Vector-boson condensates, spin-triplet superfluidity of paired neutral and charged fermions, and $3P_2$ pairing of nucleons

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After reminding of properties of the condensate of the complex scalar field in the external uniform magnetic field H, focus is made on the study of phases of the complex neutral vector-boson fields coupled with magnetic field by the Zeeman coupling and phases of the charged vector-boson fields. The systems may behave as nonmagnetic and ferromagnetic superfluids and ordinary and ferromagnetic superconductors. Response of these superfluid and superconducting systems occupying half of space on the external uniform static magnetic field H is thoroughly studied. Then the spin-triplet pairing of neutral fermions at conserved spin is considered. Novel phases are found. In external magnetic field, the phase with zero mean spin proves to be unstable to the formation of a phase with a nonzero spin. For a certain parameter choice ferromagnetic superfluid phases are formed already for $H = 0$, characterized by an own magnetic field h. For $H > H_{\rm cr2}$, the spin-triplet pairing and ferromagnetic superfluidity continue to exist above the "old" phase transition critical temperature $T_{\rm cr}$. Formation of domains is discussed. Next, spin-triplet pairing of charged fermions is studied. Novel phases are found. Then, the $3P_2$ nn pairing in neutron star matter is studied. Also, a $3P_2$ pp pairing is considered. Numerical estimates are performed in the BCS weak-coupling limit and beyond it for the $3P_2$ nn and pp pairings, as well as for the $3S_1$ np pairing.

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I. INTRODUCTION

In the condensed matter physics, the spin-ordered pairing is known from the studies of the ^{3}He liquid, heavy-fermion systems like UPt_3, some Rb-and La-based superconductors and other materials; see Refs. [1–7] for review. A Josephson supercurrent through the strong ferromagnet CrO_2 was observed in [8], from which it was inferred that it is a spin-triplet supercurrent. A long-range supercurrent in Josephson junctions containing Co (a strong ferromagnetic material) was observed [9] when one inserted thin layers of either PdNi or CuNi weakly ferromagnetic alloys between the Co and the two superconducting Nb electrodes.

In cold atomic Fermi gases, strong magnetic dipolar interaction may cause pairing in the state with orbital angular momentum L, spin S, and total angular momentum J equal to one, i.e., in the $(2S+1)L_J = 3P_1$ [10]. Isotopes ^{161}Dy and ^{163}Dy are the most magnetic fermionic atoms with magnetic moments as high as $10\mu_{e,\rm B}$, where $\mu_{e,\rm B}$ is the electron Bohr magneton. The lowest temperature reached in experiments [11] for the spin-polarized ^{161}Dy is a factor 0.2 below the Fermi temperature $\epsilon_{\rm F} = 300$ nK. The cotrapped ^{162}Dy cools to approximately critical temperature for the Bose-Einstein condensation, realizing a novel nearly quantum degenerate dipolar Bose-Fermi gas mixture. In some systems such as dilute Fermi gases, the p-wave pairing may occur even in the case of a repulsive interaction [12–14]. Conventional electron-phonon interactions induce triplet pairing in time-reversal three-dimensional (3D) Dirac semimetals, if magnetic impurities or exchange interaction is sufficiently strong, cf. [15] and refs therein. Very recently, a paramagnetic Meissner effect in an Nb-Ho-Au structure was observed [16]. In this system, superconductivity enhances the magnetic signal rather than expel it. Reference [17] demonstrated that by combining superconductors with spin-orbit coupled materials the Meissner effect can be modulated by the orientation of an external magnetic field.

Since the magnetic field is the axial vector, it is efficient pair breaker for s-wave superconductors but it should not break pairs with parallel spins. The Zeeman coupling of pairs with $S_z = \pm 1$, where S_z is the spin projection on the quantization axis, is responsible for this effect. For instance, the phase A of the p-wave spin-triplet pairing in the ^{3}He survives in the external magnetic field, which can also induce a specific A_1 phase [18,19]. The ^{3}He-A_1 phase behaves as the magnetic superfluid in the external magnetic field. Superconductivity of the spin-triplet electron pairs in unconventional superconductors in external magnetic fields was extensively studied; see Refs. [20,21] and review [22]. Interaction of the vector order parameter with the magnetic field is introduced with the help of the minimal coupling and the Zeeman coupling.

In the description of the ^{3}He, the rotation field was introduced in [23] with the help of the Galilean variable shift, similarly to that for the magnetic field. To the best of our knowledge, a possibility of the appearance of own magnetic and rotation fields in the whole volume of fermion superfluids has not been considered.

The $1S_0$ channel provides the largest nn and pp attractive interactions at low densities in the neutron star matter. Thereby, A.B. Migdal suggested the Cooper pairing and superfluidity of neutrons in neutron stars in the $1S_0$ state [24]. With an increase of the baryon density, the NN interaction in the s wave is weakened. The nucleon-nucleon (NN) phase shift in the $3P_2$ partial wave becomes the largest one among others at sufficiently high momenta owing to the strong spin-orbit NN interaction in the vacuum that allows for the nn pairing in $3P_2$ state [25–28]. Therefore, in the neutron star interiors, neutrons are supposed to be paired in the $1S_0$ state at a low baryon density $n \le (0.7–0.8)n_0$, where n_0 is the nuclear saturation density, and in $3P_2$ state for $(3–4)n_0 \ge n \ge 0.8n_0$, cf. [29] and recent works [30,31]. However, the value of the $3P_2$ nn gap in the neutron matter is poorly known and varies in various calculations from tiny values $\lesssim 10$ KeV up to values of the order of several MeV and may be more; see Refs. [28,32–37]. Uncertainties appear largely due to a lack of knowledge of the efficiency of the three-body forces in a dense baryon matter [38]. Note that the cooling history of neutron stars is appropriately described in the nuclear medium cooling scenario within an Ansatz that the $3P_2$ pairing gap has only a tiny value, cf. [39–41]. Because of all these uncertainties, and since microscopic calculations of the gap are beyond the scope of the given work, we further consider the critical temperature as an external phenomenological parameter varying in broad limits.

Mixing of $3P_2$ and $3F_2$ partial waves increases the value of the $3P_2$ gap. In some density interval, the nn $3PF_2$ pairing may coexist with the pp $1S_0$ pairing. The $1S_0$ channel is most attractive for protons, as a consequence of their small concentration in neutron stars but in the hyperon-enriched central regions of sufficiently massive neutron stars proton concentration increases and protons can be paired also in the $3PF_2$ state [42], as well as Λ hyperons [43]. Besides hyperons [44,45], $\Delta(\frac{3}{2}, \frac{3}{2})$ isobars may exist in central regions of sufficiently massive neutron stars [46,47]. Pairing in the fermion systems of spin $3/2$ was recently discussed in [48]. Moreover, a developed pion condensate may exist in the central regions of sufficiently massive neutron stars. In the presence of the developed pion condensate, there is only one Fermi sea of a mixture of the baryon quasiparticles consisted of neutrons, protons, and Δ isobars [49–53], and thereby they can be paired in the $3S_1$ state.

The phases of the $3P_2$ nn pairing were studied in [54–58] within the BCS weak-coupling approximation, when the ground state corresponds to the symmetric (magnetically neutral) phase. The order parameter for the $3P_2$ nn pairing is the 3×3 matrix. The Ginzburg-Landau free-energy functional is ordinarily considered as the expansion in the order parameter up to fourth power. However, the sixth-order term calculated within the BCS approximation proves to be negative [54] causing a possibility of the first-order phase transitions in the system. Recently, the Ginzburg-Landau free-energy functional was expanded in the order parameter up to eighth power and coefficients of expansion were found in the BCS approximation [56]. A particular role of the Zeeman and gradient terms, which are of our key interest here, was not studied, cf. [58].

Magnetic fields in ordinary pulsars, like the Crab pulsar, reach values $\sim(10^{12}–10^{13})$ Gs at their surfaces. At the surface of magnetars, magnetic fields may reach values $\gtrsim 10^{15}$ Gs. In the interior, the magnetic field might be even stronger (up to $\sim 10^{18}$ Gs) depending on the assumed (still badly known) mechanism of magnetic field formation [59]. Still stronger magnetic fields appear in noncentral heavy-ion collisions. The first estimate of the value of the magnetic field in heavy-ion collisions performed in Ref. [60] argued for the presence of the magnetic fields of the order of $\sim 10^{17} \div 10^{18}$ G at collision energies $\sim$GeV/A. Subsequent calculations [61] demonstrated that typical values of magnetic fields may reach $\sim(10^{17}–10^{19})$ Gs in heavy-ion collision experiments from GSI to LHC energies. Thus, the coupling of a spin-triplet order parameter to a magnetic field might be of importance for the description of nuclear systems prepared in peripheral heavy-ion collisions.

For low densities, the $3S_1$ channel provides the largest attractive interaction for the np pairing in the isospin-symmetrical matter. With increasing density, the $3D_2$ channel becomes most attractive, cf. [31]. One of the hypothesis for the explanation of the level structure of superdeformed (rotated) nuclei is the spin-triplet pairing [62–64]. Spin-triplet pairing in $N = Z$ nuclei with $A > 140$ may be favored, since the spin-orbit force becomes vanishing [65]. In the vicinity of the proton drip in heavier nuclei, the spin-triplet pairing also could potentially become important. The $^3SD_1$ spin-triplet np pairing in nuclei was studied in [66]. The BCS calculations for the symmetric matter with the vacuum interactions predict the np pairing gaps as large as $\simeq 12$ MeV. Even with the effect of the depletion of the Fermi surface taken into account, one estimates the np pairing gap in maximum to be as high as $\simeq 4$ MeV. Reference [67] studying the level structure of ^{92}Pd found signals of the spin aligned np-paired state with $J = 9$ and $L \neq 0$.

There exist millisecond pulsars, being fast-rotating neutron stars with the angular rotation frequencies as high as $\sim 10^4$ Hz; see Ref. [59]. The rotation frequency of the fireball in ultrarelativistic heavy-ion collisions at the freeze-out [68] is estimated as $\sim 10^{22}$ Hz. For low energies, the spectator pieces in heavy-ion collisions can rotate at a still larger frequency ($> 10^{22}$ Hz). In a sense, the rotation acts in a neutral system similar to a magnetic field in a charged system. Thereby, description of the behavior of the spin-triplet condensates in the magnetic and the rotation fields is an important issue.

Another phenomenon, which might be relevant to our study, is a condensation of the charged ρ mesons in a dense isospin-asymmetric baryon matter [69–71]. The ρ mesons, being bosons with the spin and isotopic spin equal to one, are described by the vector–isospin-vector field ρ^a_ν, where $a = 1, 2, 3$ is the isospin index and $\mu = 0, 1, 2, 3$ is the Lorentz index. In the quantum field theory, relevant phenomena are condensations of non-Abelian charged ρ and W bosons in superstrong magnetic fields $\gtrsim 10^{19}$ G in vacuum; see Refs. [72–74]. Presence of strong magnetic fields in neutron star interiors would promote the charged ρ meson condensation [75]. Gluons become massive in the hot quark-gluon plasma and may form condensates at some conditions. Thereby, a ferromagnetic superconductivity of the condensate of charged vector fields is another issue of our interest. The axial-vector–isospin-vector boson may also play a role in nuclear phenomena forming condensates at some conditions, cf. [76,77]. Finally, the order parameter in color superconductors has a matrix structure and a spin-triplet diquark pairing is allowed in some cases [78–81].

The Ginzburg-Landau description is relevant not only below critical point for the order parameters but also for their long-range fluctuations within a fluctuation region above the critical point. The width of the fluctuation region is determined by the Ginzburg number Gi following the so-called Ginzburg-Levanyuk criterion [82]. In the substances with a strong interaction between quasiparticles, the Ginzburg number Gi ~ 1 and the fluctuation region should be broad [83]. For instance, the fluctuation region might be very broad for the color superconductors and for the proton pairing in neutron stars [84]. Thereby, the consideration of a triplet paring correlations above the critical point is also an important issue.

In this work, we study nonmagnetic, diamagnetic, paramagnetic, and ferromagnetic responses of superfluid and superconductive condensates of vector bosons and spin-triplet Cooper pairs. We start with a reminding of superfluid and superconductive properties of the complex scalar field at the negative squared effective mass of the boson (in Sec. II) and then (in Sec. III) we focus on the description of the complex vector field of neutral and charged bosons at the conditions when their squared effective masses might be either negative or positive. Influence of the external magnetic field is considered. Various nonmagnetic and ferromagnetic superfluid phases and nonmagnetic, superdiamagnetic, and ferromagnetic superconducting phases will be studied. In Sec. IV, we perform a general analysis of the spin-triplet pairing of charge-neutral fermions with a magnetic moment, interacting with the magnetic field by the Zeeman coupling. First, we assume that spin-orbit forces are weak and spin of the pair is a good quantum number. The spin-triplet pairing is then described by a vector order parameter, as for a composed spin-one charge-neutral boson with an anomalous magnetic moment. Some of the phases are characterized by the spin-order parameter and a self-magnetization. In this sense, we deal with a ferromagnetic superfluidity. Note that an another type of the ferromagnetic superfluidity, when a magnetization exists already in the absence of the Cooper pairing and remains in presence of the superfluidity, as it may occur in some uranium compounds, is not of our interest here; see [85,86]. In Sec. V, we consider the spin-triplet pairing in charged fermion superconducting systems described by the vector order parameter. In Sec. VI, focus is made on the description of the $3P_2$ nn pairing in the neutron star matter. Various phases are found. Some numerical evaluations are performed in Sec. VII for the $3P_2$ nn and pp pairings and for the $3S_1$ np pairing and some physical consequences of the ferromagnetic superfluidity and superconductivity for neutron stars and heavy-ion collisions are specified. In Sec. VIII, we formulate our conclusions.

Throughout the paper, we use units $\hbar = c = 1$, Lattin indices are $i = 1, 2, 3$, Greek indices are Lorentz ones, $\mu = 0, 1, 2, 3$. For three-vectors, where it does not cause a confusion, we use the ordinary three-dimensional notations, $\vec{a} = (a_1, a_2, a_3)$. Summation over repeated indices is implied, if not presented explicitly.

II. PRELIMINARIES

A. Superfluidity and superconductivity of complex scalar fields

1. Lagrangian and equations of motion

Consider the model described by the Lagrangian density,

$$L = D_\mu \phi D^\mu \phi^* - m_{\rm sc}^2 |\phi|^2 - \lambda |\phi|^4/2 - F_{\mu\nu} F^{\mu\nu}/(16\pi), \quad (1)$$

$\phi = (\phi_1 - i\phi_2)/\sqrt{2}$ is the spin-zero complex field of a negatively charged boson, ϕ_1 and ϕ_2 are real components, $\phi_+ = \phi^* = (\phi_1 + i\phi_2)/\sqrt{2}$ is spin-zero complex field of a positively charged boson, A_μ is the electromagnetic field,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad D_\mu = \partial_\mu + ieA_\mu - i\mu\delta_{\mu 0}, \quad (2)$$

$e < 0$ is the charge of the electron, $e^2 = 1/137$, μ is the chemical potential of the negatively charged boson, e.g., in the neutron-star matter due to reactions $n + e \to n + \pi^-$, $n \to p + \pi^-$ one gets $\mu_{\pi^-} = \mu_e = \mu_n - \mu_p$. The quantity $m_{\rm sc}^2 = m^2 + U$, where $m > 0$ is the mass of the vector particle, U is a scalar potential, which we assume to be zero in vacuum and heaving a negative value in the medium. We will only use that in a deep potential, $U < -m^2$, in the medium the quantity $m_{\rm sc}^2$ becomes negative.

Equations of motion are

$$D_\mu D^\mu \phi + m_{\rm sc}^2\phi + \lambda|\phi|^2\phi = 0, \tag{3}$$

$$\partial_\mu F^{\mu\nu} = -4\pi\delta L_\phi/\delta A_\nu = 4\pi J^\nu, \tag{4}$$

with the four-current-density

$$J^\nu = -ie\phi D^{*\nu}\phi^* + \text{c.c.},$$

which is conserved, $\partial_\nu J^\nu = 0$; the abbreviation c.c. denotes complex conjugation, L_ϕ is the part of the Lagrangian density depending on ϕ.

For the case of the static field, ϕ and the static magnetic field equations of motion render

$$(\nabla - ie\vec{A})^2\phi - m_{\rm ef}^2\phi - \lambda|\phi|^2\phi = 0, \tag{5}$$

$$\Delta\vec{A} = -4\pi\vec{J}, \quad \vec{J} = ie(\phi\nabla\phi^* - \phi^*\nabla\phi) - 2e^2\vec{A}|\phi|^2, \tag{6}$$

where $m_{\rm ef}^2 = m_{\rm sc}^2 - \mu^2$ has a sense of the squared effective mass term. It is used that $\mathrm{div}\vec{A} = 0$.

We introduce the Gibbs free-energy density $G = F - \vec{M}\vec{H} - \vec{H}^2/8\pi$, F is the free-energy density, $\vec{M} = (\vec{h} - \vec{H})/4\pi$ is the density of the magnetization, $\vec{H}$ is the strength of the external uniform static magnetic field. Note that in the given paper we use the definition of G, which differs from the often used definition by the shift on the constant value $\vec{H}^2/8\pi$. Thus, the Gibbs free-energy density is

$$G = |(\nabla - ie\vec{A})\phi|^2 + m_{\rm ef}^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + \frac{(\vec{h} - \vec{H})^2}{8\pi}, \tag{7}$$

where $\vec{h} = \mathrm{curl}\vec{A}$. The condensate of the charged boson field appears provided $m_{\rm ef}^2 < 0$ in a part of the space.

A superfluid nonrelativistic motion of the system with the velocity $\vec{v}$ is described with the help of the replacement $\vec{D} \to \vec{D} + im_{\rm qp}\vec{v}$, where $m_{\rm qp}$ is a quasiparticle mass coefficient, the value which is not of our interest at present. Replacing $\phi \to \phi e^{i\chi}$ we find the contribution to the density of the momentum of the system $\vec{J}_v = 2\nabla\chi|\phi|^2$, $\vec{v} = \nabla\chi$, $\vec{p} = m_{\rm qp}\vec{v}$ is the momentum of the particle of the superfluid.

2. Neutral complex scalar field. Nonmagnetic superfluid phase

Consider a complex scalar field, which does not interact with the electromagnetic field. Thus, we put $e = 0$. Then $\mu = 0$ as well, and thereby $m_{\rm ef}^2 = m_{\rm sc}^2$.

Let us consider the half-space $x < 0$ medium, where $m_{\rm sc}^2 = m_0^2 < 0$ is the constant, and the system is placed in the external uniform static magnetic field $\vec{H}$. For $x > 0$, $m_{\rm sc}^2 = m^2 > 0$. The specific interactions, which may provide inequality $m_0^2 < 0$, are not of our interest here.

For $x \le 0$ from Eqs. (5) and (6) putting there $e = 0$, we obtain solutions

$$\phi = f_0 \mathrm{th}[(x - x_0)/(\sqrt{2}l_\phi)], \qquad \vec{h} = \vec{h}_0 = \text{const},$$
$$f_0 = \pm\sqrt{-m_0^2/\lambda}\,\theta(-m_0^2), \quad l_\phi = 1/\sqrt{|m_0|}, \quad x_0 = \text{const}, \tag{8}$$

$\theta(z)$ is the step function. For $x \ge 0$, we put $\phi = 0$, $\vec{h} = \vec{H}$. It is possible to do provided $l_\phi \gg 1/m$, i.e., $|m_0| \ll m$, that we assume for simplicity. Such an approximation in the phase transition theory is usually called the Landau approximation. From the boundary conditions for $x = 0$, we get $x_0 = 0$ and $h_0 = H$. Thus, we conclude that the magnetic field and condensate decouple.

With these solutions, we obtain the space-averaged Gibbs free-energy density,

$$\bar{G} = \frac{\int d^3x G}{\int d^3x} = -\frac{m_0^4}{2\lambda}\left(1 - \frac{4\sqrt{2}}{3}\frac{l_\phi}{d_x}\right)\theta(-m_0^2), \tag{9}$$

d_x is the length of the system in the x direction. Note that for the semi-infinite matter $d_x \to \infty$ and surface-energy term is vanishingly small. However, after the replacement $d_x \to d_x/2$, Eq. (9) holds also for the layer of a finite length provided $d_x \gg l_\phi$.

3. Charged complex scalar field. Superdiamagnetic response, superconductivity, and mixed Abrikosov state

Assume that $m_{\rm ef}^2 = m_{\rm sc}^2 - \mu^2 = m_{\rm ef,0}^2$ for $x < 0$, with $m_{\rm ef,0}^2 = \text{const} < 0$ and that $m_{\rm ef}^2 = m^2 > 0$ for $x > 0$, and the system is placed in the static uniform magnetic field parallel z. For $1/l_\phi \gg eHl_h$, where $m_{\rm ef,0}^2$ now replaces the value m_0^2 in previous example, l_h is the penetration depth of the magnetic field in the medium, assuming $l_\phi \gg 1/m$ from Eq. (5) we recover solution (8). With this solution at hand, Eq. (6) in the gauge $\mathrm{div}\vec{A} = 0$ simplifies as

$$\Delta\vec{A} - 8\pi e^2|\phi|^2\vec{A} = 0, \quad x \le 0. \tag{10}$$

For

$$l_h = 1/\sqrt{8\pi e^2 f_0^2} \gg l_\phi,$$

we may put $|\phi|^2 = f_0^2$ in (10). The value $m_\gamma = 1/l_h$ plays the role of the photon mass in the superconducting region, the quantity $\kappa = \sqrt{l_h/l_\phi}$ is the Ginzburg-Landau parameter. The inequality $1/l_\phi \gg eHl_h$ is rewritten as $H \ll H_{\rm cr}$, with the thermodynamical critical field

$$H_{\mathrm{cr}} = \sqrt{4\pi}|m_{\mathrm{ef}}^2|/\sqrt{\lambda}.$$

For $H \ll H_{\mathrm{cr}}$, the solution of Eq. (10) is

$$A_2(x \le 0) = H l_h e^{x/l_H},$$

where we used the gauge $\vec{A} = (0, A_2(x), 0)$ for $\vec{H}\| z$ and the boundary conditions $A_2'(x \to 0) = H$, $|A_2(x)| < \infty$. This solution demonstrates the Meissner-Higgs effect of the repulsion of the magnetic field from the superconducting region. For $\vec{H}\| y$, a similar Meissner effect exists for $A_3(x)$ with $\vec{A} = (0, 0, A_3(x))$.

The volume part of the space-averaged Gibbs free-energy density in the presence of the condensate, with the magnetic field being repelled from the condensate matter (phase I: $\phi = f_0$, $\bar{h} = 0$), is as follows: $\bar{G}_{\mathrm{I}} = -\frac{m_{\mathrm{ef}}^4}{2\lambda} + \frac{H^2}{8\pi}$. The volume part of the averaged Gibbs free-energy density in the absence of the condensate, with the magnetic field (phase II: $\phi = 0$, $\bar{h} = H$) is $\bar{G}_{\mathrm{II}} = 0$. Thus, for $H < H_{\mathrm{cr}}$, the condensate phase is energetically favorable, since $\bar{G}_{\mathrm{I}} < \bar{G}_{\mathrm{II}}$.

For the Ginzburg-Landau parameter, $\kappa \gg 1$ (actually it is sufficient to have $\kappa > 1/\sqrt{2}$) in a range of the fields $H_{\mathrm{c1}} < H < H_{\mathrm{c2}}$ the Abrikosov mixed phase is formed. Already for $H > H_{\mathrm{cr1}}$ (at $H_{\mathrm{cr1}} < H_{\mathrm{cr}}$) the surface energy of the system is decreased, if there appear filament vortices of the normal phase. The typical transversal size of the normal filament vortex directed parallel $\vec{H}$ is $\sim l_\phi$, whereas the magnetic field decreases at the distance $\sim l_h$ in the transversal direction. Thus, the Gibbs free-energy gain due to the appearance of the single vortex is estimated as $\sim -\pi l_h^2 d_z H^2/8\pi$ and the energy loss is $\sim \pi l_\phi^2 d_z m_{\mathrm{ef}}^4/2\lambda$. Comparing the gain and loss contributions we see that the Gibbs free energy is indeed gained for $H < H_{c1} \sim H_c/\kappa$. For $H > H_{\mathrm{cr1}}$, the vortices form the triangular lattice, which proves to be energetically more favorable compared to the quadratic lattice originally considered by Abrikosov, cf. [87]. Thus, for $H > H_{\mathrm{cr1}}$, the solution for the field ϕ should satisfy the periodic boundary conditions. Such a solution replaces the solution satisfying the boundary conditions for $x = 0$ that we had for $H < H_{\mathrm{cr1}}$. With subsequent increase of H, the distance between vortices decreases, the condensate weakens, and vanishes for $H = H_{\mathrm{cr2}}$.

For H slightly below H_{cr2}, the condensate field is weak and the equations of motion (5) and (6) can be linearized. Then the solution can be found analytically. Equation (5) for $\vec{A} = (A_1(y), A_2(x), 0)$ renders

$$-(D_1^2 + D_2^2)\phi = -m_{\mathrm{ef}}^2 \phi. \tag{11}$$

With $\vec{A} \simeq (0, Hx, 0)$, being the solution of the linearized Eq. (6), we may rewrite Eq. (11) in the form

$$-\frac{(\nabla - ie\vec{A})^2\phi}{2m_{\mathrm{aux}}} \simeq -\frac{m_{\mathrm{ef}}^2\phi}{2m_{\mathrm{aux}}} \tag{12}$$

of the Schrödinger equation for the nonrelativistic spin-less particle in the uniform magnetic field, where the quantity m_{aux} is an auxiliary mass coefficient.

The energy in the ground state is $E_{\min} = \frac{|m_{\mathrm{ef}}^2|}{2m_{\mathrm{aux}}} = |e|H_{\mathrm{cr2}}/2m_{\mathrm{aux}}$, from where we find

$$H_{\mathrm{cr2}} = |m_{\mathrm{ef}}^2/e| = H_{\mathrm{cr}}\sqrt{2}\kappa. \tag{13}$$

For the further usage, let us introduce the auxiliary condition

$$D_i\phi_i = 0, \quad \text{or} \quad D\phi = 0, \tag{14}$$

where $i = 1,\ 2$, $\phi_i = (\phi, -i\phi)$, $D = D_1 - iD_2$. Let us apply the operator $D = D_1 + iD_2$ to (14). Then, we obtain equation

$$(D_1^2 + D_2^2 - i[D_1, D_2]_-)\phi = 0, \tag{15}$$

with $[a, b]_- = ab - ba$, $i[D_1, D_2]_-\phi = eh_3\phi$, $h_3 = \partial_1 A_2 - \partial_2 A_1$. Thus,

$$-(D_1^2 + D_2^2)\phi = -eh_3\phi > 0. \tag{16}$$

With $\vec{A} \simeq (0, H_{\mathrm{c2}}x, 0)$, this equation is equivalent to Eq. (11). The solution has the form

$$\phi = \sum_{n=-\infty}^{\infty} C_n e^{ikny}\phi_n(x), \qquad \phi_n(x) = e^{-(x-x_n)^2/2l_\phi^2},$$

where $x_n = nkl_\phi^2$, $C_{n+N} = C_n$, $k = |e^*|H_{\mathrm{cr2}}x_0$, $N = 1$ corresponds to the quadratic lattice, $N = 2$, to the triangular one. This solution can be then used as the probe function to calculate the space-averaged Gibbs free-energy $\bar{G}$ within the mixed state and by variation of the free parameters to find its minimum.

In the toy model considered above, we were not interested in specification of the interactions, which provide the inequality $m_{\mathrm{sc}}^2 < 0$ for the neutral system and inequality $m_{\mathrm{ef}}^2 = m_{\mathrm{sc}}^2 - \mu^2 < 0$ for the charged system. In the neutron-star matter, there exists a fraction of protons, and one can consider a possibility of the π^- condensation, described by the negatively charged field $\phi = \phi_{\pi^-}$. Chemical potentials of particles fulfill equalities $\mu_p = \mu_n - \mu_e$ and $\mu_{\pi^-} = \mu_e$. In the approximation of the ideal pion gas, the s-wave π^- condensation would occur for $\mu_e(n) > m_\pi$, where n is the baryon density. However, it proves to be that the ideal gas approximation is hardly realized in a realistic problem due to the presence of the s-wave repulsive Weinberg-Tomozawa $\pi^- n$ interaction. The latter interaction does not allow for the s-wave π^- condensation up to high densities [88]. The π^- condensation with the field $\phi = f_0 e^{i\vec{k}_0\vec{r}}$ for $k_0 \neq 0$ in neutron star matter may appear for $n > n_c^\pi \sim (1.5-3)n_0$ due to a strong p-wave πN attraction [53,88]. The condensate π^- has properties of an unconventional superconductor of the second kind. In the external magnetic field, for $H > H_{\rm cr1}$, the vortices form the plane-layer structures rather than the filamentary structures and the value $H_{\rm cr2}$ proves to be very high [53,60]. Also, for $n > (n_c^{K^-}, n_c^{K^0}) \sim (2-4)n_0$, there may appear the s-wave [44,89] and p-wave [89,90] antikaon condensates. The condensate K^- has properties of a superconductor and $\bar{K}^0$, of a superfluid.

B. Zeeman coupling of neutral fermions and ferromagnetic state

In the quantum field theory in the famous Nambu-Jona-Lasinio (NJL) model [91,92], the $\langle(\bar\psi\psi)^2\rangle$ self-interaction of quarks represents the squared chiral condensate $\bar\psi\psi$. Angular brackets denote averaging over the equilibrium state of the fermion medium. Reference [76] considers a generalization of the NJL model with the spin-spin interaction term in the free-energy density $b_s\langle(\bar\psi\vec\gamma\gamma_5\psi)^2\rangle$ instead of $\langle(\bar\psi\psi)^2\rangle$ term in the original NJL model, γ_i, γ_5 are the Dirac matrices, the spin operator of the fermion is $\vec{S} = \frac{1}{2}\bar\psi\vec\gamma\gamma_5\psi$, $i = 1, 2, 3$. The spin-spin interaction for $b_s < 0$ causes a spontaneous magnetization. Such an interaction appears also in the model of the neutral massive fermion field ψ interacting with the own static magnetic field $\vec{h} = {\rm curl}\vec{A}$ by the Zeeman coupling, $\vec{A}$ is the vector potential of the magnetic field. The Lagrangian density is as follows:

$$L = \bar\psi(i\gamma^\nu\partial_\nu + i\gamma^0\mu - m_{\rm F})\psi - U + \eta\vec{S}\,\vec{h} - \vec{h}^2/8\pi,$$

$m_{\rm F}$ is the bare fermion mass, $\vec{\mathcal{M}} = \eta\vec{S}$ is the magnetic moment of the fermion, U is a fermion interaction term not depending on h.

The contribution to the Gibbs free-energy density dependent on h is

$$G_h = -\eta\langle(\bar\psi\vec\gamma\gamma_5\psi)\vec{h}\rangle/2 + (\vec{h} - \vec{H})^2/8\pi.$$

For $\langle\bar\psi\gamma^3\gamma_5\psi\rangle \neq 0$ and $\langle\bar\psi\gamma^{1,2}\gamma_5\psi\rangle = 0$, minimizing G_h in h (let it be parallel z), one gets

$$\vec{h} = \vec{H} + \vec{n}_3 \cdot 2\pi\eta\langle(\bar\psi\gamma^3\gamma_5\psi)\rangle,$$

where $\vec{n}_3 = (0, 0, 1)$, and

$$G_h = -\pi\eta^2\langle\bar\psi\gamma^3\gamma_5\psi\rangle^2/2 + \eta\langle(\bar\psi\gamma_3\gamma_5\psi)\rangle H_3/2.$$

The first term represents a spin-spin interaction. For the polarized spin state, this contribution to the free-energy density is negative (even for $H = 0$). However, the positive Fermi gas energy term for the polarized state is higher than that for the nonpolarized state. For the fully polarized matter $\langle\bar\psi\gamma^3\gamma_5\psi\rangle = n$, where n is the fermion density. Thus, the difference in the energy density for the fully spin-polarized matter and the nonpolarized one for $H = 0$, $T = 0$ becomes

$$E - E(h = 0) = \frac{3^{5/3}\pi^{4/3}(2^{2/3} - 1)n^{5/3}}{10m_{\rm F}^*} - \frac{\pi\eta^2 n^2}{2},$$

$m_{\rm F}^*$ is the effective fermion mass resulting from interactions not dependent on h. Thus, in this toy model, the ferromagnetic state becomes energetically favorable only for an abnormally high density $n > n_{\rm cr} = \frac{3^5\pi(2^{2/3}-1)^3}{125 m_{\rm F}^{*3}\eta^6}$, that is not realized for densities reachable in neutron stars. Only in an extremely high external magnetic field, the neutron star matter could be fully polarized. Reference [76] additionally included the axial anomaly term, a contribution of the axial-vector meson condensate and the neutral pion condensate. With these additional contributions, the critical density, above which the neutron star matter can be polarized, is strongly diminished up to the values reachable in the most massive neutron stars.

Note also that there exists a possibility of a ferromagnetic transition in quark matter interacting with one-gluon-exchange interaction [93], similarly to the ferromagnetism in electron gas. Spontaneous spin polarization due to the tensor self-energies in quark matter within the NJL model was considered in Ref. [94].

III. COMPLEX VECTOR-BOSON FIELDS. FERROMAGNETIC SUPERFLUIDITY AND SUPERCONDUCTIVITY

A. Lagrangian, equations of motion, Gibbs free energy

Let $\phi_\nu^{(j)} = (\phi_\nu^{(1)}, \phi_\nu^{(2)}, \phi_\nu^{(3)})$ be the field of a massive vector–isospin-vector boson, such as ρ meson, with $\phi_\nu^{(1)}, \phi_\nu^{(2)}, \phi_\nu^{(3)}$ as real quantities. Latin superscripts (1), (2), and (3) describe isospin, whereas the Greek index is as above the Lorentz index 0, 1, 2, 3. Instead of real fields $\phi_\nu^{(1)}$ and $\phi_\nu^{(2)}$, it is convenient to introduce complex fields,

$$\phi_\nu = (\phi_\nu^{(1)} - i\phi_\nu^{(1)})/\sqrt{2}, \qquad \phi_\nu^* = (\phi_\nu^{(1)} + i\phi_\nu^{(1)})/\sqrt{2}.$$

In our toy model, we will for simplicity put $\phi_\nu^{(3)} = 0$. Then we deal with a simpler problem of the description of the complex vector field ϕ_μ. The interaction of ϕ_μ with the electromagnetic field is described with the help of the long-derivative replacement $\partial_\mu\phi_\nu \to D_\mu\phi_\nu$, cf. (2), and the Zeeman term. Then the Lagrangian density for the interacting ϕ and the electromagnetic fields renders

$$L_{\phi,A} = -\frac{F_{\mu\nu}F^{\mu\nu}}{16\pi} - \frac{\phi_{\mu\nu}\phi^{*\mu\nu}}{2} + m_{\rm sc}^2\phi_\nu\phi^{*\nu} + L_{\phi\phi} + i\eta F_{\mu\nu}\phi^{*\mu}\phi^\nu, \quad (17)$$

$\phi_{\mu\nu} = D_\mu\phi_\nu - D_\nu\phi_\mu$, as above $m_{\rm sc}^2$ is the squared bare mass shifted by an attractive scalar potential.

The self-interaction term we take in the following form:

$$L_{\phi\phi} = -\Lambda[(\phi_\nu\phi^{*\nu})^2 + \xi_1(\phi_\nu\phi^\nu)(\phi_\mu^*\phi^{*\mu})], \quad (18)$$

where Λ is a positive coupling constant. Simplifying consideration we shall employ $\xi_1 = 0$, if the other is not mentioned. A non-Abelian form of the self-interaction was used in [95,96] in the problem of the instability of the W boson vacuum in a strong external magnetic field, in [69–71] for the description of the charged ρ meson condensation in the dense isospin-asymmetric baryon matter, and in [72,73] for the description of the instability of the ρ meson vacuum in a strong external magnetic field. At the condition $\phi_\nu^{(3)} = 0$, that we use, results of those works and ours here coincide provided $\xi_1 = -1$.

The Zeeman coupling term, $L_{\rm Zeeman} = i\eta F_{\mu\nu}\phi^{*\mu}\phi^\nu$, describes the interaction of the spin of the complex vector field with the electromagnetic field. In absence of the anomalous magnetic moment, the magnetic moment of the ρ^- meson would be $\mathcal{M}_\rho = \eta/m_\rho = e/m_\rho$, $e < 0$. With inclusion of a contribution of the anomalous magnetic moment, $\mathcal{M}_\rho \neq 2e/(2m_\rho)$. Reference [97] finds $\mathcal{M}_\rho \simeq 2.2e/(2m_\rho)$; other existing calculations give other values. An important circumstance here is only that in general case $\eta \neq e$.

Note that in a realistic problem of the behavior of the ρ meson in isospin-asymmetric nuclear matter one should include $\phi_0^{(3)}$ component, the electromagnetic interaction of the charged ρ fields, and the ρ interaction with fermions and other mesons, e.g., with the σ meson field, cf. [69–71].

Equations of motion for the fields ϕ^ν render

$$D^\mu D_\mu\phi^\nu - D^\nu D_\mu\phi^\mu - i(e+\eta)F^{\mu\nu}\phi_\mu + m_{\rm sc}^2\phi^\nu - 2\Lambda(\phi_\mu^*\phi^\mu)\phi^\nu = 0, \quad (19)$$

where we used the identity

$$[D_\mu, D_\nu]_-\psi = ieF_{\mu\nu}\psi \quad (20)$$

and

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu, \quad \text{with}$$
$$J^\nu = ieD^\nu\phi_\mu \cdot \phi^{*\mu} - ie\phi^{*\mu}D_\mu\phi^\nu + \text{c.c.} - i(e+\eta)\partial_\mu(\phi^{*\mu}\phi^\nu - \phi^{*\nu}\phi^\mu). \quad (21)$$

Now, the value $m_{\rm ef}^2 = m_{\rm sc}^2 - \mu_\phi^2$ has a sense of the squared effective mass of the complex vector field.

From (19) for $\eta = e$ neglecting $\sim\phi^3$ terms, we recover ordinary Proca equation for the Bose particle with the spin one compatible with the condition

$$D_\mu\phi^\mu = 0, \quad (22)$$

which is fulfilled identically away from the sources of the electromagnetic field. To show this, we apply the operator D_ν to the equation of motion (19) and make use of the identity (20) and that away from the sources $J_\nu = 0$. Contrary, the condition (22) is not necessarily compatible with the nonlinear equation of motion (19) and even with the linear equation of motion at $\eta \neq e$. Below [see discussion of Eq. (50)] we shall demonstrate a specific case, when the condition (22) is compatible with the linear equations of motion for the charged field at $\eta \neq e$.

For static vector fields $\phi^\nu = (0, \phi^i)$ and $A^\nu = (0, A^i)$, the Gibbs free-energy density renders (now in ordinary three-dimensional notations)

$$G = m_{\rm ef}^2|\phi_j|^2 + \Lambda(\phi_j\phi_j^*)^2 + \frac{(\vec h - \vec H)^2}{8\pi} + |D_i\phi_j|^2 - D_j\phi_i D_i^*\phi_j^* + i\eta\epsilon_{jik}h_k\phi_j^*\phi_i. \quad (23)$$

The Zeeman coupling term $i\eta\epsilon_{jik}h_k\phi_j^*\phi_i$ describes the interaction of the spin density, $S_k \propto i\epsilon_{jik}\phi_j^*\phi_i$, with the static magnetic field $\vec h = \text{curl}\vec A$. The quantity $\vec{\mathcal{M}} = \eta\vec S$ is the magnetic moment, $\eta S^2 = \mathcal{M}_3 S_3$.

The identity (20) can be then written as

$$i[D_i, D_j]_- = e\epsilon_{ijk}h_k, \quad (24)$$

where $D_j = (\nabla - ie\vec A)_j$, ϵ_{jkl} is the Levi-Civita tensor. With the identity (24) taken into account, equations of motion are simplified as

$$-D_i^2\phi_j + D_jD_i\phi_i + m_{\rm ef}^2\phi_j + 2\Lambda|\phi_i|^2\phi_j + i(e+\eta)F_{ji}\phi_i = 0 \quad (25)$$

and

$$\Delta\vec A = -4\pi\vec J \quad \text{at } \text{div}\vec A = 0, \quad \text{with}$$
$$J_i = -ie\phi_j^*D_i\phi_j + ie\phi_j^*D_j\phi_i + \text{c.c.} + i(e+\eta)\nabla_j(\phi_j^*\phi_i - \phi_i^*\phi_j). \quad (26)$$

The condition (22)

$$D_j\phi_j = 0, \quad (27)$$

cf. Eq. (14) in case of the scalar field.

B. Charge-neutral complex vector field

Consider model describing a complex vector field coupled with the electromagnetic field by the Zeeman coupling ($\eta \neq 0$) in the absence of minimal coupling (for $e = 0$). Equations (23) and (25) hold now for $e = 0$.

1. Superfluidity in nonmagnetic phase A

The simplest choice is when only one Lorentz component of the complex vector field is nonzero. Label such a choice as the phase A. The spin in this state is zero. For $m_{\rm sc}^2 > 0$, there are no solutions in this case. One can consider three subphases: A_1 $[\phi^\nu = (0, \phi_1(x), 0, 0)]$, A_2 $[\phi^\nu = (0, 0, \phi_2(x), 0)]$, and A_3 $[\phi^\nu = (0, 0, 0, \phi_3(x))]$.

In the case of the uniform matter placed in the external static uniform magnetic field $\vec{H}$, all three subphases are allowed. The magnetic field and the condensate decouple, $\vec{h} = \vec{H}$, $|\phi|^2 = -\frac{m_{\rm sc}^2}{2\Lambda}\theta(-m_{\rm sc}^2)$. The Gibbs free-energy density is $G_{\rm A} = -\frac{m_{\rm sc}^4}{4\Lambda}\theta(-m_{\rm sc}^2)$.

Let now the medium fill half-space $x < 0$, where $m_{\rm sc}^2 = m_0^2 < 0$ is a constant, placed in the external static uniform magnetic field $\vec{H}$. We will assume the vector-boson field $\vec{\phi}$ and the internal magnetic field $\vec{h} = {\rm curl}\vec{A}$ to be functions only of x (using the symmetry arguments), satisfying the boundary conditions for $x = 0$.

Subphase A_1 is not allowed. Indeed, then the boundary condition $\phi_1(x = 0) = 0$ for the vector-boson field cannot be satisfied due to the absence of $\propto \partial_1$ gradient term in Eq. (25). Also, notice that the condition $\partial_i\phi_i = 0$ is not fulfilled in this case, although the latter condition should be satisfied at least in the single particle approximation, in absence of the term $\propto \eta$ and for $e = 0$.

Subphase A_2.—Then the condition $\partial_i\phi_i = 0$ is fulfilled, and ϕ_2 and h satisfy equations of motion that follow from the variation of (23) for $e = 0$ in ϕ_2 and h, cf. Eqs. (25) and (26),

$$\partial_1^2\phi_2 - m_{\rm sc}^2\phi_2 - 2\Lambda(\phi_2\phi_2^*)\phi_2 = 0, \quad \vec{h} = \vec{H}. \tag{28}$$

Appropriate solution for the condensate field gets the form (8) for $m_{\rm sc}^2 = m_0^2 < 0$, $|m_0| \ll m$, now with $\lambda = 2\Lambda$. Then, we find

$$G_{\rm A_2} = |\partial_1\phi_2|^2 + m_{\rm sc}^2|\phi_2|^2 + \Lambda(\phi_2\phi_2^*)^2, \tag{29}$$

whereas the averaged Gibbs free energy is given by (9) (with λ replaced by 2Λ), i.e.,

$$\bar{G}_{\rm A_2} = \frac{\int d^3xG}{\int d^3x} = -\frac{m_0^4}{4\Lambda}\left(1 - \frac{4\sqrt{2}}{3}\frac{l_\phi}{d_x}\right)\theta(-m_0^2). \tag{30}$$

Subphase A_3.—Similarly, we could employ the field Ansatz $\phi^\nu = (0, 0, 0, \phi_3(x))$ with the same results as for the subphase A_2, $\bar{G}_{\rm A_2} = \bar{G}_{\rm A_3}$.

2. Ferromagnetic superfluidity in phase B

Let

$$\tilde{\Lambda} = \Lambda - 2\pi\eta^2$$

be positive. Consider the field Ansätze, which we name the phase B: $\phi^\nu = (0, 0, \phi_1(x), \phi_2(x))$ (subphase B_1); $\phi^\nu = (0, \phi_1(x), 0, \phi_2(x))$ (subphase B_2); and $\phi^\nu = (0, \phi_1(x), \phi_2(x), 0)$ (subphase B_3) with $\phi_2 = -C_0 i\phi_1$, where C_0 is real coefficient. We further take $C_0 = 1$ for $\eta < 0$ and $C_0 = -1$ for $\eta > 0$, as it follows from the minimization of the Gibbs free energy. Also, for convenience, we introduce the new variable $\tilde{\psi} = \phi_1(x)/\sqrt{2}$. We will show that now classical solutions may exist not only for $m_{\rm ef,0}^2 < 0$ but in some cases also for $m_{\rm ef,0}^2 > 0$.

Already for the uniform matter the free energy is different for the cases when the mean spin $\vec{S}$ is parallel to the external uniform static magnetic field $\vec{H}$ and perpendicular to it. For instance, for the subphase B_1 at $\vec{H}\|x$ using (23) with $\vec{A} = (0, 0, Hy \mp 4\pi\eta|\tilde{\psi}|^2 y)$, we obtain

$$h_1 = H \mp 4\pi\eta|\tilde{\psi}|^2 = {\rm const}, \qquad h_2 = h_3 = 0,$$
$$|\tilde{\psi}|^2 = \frac{-m_{\rm sc}^2 \mp \eta H}{2\tilde{\Lambda}}\theta(-m_{\rm sc}^2 \mp \eta H),$$
$$G_{\rm B_1}(\vec{H}\|x) = -\frac{(-m_{\rm sc}^2 \mp \eta H)^2}{4\tilde{\Lambda}}\theta(-m_{\rm sc}^2 \mp \eta H).$$

The upper and lower signs here correspond to two projections of the spin in the ground state for negative and positive η, respectively. For $\vec{H}\|z$, we have $\vec{A} = (0, Hx, \mp 4\pi\eta|\tilde{\psi}|^2 y)$,

$$h_1 = \mp 4\pi\eta|\tilde{\psi}|^2 = {\rm const}, \qquad h_2 = 0, \qquad h_3 = H,$$
$$|\tilde{\psi}|^2 = \frac{-m_{\rm sc}^2}{2\tilde{\Lambda}}\theta(-m_{\rm sc}^2), \qquad G_{\rm B_1}(\vec{H}\|z) = -\frac{m_{\rm sc}^4}{4\tilde{\Lambda}}\theta\left(\frac{-m_{\rm sc}^2}{4\tilde{\Lambda}}\right).$$

Similarly, it can be obtained solutions for the B_2 and B_3 subphases. We used that $\tilde{\Lambda} > 0$. It is the case, e.g., for hadrons since then $\Lambda \sim 1$ and $\eta \sim e$. Otherwise, by the first-order phase transition, there may appear a novel C phase; see below.

Let now the medium fill half-space $x < 0$, where $m_{\rm sc}^2 = m_0^2 < 0$ is a constant, placed in the external static uniform magnetic field $\vec{H}$. Again, consider the vector-boson field $\vec{\phi}$ and the internal magnetic field $\vec{h} = {\rm curl}\vec{A}$ to be functions only of x (using the symmetry arguments), satisfying the boundary conditions for $x = 0$. At least, at such assumption, there are no appropriate solutions for the vector-potential $\vec{A}$ in the case of the subphase B_1.

Subphase B_2.—Then the own magnetic field has the component $h_2(x) \neq 0$ due to the corresponding nonzero Zeeman term. The condition $\partial_i\phi_i = 0$ is not fulfilled with this field Ansatz. From (23), for $e = 0$, we have in the given case

$$G_{\mathrm{B}_2} = \frac{1}{2}|\nabla_x\tilde{\psi}|^2 + m_{\mathrm{sc}}^2|\tilde{\psi}|^2 + \Lambda|\tilde{\psi}|^4 + \frac{(\vec{h}-\vec{H})^2}{8\pi} \pm \eta h_2|\tilde{\psi}|^2. \tag{31}$$

Following the minimization of the energy, for $\eta < 0$, we should take the upper sign, and for $\eta > 0$, the lower sign, that relates to the choice $\phi_2 = \mp i\phi_1$, respectively.

Consider first $\vec{H}\|z$. Minimizing G_{B_2} in h, we obtain

$$h_2 = \mp 4\pi\eta|\tilde{\psi}(x)|^2, \qquad h_3 = H, \tag{32}$$

$h_1 = 0$. As we see, the field $\vec{h}(x)$ satisfies the necessary boundary condition $\vec{h}(0)=\vec{H}, \vec{A} = (0, Hx, \pm 4\pi\eta \int^x |\tilde{\psi}|^2 dx)$.

Equation of motion for the field $\tilde{\psi}$ is given by

$$\frac{1}{2}\partial_1^2\tilde{\psi} - m_{\mathrm{sc}}^2\tilde{\psi} - 2\Lambda|\tilde{\psi}|^2\tilde{\psi} \mp \eta h_2\tilde{\psi} = 0. \tag{33}$$

Using (32), we find for $x \le 0$,

$$\tilde{\psi}(x) = \pm\sqrt{\frac{-m_0^2}{2\tilde{\Lambda}}}\theta(-m_0^2)\mathrm{th}\frac{x - x_0}{\sqrt{2}l_\phi^{\mathrm{B}_2}}, \tag{34}$$

$l_\phi^{\mathrm{B}_2} = l_\phi/\sqrt{2}$, and assuming $|m_{\mathrm{sc}}| \ll m$, we put $x_0 = 0$ to satisfy the boundary condition $\tilde{\psi}(0) = 0$. With these solutions, we find

$$\bar{G}_{\mathrm{B}_2}(\vec{H}\|z) = -\frac{m_0^4}{4\tilde{\Lambda}}\left(1 - \frac{4\sqrt{2}l_\phi^{\mathrm{B}_2}}{3d_x}\right)\theta(-m_0^2). \tag{35}$$

Thus, at $\eta \ne 0$, comparing (30) and (35), we see that for any value of $\vec{H}\|z$ the subphase B_2 is energetically preferable compared with the subphases A.

Let now $H\|x$. We get $\vec{A} = (0, Hy, \pm 4\pi\eta \int^x |\tilde{\psi}|^2 dx)$, $h_1 = H$, $h_2 = \mp 4\pi\eta|\tilde{\psi}|^2$, $h_3 = 0$ and recover Eq. (34), and (35) now for $\bar{G}_{\mathrm{B}_2}(\vec{H}\|x)$.

Let now $H\|y$. With $\vec{A} = (0, 0, -Hx \pm 4\pi\eta \int^x |\tilde{\psi}|^2 dx)$, we obtain

$$h_1 = 0, \qquad h_2 = H \mp 4\pi\eta|\tilde{\psi}(x)|^2, \qquad h_3 = 0$$

and

$$\tilde{\psi}(x) = \pm\sqrt{\frac{-m_0^2 \mp \eta H}{2\tilde{\Lambda}}}\theta(-m_0^2 \mp \eta H)\mathrm{th}\frac{x}{\sqrt{2}l_\phi^{\mathrm{B}_2}}, \tag{36}$$

for $x \le 0$, with

$$\bar{G}_{\mathrm{B}_2}(\vec{H}\|y) = -\frac{(-m_0^2 \mp \eta H)^2}{4\tilde{\Lambda}}\left(1 - \frac{4\sqrt{2}l_\phi^{\mathrm{B}_2}}{3d_x}\right) \times \theta(-m_0^2 \mp \eta H). \tag{37}$$

For $\vec{H}\|y$ at $m_0^2 < 0$, the condensate amplitude grows with increasing value H. Thus, for $\vec{H}\|y$, the energy is gained compared to the case $\vec{H}\|x$ and $\vec{H}\|z$. The subphase B_2 is a ferromagnetic phase, since even for $H = 0$, there exists an own field $h_1 \ne 0$.

The classical vector field (36) is developed for $-m_0^2 \mp \eta H > 0$. Thus, in this case, the condensation occurs not only for $m_0^2 < 0$ (for arbitrary H) but also for

$$H > H_{\mathrm{cr}}^{\mathrm{neut}} = |m_0^2|/|\eta|, \quad \text{at } m_0^2 > 0. \tag{38}$$

For $H \ne 0$, we found that $\bar{G}_{\mathrm{B}_2}(\vec{H}\|y) < \bar{G}_{\mathrm{B}_2}(\vec{H}\|x) = \bar{G}_{\mathrm{B}_2}(\vec{H}\|z)$.

Subphase B_3.—The condition $\partial_i\phi_i = 0$ is not fulfilled with this field Ansatz. The Gibbs free-energy density renders

$$G_{\mathrm{B}_3} = \frac{1}{2}|\nabla_x\tilde{\psi}|^2 + m_{\mathrm{sc}}^2|\tilde{\psi}|^2 + \Lambda|\tilde{\psi}|^4 + \frac{(\vec{h}-\vec{H})^2}{8\pi} \pm \eta h_3|\tilde{\psi}|^2. \tag{39}$$

Equation of motion for $\tilde{\psi}$ is as follows:

$$\frac{1}{2}\partial_1^2\tilde{\psi} - m_{\mathrm{sc}}^2\tilde{\psi} - 2\Lambda(\tilde{\psi}\tilde{\psi}^*)\tilde{\psi} \mp \eta h_3\tilde{\psi} = 0. \tag{40}$$

Let $\vec{H}\|z$. With $\vec{A} = (0, Hx \mp 4\pi\eta \int^x |\tilde{\psi}|^2 dx)$, we get

$$h_3 = F_{12} = H \mp 4\pi\eta|\tilde{\psi}(x)|^2. \tag{41}$$

The appropriate solution of Eq. (40) with the boundary conditions $\tilde{\psi}(x) \to 0$ for $x \to 0$ and $\tilde{\psi}(x) \to \pm\sqrt{\frac{-m_0^2 - \eta H}{2\tilde{\Lambda}}}$ for $x \to -\infty$ coincides with Eq. (36) and

$$\bar{G}_{\mathrm{B}_3}(\vec{H}\|z) = \bar{G}_{\mathrm{B}_2}(\vec{H}\|y). \tag{42}$$

Moreover, for $H \ne 0$, we have $\bar{G}_{\mathrm{B}_3}(\vec{H}\|z) < \bar{G}_{\mathrm{B}_3}(\vec{H}\|x) = \bar{G}_{\mathrm{B}_3}(\vec{H}\|y)$.

As in case of subphase B_2 at $\vec{H}\|y$, for the subphase B_3 at $\vec{H}\|z$, the classical vector field is developed for $-m_0^2 \mp \eta H > 0$. Thus, the condensation occurs not only at $m_0^2 < 0$ for arbitrary H but also at $m_0^2 > 0$ for $H > H_{\mathrm{cr}}^{\mathrm{neut}} = |m_0^2|/|\eta|$.

Domains.—The difference in volume and surface energies of the subphases causes a possibility of existence of the domains for $H \ne 0$ and $H = 0$ with different directions of the own magnetic field $\vec{h}$ in each domain, which may merge in the presence of the external fields.

About choice of self-interaction.—With the self-interaction taken in the form (18) for $\xi_1 = 0$, that we have used, for $\vec{H}\|z$ the subphase B_3 proves to be energetically preferable compared to the other allowed subphases A_2, A_3, and B_2. For $\xi_1 \neq 0$, the situation becomes more complicated. For example, for $\xi_1 = -1$ in the A phase, the repulsive self-interaction term vanishes, whereas in the B phase the repulsive self-interaction term does not depend on the value ξ_1. Thereby, for $\xi_1 = -1$, the A phase becomes energetically favorable compared to the B phase at least for $H = 0$. Similar problems will be considered in next section on example of fermions with spin-triplet pairing.

3. Ferromagnetic superfluidity in phase C

For $\tilde{\Lambda} = \Lambda - 2\pi\eta^2 < 0$, by the first-order phase transition, there may appear a novel C phase. Since the hadron-hadron coupling $\Lambda \gg e^2$, at least for the ρ mesons, the C phase is not realized. For the triplet pairing, the C phase is possible; we shall return to this question in Sec. IV.

C. Charged complex vector field

Now, let the complex vector field be charged and interacting with the electromagnetic field by the minimal and the Zeeman couplings. Consider first the charged static complex vector field with $m_{\mathrm{ef},0}^2 = m^2 - \mu_\phi^2$ in half-space $x < 0$, placed in the external static uniform magnetic field $\vec{H}$. In this case, fields $\vec{h}$ and ϕ_i depend only on x.

1. Nonmagnetic and superdiamagnetic responses of various superfluid subphases A

Solutions exist only for $m_{\mathrm{ef},0}^2 = m^2 - \mu_\phi^2 < 0$.

Subphase A_1 is not realized, as in case of the neutral complex field considered in Sec. III B, since the appropriate boundary conditions at $x = 0$ cannot be fulfilled with the Ansatz $\phi_i = (\phi_1(x), 0, 0)$. The condition $\partial_i\phi_i = 0$ is also not satisfied, even for $\eta = e$ and for the linearized equation of motion, when it must be fulfilled.

Subphase A_2.—For $\phi_i = (0, \phi_2(x), 0)$, taking $\vec{H}\|z$, $\vec{A}_{\mathrm{ext}} = (0, Hx, 0)$, $\vec{A} = (0, A_2(x), 0)$, from (23), we obtain

$$G_{A_2}(\vec{H}\|z) = |\partial_1\phi_2|^2 + m_{\mathrm{ef}}^2|\phi_2|^2 + \Lambda|\phi_2|^4 + \frac{(\vec{h} - \vec{H})^2}{8\pi}.$$

Minimizing $\bar{G}_{A_2}$ in h we see that the magnetic field and the condensate decouple and $\vec{h} = \vec{H}$. The resulting expression for $\bar{G}_{A_2}$,

$$\bar{G}_{A_2}(\vec{H}\|z) = -\frac{m_{\mathrm{ef},0}^4}{4\Lambda}\left(1 - \frac{4\sqrt{2}\tilde{l}_\phi}{3d_x}\right)\theta(-m_{\mathrm{ef},0}^2), \qquad (43)$$

coincides with (9) after the replacement $\lambda \to 2\Lambda$, $m_0 \to m_{\mathrm{ef},0}$, and $l_\phi = 1/|m_0| \to \tilde{l}_\phi = 1/|m_{\mathrm{ef},0}|$. For $\vec{H}\|z$, the subphase A_2 is a nonmagnetic phase.

For $\vec{H}\|x$, the Gibbs free-energy density takes the form

$$G_{A_2}(\vec{H}\|x) = |\partial_1\phi_2|^2 + e^2A_3^2|\phi_2|^2 + m_{\mathrm{ef}}^2|\phi_2|^2 + \Lambda|\phi_2|^4 + \frac{(\vec{h} - \vec{H})^2}{8\pi}. \qquad (44)$$

Comparison with (7) demonstrates that after the replacement $\lambda \to 2\Lambda$ the charged complex vector field is described completely the same as the charged complex scalar field. Thus, for low H (for $H < H_{\mathrm{cr}1}$), the magnetic field h is repelled from the condensate region and

$$\bar{G}_{A_2}(\vec{H}\|x) \simeq \frac{H^2}{8\pi} - \frac{m_{\mathrm{ef},0}^4}{4\Lambda}\left(1 - \frac{4\sqrt{2}\tilde{l}_\phi}{3d_x}\right)\theta(-m_{\mathrm{ef},0}^2). \qquad (45)$$

Thus, the A_2 superconducting subphase for $\vec{H}\|x$ demonstrates a superdiamagnetic response on a weak external magnetic field, $\bar{h} = 0$. With an increase of H in the interval $H_{\mathrm{cr}1} < H < H_{\mathrm{cr}2}$, there appears the Abrikosov mixed state of vortices alternating with the condensate, for $H = H_{\mathrm{cr}2}$ the condensate disappears, and for $H > H_{\mathrm{cr}2}$ the condensate does not exist.

Subphase A_3.—For $\phi_i = (0, 0, \phi_3(x))$, choosing $\vec{H}\|z$, $\vec{A}_{\mathrm{ext}} = (0, Hx, 0)$, with $\vec{A} = (0, A_2(x), 0)$, i.e., with $\vec{h}\|z$, we are able to satisfy the boundary condition $\vec{h}(x = 0) = \vec{H}$. The Gibbs free-energy density takes the form

$$G_{A_3}(\vec{H}\|z) = |\partial_1\phi_3|^2 + e^2A_2^2|\phi_3|^2 + m_{\mathrm{ef}}^2|\phi_3|^2 + \Lambda|\phi_3|^4 + \frac{(\vec{h} - \vec{H})^2}{8\pi}. \qquad (46)$$

Comparison with (7) demonstrates that after the replacement $\lambda \to 2\Lambda$ the charged complex vector field is described completely the same as the charged complex scalar field. Thus, for low H (for $H < H_{\mathrm{cr}1}$), the magnetic field h is repelled from the condensate region and

$$\bar{G}_{A_3}(\vec{H}\|z) \simeq \frac{H^2}{8\pi} - \frac{m_{\mathrm{ef},0}^4}{4\Lambda}\left(1 - \frac{4\sqrt{2}\tilde{l}_\phi}{3d_x}\right)\theta(-m_{\mathrm{ef},0}^2). \qquad (47)$$

The subphase A_3 for a weak external magnetic field $\vec{H}\|z$ is superdiamagnetic, and $\bar{G}_{A_3}(\vec{H}\|z) = \bar{G}_{A_2}(\vec{H}\|x)$.

With an increase of H in the interval $H_{\mathrm{cr}1} < H < H_{\mathrm{cr}2}$, there appears the Abrikosov mixed state of vortices alternating with the condensate and for $H > H_{\mathrm{cr}2}$ the condensate disappears.

For $H \neq 0$, we find that $\bar{G}_{A_2}(\vec{H}\|z) < \bar{G}_{A_3}(\vec{H}\|z)$. For $H \to 0$, both quantities coincide.

With $\vec{A}_{\rm ext} = (0, 0, Hy)$, we have $\vec{h} = \vec{H}$ and

$$G_{A_3}(\vec{H}\|x) = |\partial_1\phi_3|^2 + m_{\rm ef}^2|\phi_3|^2 + \Lambda|\phi_3|^4. \quad (48)$$

Therefore, $\bar{G}_{A_3}(\vec{H}\|x) = \bar{G}_{A_2}(\vec{H}\|z)$.

Thus most energetically preferable are the subphase A_2 for $\vec{H}\|z$ and the subphase A_3 for $\vec{H}\|x$. In both cases, the subphases are nonmagnetic and the condensate of the charged vector field exists for arbitrary values of the external magnetic field.

2. Superconductivity in phase B

We will show that, as in case of the charge-neutral vector bosons, classical solutions may exist not only for $m_{\rm ef,0}^2 < 0$, when the response on a weak external magnetic field is superdiamagnetic, but in the presence of an overcritical external magnetic field also for $m_{\rm ef,0}^2 > 0$.

Subphase B_3.—Let $\vec{H}\|z$ and employ $\vec{A} = (A_1(x, y), A_2(x, y), 0)$, i.e., $\vec{h}\|z$.

Integrating by parts the gradient term in the Gibbs free energy, using the identity (24) and retaining only the volume part in the Gibbs free energy, we get

$$\int d^3x G_{B_3}(\vec{H}\|z) = \int d^3x\left[-\frac{1}{2}\tilde{\psi}^*(D_1^2+D_2^2)\tilde{\psi}\right] + \int d^3x\left[\frac{(h_3-H)^2}{8\pi}+\left[m_{\rm ef}^2+\left(\eta+\frac{e}{2}\right)h_3\right]|\tilde{\psi}|^2+\Lambda|\tilde{\psi}|^4\right], \quad (49)$$

for $\eta < 0$, $e < 0$. Varying the Gibbs free energy in $\tilde{\psi}^*$, we obtain equation of motion for the order parameter

$$-\frac{1}{2}(D_1^2 + D_2^2)\tilde{\psi} + \left[m_{\rm ef}^2 + \left(\eta + \frac{e}{2}\right)h_3\right]\tilde{\psi} + 2\Lambda|\tilde{\psi}|^2\tilde{\psi} = 0. \quad (50)$$

Setting $e = 0$, we recover Eq. (33). Choosing $A_1 = 0$ and varying (49) in A_2, we get

$$\partial_1^2 A_2 = -4\pi J_2 = 4\pi e^2|\tilde{\psi}|^2 A_2 - 4\pi\left(\eta + \frac{e}{2}\right)\partial_1|\tilde{\psi}|^2, \quad (51)$$

cf. Eq. (26) for the scalar charged bosons. There are two typical lengths characterizing solutions of these equations: $\tilde{l}_h = \sqrt{2}l_h$ characterizing the field $A_2(x)$ and $\tilde{l}_\phi = 1/(\sqrt{2}|m_{\rm ef,0}|)$, characterizing the field $\tilde{\psi}(x)$, cf. quantities l_h and l_ϕ introduced above. We will see that there are two types of solutions of these equations. One solution describes the Meissner screening effect, when the external magnetic field decreases on the length $\tilde{l}_h$ near the system boundary, whereas the condensate field reaches constant value for $-x > \tilde{l}_\phi$. In ordinary superconductors of the second kind, this solution is realized for $H < H_{\rm cr1}$. Another type of solution describes periodic structures for $H_{\rm cr1} < H < H_{\rm cr2}$. Consider first a specifics of the Meissner effect in our case. For $-x \sim \tilde{l}_h \gg \tilde{l}_\phi$, corresponding to the case $\kappa \gg 1$ that we consider, the term $4\pi(\eta + e/2)\partial_1|\tilde{\psi}|^2$ can be dropped and the solution satisfying the boundary condition $h_3(0) = H$ is $A_2(x) = H\tilde{l}_h e^{x/\tilde{l}_h}$. On the short distances, $-x \sim 1/\tilde{l}_\phi$ from the surface the y component of the vector potential, A_2, is a constant and the term $4\pi e^2|\tilde{\psi}|^2 A_2$ can be dropped for $H \ll 1/(\tilde{l}_\phi\tilde{l}_h) \sim H_{\rm cr}$. Then the solution (51), being valid for $-x \gtrsim \tilde{l}_h$, but satisfying the appropriate boundary condition for $x = 0$, $h_3(0) = H$, $\tilde{\psi}(0) = 0$, renders

$$h_3 \simeq -4\pi\left(\frac{e}{2}+\eta\right)[|\tilde{\psi}(x)|^2 - |\tilde{\psi}(-\infty)|^2](1-e^{x/\tilde{l}_h}) + He^{x/\tilde{l}_h}.$$

This solution describes the screening Meissner effect.

For $H \ll H_{\rm cr}$, using estimate done above for the scalar charged field, we can replace $D_1^2 + D_2^2 \to \partial_1^2 + \partial_2^2 \to \partial_1^2$. The solution of Eq. (50) then renders

$$\tilde{\psi}(x) \simeq \pm f_0\theta(-m_{\rm ef,0}^2){\rm th}\frac{x}{\sqrt{2}\tilde{l}_\phi}, \qquad f_0 = \sqrt{\frac{-m_{\rm ef,0}^2}{2\Lambda}}.$$

For the space-averaged Gibbs free energy, we obtain the expression

$$\bar{G}_{B_3}(\vec{H}\|z) \simeq \frac{H^2}{8\pi} - \frac{m_{\rm ef,0}^4}{4\Lambda}\left(1 - \frac{4\sqrt{2}\tilde{l}_\phi}{3d_x}\right)\theta(-m_{\rm ef,0}^2). \quad (52)$$

We see that for $H \neq 0$,

$$\bar{G}_{A_2}(\vec{H}\|z) < \bar{G}_{A_3}(\vec{H}\|z) < \bar{G}_{B_3}(\vec{H}\|z),$$

whereas for $H \to 0$, due to a smaller surface energy contribution, for the system of the finite size, we get $\bar{G}_{B_3}(\vec{H}\|z) < \bar{G}_{A_2}(\vec{H}\|z)$.

With increasing H above the value $H_{\rm cr1}$, there appears the Abrikosov lattice of vortices. For the ordinary metallic superconductors and similarly for the case of the charged scalar field, with a subsequent increase of H, the condensate weakens and for $H = H_{\rm cr2}$ it disappears. Assume that for H near the value $H_{\rm cr2}$ the condensate is weak. Then we drop the nonlinear term in Eq. (50) and put $\vec{A} = (0, H_{\rm cr2}x, 0)$. Thus, as for the case of the complex scalar field considered above at $H \simeq H_{\rm cr,2}$, we find the solution satisfying periodic boundary conditions. After dividing all terms in linearized Eq. (50) on an artificial mass coefficient, the former equation acquires the form of the Schrödinger equation for the nonrelativistic particle in the uniform magnetic field $h_3 = H$. The quantity

$$E_{\min} = -m_{\mathrm{ef},0}^2 - (\eta + e/2)H = |e|H/2$$

is the minimal eigenvalue. However, as we see, for $m_{\mathrm{ef},0}^2 < 0$, $\eta < 0$ there is no solution of this equation and there is no upper critical field $H_{\mathrm{cr}2}$, at which the condensate vanishes with increasing H.

On the other hand, the solution exists for $m_{\mathrm{ef}}^2 > 0$, $\eta < 0$ at

$$H > H_{\mathrm{cr}2} = -m_{\mathrm{ef}}^2/\eta > 0. \tag{53}$$

Note that we did not use the relation (27). Now, using condition (27) and the identity (24), we recover Eq. (16), which coincides with the linearized Eq. (50) at $h_3 = H_{\mathrm{cr}2}$ for any $\eta < 0$ at $m_{\mathrm{ef}}^2 > 0$. Let H be slightly above $H_{\mathrm{cr}2}$. Then, from (27), we find that $\partial_1|\phi_1|^2 = 2eA_2(x)|\phi_1|^2$. Setting this result in Eq. (51), we obtain

$$\partial_1^2 A_2 + 8\pi\eta e A_2|\tilde{\psi}|^2 = 0, \tag{54}$$

with the solution corresponding to the antiscreening effect, being in accordance with our observation that the superconductivity of the charged vector bosons appears at $H > H_{\mathrm{cr}2}$ for $m_{\mathrm{ef}}^2 > 0$, cf. statement of [74] that "new superconductivity" may antiscreen magnetic field.

Below I will demonstrate similarities and differences in the description of the complex vector meson fields and the spin-triplet pairing of fermions.

IV. SPIN-TRIPLET PAIRING IN NEUTRAL FERMION SYSTEM DESCRIBED BY COMPLEX VECTOR ORDER PARAMETER

A. Phenomenological Gibbs free-energy density

A formalism for description of the spin-triplet pairing in charged fermion systems, where the nonzero spin of the Cooper pair might be considered as a conserved quantum number, has been developed, cf. [20–22] and refs. therein. In this section, we employ a similar formalism for the description of the spin-triplet pairing in neutral fermion systems, where the complex vector order parameter is coupled to the magnetic field by the Zeeman term. Novel phases will be found.

Consider pairing of identical fermions. Since the total wave function of the system of identical fermions is antisymmetric under their exchange, and the spin part in the triplet state is symmetric, the angular part behaves as $(-1)^L$ with odd L. To be specific, let $L = 1$. For the description of the spin-triplet p-wave pairing of fermions, the pairing gap is as follows [20], $\hat{\Delta}(\vec{k}) = \vec{\sigma}\,\vec{d}(\vec{k}) i\sigma_2$, where $\vec{d}(\vec{k}) = -\vec{d}(-\vec{k})$ is an odd vector function and σ_j are the Pauli matrices, $j = 1, 2, 3$. If we considered pairing of nonidentical fermions, e.g., neutrons and protons, the isospin quantum number should be taken into account, $S + L + T$ (spin plus orbital momentum plus isospin) should be odd. The np $3\mathrm{S}_1$ phase shift is the largest among others at low nucleon-nucleon scattering energies. Thus, the np pairing in the $3\mathrm{S}_1$ channel is possible in the isospin-symmetric nuclear matter, also described by the complex vector order parameter.

We present $\hat{\Delta}(\vec{k}) = \psi_i \Phi_i(\vec{k})$, where Φ_i are three basis functions. Let us postpone consideration of the rotating systems (external rotation) and also disregard a possibility of an internal self-rotation. Thereby, we present the Gibbs free-energy density associated with the charge-neutral fermion pairs paired in the spin-triplet state in the form, cf. [20–22],

$$\begin{aligned}
G &= G_{\mathrm{grad}}^{\mathrm{neut}} + G_{\mathrm{hom}},\\
G_{\mathrm{grad}}^{\mathrm{neut}} &= c_1|\partial_i\psi_j|^2 + c_2|\partial_i\psi_i|^2 + c_3(\partial_i\psi_j)^*\partial_j\psi_i,\\
G_{\mathrm{hom}} &= -a|\psi_i|^2 + b_1(\psi_i\psi_i^*)^2 + b_2(\psi_i\psi_i)(\psi_j^*\psi_j^*)\\
&\quad + \mathcal{M} h_i i C\epsilon_{ijk}\psi_j^*\psi_k + (h_i - H_i)^2/(8\pi)\\
&\quad + b_3\sum_j|\psi_j|^4 + \{\gamma_k\psi_i\}^6,
\end{aligned} \tag{55}$$

where ψ_i is the complex vector order parameter with indices i, j, $k = 1, 2, 3$ transformed as a vector indices under spin rotations, cf. Eq. (23) introduced above for the complex vector-boson fields. The functional is symmetric under the U(1) phase transformations. Coefficients a, b_1, b_2, $\mathcal{M}$, c_1, c_2, c_3 are real quantities, and relations between c_1, c_2, c_3 should be such that the resulting surface term is positive. As we see at least should be

$$c_1, c_2 \geq 0. \tag{56}$$

In the quantum field theory of the vector field, cf. [73,74,95,96] and Sec. III, the gradient term is $\propto (D^\mu\phi^\nu - D^\nu\phi^\mu)^*(D_\mu\phi_\nu - D_\nu\phi_\mu)$, that corresponds to the choice $c_1 = -c_3 > 0$, $c_2 = 0$. In the BCS theory of clean materials, one employs [22] $c_1 \simeq c_2 \simeq c_3 > 0$. Reference [21] for the description of a new class of Ru-based superconductors uses the simplest choice $c_2 = c_3 = 0$, Ref. [2] employs also the choice $c_2 = c_3 \ll c_1 \sim N(0)v_{\mathrm{F}}^2/(\pi^2 T_{\mathrm{cr}}^2)$ (E_2 model), v_{F} is the Fermi velocity. Using the most general gradient contribution consistent with the U(1) gauge symmetry and the rotational symmetry, Ref. [15] calculated for the triplet superconductivity in 3D Dirac semimetals $c_3 = [u_L - u_T]/4$, $c_1 = u_T/4$, $c_2 = 0$, $u_L = u_T/32$, $u_T = \frac{7\zeta(3)N(0)v_{\mathrm{F}}^2}{15\pi^2 T_{\mathrm{cr}}^2}$, i.e., $c_1 \simeq -c_3$, $c_2 = 0$. Bearing in mind these different possibilities, we further employ general expression not asking for any relations between c_1, c_2, and c_3. Reference [15] also derives $b_1 = \frac{7\zeta(3)N(0)}{640\pi^2 T_{\mathrm{cr}}^2}$ and $b_2 = -b_1/3$. On the other hand, the heat capacity measurements performed for UPt_3 by several groups give $b_2/b_1 = (0.2\text{–}0.5)$, cf. [2,98].

The quantity $h_i = \epsilon_{ijk}\partial A_k/\partial x_j$ is the actual value of the strength of the magnetic field; ϵ_{ijk}, as above, is the Levi-Civita symbol. As in previous sections, $\vec{A}$ is the vector potential of the magnetic field and $\vec{H}$ is the strength of the uniform external static magnetic field. Simplifying consideration we neglect ψ^2 corrections to the H^2 magnetic energy terms.

The term $\propto b_3$ appears only in case of anisotropic systems. Thereby, and for simplicity, we further put $b_3 = 0$, cf. [15,22]. The term $\{\gamma_k \psi_i\}^6$ in (55) symbolically means all possible combinations of the sixth order in the order parameter. For the sake of simplicity, where it does not lead to the generation of instabilities, we put $\gamma = 0$.

Assuming that in the absence of external fields for $\gamma = 0$, we deal with the second-order phase transition; we take

$$a = \alpha_0 \varphi(t), \qquad t = (T_{\rm cr} - T)/T_{\rm cr}, \tag{57}$$

where the function $\varphi(t) = t + O(t^2)$ for small t, $T_{\rm cr}$ has the sense of the critical temperature of the pairing transition for $H = 0$, and all the parameters $a_0 > 0$, $b_1 > 0$, and b_2, b_3, c_1, c_2, c_3 can be considered as T-independent constants for a small t. Also, simplifying consideration in this work, we employ the mean-field theory. As is known, fluctuations of the order parameter prove to be significant in the vicinity of the critical point of the second-order phase transition, for T near $T_{\rm cr}$, cf. [82,83]. We will show that for certain subphases placed in the external magnetic field the mean-field solutions may exist not only for $T < T_{\rm cr}$ but also for T above $T_{\rm cr}$, i.e., below a higher value of the new critical temperature $T^{\rm H}_{\rm cr}$. Thus, for mentioned subphases, the fluctuation region is shifted to the vicinity of the critical temperature $T^{\rm H}_{\rm cr}$. As pointed out in Ref. [82], expansion in the order parameter is a primary feature in the Landau theory of phase transitions, whereas an expansion in powers of t is a secondary assumption valid for T near $T_{\rm cr}$. Therefore, at least for estimates, we may employ the functional (55) for T outside the vicinity of $T_{\rm cr}$ using $\varphi(T) = t$, $\varphi(T = 0) = 1$, cf. [99]. Below, if not mentioned another, to be specific, we suppose that the external magnetic field $\vec{H}$ is aligned parallel to z, i.e., $\vec{H}_i = \delta_{i3} H$, although the behavior of the system described by the vector order parameter is sensitive to the choice of the direction of H relatively to the surface, as we have demonstrated in the previous section.

The mean spin density is carried by the order parameter,

$$S_i = -iC\epsilon_{ijk}\psi^*_j \psi_k, \tag{58}$$

where $C > 0$ is a normalization constant. For $\vec{\psi}$ aligned along one of the axis 1, 2, 3 (x, y, or z), one has $\vec{S} = 0$.

Note that in case $b_2 = -b_1$ the self-interaction contribution to the Gibbs free-energy density, $b_1(\psi_i\psi^*_i)^2 + b_2(\psi_i\psi_i)(\psi^*_j\psi^*_j)$, is reduced to the spin-spin interaction term $b_s S_i S_i$ with $b_s = b_1/C^2$ yielding the repulsion for $b_1 > 0$, as in the Ginzburg-Landau treatment of superfluids described by a single order parameter, and the attraction for $b_1 < 0$. For $b_1 < 0$ and $b_2 = 0$, the system is unstable.

In difference with description of magnetic superconductors performed in [20–22], when dealing with neutral fermions we suppress minimal coupling with the magnetic field but retain the Zeeman term assuming that neutral fermions under consideration have magnetic moments. The orientation of the averaged spin related to the order parameter relatively the magnetic field depends on the sign of the magnetic moment of the pair. The effective magnetic moment of the pair is $\vec{\mathcal{M}}_{\rm pair} = \mathcal{M}_{\rm pair}\vec{s}_{\rm pair}$, $\vec{s}_{\rm pair}$ is the spin of the pair. Owing to the existence of the anomalous magnetic moment, the neutron pair with parallel spins gets the magnetic moment $\mathcal{M}_{nn} \simeq g_{nn}\mathcal{M}_N$, where $\mathcal{M}_N > 0$ is the nucleon Bohr magneton, $g_{nn} = -2 \times 1.91$ is the effective Lande factor. The proton pair has the magnetic moment $\mathcal{M}_{pp} \simeq g_{pp}\mathcal{M}_N$ with $g_{pp} = 2 \times 2.79$, $\mathcal{M}_N \simeq 3.15 \times 10^{-18}$ MeV/G. Note that the ratio of neutron to proton magnetic moments $\mathcal{M}_{nn}/\mathcal{M}_{pp} \simeq -0.68$ is close to the value $-2/3$ predicted by the valence quark model. In the spin-orbit Fermi superfluids, the role of the $\mathcal{M}h_i$ coefficient in the Zeeman term is played by the Rabi frequency [100]. The volume-averaged Gibbs free-energy density $\bar{G} = \bar{F} - \overline{\vec{M}\,\vec{H}}$, where $\bar{F}$ is the averaged free-energy density, $\vec{M} = (\vec{h} - \vec{H})/(4\pi)$ is the induced magnetization, $\bar{\vec{h}} = \vec{B}$ is the vector of the magnetic induction.

SO(3) symmetry is partially broken to its SO(2). Thereby, as in Ref. [21], we may present

$$\vec{\psi} = f(\vec{n}\cos\theta + i\vec{m}\sin\theta), \tag{59}$$

where f is real and $\vec{n}$ and $\vec{m}$ are arbitrary unit vectors. Let ϕ be the angle between $\vec{n}$ and $\vec{m}$. Then, for a uniform matter replacing (59) in (55), we find

$$G^{\rm hom} = -af^2 + [b_1 + b_2(\cos^2(2\theta) + (\vec{n}\,\vec{m})^2\sin^2(2\theta))]f^4 - C\mathcal{M}f^2\vec{h}[\vec{n}\times\vec{m}]\sin(2\theta) + \frac{(\vec{h}-\vec{H})^2}{8\pi} + O(f^6). \tag{60}$$

Now, we focus on the consideration of various phases in a system of fermions with the spin-triplet pairing. First, consider the case when one can neglect contribution $\propto f^6$ formally setting $\gamma = 0$. Minimization in h and f yields

$$\vec{h} = \vec{H} + 4\pi C\mathcal{M}f^2[\vec{n}\times\vec{m}]\sin(2\theta), \tag{61}$$

$$f^2 = \frac{a + C\mathcal{M}\vec{H}[\vec{n}\times\vec{m}]\sin(2\theta)}{2Y}\theta(f^2), \tag{62}$$

$$Y = b_1 + b_2(\cos^2(2\theta) + (\vec{n}\,\vec{m})^2\sin^2(2\theta)) - 2\pi C^2\mathcal{M}^2[\vec{n}\times\vec{m}]^2\sin^2(2\theta). \tag{63}$$

Stable solution exists only for $Y > 0$. For $H = 0$, the solution exists for $a > 0$, $Y > 0$.

With the solution (61)–(63), we get the Gibbs free-energy density,

$$G^{\mathrm{hom}} = -\frac{[a + C\mathcal{M}\vec{H}[\vec{n}\times\vec{m}]\sin(2\theta)]^2\theta(f^2)}{4Y}. \tag{64}$$

B. Nonmagnetic superfluidity in phase A

1. Uniform matter

The phase A with zero mean spin density (58) corresponds to the choice $\theta = 0$. Then $\vec{\psi} = f\vec{n}$, as it follows from (59).

For $\theta = \phi = 0$ in the stable phase A, Eq. (60) simplifies as

$$G_{\mathrm{A}}^{\mathrm{hom}} = -af^2 + (b_1 + b_2)f^4. \tag{65}$$

Equations (61), (62), and (64) read

$$f^2 = f_0^2 = \frac{a}{2(b_1 + b_2)}\theta(f_0^2), \tag{66}$$

$$\vec{h} = \vec{h}_0 = H, \tag{67}$$

$$G_{\mathrm{A}}^{\mathrm{hom}} = -\frac{a^2}{4(b_1 + b_2)}\theta(f_0^2), \tag{68}$$

for $T < T_{\mathrm{cr}}^{\mathrm{A}} \equiv T_{\mathrm{cr}}$ $(a > 0)$. For $T > T_{\mathrm{cr}}$, we have $f = 0$, $\vec{h} = 0$, and $G_{\mathrm{A}}^{\mathrm{hom}} = 0$. The gradient term (56) is zero for the homogeneous solution. In the critical point $G_{\mathrm{A}}^{\mathrm{hom}} = 0$, $\partial G_{\mathrm{A}}^{\mathrm{hom}}/\partial T = 0$ but $\partial^2 G_{\mathrm{A}}^{\mathrm{hom}}/\partial T^2 \neq 0$ that corresponds to the second-order phase transition at $T = T_{\mathrm{cr}}$.

Consider stability of the phase A, respectively, to the formation of a small spin density in the system for $H = 0$. Taking $|\theta| = \delta\theta \ll 1$ and allowing $\phi \neq 0$ in Eq. (63), we obtain

$$\begin{aligned} Y &= b_1 + b_2[1 - 4(\delta\theta)^2(1 - (\vec{n}\,\vec{m})^2)] \\ &\quad - 8\pi C^2\mathcal{M}^2[\vec{n}\times\vec{m}]^2(\delta\theta)^2 \\ &= b_1 + b_2 - 4(\delta\theta)^2\sin^2\phi[b_2 + 2\pi C^2\mathcal{M}^2] \end{aligned} \tag{69}$$

that demonstrates stability of the phase A only provided

$$b_1 + b_2 > 0 \tag{70}$$

(otherwise one should incorporate $\gamma \neq 0$ terms) and for

$$b_2 + 2\pi C^2\mathcal{M}^2 < 0 \tag{71}$$

(otherwise the A phase is unstable to the appearance of $\theta \neq 0$ and $\phi \neq 0$ in the ground state). Thus, for $H = 0$, the phase A is stable to appearance of a nonzero spin density in the system. Note that for $b_2 = 0$, that corresponds to $\xi_1 = 0$ in the vector-boson case considered in Sec. III, the condition (71) is not fulfilled. In the vector-boson case, it was reflected in the fact that for $\xi_1 = 0$ in the B phase the Gibbs free energy is smaller than in the A phase.

2. *Subphases* A_1, A_2, A_3. *Gradient term. Domains*

Subphases A_1, A_2, A_3.—Since $\vec{n}$ is fully characterized by its three projections, we may consider three specific choices $\vec{n} = (1,0,0)$, $\vec{n} = (0,1,0)$, and $\vec{n} = (0,0,1)$: the A_1 subphase $(\psi_1 = \psi \neq 0, \psi_2 = \psi_3 = 0)$, A_2 subphase $(\psi_2 = \psi \neq 0, \psi_1 = \psi_3 = 0)$, and A_3 subphase $(\psi_3 = \psi \neq 0, \psi_1 = \psi_2 = 0)$, which we have introduced in Sec. III. In the uniform neutral superfluid, these states are degenerate and correspond to the same Gibbs free energies.

Gradient term. Stability of A subphases.—We focus now on the role of the gradient contribution to the free energy (55). Let the medium fill the half-space $x < 0$. Then $f = f(x)$ and does not depend on y and z due to the uniformity of the system in these directions. The gradient contributions for subphases A_1, A_2, and A_3 are different,

$$G_i^{\mathrm{grad}} = C_i(\partial_1 f)^2, \quad i = \mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3. \tag{72}$$

In the subphase A_1, $\psi_1(x) \neq 0$, $\psi_2 = \psi_3 = 0$, and $C_{\mathrm{A}_1} = c_1 + c_2 + c_3$. For such a solution, $\mathrm{div}\,\vec{\psi} \neq 0$. In the subphases A_2 and A_3, $C_{\mathrm{A}_2} = C_{\mathrm{A}_3} = c_1$. Here only $\psi_2(x) \neq 0$ or $\psi_3(x) \neq 0$, respectively, and the condition $\mathrm{div}\,\vec{\psi} = 0$ is fulfilled.

Thus, the stability conditions are

$$c_1 + c_2 + c_3 \geq 0, \quad c_1 \geq 0. \tag{73}$$

Now, let us check the stability of the phase A in the presence of the gradient contribution to the Gibbs free energy relatively the appearance of a small $\theta(x)$. For $\vec{m}\|\vec{n}$ in Eq. (72), there appear extra terms $(c_1 + c_2 + c_3)f^2(\partial_1\theta)^2$ for $\vec{n} = (1,0,0)$ and $c_1 f^2(\partial_1\theta)^2$ for $\vec{n} = (0,1,0)$ or $\vec{n} = (0,0,1)$. For $\vec{m}\perp\vec{n}$, with $\vec{n} = (1,0,0)$ and $\vec{m} = (0,1,0)$ or $\vec{m} = (0,0,1)$ in Eq. (72) appears extra term $c_1 f^2(\partial_1\theta)^2$. For $\vec{n} = (0,1,0)$ for $\vec{m} = (0,0,1)$, there appears the term $c_1 f^2(\partial_1\theta)^2$ and for $\vec{m} = (1,0,0)$, the term $(c_1 + c_2 + c_3)f^2(\partial_1\theta)^2$. As we see, in all these cases, an increase of θ is energetically not profitable. Thus, the phase A is stable in respect to the growth of weak perturbations both in the uniform and the nonuniform matter.

Variation of the Gibbs free energy (55) in the field f yields equations of motion

$$C_i\partial_1^2 f + af - 2(b_1+b_2)f^3 = 0, \qquad (74)$$

with the solutions satisfying the boundary condition $f(x=0)=0$,

$$f(x) = f_0 \mathrm{th}\frac{x}{\sqrt{2}\xi_{A_i}}, \qquad \xi_{A_i} = \sqrt{C_i/a}. \qquad (75)$$

f_0 is given by Eq. (66). Replacing (75) in the expression for the Gibbs free energy, $\int d^3xG = \mathcal{G}^{\mathrm{vol}} + \mathcal{G}^{\mathrm{surf}}$, we find that the surface contribution is $\mathcal{G}_i^{\mathrm{surf}} \propto \xi_i S$, S is the square in y, z plane. $\mathcal{G}_i^{\mathrm{surf}}$ gets minimum for the subphases A_2 and A_3, if $0 < c_1 < c_1+c_2+c_3$, and for the subphase A_1, if $c_1 > c_1+c_2+c_3 > 0$.

Domains.— Depending on how the system was prepared, it can consist of domains with different directions of the order parameter $\vec{\psi}$ in each domain. Due to the difference in the contributions to the surface energies in the longitudinal and transversal directions relatively the surface, for a domain of a fixed volume, it is profitable to become oblate or prolate in dependence on the sign of c_2+c_3.

For a slab of the subphase A_1 surrounded by the matter in the subphase A_2 due to the presence of the phase boundary, there appears a contribution to the surface energy, $\delta\mathcal{G}^{\mathrm{surf}}_{A_1,A_2} = \mathcal{G}^{\mathrm{surf}}_{A_1} + \mathcal{G}^{\mathrm{surf}}_{A_2} > 0$. However, as we have demonstrated, the solution for the order parameter in the phase A characterized by a direction $\vec{n}$ is stable. Thus, to melt the domain should overcome the energy barrier $\delta\mathcal{G}^{\mathrm{surf}}_{A_1,A_2}$. Necessary energy to overcome the barrier can be extracted, e.g., from thermal fluctuations, or from external magnetic field, or for the system subjected to the external rotation the required energy can be taken from the energy of the rotation.

Notice that in difference with the case $c_1 = -c_3 \neq 0$, $c_2 = 0$ considered in Sec. III, where the A_1 phase was not realized and the subphases A_2 and A_3 had the same volume and surface energies, here for $c_1+c_2+c_3 \neq 0$, $c_1 \neq 0$ and $c_1 \neq c_1+c_2+c_3$ all three subphases can be realized and the surface energy in the A_1 subphase differs from those in A_2 and A_3 subphases.

C. Instability of A phase in external magnetic field, AH phase

Above we have demonstrated stability of the phase A (at zero mean spin density) to formation of a nonzero spin state in the absence of the external magnetic field. Let us study stability of the ground state of the A phase (conditions (70) and (71) are supposed to be fulfilled) relatively the growth of θ and ϕ, i.e., to the formation of a mean spin density in the system, for $H \neq 0$. Further, we consider energetically favorable cases, one corresponding to $\vec{\mathcal{M}}\vec{H} > 0$ for $\mathcal{M} > 0$ (for protons) and another for $\vec{\mathcal{M}}$ aligned antiparallel z for $\mathcal{M} < 0$ (for neutrons). Rewrite (64) as

$$G^{\mathrm{hom}}_{\mathrm{AH}} = -\frac{[a + C|\mathcal{M}H\zeta|]^2\theta(a + C|\mathcal{M}H\zeta|)}{4[b_1+b_2-\zeta^2(b_2+2\pi C^2\mathcal{M}^2)]}, \qquad (76)$$

for $\mathcal{M}H > 0$ with $\zeta = \sin\phi\sin(2\theta) > 0$ and for $\mathcal{M}H < 0$ with $\zeta = \sin\phi\sin(2\theta) < 0$. The denominator is positive provided conditions (70) and (71) are fulfilled. As we can see, for $H \neq 0$ the phase A proves to be unstable in respect to production of a spin density, since it is energetically profitable to have $\zeta \neq 0$. Accordingly, cf. (61), in the presence of the external magnetic field, the strength of the magnetic field becomes

$$h = H + 4\pi C\mathcal{M}f^2\zeta.$$

1. Paramagnetic response of superfluid in AH phase for $T < T_{\mathrm{cr}}$

For $a > 0$, i.e., $T < T_{\mathrm{cr}}$, minimizing the Gibbs free-energy density in $\zeta = \sin\phi\sin(2\theta)$, we get at the extremum

$$\zeta_m = -\frac{C\mathcal{M}H(b_1+b_2)}{a(b_2+2\pi C^2\mathcal{M}^2)}, \qquad (77)$$

valid for $|\zeta_m| \leq 1$, with $\zeta_m \to 0$ for $H \to 0$. Note that with $\zeta = \zeta_m \neq 1$ we obtain $f^2 > 0$ in (62) only for $a > 0$, i.e., for $T < T_{\mathrm{cr}}$. Thus, for $H \neq 0$, not all spins in the condensate are aligned in one direction at $T < T_{\mathrm{cr}}$. We deal with the novel phase, which we name the AH phase when the conditions (70) and (71) are fulfilled but not all spins of the paired fermions are aligned in one direction. For $b_2 < 0$ and $|C\mathcal{M}\zeta_m|H \ll a$, we find $h = H[1 + 2\pi C^2\mathcal{M}^2/(|b_2|(b_1-|b_2|)]$. Also, from (77), we find an additional constraint,

$$H \leq H^{\mathrm{AH}}_{\mathrm{cr}}(T < T_{\mathrm{cr}}) = \frac{a(|b_2| - 2\pi C^2\mathcal{M}^2)}{|C\mathcal{M}|(b_1-|b_2|)} \qquad (78)$$

for $a > 0$.

The Gibbs free-energy density in the ground state for $T < T_{\mathrm{cr}}$ ($a > 0$) can be presented as

$$G^{\mathrm{hom}}_{\mathrm{AH}} \simeq -\frac{a^2}{4(b_1+b_2)} + \frac{C^2\mathcal{M}^2H^2}{4(b_2+2\pi C^2\mathcal{M}^2)}. \qquad (79)$$

Although for $H \neq 0$, the resulting magnetic field $h \neq 0$, for $H \to 0$ we obtain $h \to 0$.

2. Instability of AH phase for $T > T_{\rm cr}$. Transition to a ferromagnetic superfluid phase

Now, consider the case $a < 0$, i.e., $T > T_{\rm cr}$. The actual critical temperature is determined from the condition $a + C\mathcal{M}H\zeta = 0$ for $\mathcal{M}H > 0$ and $\zeta = 1$, and from $a - C\mathcal{M}H\zeta = 0$ for $\mathcal{M}H < 0$ and $\zeta = -1$. For favorably aligned spins, we obtain

$$T_{\rm cr}^{\rm AH} = T_{\rm cr}(1 + |C\mathcal{M}H|/\alpha_0) > T_{\rm cr}, \quad \text{for } a < 0. \tag{80}$$

Thus, the AH phase may exist not only for $T < T_{\rm cr}$ but also in the temperature interval $T_{\rm cr} < T < T_{\rm cr}^{\rm AH}$ and the critical temperature $T_{\rm cr}^{\rm AH}$ is increased with increasing H. In this respect, the AH phase is similar to the A_1 phase of the ^{3}He, cf. [18].

The Gibbs free-energy density in the ground state for $T_{\rm cr} < T < T_{\rm cr}^{\rm AH}$ is as follows:

$$G_{\rm AH}^{\rm hom} = -\frac{[a + |C\mathcal{M}H|]^2\theta(a + |C\mathcal{M}H|)\theta(-a)}{4(b_1 - 2\pi C^2\mathcal{M}^2)}. \tag{81}$$

As we see, for $a < 0$ still the condition $b_1 - 2\pi C^2\mathcal{M}^2 > 0$ should be satisfied for the stability of the phase. In next Sec. IV D, such a phase will be named the B phase. Thus, for $T_{\rm cr} < T < T_{\rm cr}^{\rm AH}$, the AH phase coincides with the B phase, if besides the conditions (70) and (71) also the condition $b_1 - 2\pi C^2\mathcal{M}^2 > 0$ is satisfied. For $T_{\rm cr} < T < T_{\rm cr}^{\rm AH}$, (for $a < 0$) we put in Eq. (77) $\zeta = 1$ for $C\mathcal{M} > 0$ and $\zeta = -1$ for $C\mathcal{M} < 0$, that corresponds to the fact that all spins are aligned in one direction. We find that the condensate exists now for

$$H > H_{\rm cr}^{\rm AH}(T_{\rm cr} < T < T_{\rm cr}^{\rm AH}) = |a|/|C\mathcal{M}|. \tag{82}$$

D. Ferromagnetic superfluidity in phases B and C

1. Stability conditions

We name the phase B or C the choice $\theta = \pi/4$, $\vec{n}\perp\vec{m}$, H is arbitrary. Setting $\theta = \pi/4 - \delta\theta$ in Eq. (63), we find

$$\begin{aligned} Y = b_1 &+ b_2[4(\delta\theta)^2(1 - (\vec{n}\,\vec{m})^2) + (\vec{n}\,\vec{m})^2] \\ &-2\pi C^2\mathcal{M}^2[\vec{n}\times\vec{m}]^2(1 - 4(\delta\theta)^2). \end{aligned} \tag{83}$$

We deal with the phase B, if

$$b_1 - 2\pi\mathcal{M}^2C^2 > 0 \tag{84}$$

and with the phase C, if

$$b_1 - 2\pi\mathcal{M}^2C^2 < 0. \tag{85}$$

These conditions together with condition

$$b_2 + 2\pi\mathcal{M}^2C^2 > 0 \tag{86}$$

replace the stability conditions (70) and (71), being fulfilled in case of the A phase. Favorable direction of $\vec{H}$ is parallel to $[\vec{n}\times\vec{m}]$, as follows from (62). This is in agreement with our observation done in previous section that the subphase B_3 with $\vec{H}\| z$ corresponds to the lowest Gibbs free energy.

For the phase C (at $b_1 - 2\pi C^2\mathcal{M}^2 < 0$), one needs to include at least the sixth-order term ($\gamma \neq 0$) in the free energy. In order not to complicate consideration, we further choose the simplest form of the $\{\gamma_i\psi_i\}^6$ term ($\gamma(\psi_i^*\psi_i)^3$) assuming $\gamma > 0$. Note here that in the BCS weak-coupling theory, one obtains $\gamma < 0$ and expansion of the Gibbs free energy should be continued up to the eighth order [56].

For simplicity, we put $T_{\rm cr}^{\rm A} = T_{\rm cr} = T_{\rm cr}^{\rm B}$; on the other hand, $T_{\rm cr}^{\rm C} \neq T_{\rm cr}$ since the phase transition to the phase C proves to be of the first order.

2. Subphases B_1, B_2, B_3, C_1, C_2, C_3

In general, we may consider the following three choices:

$$\begin{aligned} &\psi_1 = 0, \qquad \psi_2 = \mp\, i\psi_3 \equiv \frac{1}{\sqrt{2}}\tilde{\psi}, \\ &\psi_2 = 0, \qquad \psi_1 = \mp\, i\psi_3 \equiv \frac{1}{\sqrt{2}}\tilde{\psi}, \\ &\psi_3 = 0, \qquad \psi_1 = \mp\, i\psi_2 \equiv \frac{1}{\sqrt{2}}\tilde{\psi}, \end{aligned} \tag{87}$$

for subphases B_1 (or C_1), B_2 (or C_2), and B_3 (or C_3), respectively. In all these cases, $\psi_i\psi_i = 0$.

With the simplest $\gamma(\psi_i^*\psi_i)^3$ term taken into account, we have

$$G^{\rm hom} = -a|\tilde{\psi}|^2 + b_1|\tilde{\psi}|^4 - C\vec{\mathcal{M}}\,\vec{h}|\tilde{\psi}|^2 + \gamma|\tilde{\psi}|^6 + \frac{(\vec{h} - \vec{H})^2}{8\pi}. \tag{88}$$

The gradient contribution to the Gibbs free energy does not depend on h. Thus, varying (88) in h, we obtain

$$\vec{h} - \vec{H} = 4\pi|\tilde{\psi}|^2C\vec{\mathcal{M}}. \tag{89}$$

In the B_3 and C_3 subphases, the averaged spin density and $\vec{h}$ are directed parallel or antiparallel z and for $\vec{H}$ directed in z, we get $h_3 = H + 4\pi|\tilde{\psi}|^2C\mathcal{M}$.

In subphases B_1 and C_1, the averaged spin density is directed parallel/antiparallel x, and $\vec{H}$ directed in z, we have

$$h_1 = 4\pi|\tilde{\psi}|^2C\mathcal{M}, \qquad h_3 = H. \tag{90}$$

Similarly, in subphases B_2 and C_2, the averaged spin density is directed parallel/antiparallel y. Replacing (89) in (88), we see that for $H \neq 0$ in subphases B_3, C_3 the energy density is gained compared to subphases B_1, C_1 and

B_2, C_2. In the absence of H, the Gibbs free-energy density is the same for all the subphases B_1, B_2, B_3 and C_1, C_2, C_3, respectively. Thereby, since (90) does not depend on H, the results for subphases B_1, B_2, and C_1, C_2, can be obtained from those for subphases B_3 and C_3 by setting $H = 0$.

3. Uniform solutions for phases B and C

For the uniform phases B and C, we find the solution

$$|\tilde{\psi}|^2 = \tilde{\psi}_0^2 = -\frac{1}{3\gamma}(b_1 - 2\pi C^2\mathcal{M}^2) \pm \frac{1}{3\gamma}\sqrt{(b_1 - 2\pi C^2\mathcal{M}^2)^2 + 3\gamma(a + C\vec{\mathcal{M}}\vec{H})}. \quad (91)$$

We need to retain the solution corresponding to $|\tilde{\psi}|^2 > 0$. In the case of phase B, it is the solution corresponding to the upper sign in (91) and in the case of phase C, it is the solution corresponding to the lower sign. From Eqs. (88), (89), and (91), we obtain

$$G^{\mathrm{hom}}_{\mathrm{B,C}} = -\frac{\tilde{\psi}_0^2}{3}[2(a + C\vec{\mathcal{M}}\vec{H}) - (b_1 - 2\pi C^2\mathcal{M}^2)\tilde{\psi}_0^2]. \quad (92)$$

We see that the energetically preferable direction of the spin is such that $\vec{\mathcal{M}}\vec{H} > 0$. Thus, we may replace $\vec{\mathcal{M}}\vec{H}$ to $|\vec{\mathcal{M}}\vec{H}|$.

Note that the Ansatz $\psi_1 = \pm i\psi_2$ has been exploited previously in description of the unconventional superconductors, cf. [21,22,74], but a possibility of appearance of an own magnetic field $h \neq 0$ was not considered. Therefore, phases B and C are novel magnetic phases: already in the absence of the external magnetic field the matter in these phases represents a ferromagnetic superfluid.

Uniform solutions for phase B.—Setting $\gamma \to 0$ in (91) with the plus-sign solution, we find

$$\tilde{\psi}_0^2 = \frac{a + |C\vec{\mathcal{M}}\vec{H}|}{2(b_1 - 2\pi C^2\mathcal{M}^2)}\theta\left(\frac{a + |C\vec{\mathcal{M}}\vec{H}|}{b_1 - 2\pi C^2\mathcal{M}^2}\right), \quad (93)$$

for $a + C\vec{\mathcal{M}}\vec{H} > 0$ provided the condition (84) is fulfilled. Using (93), we obtain the own magnetic field $\vec{h}$,

$$\vec{h} = \vec{H} + 2\pi C\vec{\mathcal{M}}\frac{(a + |C\vec{\mathcal{M}}\vec{H}|)}{b_1 - 2\pi C^2\mathcal{M}^2}. \quad (94)$$

Choosing "−" sign solution of Eq. (91) would lead to the positive value of G.

Replacing (93) in (92), we find the expression for the Gibbs free-energy density,

$$G^{\mathrm{hom}}_{\mathrm{B}} = -\frac{(a + |C\vec{\mathcal{M}}\vec{H}|)^2}{4(b_1 - 2\pi C^2\mathcal{M}^2)}\theta\left(\frac{a + |C\vec{\mathcal{M}}\vec{H}|}{b_1 - 2\pi C^2\mathcal{M}^2}\right), \quad (95)$$

cf. Eq. (37) for the B_1 subphase for vector bosons and Eq. (42) for B_3 subphase. For the B_1 subphase here, $\vec{\mathcal{M}}\vec{H} = 0$ and for B_3 subphase $\vec{\mathcal{M}}\vec{H} = \pm\mathcal{M}H$. Setting $H = 0$ in (95), we recover the Gibbs free energies for the B_1 and B_2 subphases,

$$G^{\mathrm{hom}}_{\mathrm{B_1,B_2}} = G^{\mathrm{hom}}_{\mathrm{B_3}}(H = 0),$$

cf. Eq. (35) for neutral vector bosons. For $H = 0$, in all three subphases B_i, there appears an internal magnetic field,

$$\vec{h}(H = 0) = \frac{2\pi aC\vec{\mathcal{M}}}{b_1 - 2\pi C^2\mathcal{M}^2}. \quad (96)$$

Thus, we found that in subphases B_i, superfluidity ($\tilde{\psi} \neq 0$) coexists with ferromagnetism ($h(H = 0) \neq 0$). With increasing H, the amplitude of the condensate grows.

We see that in the presence of an external magnetic field the subphase B_3, where $\vec{\mathcal{M}}\|z$ (for $\mathcal{M} > 0$), becomes energetically preferable compared to subphases B_1 and B_2. For $\mathcal{M} > 0$, the preferable orientation of the averaged spin density $\vec{S}$ is parallel to $\vec{H}$. For $\mathcal{M} < 0$, the preferable orientation of the averaged spin density $\vec{S}$ is antiparallel to $\vec{H}$. Note that superfluidity may arise even in the state, where $\vec{\mathcal{M}}$ is antiparallel to $\vec{H}$ (for $\mathcal{M} > 0$) provided $a - |\mathcal{M}H| > 0$; however, this state corresponds to a higher Gibbs free energy than the state with $\vec{\mathcal{M}}$ parallel to $\vec{H}$.

In the external magnetic field, $H \neq 0$, the actual value of the critical temperature for the subphase B_3 is

$$T^{\mathrm{B_3H}}_{\mathrm{cr}} = T_{\mathrm{cr}}(1 + |C\mathcal{M}H|/\alpha_0), \quad (97)$$

where T_{cr} is the critical temperature for $H = 0$, provided one may use the parametrization $a = \alpha_0 t$ with $t = (T_{\mathrm{cr}} - T)/T_{\mathrm{cr}}$. Thus, for $H \neq 0$ ($\vec{H}\|z$), the subphase B_3 continues to exist above T_{cr} up to $T = T^{\mathrm{B_3H}}_{\mathrm{cr}}$. For $T > T^{\mathrm{B_3H}}_{\mathrm{cr}}$, we have $\tilde{\psi} = 0$. Notice that Eq. (95) coincides with Eq. (81), which we have derived considering AH phase. However, Eq. (95) is valid for all $T < T^{\mathrm{B_3H}}_{\mathrm{cr}}$ provided condition (86) is fulfilled, whereas Eq. (81) is valid for $T_{\mathrm{cr}} < T < T^{\mathrm{AH}}_{\mathrm{cr}}$ and at the condition (71) satisfied. For $T^{\mathrm{B}}_{\mathrm{cr}} = T^{\mathrm{A}}_{\mathrm{cr}} = T_{\mathrm{cr}}$, that we for simplicity postulated, Eq. (97) coincides with Eq. (80). We find that in the temperature interval $T_{\mathrm{cr}} < T < T^{\mathrm{B_3H}}_{\mathrm{cr}}$ the condensate exists now for

$$H > H^{\mathrm{BH}}_{\mathrm{cr}} = |a|/|C\mathcal{M}|. \quad (98)$$

For $H \gtrsim \alpha_0/|\mathcal{M}|$, the parametrization $a = \alpha_0 t$ might become invalid. Using another popular parametrization $\varphi = \ln(T/T_{\mathrm{cr}})$ in Eq. (57), we find

$$T_{\rm cr}^{\rm B_3H} = T_{\rm cr} e^{C|\mathcal{M}H|/\alpha_0}. \quad (99)$$

Here we should notice that although expression (99) allows for $T_{\rm cr}^{\rm B_3H} \gg T_{\rm cr}$, the Ginzburg-Landau mean-field approach itself becomes invalid for such temperatures.

At the critical point $G_{\rm B_3}^{\rm hom} = 0$, $\partial G_{\rm B_3}^{\rm hom}/\partial T = 0$ but $\partial^2 G_{\rm B_3}^{\rm hom}/\partial T^2 \neq 0$ that, as in case of the phase A, corresponds to the second-order phase transition.

Note that the quantity $b_1 - 2\pi C^2\mathcal{M}^2$ should not be too small. Otherwise, terms $\propto \gamma\psi^6$ must be taken into account.

Also note that above we considered only contributions to the Gibbs free energy, which depend on the pairing order parameter. However, the total Gibbs free energy contains also a normal contribution of unpaired fermions. Owing to the normal term, there appears a small paramagnetic contribution proportional to h^2. Simplifying consideration, we disregarded this small correction term in our calculations.

Uniform solutions for phase C.—Now we assume that conditions (85) and (86) are fulfilled. The $\tilde{\psi}^4$ term in the Gibbs free energy proves to be negative, and the problem should be reconsidered with taking into account $\{\gamma_i\psi_i\}^6$ term, which provides stability (for $\gamma > 0$).

Let us perform expansion of (91) in a small γ. The minimum of the Gibbs free-energy density is realized for the choice of "+" sign solution in Eq. (91). Then, from Eqs. (92)–(107), we find

$$\tilde{\psi}_0^2 \simeq \frac{2}{3\gamma}(2\pi C^2\mathcal{M}^2 - b_1) + \frac{(a + C\vec{\mathcal{M}}\vec{H})}{2(2\pi C^2\mathcal{M}^2 - b_1)} > 0, \quad (100)$$

$$h \simeq H + \frac{8\pi C\mathcal{M}}{3\gamma}(2\pi C^2\mathcal{M}^2 - b_1) + \frac{2\pi C\mathcal{M}(a + C\vec{\mathcal{M}}\vec{H})}{(2\pi C^2\mathcal{M}^2 - b_1)}, \quad (101)$$

$$G_{\rm C}^{\rm hom} \simeq -\frac{4}{27\gamma^2}(2\pi C^2\mathcal{M}^2 - b_1)^3, \quad (102)$$

again with the energetically preferable direction of $\vec{\mathcal{M}}$ corresponding to $\vec{\mathcal{M}}\vec{H} > 0$. Expansion is valid for

$$0 < \gamma \ll (2\pi C^2\mathcal{M}^2 - b_1)^2/(a + |C\mathcal{M}H|). \quad (103)$$

The condensate amplitude grows with increasing H. For the case $H \neq 0$ parallel z, which we consider, the subphase C_3 proves to be the most energetically profitable. The results for C_1 and C_2 follow, if one puts $H = 0$.

The value of the new critical temperature is determined by the condition of the vanishing of the square root in Eq. (91),

$$T_{\rm cr}^{\rm C_3H} = T_{\rm cr}\left[1 + \frac{(2\pi C^2\mathcal{M}^2 - b_1)^2}{3\gamma\alpha_0} + \frac{|C\mathcal{M}H|}{\alpha_0}\right], \quad (104)$$

that holds provided the validity of the relation $a = \alpha_0 t$, $t = (T_{\rm cr} - T)/T_{\rm cr}$. Thus, $T_{\rm cr}^{\rm CH} \geq T_{\rm cr}^{\rm C}$, where

$$T_{\rm cr}^{\rm C} = T_{\rm cr}\left[1 + \frac{(2\pi C^2\mathcal{M}^2 - b_1)^2}{3\gamma\alpha_0}\right] > T_{\rm cr}. \quad (105)$$

For the subphases C_1 and C_2, we have $T_{\rm cr}^{\rm C_2}(H=0) = T_{\rm cr}^{\rm C_1}(H=0) = T_{\rm cr}^{\rm C_3}(H=0) = T_{\rm cr}^{\rm C}$.

At the critical point, $G_{\rm C}$ changes discontinuously, that corresponds to the first-order phase transition. Ferromagnetic superfluid solution (100)–(102) holds provided conditions (85), (86), and (103) are fulfilled.

4. Gradient term. Domains

For the system of a large but finite size already at $H = 0$, the degeneracy of the subphases is removed because of a difference in the gradient contributions in the Gibbs free-energy density of various subphases. As in case of the phase A studied above, to be specific, let us consider sample filling the half-space $x < 0$. Then,

$$G_i^{\rm grad} = C_i|\partial_1\tilde{\psi}|^2, \qquad i = {\rm B_1(C_1), B_2(C_2), B_3(C_3)}. \quad (106)$$

For the subphases $\rm B_3$ ($\rm C_3$) and $\rm B_2$ ($\rm C_2$), the coefficient $C_i = c_1 + (c_2 + c_3)/2$, and $\mathrm{div}\vec{\psi} \neq 0$. For subphases $\rm B_1$ and $\rm C_1$, $C_i = c_1$ and $\mathrm{div}\vec{\psi} = 0$. The stability conditions render

$$c_1 + c_2/2 + c_3/2 > 0, \qquad c_1 > 0. \quad (107)$$

Consider the phase B and put $\gamma = 0$. Variation of the Gibbs free energy in fields, cf. (55), yields the equation of motion,

$$C_i\partial_1^2\tilde{\psi} + (a + |C\vec{\mathcal{M}}\vec{H}|)\tilde{\psi} - 2(b_1 - 2\pi C^2\mathcal{M}^2)|\tilde{\psi}|^2\tilde{\psi} = 0, \quad (108)$$

with the solution satisfying the boundary condition $\tilde{\psi}(x=0) = 0$,

$$\tilde{\psi}(x) = \tilde{\psi}_0 \mathrm{th}\frac{x}{\sqrt{2}\xi_{\rm B_i}}, \qquad \xi_{\rm B_i} = \sqrt{\frac{C_i}{a + |C\vec{\mathcal{M}}\vec{H}|}}, \quad (109)$$

instead of Eq. (75) for the A phase. The value $\tilde{\psi}_0$ is determined by Eq. (93). At the fulfilled condition (84), the solution exists provided $a + |C\vec{\mathcal{M}}\vec{H}| > 0$.

Replacing (109) in the expression for the Gibbs free energy, $\int d^3xG = \mathcal{G}^{\rm vol} + \mathcal{G}^{\rm surf}$, we find that the surface contribution is $\mathcal{G}_i^{\rm surf} \propto \xi_i S$, S is the square in y, z plane. $\mathcal{G}_i^{\rm surf}$ gets minimum for the subphases B_3 and B_2, if $0 < c_1 + c_2/2 + c_3/2 < c_1$, and for the subphase B_1, if $0 < c_1 < c_1 + c_2/2 + c_3/2$. Gradient terms do not contribute to the minimization of G in h and Eq. (89) continues to hold, from where using the boundary condition $\tilde{\psi}(0) = 0$ we find that $h(x \to 0) \to H$.

The coordinate dependence of the field $\tilde{\psi}$ in the phase C is more involved, since one needs to include at least $\tilde{\psi}^5$ term in the equation of motion to provide stability.

Domains.—At the phase transition to phase B or C, there can be formed domains with different directions of $\vec{h}\|\vec{\mathcal{M}}$ and $\vec{\psi}$ in each domain. As we have argued, when have considered the subphase A, an extra energy is needed to merge the domains. For finite T, the required energy can be taken from thermal fluctuations. In the presence of the external magnetic field or the external rotation, the extra energy can be taken from the energy of the magnetic and rotation fields, respectively.

V. SPIN-TRIPLET PAIRING IN CHARGED FERMION SYSTEM DESCRIBED BY COMPLEX VECTOR ORDER PARAMETER

The spin-triplet pairing in the condensed matter, e.g., in systems with heavy fermions, is described by the vector order parameter at the effective charge of the pair $e_* = 2e < 0$, e.g., cf. [20–22]. In the nuclear systems, the np pairing in the $3\mathrm{S}_1$ channel is allowed for the case of the isospin-symmetric nuclear matter. The $3\mathrm{S}_1$ np phase shift is the largest among others at low energies, cf. [31]. The np pairing in the $3\mathrm{S}_1$ channel in the absence of the spin-orbital interaction is described by the vector order parameter at $e_* = -e > 0$.

A. Gibbs free-energy density

For the description of the charged superconductors, we may use Eq. (55) for the Gibbs free-energy density [21,22,74] with $G_{\rm grad}^{\rm neut}$ replaced by $G_{\rm grad}^{\rm ch}$,

$$G_{\rm grad}^{\rm ch} = c_1|D_i\psi_j|^2 + c_2|D_i\psi_i|^2 + c_3(D_i\psi_j)^* D_j\psi_i, \quad (110)$$

where $D_i = \partial_i - ie_*A_i$, $A_i = (A_x, A_y, A_z)$, e_* is the charge of the fermion pair. The D_i operators fulfill the relation for the commutator $i[D_i, D_j]_- = e_*\epsilon_{ijk}h_k$, cf. Eq. (24) above. In case when $\vec{h} = h\vec{n}_3$ with $\vec{n}_3 = (0, 0, 1)$, we have

$$i[D_1, D_2]_- = e_* h, \qquad h = F_{12} = \partial_1 A_2 - \partial_2 A_1. \quad (111)$$

B. Nonmagnetic phase A in the medium filling half of space placed in uniform magnetic field

We deal with the phase A provided conditions (70) and (71) are fulfilled. For this case, a difference with the standard description of the superconductivity of spin-zero pairs is only in the specificity of the gradient terms. In the absence of the external magnetic field, the description of the charged uniform system within the A phase remains the same as for the neutral system performed above. In the presence of the external magnetic field, the properties of the subphase A of a neutral spin-triplet superfluid and the charged one are different similarly to that we have demonstrated in previous section on example of the vector-boson field.

Further consider a superconductor filling half of space $x < 0$, placed in a homogeneous external magnetic field $\vec{H}$ parallel z, for $H > 0$. We may choose the gauge, where $\vec{A}$ has only one nonzero component $A_2(x)$ for $x < 0$. We choose $\vec{A}_{\rm ext} = (0, Hx, 0)$, satisfying the gauge condition $\mathrm{div}\vec{A}_{\rm ext} = 0$ and yielding $\mathrm{rot}\vec{A}_{\rm ext} = \vec{H}$.

Consider the phases A_1, A_2, and A_3, which are now not degenerate.

1. Subphase A_1 *for* $b_2 > 0$

Consider first phase A_1, where $\psi_1 = \psi(x)$ is real, $\psi_1(x \to -\infty) \to \psi_0 = \sqrt{\frac{a}{2(b_1+b_2)}}$ for $T < T_{\rm cr}^{\rm A}$ (to be specific we choose + sign solution), and allow for small perturbations of the fields $\psi_2 = -if_2(x)$, $\psi_3 = -if_3(x)$, and $A_2 = A_2(x)$, where f_2 and f_3 are real quantities. The field f_2 is introduced to check stability of the A phase in the presence of the external field H. Without loss of the generality, one may put $f_3 = 0$. For simplicity, assume that A_2 and f_2 are weak fields.

The gradient part of the Gibbs free-energy density can be presented as

$$G_{\rm grad}^{\rm ch} = (c_1 + c_2 + c_3)(\partial_1\psi)^2 + c_1 e_*^2 A_2^2(x)\psi^2 + 2c_3 e_* A_2(x)\psi\partial_1 f_2 + c_1(\partial_1 f_2)^2, \quad (112)$$

written in quadratic approximation over the perturbative fields A_2 and f_2 and the derivatives ∂_1. Stability conditions imply that $c_1 + c_2 + c_3 > 0$, $c_1 > 0$.

Variation of the Gibbs free energy in the fields ψ, A_2, and f_2 yields equations of motion,

$$(c_1 + c_2 + c_3)\partial_1^2\psi + a\psi - 2(b_1 + b_2)\psi^3 = 0, \quad (113)$$

$$\partial_1^2 A_2(x) - 8\pi c_1 e_*^2\psi^2 A_2(x) + 8\pi(C\mathcal{M}_3 - e_* c_3)\psi\partial_1 f_2 = 0, \quad (114)$$

$$c_1\partial_1^2 f_2 + (c_3 e_* - C\mathcal{M}_3)\psi\partial_1 A_2 + c_3 e_* A_2\partial_1\psi + (a - 2\psi^2(b_1 - b_2))f_2 = 0, \quad (115)$$

written in linear approximation over perturbative fields. The solution of Eq. (113) for $a > 0$ is given by

$\psi = \psi_0 \mathrm{th}(x/\sqrt{2}\xi_{A_1})$ with the coherence length $\xi_{A_1} = \sqrt{(c_1+c_2+c_3)/a}$, cf. Eq. (75).

In the absence of the external magnetic field ($H = 0$), minimization in fields leads us to solutions (66) and (68) and $h = 0$ for $T < T_{cr}$ in the region $x < 0$ everywhere except a surface layer. In the presence of a weak external magnetic field, there exists complete Meissner effect. We assume $d_{A_1}/\xi_{A_1} \gg 1$, where $d_{A_1} > 0$ is the penetration depth for the magnetic field determined by Eq. (114). Then, we may put $\psi = \psi_0$ in Eqs. (114) and (115). In the theory of ordinary superconductors and for the case of the charged scalar bosons considered in Sec. II for $\vec{H}\|z$, the quantity d_{A_1}/ξ_{A_1} is called the Ginzburg-Landau parameter, which value determines the behavior of the system. In the case under consideration, situation is a more involved. Explicit solution of Eq. (114) matched with that valid for $x \geq 0$ at the boundary $x = 0$ is given by

$$A_2(x) = H d_{A_1} e^{x/d_{A_1}}. \tag{116}$$

We search f_2 as

$$f_2(x) = D e^{x/d_{A_1}}, \tag{117}$$

with a constant D. Since $D \neq 0$, to fulfill Eq. (115) for $A_2(x) \neq 0$, in case of the subphase A_1 there appears a spin density in a surface layer. Dependence on $\psi(x)$ allows to fulfill the condition $f_2(x=0) = 0$. Since ψ is dropping to zero on a scale $\xi_{A_1} \ll d_{A_1}$, for $-x \sim d_{A_1} \gg \xi_{A_1}$ we may put $\psi = \psi_0$ in equation for f_2. Substituting (116) and (117) in (114) and (115), we find two solutions for $d^{\pm}_{A_1}$,

$$1/d^2_{A_1} = \psi_0^2\left[\lambda_{A_1} \pm \sqrt{\lambda^2_{A_1} + 32\pi e_*^2 b_2}\right],$$
$$\lambda_{A_1} = 4\pi c_1 e_*^2 - 2[b_2 + 2\pi(C\mathcal{M}_3 - c_3 e_*)^2]/c_1. \tag{118}$$

We should retain + sign square root. Solution with other sign does not satisfy boundary condition $A_2'(x=0) = H$. The roots of Eq. (118) are positive (in accordance with the Meissner effect) for

$$b_2 < 0 \quad \text{at} \; -b_2 - 2\pi(C\mathcal{M}_3 - e_* c_3)^2 + 2\pi c_1^2 e_*^2 > 0,$$

cf. condition (71) for neutral systems. For $c_1 = \pm c_3$, the latter inequality is simplified as $-b_2 - 2\pi C^2\mathcal{M}^2 - 4\pi c_1 e_* C\mathcal{M} > 0$. If the term $\propto e_*$ is small compared to the term $\propto (-b_2 - 2\pi C^2\mathcal{M}^2)$, for $b_2 + 2\pi C^2\mathcal{M}^2 < 0$, the minimal among two lengths, $d^{\pm}_{A_1}$, becomes $d^+_{A_1} \simeq \sqrt{c_1/(4(-b_2 - 2\pi C^2\mathcal{M}^2)\psi_0^2)}$. A larger length then is $d^-_{A_1} \simeq \sqrt{(1 + 2\pi C^2\mathcal{M}^2/b_2)/(8\pi e_*^2 c_1 \psi_0^2)}$.

To be specific, let us further use that $d^-_{A_1} >^+_{A_1}$. Then we may introduce the Ginzburg-Landau parameter as the ratio of the maximum among the lengths $d^-_{A_1}$ and $d^+_{A_1}$ to ξ_{A_1}, i.e.,

$$\kappa_{1,A_1} = \frac{d^-_{A_1}}{\xi_{A_1}} = \sqrt{\frac{(1 + 2\pi C^2\mathcal{M}^2/b_2)(b_1 + b_2)}{4\pi e_*^2 c_1 (c_1 + c_2 + c_3)}}. \tag{119}$$

Also, we further suppose that parameters are such that $\kappa_{1,A_1} \gg 1$, cf. estimates performed below in Sec. VII in the BCS approximation. For $\kappa > 1/\sqrt{2}$, the superconductor proves to be of the second kind, cf. [87,99], and with increasing H in the interval $H^{A_1}_{c1} < H < H^{A_1}_{c2}$ there appears a triangular Abrikosov lattice of vortices. The value $H^{A_1}_{c1} \sim \frac{H_{cr}}{\kappa_{1,A_1}}$ is the lower critical field, such that for $H > H^{A_1}_{c1}$ appearance of filament vortices is energetically profitable,

$$H^{A_1}_{c1} = \frac{H_{cr}}{\sqrt{2}\kappa_{1,A_1}}, \qquad H_{cr} = \sqrt{\frac{2\pi a^2}{b_1 + b_2}},$$

where H_{cr} has a sense of the thermodynamical critical field, at which the Gibbs free energy of the phase with $\bar{h} = 0$, $\psi = \psi_0$ coincides with that for $\bar{h} = H$, $\psi = 0$. The overline, as above, means averaging over the volume.

To find the upper critical magnetic field, one assumes ψ to be tiny and $A_2 \simeq Hx + O(\psi^2)$. As follows from Eq. (115), for fields nearby $H^{A_1}_{c2}$, the field f_2 is of the second-order smallness and can be dropped in equation for ψ. Then, equation of motion for ψ becomes

$$(c_1 + c_2 + c_3)\partial_1^2\psi + c_1\tilde{D}_2^2\psi + a\psi = 0, \tag{120}$$

with $\tilde{D}_2 = \partial_2 - ie_* Hx$, $f_2 = 0$. From here, we find

$$H^{A_1}_{c2} = \frac{a}{\sqrt{c_1(c_1 + c_2 + c_3)e_*^2}} \equiv H_{cr}\sqrt{2}\kappa_{2,A_1}, \tag{121}$$

for $a > 0$, being the upper critical field, at which the pairing is completely destroyed. Here we introduced the quantity

$$\kappa_{2,A_1} = \sqrt{\frac{b_1 + b_2}{4\pi c_1(c_1 + c_2 + c_3)e_*^2}}. \tag{122}$$

We see that $\kappa_{2,A_1} \neq \kappa_{1,A_1}$. For $b_2 < 0$ with above simplified estimate for $d^-_{A_1}$, we find that $\kappa_{2,A_1} > \kappa_{1,A_1}$.

Recall that for $c_1 + c_2 + c_3 = 0$ Eq. (113) has no solution satisfying appropriate boundary condition for $x = 0$ and subphase A_1 is not realized, cf. discussion in Sec. III.

2. Instability of subphase $\mathbf{A_1}$ *for* $\mathbf{b_2 > 0}$

For $b_2 > 0$, one of the roots, $(d^+_{A_1})^2$ or $(d^-_{A_1})^2$, is negative that means existence of the oscillating solution corresponding to the penetration of the external magnetic field in the interior of the system. Also, even in the absence of the external magnetic field, an own magnetic field h is produced, as we will show.

Let us first put $H = 0$ and search the fields in the form

$$A_2(x) = h_0 k_0^{-1} \sin(k_0 x + \chi), \qquad f_2 = D\cos(k_0 x + \chi),$$

with h_0 and D being small constants and χ is a constant phase. Also, assume that $1/k_0 \gg \xi_{\mathrm{A}_1}$. Then, in Eqs. (114) and (115), we may put $\psi = \psi_0$. The spatially averaged Gibbs free energy becomes

$$\bar{G}^{\mathrm{tot}}_{\mathrm{A}_1} = -\frac{a^2}{4(b_1+b_2)} + \frac{c_1 e_*^2 \psi_0^2 h_0^2}{2k_0^2} + \frac{c_1 k_0^2 D^2}{2} + \frac{h_0^2}{16\pi} + (C\mathcal{M}_3 - e_* c_3)\psi_0 h_0 D - \frac{aD^2}{2} + (b_1 - b_2)\psi_0^2 D^2. \tag{123}$$

This expression can be rewritten as

$$\bar{G}^{\mathrm{tot}}_{\mathrm{A}_1} = -\frac{a^2}{4(b_1+b_2)} - \left[\frac{4\pi(C\mathcal{M}_3 - e_* c_3)^2\psi_0^2}{1 + 8\pi c_1 e_*^2\psi_0^2/k_0^2} + 2b_2\psi_0^2 - \frac{c_1 k_0^2}{2}\right]D^2 + \frac{1 + 8\pi c_1 e_*^2\psi_0^2/k_0^2}{16\pi}\left[h_0 + \frac{8\pi(C\mu_3 - e_* c_3)\psi_0 D}{1 + 8\pi c_1 e_*^2\psi_0^2/k_0^2}\right]^2. \tag{124}$$

Minimum of $\bar{G}^{\mathrm{tot}}_{\mathrm{A}_1}$ corresponds to

$$h_0 = -\frac{8\pi(C\mathcal{M}_3 - e_* c_3)\psi_0 D}{1 + 8\pi c_1 e_*^2\psi_0^2/k_0^2}. \tag{125}$$

The occurrence of the oscillating fields is energetically profitable provided

$$\frac{4\pi(C\mathcal{M}_3 - e_* c_3)^2\psi_0^2}{1 + 8\pi c_1 e_*^2\psi_0^2/k_0^2} + 2b_2\psi_0^2 - \frac{c_1 k_0^2}{2} > 0. \tag{126}$$

This is so for k_0 varying in the range,

$$\nu_{\mp} < k_0^2 < \nu_{\pm}, \tag{127}$$

the upper sign solution is here for $\nu = -\lambda_{\mathrm{A}_1} > 0$ and lower sign one, for $\nu = -\lambda_{\mathrm{A}_1} < 0$,

$$\nu_{\pm} = \left(\nu \pm \sqrt{\nu^2 + 32\pi e_*^2 b_2}\right)\psi_0^2. \tag{128}$$

Thus, we have shown that for $b_2 > 0$, there exists an interval of values k_0 corresponding to the growing fields h and f_2. Thereby, the linear approximation that we used becomes invalid. As we show below, stable solutions then correspond to the phase B or C.

3. Subphase $\mathbf{A_2}$

Consider the subphase A_2, where $\psi_2 = \psi(x) \neq 0$ is real. Assume that fields $A_2(x)$ and $\psi_1 = -if_1(x)$ are small real quantities, and we assume $d_{\mathrm{A}_2} \gg \xi_{\mathrm{A}_2}$. Without loss of the generality, one may put $f_3 = 0$. Then the gradient part of the Gibbs free-energy density in the quadratic approximation in perturbative fields can be presented as

$$G^{\mathrm{ch}}_{\mathrm{grad}} = c_1(\partial_1\psi)^2 + (c_1 + c_2 + c_3)[(\partial_1 f_1)^2 + e_*^2 A_2^2(x)\psi^2] + 2c_3 e_* A_2(x)\psi\partial_1 f_1. \tag{129}$$

As in case of the subphase A_1, the stability conditions imply that $c_1 + c_2 + c_3 > 0$, $c_1 > 0$. Equations of motion for the perturbative fields in the linear approximation become

$$c_1\partial_1^2\psi + a\psi - 2(b_1 + b_2)\psi^3 = 0, \tag{130}$$

$$\partial_1^2 A_2(x) - 8\pi(c_1 + c_2 + c_3)e_*^2\psi^2 A_2(x) + 8\pi(C\mathcal{M}_3 - e_* c_3)\psi\partial_1 f_1 = 0, \tag{131}$$

$$(c_1 + c_2 + c_3)\partial_1^2 f_1 + (c_3 e_* - C\mathcal{M}_3)\psi\partial_1 A_2 + c_3 e_* A_2 \partial_1\psi + (a - 2\psi^2(b_1 - b_2))f_1 = 0. \tag{132}$$

We may put $\psi = \psi_0$ in Eq. (132). Solution of Eq. (130) reads $\psi = \psi_0 \mathrm{th}[x/(\sqrt{2}\xi_{\mathrm{A}_2})]$ for $a > 0$, and the coherence length $\xi_{\mathrm{A}_2} = \sqrt{c_1/a}$. Equation (118) for the spectrum holds after the replacement $c_1 \leftrightarrow c_1 + c_2 + c_3$,

$$1/d^2_{\mathrm{A}_2} = \psi_0^2\left[\lambda_{\mathrm{A}_2} \pm \sqrt{\lambda^2_{\mathrm{A}_2} + 32\pi e_*^2 b_2}\right], \quad \lambda_{\mathrm{A}_2} = 4\pi(c_1 + c_2 + c_3)e_*^2 - \frac{2[b_2 + 2\pi(C\mathcal{M}_3 - c_3 e_*)^2]}{(c_1 + c_2 + c_3)}. \tag{133}$$

We deal with the superconductor of the second kind for $d_{\mathrm{A}_2} \gg \xi_{\mathrm{A}_2}$. The value of the critical field $H^{\mathrm{A}_2}_{c1} = H_{\mathrm{cr}}/(\sqrt{2}\kappa_{1,\mathrm{A}_2})$, $\kappa_{1,\mathrm{A}_2} = d_{\mathrm{A}_2}/\xi_{\mathrm{A}_2}$, with d_{A_2} corresponding to the maximum length among $d^+_{\mathrm{A}_2}$ and $d^-_{\mathrm{A}_2}$. Assume $c_1 < c_1 + c_2 + c_3$. Then, $\xi_{\mathrm{A}_2} < \xi_{\mathrm{A}_1}$. For $b_2 < 0$, assuming that the terms $\propto e_*^2$ in (133) are small, we find $d^+_{\mathrm{A}_2} \simeq \sqrt{(c_1 + c_2 + c_3)/(4(-b_2 - 2\pi C^2\mathcal{M}^2)\psi_0^2)} > d^+_{\mathrm{A}_1} \simeq \sqrt{c_1/(4(-b_2 - 2\pi C^2\mathcal{M}^2)\psi_0^2)}$. For $d^-_{\mathrm{A}_2}$, we get $d^-_{\mathrm{A}_2} \simeq \sqrt{(1 + 2\pi C^2\mathcal{M}^2/b_2)/(8\pi e_*^2(c_1 + c_2 + c_3)\psi_0^2)} < d^-_{\mathrm{A}_1}$. Assuming that $d^-_{\mathrm{A}_2} > d^+_{\mathrm{A}_2}$, we find that the Ginzburg-Landau parameter related to the maximum among d lengths is $\kappa_{\mathrm{A}_2} = \kappa_{\mathrm{A}_1}$. Also, $\kappa_{2,\mathrm{A}_2} = \kappa_{2,\mathrm{A}_1}$ and $H^{\mathrm{A}_2}_{c2} = H^{\mathrm{A}_1}_{c2}$.

As the subphase A_1, the subphase A_2 proves to be unstable for $b_2 > 0$ in respect to the growing of the oscillating fields h and f_1. Equations (123)–(128) continue to hold after the replacement $c_1 \leftrightarrow c_1 + c_2 + c_3$.

In the particular case $c_1 + c_2 + c_3 = 0$, the subphase A_2 for $H \| z$ is nonmagnetic, cf. discussion in Sec. III.

4. Subphase A_3

Now, consider the subphase A_3, where $\psi_3 = \psi(x)$. In this case,

$$G^{\rm ch}_{\rm grad} = c_1|\partial_1\psi|^2 + c_1 e_*^2 A_2^2(x)|\psi|^2. \tag{134}$$

In the quadratic order in the perturbative fields $\psi_1(x)$, $\psi_2(x)$, their contribution to the Gibbs free energy decouples with that for the fields $\psi_3 = \psi(x)$ and A_2. The stability conditions imply that $c_1 > 0$. In this subphase, we have

$$\xi_{A_3} = \sqrt{c_1/a}, \quad d_{A_3} = 1/\sqrt{8\pi c_1 \psi_0^2 e_*^2}, \quad \kappa_{1,A_3} = \sqrt{\frac{b_1+b_2}{4\pi c_1^2 e_*^2}}, \tag{135}$$

from where it follows that $\kappa_{1,A_3} = \kappa_{2,A_3}$. As above, we suppose that $\kappa_{A_3} \gg 1$ (although it is sufficient to have $\kappa_{A_3} > 1/\sqrt{2}$).

Equations of motion for the fields $\psi_1 = if_1$ and $\psi_2 = if_2$ decouple in the linear approximation, e.g., we have

$$(c_1 + c_2 + c_3)\partial_1^2 f_1 + 4b_2\psi_0^2 f_1 = 0. \tag{136}$$

Equation for f_2 appears after the replacement $c_1 + c_2 + c_3 \to c_1$. For $b_2 < 0$, the energetically profitable solutions correspond to $f_1, f_2 = 0$.

For $b_2 > 0$, there are oscillating solutions indicating on instability of the subphase A_3.

5. Which A subphase is energetically most preferable? Domains

If $0 < c_1 < c_1 + c_2 + c_3$, then $\xi_{A_3} = \xi_{A_2} < \xi_{A_1}$. Using above done estimates for d_{A_2}, d_{A_1}, we have (for $b_2 < 0$) $d_{A_3} > d^-_{A_1} > d^-_{A_2}$, and the subphase A_3 proves to be energetically favorable compared to the subphases A_1 and A_2 for all H at $T < T^{\rm A}_{\rm cr}$ under consideration. Since $\kappa_{2,A_3} > \kappa_{2,A_1} = \kappa_{2,A_2}$,

$$H^{A_3}_{c2} = H_{\rm cr}\sqrt{2}\kappa_{2,A_3} = \frac{a}{c_1|e_*|} \tag{137}$$

is higher than $H^{A_1}_{c2} = H^{A_2}_{c2}$ and the subphases A_1 and A_2 are thus destroyed at a smaller value of the external magnetic field compared to that for the subphase A_3.

If $0 < c_1 + c_2 + c_3 < c_1$, then $\xi_{A_1} < \xi_{A_3} = \xi_{A_2}$, for $b_2 < 0$ we have $d^-_{A_1} < d_{A_3} < d^-_{A_2}$, and the subphase A_1 is energetically favorable for low H, then with increase of H above the value H_{c1}, the subphase A_2 might become preferable one, and for H near the value $H^{A_1}_{c2}$, the subphase A_1 again becomes most favorable.

Assume that a domain is in a certain subphase A_i, with $i = 1$, either 2 or 3. Since for $b_2 < 0$, each subphase A_i is stable to weak perturbations, in the absence of an external force the domain remains in the same subphase. In the presence of the magnetic field or the rotation of the system as the whole, or due to a temperature fluctuation the domain, being in one of subphases, after a while may undergo transition to another subphase.

Thus, we demonstrated that even, being in the mean spin-zero phase A, the spin-triplet superconductor has unconventional properties in the presence of the external magnetic field.

C. Ferromagnetic superconductive phases B and C for $b_2 > 0$ in the medium filling half of space

1. Subphases B_3 and C_3. General consideration

Above on example of the subphase A_1 we demonstrated that for $b_2 > 0$ the phases A_i are unstable. Let $b_2 > 0$, the superconductor fills half-space $x < 0$ and as above assume $\vec{H}$ to be directed parallel z. To be specific, let us focus on the consideration of the subphase B_3 (or C_3), then $\vec{h}$ is directed parallel or antiparallel z.

The gradient contribution to the Gibbs free-energy density (110) can be rewritten as

$$G^{\rm ch}_{\rm grad} = c_1|D_i\psi_j|^2 + \frac{c_2+c_3}{2}[|D_i\psi_i|^2 + (D_i\psi_j)^* D_j\psi_i] - \frac{c_3-c_2}{2}[|D_i\psi_i|^2 - (D_i\psi_j)^* D_j\psi_i], \tag{138}$$

cf. Ref. [22]. Integrating by parts the gradient term in the Gibbs free energy, using the commutator (111), and retaining only the volume part of the free energy, we get

$$\begin{aligned}
&\int d^3x(G^{\rm ch}_{\rm grad} + G^{\rm ch}_{\rm hom}) \\
&= \int d^3x\left[-\frac{2c_1+c_2+c_3}{2}\tilde{\psi}^*(D_1^2+D_2^2)\tilde{\psi}\right] \\
&\quad + \int d^3x\left[e_*\frac{c_3-c_2}{2}\vec{n}_3\vec{h}|\tilde{\psi}|^2\right] \\
&\quad + \int d^3x\left[\frac{(\vec{h}-\vec{H})^2}{8\pi} - (a + C\vec{\mathcal{M}}\vec{h})|\tilde{\psi}|^2 + b_1|\tilde{\psi}|^4 + \gamma|\tilde{\psi}|^6\right],
\end{aligned} \tag{139}$$

where as above we have chosen simplest form of the sixth-order term and used that $\tilde{\psi}$ does not depend on z. To be specific, we took $\psi_1 = -\psi_2$ for the B_3 and C_3 subphases in (87). The gradient term is positive due to the stability conditions (107). The presence of the term $\propto i[D_1, D_2]_-$ in the gradient contribution to the Gibbs free energy resulted in appearance of the contribution,

$$\int d^3x \delta G_{\mathrm{intr},1} = -\int d^3x \vec{M}_{\mathrm{intr},1}\vec{h}|\tilde{\psi}|^2, \qquad (140)$$

with the quantity $\vec{M}_{\mathrm{intr},1} = -\vec{n}_3 \frac{1}{2} e_*(c_3 - c_2)$ associated in [22,23] with an intrinsic magnetic moment of the fermion pair in the spin-triplet superconductor, $\vec{n}_3$ is the unit vector aligned in the z direction. In [22], this contribution was considered as the total contribution to the intrinsic magnetic moment density. However, an extra contribution to the effective magnetic moment of the pair may still appear due to the presence of the terms $\propto (D_1^2 + D_2^2)$ in the Gibbs free energy.

Varying the Gibbs free energy in $\tilde{\psi}$, we obtain equation of motion for the order parameter,

$$-\left(c_1 + \frac{c_2 + c_3}{2}\right)(D_1^2 + D_2^2)\tilde{\psi} - [a + \vec{M}\,\vec{h}]\tilde{\psi} + 2b_1|\tilde{\psi}|^2\tilde{\psi} + 3\gamma|\tilde{\psi}|^4\tilde{\psi} = 0, \qquad (141)$$

where we introduced the quantity

$$\vec{M} = C\vec{\mathcal{M}} - \vec{n}_3 e_*(c_3 - c_2)/2. \qquad (142)$$

$\vec{n}_3$ is the unit vector aligned in the z direction. If we used $\psi_1 = +\psi_2$, we would get expression with $-\vec{M}$ instead of $\vec{M}$. The direction of $\vec{M}$ (the direction of the spin) is selected to minimize the energy, cf. Eq. (50).

We note that, if we artificially suppressed the gradient term $\propto (D_1^2 + D_2^2)$ in (139) and performed variation of the resulting Gibbs free energy in h and $\tilde{\psi}$, we would recover (in dependence of the sign of the term $b_1 - 2\pi M^2$) either Eqs. (93), (94), and (95) or Eqs. (100), (101), and (102), now with $\vec{M}$ instead of $C\vec{\mathcal{M}}$.

Equation (141) is supplemented by the Maxwell equation determining the A_i, h_i fields,

$$\partial_i F_{ik} = -4\pi J_k, \qquad (143)$$

where $\vec{J}$ is the corresponding current density, cf. Eq. (26) for the case of the charged vector field.

Multiplying (141) by $\tilde{\psi}^*$ and replacing result back to the expression for the Gibbs free energy, we obtain

$$\int G^{\mathrm{ch}} d^3x = \int d^3x \left[-b_1|\tilde{\psi}|^4 - 2\gamma|\tilde{\psi}|^6 + \frac{(\vec{h} - \vec{H})^2}{8\pi}\right]. \qquad (144)$$

From (141), we can immediately recover the value of the upper critical field H_{c2} taking $\tilde{\psi} \to 0$. This is valid for the consideration of the B phase where the phase transition is of the second order. Neglecting $O(|\tilde{\psi}|^2)$ terms in (141) and setting $\vec{h} = \vec{H}$, we get

$$-\left(c_1 + \frac{c_2 + c_3}{2}\right)(D_1^2 + D_2^2)\tilde{\psi} = E\tilde{\psi}, \qquad (145)$$

with $E = a + M_3 H$, cf. Eq. (142). Directions of the fields should be chosen such that the value H_{c2} be maximum. Equation (145) can be interpreted as the nonrelativistic Schrödinger equation in the homogeneous magnetic field H for the particle with the mass $m = 1/(2c_1 + c_2 + c_3) > 0$ and the energy E. The maximum/minimum magnetic field, when there still exists/appears the solution, corresponds to $E = E(n = 0, p_z = 0) = |e_*|H_{c2}^{\mathrm{B}}/(2m)$. Thus, we find

$$H_{c2}^{\mathrm{B}} = -a/M_{\pm}. \qquad (146)$$

Here $M_+ = C\mathcal{M}_3 - e_*(c_1 + c_3)$ corresponds to $e_* > 0$, and $M_- = C\mathcal{M}_3 - |e_*|(c_1 + c_2)$ relates to $e_* < 0$. For $M_\pm < 0$, solution with $\psi \neq 0$ exists for $H < H_{c2}^{\mathrm{B}}$ at $a > 0$ (i.e., for $T < T_{\mathrm{cr}}$). For $M_\pm > 0$, solution with $\psi \neq 0$ exists for $H > H_{c2}^{\mathrm{B}}$ at $a < 0$ (i.e., for $T_{\mathrm{cr}}^{\mathrm{BH,CH}} > T > T_{\mathrm{cr}}$), and for any H at $a > 0$ (i.e., for $T < T_{\mathrm{cr}}$).

Inverting Eq. (146) we may find the critical temperature $T_{\mathrm{cr}}^{\mathrm{BH}}$ as a function of H. We see that the value of this critical temperature coincides with that follows from Eq. (97) [or (99)], but with $M_\pm$ instead of $|C\mathcal{M}|$, provided $M_\pm > 0$. For $e_* > 0$ and $c_1 = -c_3$, the mentioned values of the critical temperatures coincide completely.

2. *Subphases* B_3 *and* C_3. *Abrikosov Ansatz*

We did not succeed to solve a general problem. Therefore, let us consider the matter far from the boundary and employ the variational approach. Let the probe functions satisfy the so-called Abrikosov Ansatz, cf. [73,74],

$$D_i \psi_i = 0. \qquad (147)$$

As we have seen in Sec. III in the problem of the description of the complex vector-boson fields, the condition (147), cf. (22) and (27), was required to recover correct interpretation of the single-particle problem for $\eta = e$. Also in Sec. III, we have shown that the condition (147) is fulfilled for arbitrary η at the consideration of the behavior of the vector field interacting with the static uniform magnetic field at $h \simeq H_{\mathrm{cr2}}$. Here, in the problem of the spin-triplet pairing of charged fermions, the fulfilment of the condition (147) is not necessary even for $\eta = C\mathcal{M} = e^*$, but making use of this condition allows to develop a variational treatment of the problem. Besides that, below we show that solution of Eq. (147) coincides with exact solution of the problem for the value of the external magnetic field $H = H_{\mathrm{cr2}}$.

From Eq. (147) in the gauge $\vec{A} = (A_1(y), A_2(x), 0)$, we obtain

$$e_*(A_2 - iA_1) = -(\partial_1 + i\partial_2)\ln\psi_1. \tag{148}$$

We find

$$\tilde{\psi} = e^{-e_*\int^x A_2(x')dx' + e_*\int^y A_1(x')dx'}F(x+iy), \tag{149}$$

where F is an arbitrary analytical function. On the other hand, from (148), we find

$$\frac{1}{2}\epsilon_{ki}\partial_i|\tilde{\psi}|^2 + |\tilde{\psi}|^2\partial_k\chi = e_* A_k|\tilde{\psi}|^2, i, \quad k = 1, 2, \tag{150}$$

$\epsilon_{12} = 1$, $\epsilon_{21} = -1$, $\epsilon_{11} = \epsilon_{22} = 0$, $\tilde{\psi} = |\tilde{\psi}|e^{i\chi}$, and for simplicity choosing $\chi = 0$, we get

$$-(\partial_x^2 + \partial_y^2)\ln|\tilde{\psi}| = e_*\vec{h}\vec{n}_3. \tag{151}$$

Using Eqs. (111) and (147), we derive a helpful relation

$$-(D_1^2 + D_2^2)\tilde{\psi} = e_*\vec{h}\vec{n}_3\tilde{\psi}, \tag{152}$$

cf. Eq. (50) in Sec. III.

For the current from (139) and (143) using (147), we obtain

$$J_k = -\tilde{M}_3\epsilon_{ki}\partial_i|\tilde{\psi}|^2. \tag{153}$$

Here we introduced the effective magnetic moment of the Cooper pair,

$$\vec{\tilde{M}} = C\vec{\mathcal{M}} + \vec{M}_{\rm intr}, \tag{154}$$

where $\vec{M}_{\rm intr} = -\vec{n}_3 e_*(c_1 + c_3)$ is an intrinsic magnetic moment of the fermion pair, which however differs from the contribution $\vec{M}_{\rm intr,1}$, cf. (140).

Replacing (152) in (141), we find

$$\vec{\tilde{M}}\vec{h} = -a + 2b_1|\tilde{\psi}|^2 + 3\gamma|\tilde{\psi}|^4. \tag{155}$$

Setting (152) in the gradient term in (139), we get

$$\int d^3x G^{\rm ch}_{\rm grad} = \int d^3x e_*\vec{h}\vec{n}_3(c_1 + c_3)|\tilde{\psi}|^2 \tag{156}$$

and

$$\int d^3x G^{\rm ch} = \int d^3x\left[-(a + \vec{\tilde{M}}\vec{h})|\tilde{\psi}|^2 + b_1|\tilde{\psi}|^4 + \frac{(\vec{h} - \vec{H})^2}{8\pi} + \gamma|\tilde{\psi}|^6\right]. \tag{157}$$

Since the gradient contribution to the Gibbs free energy should be non-negative, our result is valid only provided the stability condition $(c_1 + c_3)e_* h_3 \geq 0$ is fulfilled.

Minimizing (157) in h, we find the solution

$$\vec{h} = \vec{H} + 4\pi\vec{\tilde{M}}|\tilde{\psi}|^2. \tag{158}$$

Note that in general (for $H \neq H_{\rm cr2}$) the Ansatz (147) is incompatible with one of the equations of motion, which follow from the minimization of (157) in the order parameter and the electromagnetic field. Indeed, setting in (157) solution (141), where we substitute Eq. (152), in the limit $\gamma \to 0$ in dependence of the sign of $b_1 - 2\pi\tilde{M}^2$ we recover either Eqs. (93), (94), and (95) or Eqs. (100), (101), and (102), however now with $\tilde{M}$ from (154) instead of $C\mathcal{M}$.

Only for $H \simeq H_{c2}$ Ansatz (147) is compatible with the solution (141). An analogy of Eq. (152) with the Schrödinger equation in a uniform magnetic field (at $\vec{h} \simeq \vec{H}$ for $\tilde{\psi} \to 0$) demonstrates that the solutions with appropriate boundary condition $|\tilde{\psi}(x, y \to \infty)| < \infty$ exist provided $e_*\vec{h}\vec{n}_3 = |e_*|H_{\rm cr2} > 0$, i.e., for $e_* > 0$. Otherwise Abrikosov Ansatz cannot be exploited.

Let us employ the variational procedure. After substitution of $\vec{h}$ from (158) into (151), the equation for $\tilde{\psi}$ gets the form

$$-(\partial_x^2 + \partial_y^2)\ln|\tilde{\psi}| = e_*\vec{H}\vec{n}_3 + e_* 4\pi\vec{\tilde{M}}\vec{n}_3|\tilde{\psi}|^2. \tag{159}$$

For example, in case $H = 0$, the solution of this equation with periodic boundary conditions is given by the Weierstrass doubly periodic function ζ, cf. [74],

$$|\tilde{\psi}| = \frac{|\zeta\prime(x+iy)|}{\sqrt{\pi|\tilde{M}e_*|(e_2 - e_3)(e_3 - e_1)}} \times \frac{(e_3 - e_1)(e_2 - e_3)}{(e_3 - e_1)(e_2 - e_3) + |\zeta(x+iy) - e_3|^2}, \tag{160}$$

e_i are the roots of equation

$$4t^3 - g_2 t - g_3 = 0, \tag{161}$$

where the quantities g_2 and g_3 are defined in the standard presentations of the Weierstrass p function. We assume that these roots are real (that requires $g_2^3 - 27g_3^2 > 0$) and $e_2 > e_3 > e_1$. Other forms of the solution can be found in

[101,102]. If in Eq. (160) periods of ζ are $2a$, $2ib$, then $|\tilde{\psi}|$ is periodic function with periods a, ib.

Now, we substitute solution (158) in (157). With the solution of Eq. (151) presented in the form $\tilde{\psi} = \psi_0 \nu(\vec{r})$, we get

$$\overline{G^{\rm ch}} = -(a + \vec{\tilde{M}}\vec{H})\overline{\nu^2}|\psi_0|^2 + (b_1\overline{\nu^4} - 2\pi\tilde{M}^2(\overline{\nu^2})^2)|\psi_0|^4 + \gamma\overline{\nu^6}|\psi_0|^6. \quad (162)$$

Here spatial averaging, $\overline{G^{\rm ch}} = \int d^3x G^{\rm ch}/\int d^3x$, is performed with the probed function satisfying Eq. (151). For $H = 0$, we may use solution (160). Variational parameter ψ_0 is found by minimization of (162). We obtain

$$|\psi_0^2| = \frac{(2\pi\tilde{M}^2(\overline{\nu^2})^2 - b_1\overline{\nu^4})}{3\gamma\overline{\nu^6}} \pm \frac{1}{3\gamma\overline{\nu^6}}\sqrt{(2\pi\tilde{M}^2(\overline{\nu^2})^2 - b_1\overline{\nu^4})^2 + 3\gamma\overline{\nu^6}(a + \vec{\tilde{M}}\vec{H})\overline{\nu^2}}. \quad (163)$$

For the probed function describing the periodic triangular lattice at ordinary spin zero pairing, one has [87] $\tilde{\beta} = \overline{\nu^4}/(\overline{\nu^2})^2 \simeq 1.16$.

In the absence of the external magnetic field, the system of a large size may exist in a metastable state, being constructed of domains with different directions of $\vec{h}$ and $\vec{\psi}$ in each domain. Since the ground state of the uniform system corresponds to $\vec{h}$ aligned in one fixed direction, the system may undergo transitions with the flip of the domains until it will reach the state with the minimal surface energy. Note that the process of the alignment of domains should be compatible with the conservation of the magnetic flux. As we have argued, when considered the B_i and C_i phases in neutral superfluids, the spin and the $\vec{h}$ flips require an energy. In the presence of the external magnetic field or for the rotating system, a required extra energy can be taken from the energy of the external magnetic and rotation fields. Also flips of the domains are possible via thermal fluctuations.

3. Subphase B_3. *Averaged Gibbs free energy*

Let us focus on the subphase B_3. In case $H = 0$, results are valid also for subphases $\mathrm{B}_{1,2}$. Within the variational problem, the B phase arises provided

$$b_1\tilde{\beta} - 2\pi\tilde{M}^2 > 0, \quad (164)$$

where as above $\tilde{\beta} = \overline{\nu^4}/(\overline{\nu^2})^2$. We may for simplicity put $\gamma = 0$.

Assume $(a + \vec{\tilde{M}}\vec{H}) > 0$. From (163), we find

$$|\psi_0|^2 = \frac{a + \vec{\tilde{M}}\vec{H}}{2\overline{\nu^2}(b_1\tilde{\beta} - 2\pi\tilde{M}^2)}\theta(a + \vec{\tilde{M}}\vec{H}), \quad (165)$$

$$\bar{G}^{\rm ch}_{\mathrm{B_3H}} = -\frac{(a + \vec{\tilde{M}}\vec{H})^2}{4(b_1\tilde{\beta} - 2\pi\tilde{M}^2)} < 0, \quad (166)$$

with h from (158). Energetically favorable is the direction of the vector $\vec{\tilde{M}}$ parallel $\vec{H}$. Thereby, we may replace $\vec{\tilde{M}}\vec{H}$ to $|\vec{\tilde{M}}\vec{H}|$. The subphase B_3 appears for $H = 0$ by the second-order phase transition at $T = T_{\rm cr}$ and continues to exist in a certain interval of temperatures above $T_{\rm cr}$ for $H \neq 0$. The value of the new critical temperature $T_{\rm cr}^{\mathrm{B_3H}}$ is found from Eqs. (97) (99), however with $\tilde{M}$ from (154) instead of $C\mathcal{M}$. For example, with the parametrization $a = \alpha_0 t$, we get

$$T_{\rm cr}^{\mathrm{B_3H}} = T_{\rm cr}(1 + |\tilde{M}H|/\alpha_0). \quad (167)$$

4. Subphase C_3. *Averaged Gibbs free energy*

Consider subphase C_3. For $H = 0$, results are also valid for subphases $\mathrm{C}_{1,2}$. Now, we set

$$b_1\tilde{\beta} - 2\pi\tilde{M}^2 < 0. \quad (168)$$

To get stable solutions, we should retain $\gamma \neq 0$ term in (162). As above, simplifying consideration, we assume γ to be positive and small. Then from (163) in analogy with (100) and (102), we obtain

$$\psi_0^2 \simeq \frac{2(2\pi\tilde{M}^2 - b_1\tilde{\beta})}{3\gamma\tilde{\beta}_1} + \frac{(a + \vec{\tilde{M}}\vec{H})\tilde{\beta}_2}{2\tilde{\beta}_1(2\pi\tilde{M}^2 - b_1\tilde{\beta})} > 0, \quad (169)$$

$$\bar{G}^{\rm ch}_{\mathrm{C_3H}} \simeq -\frac{4}{27\gamma^2\tilde{\beta}_2^2}(2\pi\tilde{M}^2 - b_1\tilde{\beta})^3. \quad (170)$$

Here $\tilde{\beta}_1 = \overline{\nu^6}/(\overline{\nu^2})^2$, $\tilde{\beta}_2 = \overline{\nu^6}/(\overline{\nu^2})^3$. Expansion in the parameter γ is valid for

$$0 < \gamma \ll \frac{(2\pi\tilde{M}^2 - b_1\tilde{\beta})^2}{\tilde{\beta}_2(a + |\tilde{M}H|)}. \quad (171)$$

The own magnetic field is found with the help of Eqs. (158) and (169). The new phase appears by the first-order phase transition.

The new critical temperature is determined (for $a = \alpha_0 t$) by setting zero the square root in (163),

$$T_{\rm cr}^{\rm C_3H} = T_{\rm cr}\left[1 + \frac{(2\pi\tilde{M}^2 - b_1\tilde{\beta})^2}{3\gamma\tilde{\beta}_2\alpha_0} + \frac{|\tilde{M}H|}{\alpha_0}\right], \qquad (172)$$

with $T_{\rm cr}^{\rm C_3H} > T_{\rm cr}^{\rm C_3} > T_{\rm cr}$, where now

$$T_{\rm cr}^{\rm C_3} = T_{\rm cr}\left[1 + \frac{(2\pi\tilde{M}^2 - b_1\tilde{\beta})^2}{3\gamma\tilde{\beta}_2\alpha_0}\right].$$

VI. 3P$_2$ *nn* AND *pp* PAIRINGS IN NEUTRON STAR INTERIORS

A. Gibbs free-energy density

So far, we considered the spin-triplet paring in systems with negligible spin-orbital interactions, so that both orbital momentum and spin were assumed to be appropriate quantum numbers and we assumed that orbital momentum and spin can rotate independently. In nuclear matter, the spin-orbital interaction is strong and the state of a Cooper pair is described by the total angular momentum J and its projections m_J. The 3P$_2$ phase shift for identical nucleons (nn and pp) is the largest among others for the momenta $p > 1.3\ {\rm fm}^{-1}$. Thereby, cf. [25], for $n \gtrsim n_0$ neutrons in the neutron matter as well as in the beta-equilibrium matter prove to be paired in the 3P$_2$ state with $J = 2$. Protons might be paired in this channel at a higher density, if their fraction becomes rather high.

The pairing gap of the 3P$_2$ state can be written as $\hat{\Delta} = i\sigma_i\sigma_2 A_{ij} n_j$, where $\sigma_{1,2,3}$ are the Pauli spin matrices, $\vec{n}$ is the unity vector in the direction of the pairing momentum. The matrix $\hat{A}$ is symmetric and traceless for this type of paring and is determined by the expression [54] (here presented in another normalization, a more convenient one to compare with results of previous sections)

$$\hat{A} = \begin{bmatrix} \frac{a_{-2}}{2} - \frac{a_0}{\sqrt{6}} + \frac{a_2}{2} & \frac{i}{2}(a_2 - a_{-2}) & \frac{1}{2}(a_{-1} - a_1) \\ \frac{i}{2}(a_2 - a_{-2}) & -\frac{a_{-2}}{2} - \frac{a_0}{\sqrt{6}} - \frac{a_2}{2} & -\frac{i}{2}(a_{-1} + a_1) \\ \frac{1}{2}(a_{-1} - a_1) & -\frac{i}{2}(a_{-1} + a_1) & \sqrt{\frac{2}{3}}a_0 \end{bmatrix}. \qquad (173)$$

The Ginzburg-Landau free-energy density functional for the uniform matter has the form

$$F[\hat{A}] = -\bar{\alpha}{\rm Tr}(\hat{A}\hat{A}^*) + \bar{\beta}_1{\rm Tr}(\hat{A}\,\hat{A}){\rm Tr}(\hat{A}^*\hat{A}^*) + \bar{\beta}_2{\rm Tr}(\hat{A}\hat{A}^*){\rm Tr}(\hat{A}\hat{A}^*) + \bar{\beta}_3{\rm Tr}(\hat{A}\,\hat{A}\,\hat{A}^*\hat{A}^*) + \{\bar{\gamma}\hat{A}^6\}. \qquad (174)$$

The last term, $\{\bar{\gamma}\hat{A}^6\}$, represents symbolically all terms of the sixth order in A. Below we put $\bar{\gamma} = 0$, when it does not contradict to the stability condition of the phase. Values $\bar{\alpha}$, $\bar{\beta}_i$ are phenomenological parameters of the model. Assuming (for $\gamma = 0$) a second-order phase transition to the paired state, in the absence of the external fields, one may use $\bar{\alpha} = \bar{\alpha}_0 t$ for $|t| \lesssim 1$, cf. Eq. (57). As we have mentioned, being computed in BCS approximation, the $\gamma_6\hat{A}^6$ term proves to be negative [54] that implies necessity to continue the Ginzburg-Landau expansion up to $\gamma_8\hat{A}^8$ positive contribution [56]. Simplifying consideration, as in previous sections, we will employ the simplest form of the $\{\bar{\gamma}\hat{A}^6\}$ interaction with $\gamma > 0$.

To consider systems of a finite size, we should add the gradient contribution to the free-energy density. The generalization to the hypothetical 3P$_2$ pp pairing, which may be possible for $n \gg n_0$ in neutron star matter, is performed with the help of the replacement of the ordinary derivatives by the long derivatives, i.e., $\partial_i \to D_i = \partial_i + ie_* A_i + m_* v_i$, $A_i = (A_x, A_y, A_z)$, e_* is the charge of the fermion pair, for moving systems $\vec{v}$ is the velocity of the system, m_* is the effective mass of the pair. Therefore, to include the effects associated with the spatial nonuniformity, one should add the gradient terms,

$$F_{\rm grad} = c_1 D_i A_{\nu k} D_i^* A_{\nu k}^* + c_2 D_i A_{\nu i} D_j^* A_{\nu j}^* + c_3 D_i A_{\nu j} D_j^* A_{\nu i}^*. \qquad (175)$$

To include interaction of spins of the Cooper pair with the own magnetic field $\vec{h}$, we add to Eq. (174) the Zeeman term [18,22], $F_{\rm Zeeman} = -\vec{\eta}\vec{h} = -i\eta h_i \epsilon_{ijk} A_{lj} A_{lk}^*$. Also, the proper magnetic free-energy density contribution should be added. To be specific, we further assume $\vec{h} = (0, 0, h)$, $\vec{h}\|\vec{H}$, $\vec{h}\|\vec{\eta}$ (for $\eta > 0$) or $\vec{h}\| - \vec{\eta}$ (for $\eta < 0$). Other possibilities can be considered similarly to that we did in Sec. III. Thus, the resulting expression for the Gibbs free-energy density becomes

$$\begin{aligned} G &= F_{\rm grad}[A_{ij}, h, \omega] + F[A_{ij}] + G_H, \\ G_H &= -i\eta h_i \epsilon_{ijk} A_{lj} A_{lk}^* + \frac{1}{8\pi}(h - H)^2 \\ &= \frac{1}{2}\eta h(2|a_{-2}|^2 + |a_{-1}|^2 - |a_1|^2 - 2|a_2|^2) \\ &\quad + \frac{1}{8\pi}(h - H)^2. \end{aligned} \qquad (176)$$

As above, we for simplicity disregard small polarization terms $\propto h^2$, cf. [103].

If we retain only one m_J component among possible combinations $m_J = 0, -1, -2, +1$ or $+2$ in matrix (173), the Gibbs free-energy densities for these states become (for $m_J = 0, \pm 1, \pm 2$)

$$
\begin{aligned}
G_0 &= -\bar{\alpha}|a_0|^2 + \left(\bar{\beta}_1 + \bar{\beta}_2 + \frac{1}{2}\bar{\beta}_3\right)|a_0|^4 \\
&\quad + \frac{1}{8\pi}(h-H)^2 + F_0^{\mathrm{grad}}, \\
G_{\pm 1} &= -\bar{\alpha}|a_{\pm 1}|^2 + \left(\bar{\beta}_2 + \frac{1}{4}\bar{\beta}_3\right)|a_{\pm 1}|^4 \mp \frac{1}{2}\eta h|a_{\pm 1}|^2 \\
&\quad + \frac{1}{8\pi}(h-H)^2 + F_{\pm 1}^{\mathrm{grad}}, \\
G_{\pm 2} &= -\bar{\alpha}|a_{\pm 2}|^2 + \bar{\beta}_2|a_{\pm 2}|^4 \mp \eta h|a_{\pm 2}|^2 \\
&\quad + \frac{1}{8\pi}(h-H)^2 + F_{\pm 2}^{\mathrm{grad}}.
\end{aligned} \tag{177}
$$

If one assumes the symmetry among all a_m and a_{-m} amplitudes and takes into account the relations $a_{\pm 2} = \pm\tilde{a}_2 e^{\pm i\chi_2}$ and $a_{\pm 1} = \pm\tilde{a}_1 e^{\pm i\chi_1}$ with real amplitudes $\tilde{a}_2$ and $\tilde{a}_1$, the Gibbs functional G in such a symmetric subphase simplifies as

$$
G_{\mathrm{sym}} = -\bar{\alpha}[\tilde{a}_0^2 + 2(\tilde{a}_1^2 + \tilde{a}_2^2)] \tag{178}
$$

$$
\begin{aligned}
&+ \left(\bar{\beta}_1 + \bar{\beta}_2 + \frac{1}{2}\bar{\beta}_3\right)[\tilde{a}_0^2 + 2(\tilde{a}_1^2 + \tilde{a}_2^2)]^2 \\
&+ \frac{1}{8\pi}(h-H)^2 + F_{\mathrm{sym}}^{\mathrm{grad}},
\end{aligned} \tag{179}
$$

yielding in the case of the uniform matter the same value in the minimum as for the G_0, cf. Eq. (185) below.

Note that the critical temperatures for the symmetric subphase and the subphases $m_J = 0$, $m_J = \pm 1$, and $m_J = \pm 2$, respectively, might be different. However, according to [35], the difference proves to be very small. Thereby, simplifying consideration, we suppose, as we have used it in previous sections, that values T_{cr} are the same for all the subphases.

Assume

$$
\bar{\beta}_1 + \bar{\beta}_2 + \frac{1}{2}\bar{\beta}_3 > 0, \quad \bar{\beta}_2 + \frac{1}{4}\bar{\beta}_3 > 0, \quad \bar{\beta}_2 > 0, \tag{180}
$$

that is required for the stability of the symmetric and $m_J = 0$ subphases, the subphases with $m_J = \pm 1$ and the subphases with $m_J = \pm 2$, respectively. If we put $H = 0$ and disregard h-dependent terms for a moment, then for the uniform matter we find that for $\bar{\beta}_1 + \frac{1}{2}\bar{\beta}_3 < 0$ the symmetric subphase (and the subphase with $m_J = 0$) is energetically preferable compared to the subphases with $m_J = \pm 2$. For $\bar{\beta}_1 + \frac{1}{4}\bar{\beta}_3 < 0$, the former subphases are favorable compared to the subphase with $m_J = \pm 1$. For $\bar{\beta}_1 + \frac{1}{2}\bar{\beta}_3 > 0$, the subphase with $m_J = \pm 2$ is energetically preferable compared to the symmetric and $m_J = 0$ subphases and compared to the $m_J = \pm 1$ subphases provided simultaneously $\bar{\beta}_3 > 0$, whereas for $\bar{\beta}_3 < 0$ the $m_J = \pm 1$ subphases are favorable. In the BCS weak-coupling approximation [54,104], one has $\bar{\beta}_1 = 0$, $\bar{\beta}_2 = -\bar{\beta}_3 > 0$. In this case, the symmetric and $m_J = 0$ subphases prove to be energetically favorable.

To consider finite systems, we should include contributions F^{grad}. With taking into account these terms, degeneracy of the subphases $3\mathrm{P}_2(0)$ and $3\mathrm{P}_2(\mathrm{sym})$ disappears. For the matter filling the semi-infinite space $x < 0$ in the gauge where $h_3 = \partial_1 A_2$, $h_1 = h_2 = 0$, for the $3\mathrm{P}_2(0)$ subphase, we obtain

$$
F_0^{\mathrm{grad}} = \left(c_1 + \frac{c_2 + c_3}{6}\right)[|\partial_1 a_0|^2 + e_*^2 A_2^2 |a_0|^2], \tag{181}
$$

cf. (112), (129), and (134),

$$
\begin{aligned}
F_{\pm 1}^{\mathrm{grad}} &= \left(c_1 + \frac{c_2 + c_3}{4}\right)[|\partial_1 a_{\pm 1}|^2 + e_*^2 A_2^2 |a_{\pm 1}|^2] \\
&\quad \pm \frac{c_2 - c_3}{2} e_* A_2 \partial_1 |a_{\pm 1}|^2,
\end{aligned} \tag{182}
$$

$$
\begin{aligned}
F_{\pm 2}^{\mathrm{grad}} &= \left(c_1 + \frac{c_2 + c_3}{2}\right)[|\partial_1 a_{\pm 2}|^2 + e_*^2 A_2^2 |a_{\pm 2}|^2] \\
&\quad \pm \frac{c_2 - c_3}{2} e_* A_2 \partial_1 |a_{\pm 2}|^2,
\end{aligned} \tag{183}
$$

cf. (139).

Difference in the volume and surface energies for various subphases leads to a possibility of domains; see in Sec. III.

B. Subphases $3\mathrm{P}_2(0)$ and $3\mathrm{P}_2(\mathrm{sym})$ of *nn* pairing in external uniform static magnetic field

Expressions for the Gibbs free-energy densities for the symmetric subphase and the $m_J = 0$ subphase are similar to those for the phase A at the pairing of the neutral fermions considered above in Sec. IV B.

Let $T < T_{\mathrm{cr}}$ and $\bar{\beta}_1 + \bar{\beta}_2 + \frac{1}{2}\bar{\beta}_3 > 0$. The order parameter in subphases $3\mathrm{P}_2(0)$ and $3\mathrm{P}_2(\mathrm{sym})$ of *nn* pairing decouples with the magnetic field. Thereby, we get $\vec{h} = \vec{H}$. The order parameters $\tilde{a}_0^2 + 2(\tilde{a}_1^2 + \tilde{a}_2^2)$ and $|a_0|^2$ are found by the minimization of G_{sym} and G_0, respectively. In case of the infinite matter in the minimum, we get

$$
|a_0|^2 = \tilde{a}_0^2 + 2(\tilde{a}_1^2 + \tilde{a}_2^2) = \frac{\bar{\alpha}\theta(t)}{2(\bar{\beta}_1 + \bar{\beta}_2 + \frac{1}{2}\bar{\beta}_3)} \tag{184}
$$

and

$$
G_0^{\mathrm{hom}} = G_{\mathrm{sym}}^{\mathrm{hom}} = -\frac{\bar{\alpha}^2\theta(t)}{4(\bar{\beta}_1 + \bar{\beta}_2 + \frac{1}{2}\bar{\beta}_3)}. \tag{185}
$$

Thus, for the uniform matter, these subphases prove to be degenerate. With a decreasing temperature, they appear at $T = T_{\mathrm{cr}}$ by the second-order phase transition and exist for $T < T_{\mathrm{cr}}$. The order parameter and external magnetic field decouple. The subphases are stable relatively transitions to $m_J = \pm 2$ subphases provided $\bar{\beta}_1 + \frac{1}{2}\bar{\beta}_3 < 0$ and to $m_J = \pm 1$ subphases for $\bar{\beta}_1 + \frac{1}{4}\bar{\beta}_3 < 0$.

C. Subphases $3P_2$ (± 2)-B and $3P_2$ (± 2)-C of *nn* pairing in external uniform magnetic field

The problem is reduced to that considered above in Sec. IV B on example of the phases B and C for the vector order parameter, provided one puts now $\tilde{\psi} = a_{+2}$ or $\tilde{\psi} = a_{-2}$. We label the phase "B" provided $\bar{\beta}_2 - 2\pi\eta^2 > 0$ and "C", if $\bar{\beta}_2 - 2\pi\eta^2 < 0$.

1. Subphases $3P_2$ *(± 2)-*B_3

Let us focus on the $3P_2$ (± 2)-B subphases. For $\vec{h}$ and $\vec{\eta}$ directed parallel or antiparallel to $\vec{H}$, we deal with the subphase B_3. Consider the case of the infinite matter. Minimization of Eq. (177) yields in case of the phase $3P_2(\pm 2)$-B_3,

$$h = H \mp 4\pi\eta|a_{\pm 2}|^2, \tag{186}$$

cf. (94), and

$$|a_{\pm 2}^{\mathrm{B_3H}}|^2 = \frac{(\bar{\alpha} \pm \eta H)\theta(\bar{\alpha} \pm \eta H)}{2(\bar{\beta}_2 - 2\pi\eta^2)}, \tag{187}$$

$$G_{\pm 2}^{\mathrm{B_3H}} = -\frac{(\bar{\alpha} \pm \eta H)^2\theta(\bar{\alpha} \pm \eta H)}{4(\bar{\beta}_2 - 2\pi\eta^2)}, \tag{188}$$

for $\bar{\alpha} \pm \eta H > 0$, and for $\bar{\gamma} \to 0$, cf. (93), (95). Thus, even for $H = 0$ in this subphase, there appears the internal magnetic field $h(H = 0)$.

The critical temperature found from the condition $\bar{\alpha} \pm \eta H = 0$ is shifted up in the presence of the external field H and the new critical temperature equals to

$$T_{\mathrm{cr}}^{\mathrm{B_3H}} = T_{\mathrm{cr}}(1 \pm \eta H/\bar{\alpha}_0), \tag{189}$$

provided $\bar{\alpha} = \bar{\alpha}_0 t$, cf. Eq. (97). For $\eta > 0$, the state $m_J = +2$ is profitable and for $\eta < 0$ the state $m_J = -2$.

At $T < T_{\mathrm{cr}}$ for $\eta > 0$, solutions exist at arbitrary H for the state $m_J = +2$ and at $H < H_{\mathrm{cr2}} = -\bar{\alpha}/\eta$ they exist provided $\eta < 0$. For $\eta < 0$, solutions exist at arbitrary H for $m_J = -2$ and they exist for $H < H_{\mathrm{cr2}} = \bar{\alpha}/\eta$ provided $\eta > 0$.

For $T_{\mathrm{cr}}^{\mathrm{B_3H}} > T > T_{\mathrm{cr}}$, solutions exist for $\eta > 0$ at $H > H_{\mathrm{cr2}} = -\bar{\alpha}/\eta > 0$ for the state $m_J = 2$ and at $H > H_{\mathrm{cr2}} = \bar{\alpha}/\eta > 0$ for $\eta < 0$ for the state $m_J = -2$.

2. Subphases $3P_2$ *(± 2)-*C_3

In the subphase $3P_2(\pm 2)$-C_3 for small $\bar{\gamma} > 0$, we find

$$|a_{\pm 2}^{\mathrm{C_3H}}|^2 \simeq \frac{2(2\pi\eta^2 - \bar{\beta}_2)}{3\bar{\gamma}} + \frac{\bar{\alpha} \pm \eta H}{2(2\pi\eta^2 - \bar{\beta}_2)}, \tag{190}$$

$$G_{\pm 2}^{\mathrm{C_3H}} \simeq -\frac{4(2\pi\eta^2 - \bar{\beta}_2)^3}{27\bar{\gamma}^2}, \tag{191}$$

cf. Eq. (102). The critical temperature is increased in the presence of the external magnetic field H and the new critical temperature is as follows:

$$T_{\mathrm{cr}}^{\mathrm{C_3H}} = T_{\mathrm{cr}}\left(1 + \frac{(2\pi\eta^2 - \bar{\beta}_2)^2}{3\bar{\gamma}\bar{\alpha}_0} + \frac{|\eta|H}{\bar{\alpha}_0}\right), \tag{192}$$

provided $\bar{\alpha} = \bar{\alpha}_0 t$, cf. Eq. (104).

3. Subphases $3P_2(\pm 1)$-B *and* $3P_2(\pm 1)$-C *of nn pairing*

Expressions for $m_J = \pm 1$ can be found from those for $m_J = \pm 2$ with the help of the replacements $\bar{\beta}_2 \to \bar{\beta}_2 + \frac{1}{4}\bar{\beta}_3$, $\eta \to \frac{1}{2}\eta$, cf. Eqs. (177).

D. Subphases of $3P_2$ *pp* pairing

As above, consider medium filling half-space $x < 0$ under the action of the external uniform magnetic field $\vec{H}\| z$. Our consideration is completely similar to that performed in Sec. V.

1. Subphases $3P_2(0)$ *and* $3P_2(\mathrm{sym})$

Penetration of the external static magnetic field in case of the $3P_2(0)$ and $3P_2(\mathrm{sym})$ subphases is described similar to that for the A phase in the superconducting matter described by the vector order parameter in Sec. V B. Using (177) and (181) for simplicity at $\bar{\gamma} \to 0$, we obtain

$$\left(c_1 + \frac{c_2 + c_3}{6}\right)[\partial_1^2 a_0 - e_*^2 A_2^2 a_0] + \bar{\alpha} a_0 - 2\left(\bar{\beta}_1 + \bar{\beta}_2 + \frac{1}{2}\bar{\beta}_3\right)|a_0|^2 a_0 = 0, \tag{193}$$

$$\partial_1^2 A_2 - 8\pi e_*^2\left(c_1 + \frac{c_2 + c_3}{6}\right)A_2|a_0|^2 = 0. \tag{194}$$

Thus, for $m_J = 0$ subphase at low H, there appears Meissner effect and for $\kappa = \sqrt{\frac{\bar{\beta}_1 + \bar{\beta}_2 + \frac{1}{2}\bar{\beta}_3}{4\pi e_*^2}} > 1/\sqrt{2}$ with increasing H for $H_{\mathrm{cr1}} < H < H_{\mathrm{cr2}}$ there exists the Abrikosov mixed state. The question about stability of the subphase and a coupling between various subphases can be considered, as it has been done in Sec. V.

2. B and C phases of pp pairing. The $m \pm 1, \pm 2$ subphases

Using (177) and (181), we obtain

$$\left(c_1+\frac{c_2+c_3}{4}\right)[\partial_1^2 a_{\pm1}-e_*^2A_2^2a_{\pm1}]\pm\left(\frac{c_2-c_3}{2}e_*+\frac{\eta}{2}\right)ha_{\pm1}+\bar{\alpha}a_{\pm1}-\left(\bar{\beta}_2+\frac{1}{4}\bar{\beta}_3\right)|a_{\pm1}|^2a_{\pm1}=0, \tag{195}$$

$$\partial_1^2A_2-8\pi e_8^2\left(c_1+\frac{c_2+c_3}{4}\right)|a_{\pm1}|^2A_2\mp 4\pi\left(\eta+e_*\frac{c_2-c_3}{2}\right)\partial_1|a_{\pm1}|^2=0 \tag{196}$$

and

$$\left(c_1+\frac{c_2+c_3}{2}\right)[\partial_1^2 a_{\pm2}-e_*^2A_2^2a_{\pm2}]\pm\left(\frac{c_2-c_3}{2}e_*+\eta\right)ha_{\pm2}+\bar{\alpha}a_{\pm2}-2\bar{\beta}_2|a_{\pm2}|^2a_{\pm2}=0, \tag{197}$$

$$\partial_1^2A_2-8\pi e_8^2\left(c_1+\frac{c_2+c_3}{2}\right)|a_{\pm2}|^2A_2\mp 4\pi\left(\eta+e_*\frac{c_2-c_3}{2}\right)\partial_1|a_{\pm2}|^2=0, \tag{198}$$

cf. (141).

Instead of solving exact equations of motion, let us consider the variational problem. For that, we employ the Abrikosov Ansatz (147), which for our case of the $3P_2$ pairing reads as

$$(\partial_i+ie_*A_i)A_{\nu i}=0. \tag{199}$$

Expressions for the averaged Gibbs free-energy densities for the $m_J=\pm1$ and $m_J=\pm2$ subphases are similar to those for the subphase B_3, cf. Sec. V. We deal with $3P_2$ (±1)-B_3 subphases provided

$$\left(\bar{\beta}_2+\frac{1}{4}\bar{\beta}_3\right)\tilde{\beta}-2\pi\tilde{\eta}_{\pm1}^2>0,$$

cf. (164), where now

$$\tilde{\eta}_{+1}=\frac{\eta}{2}-e_*(c_1+c_2),\qquad \tilde{\eta}_{-1}=\frac{\eta}{2}-e_*(c_1+c_3),$$

and with $3P_2$ (±2)-B_3 subphases, if

$$\bar{\beta}_2\tilde{\beta}-2\pi\tilde{\eta}_{\pm2}^2>0, \tag{200}$$

where

$$\tilde{\eta}_{+2}=\eta-e_*(c_1+c_2),\quad \tilde{\eta}_{-2}=\eta-e_*(c_1+c_3). \tag{201}$$

If opposite inequalities are fulfilled, we deal with the corresponding C_3 subphases.

Subphases $3P_2(\pm2)$-B_3.—We employ Eq. (152) in the gauge $\vec{A}=(0,A_2(x),0)$. The minimization of the Gibbs free-energy $\mathcal{G}=\int d^3xG$, cf. Eq. (177), yields for $\bar{\gamma}\to 0$,

$$|a_{\pm2,\mathrm{B_3H}}|^2=\frac{\bar{\alpha}\pm\tilde{\eta}_{\pm2}H}{2\overline{\nu^2}(\bar{\beta}_2\tilde{\beta}-2\pi\tilde{\eta}^2)}, \tag{202}$$

$$\bar{G}_{\pm2,\mathrm{B_3H}}=-\frac{(\bar{\alpha}\pm\tilde{\eta}_{\pm2}H)^2}{4(\bar{\beta}_2\tilde{\beta}-2\pi\tilde{\eta}^2)}. \tag{203}$$

For $T<T_{\rm cr}$ at $e_*=0$ and $\eta>0$, the phase $m_J=+2$ is energetically preferable, whereas for $e_*=0$ and $\eta<0$ wins the phase $m_J=-2$. Equality

$$\bar{\alpha}\pm\tilde{\eta}_{\pm2}\vec{H}=0$$

determines the critical point for the second-order phase transition,

$$\vec{h}=\vec{H}\pm4\pi\tilde{\eta}_{\pm2}|a_{\pm2,\mathrm{B_3}H}|^2. \tag{204}$$

Even for $H=0$ in this phase, there exists an own magnetic field $h(H=0)$.

The critical temperature is shifted up in the presence of the external magnetic field H and the new critical temperature becomes (for $\bar{\alpha}=\bar{\alpha}_0t$)

$$T_{\rm cr}^{\mathrm{B_3}H}=T_{\rm cr}(1\pm\tilde{\eta}_{\pm2}H/\bar{\alpha}_0). \tag{205}$$

Upper sign is for $m_J=2$ and $\tilde{\eta}_2>0$ and lower sign is for $m_J=-2$ and $\tilde{\eta}_2<0$.

Subphases $3P_2(\pm2)$-C_3.—We deal with $3P_2(\pm2)$-C_3 subphase provided

$$\bar{\beta}_2\tilde{\beta}-2\pi\tilde{\eta}_{\pm2}^2<0.$$

For $\bar{\gamma}\le0$, the ground state is unstable. For $\bar{\gamma}>0$, we deal with the first-order phase transition. For a small $\bar{\gamma}>0$, we find

$$|a_{\pm2,\mathrm{C_3}H}|^2\simeq\frac{2(2\pi\tilde{\eta}_{\pm2}^2-\bar{\beta}_2\tilde{\beta})}{3\bar{\gamma}\bar{\beta}_1}+\frac{(\bar{\alpha}\pm\tilde{\eta}_{\pm2}H)\bar{\beta}_2}{2\bar{\beta}_1(2\pi\tilde{\eta}_{\pm2}^2-\bar{\beta}_2\tilde{\beta})}, \tag{206}$$

$$\bar{G}_{\pm2,\mathrm{C_3}H}\simeq-\frac{4(2\pi\tilde{\eta}_{\pm2}^2-\bar{\beta}_2\tilde{\beta})^3}{27\bar{\gamma}^2\bar{\beta}_2^2}, \tag{207}$$

and the own magnetic field is determined by Eq. (186).

The critical temperature is shifted up in the presence of the external magnetic field H, and the new critical temperature is given by

$$\frac{T_{\rm cr}^{\rm C_3H}}{T_{\rm cr}} = 1 + \frac{(2\pi\tilde{\eta}_{\pm 2}^2 - \bar{\beta}_2\tilde{\beta})^2}{3\bar{\gamma}\bar{\alpha}_0\tilde{\beta}_2} \pm \frac{\tilde{\eta}_{\pm 2}H}{\bar{\alpha}_0}. \tag{208}$$

VII. NUMERICAL EVALUATIONS: BCS APPROXIMATION AND BEYOND

As we have mentioned, existing in the literature, estimates of the typical value of $T_{\rm cr}$ for the 3P$_2$ nn pairing in a dense neutron star matter are controversial. Following BCS estimates [25], typical value of $T_{\rm cr}$ for the 3P$_2$ nn pairing is $T_{\rm cr}^{\rm BCS} \sim 0.1$–$1$ MeV, cf. [28]. Contrary, Ref. [32] with taking into account of the polarization effects estimated $T_{\rm cr} \lesssim 10$ keV for the 3P$_2$ nn pairing. Values of the Fermi liquid parameters for the isospin-asymmetric nuclear matter in the pairing channel at $n \neq n_0$, as well as their density dependence, are poorly known. Only rough estimates were performed, cf. [57]. Bearing this in mind, in our estimates we consider $T_{\rm cr}$ as a free parameter, which we vary in the range $T_{\rm cr}^{\rm BCS} \sim (0.01$–$1)$ MeV.

Values of the parameters used in Eq. (174) were calculated in the weak-coupling limit (BCS) [54,104,105],

$$\bar{\alpha}_0^{\rm BCS} = N(0)/3, \qquad \bar{\beta}_1^{\rm BCS} = 0,$$
$$\bar{\beta}_2^{\rm BCS} = -\bar{\beta}_3^{\rm BCS} = 4|\beta| = \frac{7\zeta(3)N(0)}{60\pi^2 T_{\rm cr}^2}. \tag{209}$$

Here $N(0) = m_{\rm F}^* p_{\rm F}/(2\pi^2)$ is the density of states with $m_{\rm F}^*$ standing for the effective fermion mass and $p_{\rm F}$ is the Fermi momentum, $\zeta(x)$ is the Riemann function, and $\zeta(3) \approx 1.202$. In the approximation of a symmetry of the particles and holes on the Fermi surface, the coefficients c_1, c_2, and c_3 approximately coincide [23]

$$c_1^{\rm BCS} \simeq c_2^{\rm BCS} \simeq c_3^{\rm BCS} \simeq \frac{7\zeta(3)N(0)}{120\pi^2}\frac{\epsilon_{\rm F}}{m_{\rm F}^* T_{\rm cr}^2} \equiv c, \tag{210}$$

where $\epsilon_{\rm F}$ is the Fermi energy, $\epsilon_{\rm F} = p_{\rm F}^2/2m_{\rm F}^*$. Exploiting the presence of a slight asymmetry of the particles and holes near the Fermi surface ($|c_2 - c_3|/c_2 \ll 1$), Refs. [23,106] estimated

$$c_2 - c_3 \simeq c(T_{\rm cr}/\epsilon_{\rm F})^2 \ln(\epsilon_{\rm F}/T_{\rm cr}). \tag{211}$$

As for T in the vicinity of $T_{\rm cr}$ as for $T \ll T_{\rm cr}$, with a logarithmic accuracy [18,22], we obtain

$$\eta_{\pm 2}^{\rm BCS} = \frac{1}{3}\mu_{\rm pair}N'(0)\ln\frac{\epsilon_{\rm F}}{T_{\rm cr}}. \tag{212}$$

We used that $\Delta^2 = 2|a_{\pm 2}|^2/3$ for the pairing in $m_J = \pm 2$ states, Δ is the pairing gap. The quantity $N'(0)$ is the derivative of the density of states with respect to the energy, $N'(0) = N(0)/2\epsilon_{\rm F}$, $\mu_{\rm pair}$ is the magnetic moment of the Cooper pair.

Following an estimate [54], in the BCS theory $\bar{\gamma}^{\rm BCS} < 0$ for the $m_J = 0$ phase. The coefficients in the $\{A^6\}$ contribution to the Gibbs free energy are $\delta G_6(m_J = 0) = \bar{\gamma}_1({\rm Tr}A^2)^3 + \bar{\gamma}_2{\rm Tr}(A^6)$, and

$$\bar{\gamma}_2^{\rm BCS} = \frac{4}{5}\bar{\gamma}_1^{\rm BCS} = -\frac{31}{16}\frac{\zeta(5)}{35}\frac{N(0)}{\pi^4 T_{\rm cr}^4}, \tag{213}$$

where $\zeta(5) \approx 1.0369$, that forces to keep A^8 term in expansion of G. Reference [56] uses a more complicated structure of the A^6 term and keeps A^8 term in expansion of G. To avoid these complications in our rough numerical estimates, e.g., for $m_J = -1$ and -2, we take $\delta G_6(m_J = -1) = \bar{\gamma}^{\rm BCS}(m_J = -1)|a_{-1}|^6$, $\delta G_6(m_J = -2) = \bar{\gamma}^{\rm BCS}(m_J = -2)|a_{-2}|^6$ with $\bar{\gamma}_i^{\rm BCS} = |\bar{\gamma}_2^{\rm BCS}|$ taken in modulus. Certainly, within such a simplified analysis, we disregard a possibility of existence of some other phases, which may appear in a sequence of the first-order phase transitions. Using (171), we find that $|\bar{\gamma}^{\rm BCS}| \ll 1$ for $|t| \ll 1$. Then, dealing with the phase B, we may use expressions for $|\bar{\gamma}^{\rm BCS}| \to 0$.

With the BCS parameters (209), stability conditions (180) are fulfilled in a case of a weak external magnetic field H for all the subphases 3P$_2(0)$, 3P$_2$(sym), 3P$_2(\pm 1)$, and 3P$_2(\pm 2)$ considered above. Also, in the BCS approximation, the value $T_{\rm cr}$ is the same for all these phases.

Actual values of the parameters of the Ginzburg-Landau functional in the strong coupling theory are poorly known. Only rough estimations have been performed [57]. Existing estimates of the gradient terms are controversial. Reference [15] calculated for the triplet superconductivity in 3D Dirac semimetals $c_3 = [u_L - u_T]/4$, $c_1 = u_T/4$, $c_2 = 0$, $u_L = u_T/32$, $u_T = \frac{7\zeta(3)N(0)v_{\rm F}^2}{15\pi^2 T_{\rm cr}^2}$, i.e., $c_1 \simeq -c_3$, $c_2 = 0$, and derived values $b_1 = \frac{7\zeta(3)N(0)}{640\pi^2 T_{\rm cr}^2}$ and $b_2 = -b_1/3$, $\Delta_i = \psi_i/2$, $v_{\rm F}$ is the Fermi velocity. As one of the choices (E$_2$ model), Ref. [2] employs $c_2 = c_3 \ll c_1 \sim N(0)v_{\rm F}^2/(\pi^2 T_{\rm cr}^2)$ that does not contain a small numerical prefactor appeared in estimate [23,106]. On the other hand, the heat capacity measurements performed for UPt$_3$ by several groups give $b_2/b_1 = (0.2$–$0.5)$, cf. [2,98]. We remind that neglecting u_L contribution one recovers the relation $c_1 = -c_3$, as follows from the microscopical consideration of the W boson fields [74].

If there were $\bar{\beta}_3 > 0$, with the same BSC estimates for other parameters the phase B would be preferable even for $H = 0$ and for $T < T_{\rm cr}$.

A. 3P$_2$ *nn* pairing in neutron stars

For the *nn* pairing in the subphase 3P$_2(0)$-A following (184) and (185) with values of the parameters estimated within the BCS approximation (209), we find

$$|a_0^{\mathrm{BCS}}|^2 = \frac{20\pi^2}{7\zeta(3)} T_{\mathrm{cr}}^2 t, \qquad G_0^{\mathrm{BCS}} = -\frac{10\pi^2 N(0)}{21\zeta(3)} T_{\mathrm{cr}}^2 t^2. \tag{214}$$

For the *nn* pairing in the subphase 3P$_2(0)$-B using $\mu_{\mathrm{pair}} = \mu_{nn} = -3.83\mu_N$, $\mu_N \approx 4.5\times 10^{-5}/\mathrm{MeV}$, for $n \sim n_0$ and $T_{\mathrm{cr}} \sim (0.01\text{–}1)$ MeV following (212), we estimate $\eta^{\mathrm{BCS}} \sim \mp(10^{-2}\text{–}10^{-1})$, $\mp\eta_{\pm 2}^{\mathrm{BCS}} H/\bar{\alpha} \sim \pm 3\times(10^{-2}\text{–}10^{-1})H/m_\pi^2$, and we obtain $(T_{\mathrm{cr}}^{\mathrm{B_3H}} - T_{\mathrm{cr}})/T_{\mathrm{cr}} \lesssim 3\times 10^{-1}$ for $H \lesssim m_\pi^2$ for all relevant values T_{cr}. From Eqs. (187) and (188), we obtain (for $\bar{\beta}_2 > 2\pi\tilde{\eta}^2$ which is indeed fulfilled)

$$|a_{-2,\mathrm{BCS}}^{\mathrm{B_3H}}|^2 = \frac{\frac{1}{2}|a_0^{\mathrm{BCS}}|^2 \nu_{-2}}{1 - \frac{T_{\mathrm{cr}}^2}{T_{\mu,-2}^2}}, \tag{215}$$

$$G_{-2,\mathrm{BCS}}^{\mathrm{B_3H}} = \frac{\frac{1}{2}G_0^{\mathrm{BCS}} \nu_{-2}^2}{1 - \frac{T_{\mathrm{cr}}^2}{T_{\mu,-2}^2}}, \tag{216}$$

cf. also Eqs. (202) and (203) at $e^* = 0$. We used that

$$\nu_{-2} = 1 - \eta H/\bar{\alpha} \tag{217}$$

is positive for relevant values $H \lesssim m_\pi^2$ and that

$$T_{\mu,-2} = \left(\frac{21\zeta(3)}{20\pi}\right)^{1/2} \frac{v_{\mathrm{F}}^{3/2}}{|\mu_{nn}| \ln\frac{\epsilon_{\mathrm{F}}}{T_{\mathrm{cr}}}}, \tag{218}$$

$v_{\mathrm{F}} = p_{\mathrm{F}}/m_n^*$. We estimate $T_{\mu,-2} \simeq 2\times 10^3 (v_{\mathrm{F}}^{3/2}/\ln\frac{\epsilon_{\mathrm{F}}}{T_{\mathrm{cr}}})$ MeV. At $n \sim n_0 \simeq 0.16/\mathrm{fm}^3$, we have $v_{\mathrm{F}} \sim 0.4$ and $T_{\mu,-2} \simeq 500/\ln\frac{\epsilon_{\mathrm{F}}}{T_{\mathrm{cr}}}$ MeV $\gg T_{\mathrm{cr}}$. Thereby, $\bar{\beta}_2 > 2\pi\tilde{\eta}^2$, as we have used deriving (215) and (216) and we indeed deal with the B phase rather than with the C phase.

We see that $G_{-2,\mathrm{B_3}}^{\mathrm{BCS}} \simeq \frac{1}{2}G_0$. Similarly, $G_{-1,\mathrm{B}}^{\mathrm{BCS}} \simeq \frac{2}{3}G_0$. Therefore, if the BCS estimates (209) were correct, the subphases 3P$_2(-2)$-B and 3P$_2(-1)$-B of the *nn* pairing would not be realized in the neutron stars for $T < T_{\mathrm{cr}}$ till $\nu_{-2} > 1/\sqrt{2}$ and $\nu_{-1} > \sqrt{2/3}$. However, in the temperature interval, $T_{\mathrm{cr}} < T < T_{\mathrm{cr}}^{\mathrm{B_3H}}$, where the A phase is impossible, the 3P$_2(-1)$-B$_3$ subphase is realized in any case.

Using the relation $b_2 = -b_1/3$ derived in [15] for the description of the superconductivity in 3D semimetals, that corresponds to the relation $\bar{\beta}_3 = -\bar{\beta}_2/3$ in the functional (176), with $\bar{\beta}_1 = 0$, we evaluate $G_{-1,\mathrm{B}}^{\mathrm{BCS}} \simeq \frac{11}{12}G_0$, i.e., $G_{-1,\mathrm{B}}^{\mathrm{BCS}}$ is only slightly larger than G_0.

On the other hand, with $\bar{\beta}_3/\bar{\beta}_2 > 0$, as follows from experiments on UPt$_3$, for $T < T_{\mathrm{cr}}$ and in the temperature interval $T_{\mathrm{cr}} < T < T_{\mathrm{cr}}^{\mathrm{B_3H}}$, the 3P$_2(-1,-2)$-B$_3$ subphases are energetically favorable compared to the A phase.

As we have mentioned, the heat capacity measurements performed for UPt$_3$ by several groups give $b_2/b_1 = (0.2\text{–}0.5)$, cf. [2,98]. Choosing estimate of $b_2 = b_1/2$, that corresponds to $\bar{\beta}_3 = +\bar{\beta}_2/2$, we find that $G_{-2,\mathrm{B_3}}^{\mathrm{BCS}} \simeq \frac{5}{4}G_0$ and $G_{-1,\mathrm{B}}^{\mathrm{BCS}} \simeq \frac{9}{8}G_0$. With these estimates, the subphase 3P$_2(-2)$-B of the *nn* pairing would be realized in the neutron stars for $0 < T < T_{\mathrm{cr}}^{\mathrm{B_3H}}$.

With the help of Eqs. (186) and (188) and making use of the estimate (215), we find

$$h_{-2}^{\mathrm{B_3}} = \frac{\epsilon_{\mathrm{F}}}{|\mu_{nn}|} \frac{2|t| \frac{T_{\mathrm{cr}}^2}{T_{\mu,-2}^2 \ln\frac{\epsilon_{\mathrm{F}}}{T_{\mathrm{cr}}}}}{(1 - \frac{T_{\mathrm{cr}}^2}{T_{\mu,-2}^2})}. \tag{219}$$

We have put $\nu_{-2} \simeq 1$, $\epsilon_{\mathrm{F}}/|\mu_{nn}| \approx 5.7\times 10^3 [\epsilon_{\mathrm{F}}/\mathrm{MeV}]\,\mathrm{MeV}^2 \approx 8.3\times 10^{16}[\epsilon_{\mathrm{F}}/\mathrm{MeV}]$ Gs. Thus, for $T_{\mathrm{cr}}^2/T_{\mu,-2}^2 \ll 1$, we estimate $h_{-2}^{\mathrm{B_3}} \sim 10^{11}|t|(\frac{T_{\mathrm{cr}}}{\mathrm{MeV}})^2 \frac{\epsilon_{\mathrm{F}}}{\mathrm{MeV}}$ Gs for $n = n_0$. For $T_{\mathrm{cr}} \sim 1$ MeV, we estimate $h \sim 10^{13}$ Gs. Note that, as we have shown, $h(H=0) \propto |a_{-2}^{\mathrm{B_3}}(x)|^2$ and thereby it vanishes at the superfluid—normal matter boundary. If by some reason the field h had a magnetic-dipole component outside the superfluid star interior, the neutron star would substantially diminish its rotation during first $\sim(10^3\text{–}10^4)$ years of its evolution. At least millisecond pulsars should not have such a strong magnetic-dipole fields. For $T_{\mathrm{cr}} \lesssim 10$ keV, cf. [32], we estimate $h \lesssim 10^9$ Gs, which value in any case does not contradict to the data on millisecond pulsars.

B. Estimates for hypothetical 3P$_2$ *pp* pairing in neutron stars

Since $e_*(c_1 + c_3) \sim 4\times 10^{-5}(m_\pi/T_{\mathrm{cr}})^2$ for $n \sim n_0$, for $T_{\mathrm{cr}} \sim (0.01\text{–}1)$ MeV using (201), we estimate $\tilde{\eta}_{\pm 1} \sim \tilde{\eta}_{\pm 2} \sim -(1\text{–}10^4)$, being valid for the 3P$_2(2)$-B and 3P$_2(1)$-B subphases of the *pp* pairing. Making use of this estimate and (209), we see that the condition (200) is fulfilled for all values T_{cr} of our interest. Thus, C phase is not realized. The value $T_{\mathrm{cr}}^{\mathrm{B_3H}}$ proves to be significantly shifted up for $T_{\mathrm{cr}} \sim 0.01$ MeV already for $H \gtrsim 10^{14}$ Gs. For a higher value H, the B$_3$ subphase becomes energetically profitable compared to the A phase, as it follows from Eqs. (203) and (185). With the help of (202) and (204), we roughly estimate the value of the own magnetic field as $h \sim 10^{16}$ Gs. Recall that at the surfaces of the magnetars the strength of the magnetic field reaches values $h \lesssim 10^{16}$ Gs. For $T_{\mathrm{cr}} \sim 1$ MeV, the value $T_{\mathrm{cr}}^{\mathrm{B_3H}}$ is significantly shifted up, respectively, T_{cr} only for $H \gtrsim 10^{18}$ Gs.

Reference [42] expressed an idea about a possibility of the triplet $3PF_2$ pp pairing in the hyperon enriched dense region. Then one should study a coexistence of the considered above phases of the nn pairing and those available for the pp pairing.

C. Estimates for $3S_1$ *np* pairing in isospin-symmetrical systems

The $3S_1$ channel provides the largest attractive interaction for the triplet np pairing in the isospin-symmetrical matter for $n \lesssim n_0$. With increasing density, the $3D_2$ channel becomes most attractive, cf. [31]. The BCS calculations for the symmetric matter with polarization effects included [66] predict the np pairing gaps $\sim$(several $-$ 10) MeV. As for the $3P_2$ pp pairing, the own magnetic field in the B_3 subphase is estimated as $h \sim 10^{16}$ Gs. In this phase, the nucleon matter is spin polarized that might be checked experimentally. For example, in peripheral heavy-ion collisions of approximately isospin-symmetric nuclei, where the temperature is rather low, the spin-triplet np pairing in the $3S_1$ channel can be formed. Moreover, in peripheral heavy-ion collisions, the external magnetic field reaches values 10^{17}–10^{19} Gs, cf. [60,61]. In such strong fields, the value $T_{\rm cr}^{B_3}$ might be significantly larger than $T_{\rm cr}$, favoring np pairing in the $3S_1$ channel. Also, the np pairing in the $3SD_1$ state is possible in the nuclei [65,66].

VIII. CONCLUSION

This paper studies effects of the vector-boson condensation and spin-triplet superfluidity and superconductivity, such as ferromagnetic superfluidity, as well as the effects of the $3P_2$ nn and pp pairing in the neutron-star matter and the $3S_1$ np pairing in the isospin-symmetrical matter in the absence and in the presence of the external static uniform magnetic field. Possible effects of the self-rotation and response of the system on "external" rotation were for simplicity disregarded and will be considered elsewhere.

We started in Sec. II with the description of the condensation of the complex scalar field characterized by a negative squared effective mass inside a half-space medium $x < 0$, placed in an external static uniform magnetic field. Next, we considered a role of the Zeeman coupling for neutral fermions and discussed a possibility of the existence of the ferromagnetic state in the fermion matter (e.g., in the neutron star matter).

In Sec. III, focus was made on the study of the complex neutral and charged vector-boson fields with negative and positive squared effective mass. A possibility of existence of the A, B, and C phases was found. In the phase A, the mean spin density is zero and in the phase B spins are aligned in one direction. The simplest choice to describe the phase A is to chose only one Lorentz component of the complex vector field to be nonzero. The C phase is not realized provided the hadron-hadron coupling constant $\Lambda \gg e^2$.

The behavior of the charge-neutral complex vector-boson field inside the half-space medium $x < 0$ was studied in the presence of the uniform static external magnetic field. Two A subphases are then permitted for $m_{\rm sc}^2 < 0$: A_2 provided the y component of the vector-boson field is nonzero and A_3, provided z component is nonzero. The vector-boson field and the magnetic field decouple and the Gibbs free energies in the subphases are the same. Thus, the A phase of the neutral vector bosons is nonmagnetic. For $m_{\rm sc}^2 > 0$, there is no condensate.

In the phase B, which is described by two nonzero complex components of the neutral vector-boson field, the system behaves as a ferromagnetic superfluid. In the condensate region, there appears an own static magnetic field. We considered the matter filling half-space $x < 0$ in the presence of the external uniform static magnetic field either directed parallel to the system boundary or perpendicular to it.

In the subphase B_2 for $\vec{H} \| y$ (i.e., parallel to the system boundary and to the direction of the spin) and in the subphase B_3 for $\vec{H} \| z$ and spin parallel z, the condensate amplitude grows with H. At $H > H_{\rm cr}^{\rm neut}$, cf. Eq. (38), the superfluid condensate exists not only for $m_{\rm sc}^2 < 0$ but also for $m_{\rm sc}^2 > 0$. Which phase A or B is energetically favorable depends on the form of the self-interaction term in the Lagrangian. For the very same values $m_{\rm sc}^2$, with the self-interaction in the form (18) for $\xi_1 = 0$, the phase B proves to be energetically preferable in comparison with the phase A.

We demonstrated that the difference in the volume and surface energies for the subphases motivates a possibility of the existence of domains with different directions of the magnetic moment in each domain. Domains may merge in the presence of the external fields.

Then we studied the behavior of the charged complex vector field interacting with the electromagnetic field by the minimal and the Zeeman couplings. As for neutral vector bosons, we first considered charged complex vector field with the negative squared effective mass, $m_{\rm ef}^2 < 0$, in the half-space $x < 0$ under the action of the external static uniform magnetic field $\vec{H}$.

For the state with zero spin density (A phase) for $\vec{H}$ perpendicular to the system boundary ($\vec{H} \| x$), the subphase A_2 demonstrates superdiamagnetic response on a weak external magnetic field, as at $\vec{H} \| z$ for the charged scalar boson field, and for $\vec{H}$ parallel to the system boundary ($\vec{H} \| z$) the subphase A_2 is nonmagnetic, as for a neutral complex vector-boson field. The phase A_3 demonstrates superdiamagnetic response for a weak external magnetic field $\vec{H} \| z$ and it is nonmagnetic for $\vec{H} \| x$. The Gibbs free energies for the subphase A_2 at $\vec{H} \| z$ and for the subphase

A_3 at $\vec{H}\|x$ are equal, and they are lower than those for the subphase A_2 at $\vec{H}\|x$ and for the subphase A_3 at $\vec{H}\|z$. There are no solutions in case of the charged complex vector field in the phase A at $m_{\mathrm{ef}}^2 > 0$.

Then we found solution for the subphase B_3 at $\vec{H}\|z$. In this case for $H < H_{\mathrm{cr1}}$, there exists ordinary Meissner effect. However, for increasing H, we did not find a solution with $H = H_{\mathrm{cr2}}$, such that the condensate vanishes for $H \to H_{\mathrm{cr2}}$ from below. For $m_{\mathrm{ef}}^2 > 0$, the superconductive condensate appears for $H > H_{\mathrm{cr2}}$. For $H \neq 0$, the nonmagnetic A_2 subphase for $\vec{H}\|z$ and A_3 subphase for $\vec{H}\|x$ are more energetically preferable compared to the B_3 subphase for $\vec{H}\|z$, whereas for $H \to 0$ the subphase B_3 wins due to a smaller surface energy, if the system occupies a finite size layer.

In Secs. IV–VI, the focus is made on the description of the spin-triplet pairing of neutral and charged fermions coupled with the magnetic field by the Zeeman coupling. First, in Secs. IV and V, we considered the case, when the spin of the pair can be treated as a conserved quantity. This is the case for a negligibly small spin-orbit interaction (as for $3S_1$ np pairing in isospin-symmetrical nuclear matter). Then the order parameter is a vector with complex components and the description is similar to that for the spin 1 vector bosons considered in Sec. III: the vector order parameter is characterized by the two complex vectors of different amplitudes.

In Sec. IV, we consider triplet pairing of neutral fermions. In the p-wave triplet phase with zero projection of the spin of the pair on a quantization axis (the phase A), the two unit vectors $\vec{n}$ and $\vec{m}$ characterizing the vector order parameter are collinear. The A phase appears for $b_1 + b_2 > 0$ by the second-order phase transition for the temperature $T < T_{\mathrm{cr}}$ [b_1 and b_2 are coefficients at the ψ^4 terms in the free energy, cf. Eq. (55)]. In the absence of the external magnetic field [for $b_2 + 2\pi C^2\mathcal{M}^2 < 0$, where $C\mathcal{M}$ is the appropriately normalized effective magnetic moment of the fermion pair, cf. Eq. (55)], the A phase proves to be stable. In difference with the case of the vector bosons considered in Sec. III, where the A_1 subphase is not realized, for the triplet pairing of fermions all three subphases can be realized, with the same volume contribution to the energy. The surface energies in subphases A_2 and A_3 are the same, whereas the surface energy in the A_1 subphase is another. The vector $\vec{n}\|\vec{m}$ may change the direction depending on the spatial point, since the surface contributions to the Gibbs free energy depend on the direction of the vector order parameter relatively to the surface boundary. Owing to this property, there may appear domains with different directions of $\vec{n}$ in each domain. In the presence of the domains, the system remains for a while in a metastable state. The system may transform to the uniform state under the action of the external magnetic field and in the presence of the external rotation, or the energetic barrier can be overcame by a heating of the system.

We have shown that with an increase of the external magnetic field the system from the phase A transforms to another phase (labeled as AH), such that there appears an angle between vectors $\vec{n}$ and $\vec{m}$ growing with increase of H. The critical temperature of the phase transition also is increased with the growth of H. For $T < T_{\mathrm{cr}}$, not all spins of Cooper pairs are aligned in the direction parallel $\vec{H}$, and in the temperature interval $T_{\mathrm{cr}} < T < T_{\mathrm{cr}}^{\mathrm{AH}}$ all spins prove to be aligned in the direction parallel $\vec{H}$. The AH phase exists at $H \leq H_{\mathrm{cr}}^{\mathrm{AH}}$ for $T < T_{\mathrm{cr}}$, cf. Eq. (78), and at $H \geq H_{\mathrm{cr}}^{\mathrm{AH}}$ for $T_{\mathrm{cr}} < T < T_{\mathrm{cr}}^{\mathrm{AH}}$, cf. Eq. (82).

Besides the A phase, we found a possibility of the ferromagnetic superfluid phases B and C in neutral superfluids characterized even for $H = 0$ by the $+1$ or -1 projections of the spin of the Cooper pair on the quantization axis. Here, the vector order parameter is the sum of two perpendicular vectors, i.e., here $\vec{n}\perp\vec{m}$. The phase B appears, if $b_2 > 2\pi C^2\mathcal{M}^2$ and $b_1 > 2\pi C^2\mathcal{M}^2$, cf. Eq. (84), and the C phase occurs, if $b_2 > 2\pi C^2\mathcal{M}^2$ but $b_1 < 2\pi C^2\mathcal{M}^2$, cf. Eq. (85). The A and B phases arise by the second-order phase transitions, whereas the C phase appears by the first-order phase transition. For simplicity, we put $T_{\mathrm{cr}}^{\mathrm{A}} = T_{\mathrm{cr}}^{\mathrm{B}} = T_{\mathrm{cr}}$, whereas $T_{\mathrm{cr}}^{\mathrm{C}} \neq T_{\mathrm{cr}}$, since the phase transition to the phase C proves to be of the first order. The B and C phases of neutral superfluids are characterized by an own uniform magnetic field. For some values of parameters at $T < T_{\mathrm{cr}}$, the B or C phase wins a competition with the phase A; for other values of parameters, the A phase wins. For $T < T_{\mathrm{cr}}$, the condensate amplitude grows with increasing H. The subphases B and C may exist for $T_{\mathrm{cr}} < T < T_{\mathrm{cr}}^{\mathrm{B_3H}}, T_{\mathrm{cr}}^{\mathrm{C_3H}}$, where $T_{\mathrm{cr}}^{\mathrm{B_3H}}, T_{\mathrm{cr}}^{\mathrm{C_3H}} > T_{\mathrm{cr}}$, cf. Eqs. (97) and (104) provided $H > H_{\mathrm{cr2}}^{\mathrm{BH}}$, cf. Eq. (98).

Then, in Sec. V, we studied the spin-triplet pairing of charged fermions. Here, as in case of neutral superfluids, there may exist the A, B, and C phases. In the A_3 subphase, the spin-triplet superconductor, occupying half-space $x < 0$, placed in uniform external magnetic field parallel z behaves as an ordinary second-order superconductor characterized by the Ginzburg-Landau parameter κ_{A_3} (we considered the case $\kappa_{A_3} \gg 1$). The subphases A_1 and A_2 have some peculiarities. The critical values of the magnetic field, $H_{\mathrm{cr1}}^{\mathrm{A_1}}$, $H_{\mathrm{cr1}}^{\mathrm{A_2}}$ and $H_{\mathrm{cr2}}^{\mathrm{A_1}}$, $H_{\mathrm{cr2}}^{\mathrm{A_2}}$, are characterized by the two Ginzburg-Landau parameters $\kappa_{1,A_1} \gg 1$, $\kappa_{1,A_2} \gg 1$ and κ_{2,A_1}, κ_{2,A_2} in each case.

Then focus was made on the description of the B_3 subphase. We solved the variational problem using the Abrikosov Ansatz (147) for the probe functions. It was demonstrated that the value $\vec{\tilde{M}} = C\vec{\mathcal{M}} - \vec{n}_3 e_*(c_1 + c_3)$, cf. (154), gets sense of the effective magnetic moment, e_* is the effective charge of the fermion pair, c_1 and c_3 are the coefficients at the gradient contributions to the free energy,

$\vec{n}_3$ is the unit vector parallel z. For $T < T_{\rm cr}$ and $\tilde{M} > 0$, the condensate exists for any value of H and for $T_{\rm cr} < T < T_{\rm cr}^{\rm BH}$ the condensate exists also for $H > H_{\rm cr2}^{\rm B}$, cf. Eq. (146) and (154). For $\tilde{M} < 0$, the condensate exists only for $T < T_{\rm cr}$ and $H < H_{\rm cr2}^{\rm B}$. Similarly, the subphase C_3 may exist in a certain temperature interval above $T_{\rm cr}$.

Then, in Sec. VI, we studied the $3\mathrm{P}_2$ pairing in nuclear systems. Due to a strong spin-orbit nn interaction, $3\mathrm{P}_2$ phase of the nn pairing is supposed to exist in the baryon density interval $0.8n_0 \lesssim n \lesssim (3 \div 4)n_0$ in the neutron star interiors. We focused on cases when the projection of the total angular momentum on quantization axis is fixed as $m_J = 2, 1, 0, -1$, or -2, and also we considered a symmetric phase. It was demonstrated that the subphase $m_J = 0$ and symmetric subphase (labeled $3\mathrm{P}_2(0)$-A and $3\mathrm{P}_2(\mathrm{sym})$-A) are described similarly to the subphases of the phase A. The subphases $m_J = 1$ and 2 (and -1 and -2) are described similarly to the subphase B_3, and then we label them as $3P_2(\pm 1)$-B_3, $3P_2(\pm 2)$-B_3, or to the subphase C_3. For the nn pairing in the mentioned subphases, the description is similar to that for the neutral complex vector-boson field and for the pp pairing it is similar to the case of the charged complex vector-boson field.

The values of the parameters of the Gibbs free-energy functional for strongly interacting systems are unknown because of the absence of sound microscopic calculations with inclusion of the polarization effects. However, these parameters can be easily evaluated in the BCS weak-coupling approximation exploiting the bare pairing potentials. In Sec. VII, within the BCS approximation and beyond it we performed some estimates relevant for the $3P_2$ nn and pp pairings in the neutron star matter and for the $3\mathrm{S}_1$ np pairing in the isospin-symmetric nuclear matter. We found at which conditions the ferromagnetic superfluid phases characterized by own magnetic field prove to be energetically favorable.

A lot of work remains to be done. Let me list some problems related to the spin-triplet superfluidity in nuclear systems. In the paper body, only simplest available phases of the $3\mathrm{P}_2$ nn pairing were studied, whereas some other phases may also exist. Calculations of parameters of the Ginzburg-Landau functional are very desirable. A possibility of the ferromagnetic color superconductivity in hybrid stars should be studied. Gluons become massive in the hot quark-gluon plasma and may form vector field condensates at some conditions. Question about a possibility of a self-rotation in ferromagnetic superfluids was not considered, as well as the response of the spin-triplet superfluid sub-system on the rotation of the normal component. Another interesting issue is the problem of the neutron star cooling with taking into account a possibility of the ferromagnetic superfluidity and superconductivity including effects on the cooling of millisecond pulsars, cf. [107], and strong magnetic fields for magnetars. If the $3\mathrm{P}_2$ nn pairing were realized in the $m_J \neq 0$ state, the neutron specific heat and the neutrino emissivity of the nucleon involved processes would decrease with decrease of the temperature as a power of the temperature rather than exponentially, since the gap vanishes at the Fermi sphere poles. This was noticed in [108] and in [53], and then considered in a more detail in [109]. However, all presently existing neutron star cooling scenarios explored $3P_2$ nn pairing in $m_J = 0$ state, since mechanisms for the formation of the nn pairs in the $m_J \neq 0$ states were not yet explored, cf. [40,41,53,109]. Possibilities of the $3P_2$ pp, hyperon-hyperon and Δ isobar—Δ isobar pairings in interiors of sufficiently massive neutron stars should be additionally investigated. Triplet pairing in nonequilibrium systems should be studied. Spin-polarization effects owing to the possibility of a feasible $3\mathrm{S}_1$ np pairing in peripheral heavy-ion collisions were not yet considered. The presence of magnetic fields of the order of $(10^{17}$–$10^{19})$ Gs, cf. [60,61], and of high angular momenta in peripheral heavy-ion collisions may act in favor of the spin-triplet pairing. Novel spin-triplet subphases can be formed during very low energy collisions of normal and superfluid nuclei and in the rotating nuclei. Energetically, favorable transitions from one phase to another one may result in an increase of the duration of the process of the collision of nuclei. In neutron star interiors, the magnetic field may reach values $\sim 10^{18}$ Gs. At such conditions, the charged ρ meson condensates may appear, may be forming a ferromagnetic superfluid. A further more detailed quantitative study is welcome to answer these and other intriguing questions.

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