot{H} and $\dot{\phi}$. Indeed, the stability analysis performed in Sec. IV leads to the condition $q_s > 0$ to avoid the ghost associated with the scalar perturbation. We also identified the total CDM density $\hat{\rho}_c$ and pressure \hat{P}_c containing the effect of interactions with the scalar field, as Eqs. (2.32) and (2.33). We showed that CDM interacts with dark energy according to Eqs. (2.35) and (2.40), whose signs on the right-hand sides are opposite to each other.

In Sec. III, we expanded the action up to second order in scalar perturbations without fixing any gauge conditions. We explicitly computed the quadratic-order actions arising from \mathcal{S}_H , \mathcal{S}_M , \mathcal{S}_{int} and finally took the sum of them on account of the background equations. The final second-order action $\mathcal{S}_s^{(2)}$ is of the form (3.37), where L_{int} is the Lagrangian arising from the couplings f_1 and f_2 . In Sec. III B, we also obtained the full linear perturbation equations of motion in the gauge-ready form, i.e., ready for choosing any particular gauges.

In Sec. IV, we first introduced a number of gauge-invariant perturbed quantities and discussed several different gauge choices. In the flat and unitary gauges, we derived the second-order actions of dynamical scalar perturbations after eliminating nondynamical quantities and identified stability conditions in the small-scale limit. The conditions for the absence of ghosts and Laplacian instabilities are independent of the gauge choices, i.e., Eqs. (4.17), (4.18), and (4.21), where \hat{c}_c^2 and c_s^2 are given, respectively, by Eqs. (4.19) and (4.29).

In Sec. V, we derived the effective gravitational couplings for CDM and baryon density perturbations by using the quasistatic approximation for the modes deep inside the sound horizon. We obtained the Bardeen gravitational potentials Ψ , Φ , and the field perturbation $\delta\phi_N$ in the gauge-independent manner. The difference from the uncoupled case is that the time derivative $\dot{\delta}_{cN}$ appears in the expressions of Ψ , Φ , $\delta\phi_N$ given by Eqs. (5.15)–(5.17). Taking the time derivative of $\delta\phi_N$ in Eq. (5.17) gives rise to the second derivative $\ddot{\delta}_{cN}$, so we need to solve Eq. (5.7) for $\ddot{\delta}_{cN}$ to obtain the closed-form equation for δ_{cN} . As a result, the CDM and baryon density contrasts obey Eqs. (5.22) and (5.23), respectively, with the effective gravitational couplings (5.24)–(5.27).

We applied our general formulas of G_{cc} , G_{cb} , G_{bc} , G_{bb} to three different forms of couplings and discussed how they

reproduce the effective gravitational couplings known in the literature. In particular, the n_c or X dependence in f_2 offers an interesting possibility for realizing G_{cc} and G_{cb} smaller than G . In uncoupled Horndeski theories, the gravitational couplings are usually larger than G . Moreover, the presence of matter couplings f_1 and f_2 gives rise to the values of G_{cc} , G_{cb} , G_{bc} , G_{bb} generally different from each other, while this is not the case in uncoupled Horndeski theories. These properties show that the coupled theories with weak gravitational interactions can be observationally distinguished from uncoupled theories.

In this paper we did not construct particular models of coupled dark energy, but it is of interest to do so to observationally probe the signature of interactions with CDM. First of all, theoretically consistent models need to satisfy all the small-scale stability conditions derived in

Sec. IV. The next step is to predict observational signatures of models both at the background and perturbation levels, e.g., the dark energy and CDM equations of state and the growth of perturbations. Then, the models should be confronted with numerous observational data associated with the cosmic expansion and growth histories. These interesting issues are left for future works.

ACKNOWLEDGMENTS

R.K. is supported by the Grant-in-Aid for Young Scientists B of the JSPS No. 17K14297. S.T. is supported by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 19K03854 and MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas ‘‘Cosmic Acceleration’’ (No. 15H05890).

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