Towards the understanding of $Z_c(3900)$ from lattice QCD

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Within the framework of three-channel Ross-Shaw effective range theory, we derive the constraints among different parameters of the theory in the case of a narrow resonance close to the threshold of the third channel, which is relevant for the resonancelike structure $Z_c(3900)$. The usage of these constraint relations, together with the multichannel Lüscher formula in lattice QCD calculations are also discussed and the strategies are outlined.

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I. INTRODUCTION

In the past decade, exotic hadronic resonancelike structures, known as XYZ particles, have been discovered by various experiments, with $Z_c(3900)$ being a typical example [1–3]. The exotic structures have been discovered in both charm and bottom sectors which necessarily bear a four valence quark structure $\bar{Q}q\bar{q}'Q$ with Q being a heavy-flavor quark while q and q' are two different light flavored ones. They also tend to appear close to the threshold of two heavy mesons with valence structures $\bar{Q}q$ and $\bar{q}'Q$. The physical nature of these structures have been contemplated and discussed in many phenomenological studies. For recent reviews on these matters, see, e.g., Refs. [4–7]. Despite many studies, the nature of $Z_c(3900)$ remains unclear. It is therefore highly desirable that nonperturbative studies like lattice QCD could provide some useful information.

Contrary to the phenomenological studies, lattice studies on these states remain relatively scarce. A lattice study was performed by S. Prelovsek *et al.* who investigated the energy levels of the two charmed meson system in the channel where Z_c appeared in a finite volume [8]. They used quite a number of operators, including two-meson operators in the channel of $J/\psi\pi$, $D\bar{D}^*$, etc. and even tetraquark operators. However, there was no indication of extra new energy levels apart from the almost free scattering states of the two mesons. Taking $D\bar{D}^*$ as the main relevant channel, which is also supported by experimental facts, CLQCD utilized single-channel Lüscher scattering formalism [9–13] to tackle the problem within a single-channel approximation. For a comprehensive review on the Lüscher formalism, interested readers could consult Ref. [14]. CLQCD found slightly repulsive interaction between the two charmed mesons [15,16], making them unlikely to form bound states. A similar study using staggered quarks also finds no clue for the existence of the state [17].

On the other hand, HALQCD studied the problem using the so-called HALQCD approach [18] which is different from Lüscher's adopted by the other groups mentioned above. An effective potential is first extracted from lattice data which is then substituted into the Schrödinger-like equation to solve for the scattering. They claimed that $Z_c(3900)$ can be reproduced and it is a structure formed due to strong cross-channel interactions among three channels, $J/\psi\pi$, $\eta_c\rho$, and $D\bar{D}^*$; see Ref. [19,20] and references therein. This scenario will be referred to as the HALQCD scenario in the following.

Recently, in order to clarify this mismatch of the two types of approaches, CLQCD performed a two-channel lattice study using the two-channel Ross-Shaw effective range expansion [21]. They took the two channels $J/\psi\pi$ and $D\bar{D}^*$ that are most strongly coupled to $Z_c(3900)$. It is found that, in this two-channel approach, the parameters of the Ross-Shaw matrix do not seem to support the HALQCD scenario. The parameters turn out to be large and the Ross-Shaw M matrix is far from singular, which is required for a resonance close to the threshold. However, since only two channels are studied, it is still not a direct comparison with the HALQCD approach in which three channels have been studied. In this paper, we move one step further to close this

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gap. We take exactly the same three channels as HALQCD did, namely, $J/\psi\pi$, $\eta_c\rho$ and $D\bar{D}^*$. BESIII experiments have investigated several channels and these three channels are found to couple to $Z_c(3900)$ strongly, all of which are s-wave two-particle channels, with $D\bar{D}^*$, $\rho\eta_c$ and $J\psi\pi$ in decreasing order of the effective coupling and threshold. In fact, they have not seen an indication of $Z_c(3900)$ in other channels, e.g., higher partial wave two-particle channels [22,23]. Therefore, in this paper, we utilize the Ross-Shaw effective range theory [24,25] for the above mentioned three channels and derive the constraint relations among the parameters of Ross-Shaw matrix M in the zero-range approximation, assuming that there is a resonance close to the threshold of the third channel, i.e., that of $D\bar{D}^*$. Similar constraint relations in the two-channel case have been discussed in detail by Ross and Shaw long time ago; see, e.g., Ref. [25]. However, to our knowledge, the corresponding constraint relations in the three-channel case, which will be established in this paper, are still lacking. These constraint relations can be further utilized in future lattice simulations, the strategy of which will also be outlined in this paper.

The difficulty with the multichannel Lüscher approach, which we briefly outline below, is twofold:

(i) First, with the number of channels n increasing, the number of unknown functions entering the S matrix also increases rapidly. For example, in the case of two channels, there are three functions, while in the case of three channels, six functions are needed to describe the full S matrix. On the other hand, the Lüscher formula only offers a single relation among these functions at a particular energy level, which is extracted from lattice simulation. Therefore, we aim to investigate the already verified two-particle channels suggested by the experiments and take the Ross-Shaw effective range expansion as our parametrization for the scattering phases, which is a special case of the K-matrix parametrization.

(ii) Second, the number of constant parameters needed to parametrize the S matrix grows quadratically fast when the number of channels n is increased. Based on the experimental facts and also to make a direct comparison with the HALQCD study, we focus on the three-channel Lüscher approach in this paper. To be more specific, we will single out the following three channels for $Z_c(3900)$: $J/\psi\pi$, $\eta_c\rho$, and $D\bar{D}^*$, the first being the discovery channel for $Z_c(3900)$ and the second and the third have been shown to be dominant channels that couple to $Z_c(3900)$ by BESIII experiments [26]. Similar to the single-channel effective range expansion which, to the second order, is characterized by two real parameters, namely, the scattering length a_0 and the effective range r_0, in a three-channel situation, one needs nine real parameters to describe the so-called Ross-Shaw matrix M: six for the scattering length matrix, and three for the effective range parameters.

This paper is organized as follows. In Sec. II, we briefly review the Ross-Shaw effective range expansion that is needed to parametrize S-matrix elements. In Sec. III, within the zero-range approximation of Ross-Shaw theory, we derive the constraint conditions that need to be satisfied in order to have a narrow resonance behavior close to the third threshold. These conditions are derived first in the limit where the coupling of the first two channels are switched off and then generalized to the case where it is turned on. In Sec. IV, we briefly outline the strategies of the lattice computations and discuss how the constraints derived in Sec. III can be tested. In Sec. V, we will conclude with some general remarks.

II. THE ROSS-SHAW EFFECTIVE RANGE THEORY

In this section, we briefly recapitulate the Ross-Shaw effective range theory which is a generalization of the usual effective range expansion to multichannels. As already mentioned in the previous section, in order to utilize the multichannel Lüscher formula, it is crucial to have a parametrization of the S-matrix elements in terms of constants instead of functions of the energy and the multichannel effective range expansion developed by Ross and Shaw [24,25] serves this purpose.

In the single-channel case, this theory is just the well-known effective range expansion for low-energy elastic scattering,

$$k \cot \delta(k) = \frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \cdots, \tag{1}$$

where $\cdots$ designates higher order terms in k^2 that vanish in the limit of $k^2 \to 0$. Therefore, in low-energy elastic scattering, the scattering length a_0 and the effective range r_0 completely characterize the scattering process. Ross-Shaw theory simply generalizes the above theory to the case of multichannels. For that purpose, they define a matrix M via

$$M = k^{1/2} \cdot K^{-1} \cdot k^{1/2}, \tag{2}$$

where k and K are both matrices in channel space. Here, and in what follows, we will be working in the case of s-wave scattering. In principle, Ross-Shaw effective range expansion also works for higher partial waves. For example, one simply replaces the kinematic factors $k^{1/2}$ by $k^{l+1/2}$ for a partial wave with orbital angular momentum l. However, since no higher partial waves are observed in the experiments [22,23,26], we therefore will not work out the explicit formula in the general case. The matrix k is the kinematic matrix which is a diagonal matrix given by

$$k = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \tag{3}$$

and k_1, k_2, and k_3 are related to the scattering energy E. The matrix K is called the K matrix in scattering theory whose relation with the S matrix is given by[1]

$$S = \frac{1 + iK}{1 - iK}. \tag{4}$$

Another useful formal expression for the matrix K is

$$K = \tan\delta, \tag{5}$$

where both sides are interpreted as matrices in channel space. From the above expressions, it is easily seen that the K^{-1} that appears in Eq. (2) is simply the matrix $\cot\delta$ and without cross-channel coupling, the M matrix is also diagonal with entries $M \sim \mathrm{Diag}(k_1\cot\delta_1, k_2\cot\delta_2, k_3\cot\delta_3)$. Thus, it is indeed a generalization of the single-channel case in Eq. (1). In their original paper, Ross and Shaw showed that the M matrix as function of energy E can be Taylor expanded around some reference energy E_0 as

$$M_{ij}(E) = M_{ij}(E_0) + \frac{1}{2}R_i\delta_{ij}[k_i^2(E) - k_i^2(E_0)], \tag{6}$$

where we have explicitly written out the channel indices i and j. The matrix $M_{ij}(E_0) \equiv M_{ij}^{(0)}$ is a real symmetric matrix in channel space that we will call the inverse scattering length matrix; $R \equiv \mathrm{Diag}(R_1, R_2, R_3)$ is a diagonal matrix which we shall call the effective range matrix. k_i^2 are the entries for the kinematic matrix defined in Eq. (3). Therefore, for three channels, there are altogether nine parameters to describe the scattering close to some energy E_0: six in the inverse scattering length matrix $M^{(0)}$ and three in the effective range matrix R. One could further reduce the number of parameters to six by neglecting terms associated with effective ranges. This is called the zero-range approximation [24]. For convenience, we usually take E_0 to be the threshold of the third channel, i.e., that of $D\bar{D}^*$.

The Ross-Shaw parametrization in Eq. (6) is a special form of the more general K-matrix parametrization. In this K-matrix representation, assuming there are altogether n open channels, the $n \times n$ K-matrix element $K(E)_{ij}$ is parametrized as

$$K(E)_{ij} = \sum_{\alpha=1}^{m} \frac{\gamma_i^\alpha \gamma_j^\alpha}{E - E_\alpha} + \sum_k c_{ij}^{(k)} E^k. \tag{7}$$

[1]The K matrix is symmetric so that the S matrix is unitary.

Here the label $\alpha = 1, 2, \ldots, m$ designates different poles that enter the modeling and γ_i^α with $i = 1, 2, \ldots, n$ are the couplings of each individual channel to these poles. Possible polynomials in the energy are also added with symmetric coefficients: $c_{ij}^{(k)} = c_{ji}^{(k)}$. In this paper, we only focus on the Ross-Shaw parametrization, i.e., a multichannel generalization of the effective range expansion with zeroth or first order for three channels.

III. RESONANCE SCENARIO IN ROSS-SHAW THEORY

In this section, we investigate the possibility of a narrow peak just close to the threshold of the third channel. In particular, this will be studied within the framework of the three-channel Ross-Shaw theory. It turns out that this requirement will implement some constraints among the different parameters in Ross-Shaw theory.

It is convenient to inspect the resonance scenario using the so-called T matrix which is continuous across the threshold. Formally, it is related to the K matrix via

$$K^{-1} = T^{-1} + i, \tag{8}$$

or equivalently as $T = K(1 - iK)^{-1}$. The relation between the S matrix and the T matrix is given by

$$S = 1 + 2iT, \tag{9}$$

where both S and T now are 3×3 matrices in channel space. Since the scattering cross section σ_{ij} is essentially proportional to $|T_{ij}|^2$, the so-called elastic cross section in a particular channel i is given by

$$\sigma_{ii} = \frac{4\pi}{k_i^2}|T_{ii}|^2. \tag{10}$$

Therefore, if we denote

$$w_{ii} \equiv \frac{T_{ii}}{k_i} = \frac{1}{\alpha_i(E) - i\beta_i(E)}, \tag{11}$$

with α_i and β_i being real functions of the energy, then the elastic cross section in channel i reads

$$\sigma_{ii} = 4\pi|w_{ii}|^2 == \frac{4\pi}{\alpha_i^2 + \beta_i^2}. \tag{12}$$

Normally, the imaginary part of w_{ii}, namely, $\beta_i(E)$, is a positive, smooth function of the energy in the energy region to be studied. In fact, if there were no coupling among different channels, we have $\beta_i = k_i$. The real part (i.e., α_i), however, could develop a zero in the corresponding energy range, which then leads to a resonance peak structure. To be more specific, a resonance peak happens when $\alpha_i(E) = 0$

and the half-width positions for this peak can be obtained by the condition $\alpha_i(E)/\beta_i(E) = \pm 1$, respectively.

To be more specific, the T matrix in channel space looks like

$$T = k^{1/2}(M - ik)^{-1}k^{1/2}. \tag{13}$$

Therefore, if we define the matrix w in channel space as

$$w = (M - ik)^{-1}, \tag{14}$$

the elements of which will be denoted by w_{ij}, then the following expression for T_{11} can be obtained:

$$w_{11} \equiv \frac{T_{11}}{k_1} = \frac{1}{D}\begin{vmatrix} M_{22} - ik_2 & M_{23} \\ M_{23} & M_{33} - ik_3 \end{vmatrix}, \tag{15}$$

where D is the determinant of the 3×3 matrix,

$$D = \begin{vmatrix} M_{11} - ik_1 & M_{12} & M_{13} \\ M_{12} & M_{22} - ik_2 & M_{23} \\ M_{13} & M_{23} & M_{33} - ik_3 \end{vmatrix}. \tag{16}$$

Similar expressions are obtained for w_{22} and w_{33}. We get the following expression for w_{ii} with $i = 1, 2, 3$:

$$\begin{aligned}
w_{11}^{-1} &= \alpha_1 - i\beta_1 = M_{11} - ik_1 - M_{12}\frac{\begin{vmatrix} M_{12} & M_{23} \\ M_{13} & M_{33} - ik_3 \end{vmatrix}}{\begin{vmatrix} M_{22} - ik_2 & M_{23} \\ M_{23} & M_{33} - ik_3 \end{vmatrix}} + M_{13}\frac{\begin{vmatrix} M_{12} & M_{22} - ik_2 \\ M_{13} & M_{23} \end{vmatrix}}{\begin{vmatrix} M_{22} - ik_2 & M_{23} \\ M_{23} & M_{33} - ik_3 \end{vmatrix}}, \\
w_{22}^{-1} &= \alpha_2 - i\beta_2 = M_{22} - ik_2 - M_{12}\frac{\begin{vmatrix} M_{12} & M_{13} \\ M_{23} & M_{33} - ik_3 \end{vmatrix}}{\begin{vmatrix} M_{11} - ik_1 & M_{13} \\ M_{13} & M_{33} - ik_3 \end{vmatrix}} - M_{23}\frac{\begin{vmatrix} M_{11} - ik_1 & M_{12} \\ M_{13} & M_{23} \end{vmatrix}}{\begin{vmatrix} M_{11} - ik_1 & M_{13} \\ M_{13} & M_{33} - ik_3 \end{vmatrix}}, \\
w_{33}^{-1} &= \alpha_3 - i\beta_3 = M_{33} - ik_3 + M_{13}\frac{\begin{vmatrix} M_{12} & M_{22} - ik_2 \\ M_{13} & M_{23} \end{vmatrix}}{\begin{vmatrix} M_{11} - ik_1 & M_{12} \\ M_{12} & M_{22} - ik_2 \end{vmatrix}} - M_{23}\frac{\begin{vmatrix} M_{11} - ik_1 & M_{12} \\ M_{13} & M_{23} \end{vmatrix}}{\begin{vmatrix} M_{11} - ik_1 & M_{12} \\ M_{12} & M_{22} - ik_2 \end{vmatrix}}.
\end{aligned} \tag{17}$$

In the above formulas, below a specific threshold, the corresponding momentum becomes purely imaginary. For example, below the threshold of the third channel, we have $-ik_3 = \kappa_3$ with κ_3 being a positive real number.

On the other hand, it is known from BESIII experiments [1,23] that, close to the threshold of the third channel, all three elastic channels show resonant peaks. If we assume that these three peaks correspond to a single resonance structure, constraint equations can be obtained from Eq. (17). In the following, using Eq. (17), we will derive these equations that needs to be satisfied among the parameters. The corresponding conditions in the two-channel case have been studied a long time ago by Ross and Shaw, e.g., Refs. [24,25]. However, to our knowledge, the case of three channels has not been studied explicitly, which will be done within this paper.

A. Resonance scenario in Ross-Shaw theory: $M_{12} = 0$ case

It is worthwhile to work in a somewhat simpler situation, namely, that the coupling between channel 1 and 2 is negligible. This turns off the coupling between channels 1 and 2 completely, so we have $M_{12} = 0$. Suppose that such a close to the third threshold resonance structure arises from a single pole structure of the T matrix in the complex plane, then we could demand that the position of the pole to the be same; i.e., they correspond to the same structure. As we will see, this then leads to a relation among different matrix elements of the M matrix.

In the limit where $M_{12} = 0$, the condition $w_{11}^{-1} = w_{22}^{-1} = w_{33}^{-1} = 0$ turns out to yield a single equation (not three, but only one) for the parameters,

$$M_{33} - ik_3 = \frac{M_{13}^2}{M_{11} - ik_1} + \frac{M_{23}^2}{M_{22} - ik_2}, \tag{18}$$

where k_1, k_2, and k_3 are all related to the energy via

$$\begin{aligned}
E &= \sqrt{m_{J/\psi}^2 + k_1^2} + \sqrt{m_\pi^2 + k_1^2}, \\
&= \sqrt{m_{\eta_c}^2 + k_2^2} + \sqrt{m_\rho^2 + k_2^2}, \\
&= \sqrt{m_D^2 + k_3^2} + \sqrt{m_{D^*}^2 + k_3^2}.
\end{aligned} \tag{19}$$

Here, $m_{J/\psi}$, m_π, etc. are the masses of the corresponding mesons and k_i's with $i = 1, 2, 3$ being the scattering momenta in various channels. Now, viewing the k_i^2's, $i=1$, 2, 3 as complex variables that are related to each other by Eq. (19), one can solve Eq. (18) in some Riemann sheet to yield the pole position for the complex k_i^2's. This pole then manifests itself as peaks in elastic cross sections in all three channels. Therefore, in the limit of $M_{12}=0$, the so-called HALQCD scenario is fully represented by Eq. (18) in the Ross-Shaw theory.

To search for such solutions, we utilize the following notations. We assume that $k_3 \equiv z$ is small in magnitude compared with the typical energy scale of the problem, say $k_1^{(0)}$, the magnitude of the momentum for the two particles in the first channel (i.e., $J/\psi\pi$) at the threshold of the third channel (i.e., $D\bar{D}^*$). Thus, we have

$$\delta E \equiv E - (m_{D^*} + m_D) = \frac{z^2}{2\mu_{DD^*}}, \tag{20}$$

with μ_{DD^*} being the reduced mass of D and D^*. Similarly, k_1 and k_2 will assume their values at the third threshold, namely, $k_1^{(0)}$ and $k_2^{(0)}$, plus small corrections that are linear in z^2.

$$\begin{aligned} \delta k_1 &= \frac{z^2}{2\mu_{DD^*}(v_{J/\psi}^{(0)} + v_\pi^{(0)})} = \gamma_1 z^2, \\ \delta k_2 &= \frac{z^2}{2\mu_{DD^*}(v_{\eta_c}^{(0)} + v_\rho^{(0)})} = \gamma_2 z^2, \end{aligned} \tag{21}$$

where $v_{J/\psi}^{(0)}$, $v_\pi^{(0)}$, $v_{\eta_c}^{(0)}$, and $v_\rho^{(0)}$ are the speed of the corresponding mesons at the threshold. To be specific, we have $v_{J/\psi}^{(0)} = k_1^{(0)}/E_{J/\psi}(k_1^{(0)})$, etc. Therefore, the solution z_0, where all w_{ii} diverge satisfy the following equation:

$$M_{33} - iz_0 = \frac{M_{13}^2}{M_{11} - ik_1^{(0)} - i\gamma_1 z_0^2} + \frac{M_{23}^2}{M_{22} - ik_2^{(0)} - i\gamma_2 z_0^2}. \tag{22}$$

This equation should be solved for small $|z_0|$ near the origin in the complex z plane. Here, smallness could be measured in some reasonable unit. A convenient choice is to use a unit system in which $k_1^{(0)} = 1$ adopted in Ref. [21]. In such a system, every quantity in Eq. (22) becomes dimensionless and we are searching for $|z_0| \ll 1$ in the complex plane.

Now, note that the lhs of Eq. (22) is linear in z_0 while the rhs depends on z_0^2; therefore, we could write the solution z_0 as

$$z_0 = z_0^{(1)} + z_0^{(2)} + \cdots, \tag{23}$$

where $z_0^{(i)}$ for different i designates different orders of z_0, all of which are small, but the higher the index i is, the smaller the $z_0^{(i)}$ becomes. Taylor expanding both sides of Eq. (22), order by order, we obtain the following equations:

$$iz_0^{(1)} = \varepsilon \equiv M_{33} - \frac{M_{13}^2}{M_{11} - ik_1^{(0)}} - \frac{M_{23}^2}{M_{22} - ik_2^{(0)}} \tag{24}$$

$$z_0^{(2)} = \left[\frac{M_{13}^2\gamma_1}{(M_{11} - ik_1^{(0)})^2} + \frac{M_{23}^2\gamma_2}{(M_{22} - ik_2^{(0)})^2}\right]\varepsilon^2 \tag{25}$$

$$\begin{aligned} z_0^{(3)} &= 2i\left[\frac{M_{13}^2\gamma_1}{(M_{11} - ik_1^{(0)})^2} + \frac{M_{23}^2\gamma_2}{(M_{22} - ik_2^{(0)})^2}\right]^2 \varepsilon^3 \\ z_0^{(4)} &= \cdots. \end{aligned} \tag{26}$$

It is seen that the leading order equation (24) demands that the quantity ε thus defined needs to be a complex number that is small in magnitude. Otherwise, there is no consistent small z solution for Eq. (22). This implies that both the real part and the imaginary part have to be small. If we denote

$$\varepsilon = \varepsilon_1 - i\varepsilon_2, \tag{27}$$

with both ε_1 and ε_2 being real, it is easy to work out the explicit expressions. It is also found that, the imaginary part parameter $\varepsilon_2 > 0$ at the threshold of the third channel. The sign of ε_1, however, is not definite, depending on other parameters. In order for them to be small, we have

$$\begin{aligned} &\left| M_{33} - \frac{M_{13}^2 M_{11}}{M_{11}^2 + (k_1^{(0)})^2} - \frac{M_{23}^2 M_{22}}{M_{22}^2 + (k_2^{(0)})^2} \right| \ll 1, \\ &\frac{M_{13}^2 k_1^{(0)}}{M_{11}^2 + (k_1^{(0)})^2} + \frac{M_{23}^2 k_2^{(0)}}{M_{22}^2 + (k_2^{(0)})^2} \ll 1. \end{aligned} \tag{28}$$

To leading order, the solution of the pole reads

$$z_0^{(1)} = -i\varepsilon = -\varepsilon_2 - i\varepsilon_1, \tag{29}$$

which points out the approximate location of the pole position in the complex plane. To be more precise, the location is given by

$$z_0 = -\varepsilon_2 - i\varepsilon_1 + z^{(2)} + z^{(3)} + \cdots, \tag{30}$$

where $z^{(2)}$ and $z^{(3)}$ are given by Eq. (25) and Eq. (26). More iterates can be obtained if necessary.

We can now work out the elastic scattering cross sections close to the threshold of the third channel. These are given by Eq. (17) by taking $M_{12}=0$. Taking, e.g., the first channel, we have

$$w_{11}^{-1} = M_{11} - ik_1 - \frac{M_{13}^2}{M_{33} - ik_3 - \frac{M_{23}^2}{M_{22} - ik_2}}, \tag{31}$$

where k_i takes real or pure imaginary values, depending on whether it is above or below the thresholds. Since the k_i's are related to the total energy via Eq. (19), we know that the rhs vanishes when the k_i's take complex values at $k_3 = z_0$:

$$M_{11} - ik_1(z_0) = \frac{M_{13}^2}{M_{33} - ik_3(z_0) - \frac{M_{23}^2}{M_{22} - ik_2(z_0)}}, \tag{32}$$

which is consistent with Eq. (22). Therefore, we introduce the function

$$w_{ii}^{-1} = F_i(z), \tag{33}$$

where in $F_i(z)$ the k_i's are viewed as complex functions of z, which we still take as the complex $k_3 = z$. We know that the function $F_i(z)$ has a zero at the location z_0 which is given by Eq. (30), and that z_0 is close to the origin. Therefore, we may expand

$$\begin{aligned} F_i(z) &= F_i(z_0) + F_i'(z_0)(z - z_0) + \cdots \\ &\approx F_i'(0)(z - z_0), \end{aligned} \tag{34}$$

where we have utilized the condition $F_i(z_0) = 0$ and $F_i'(z_0) \approx F_i'(0)$ since z_0 is rather close to the origin. Thus, the elastic cross section in channel i reads

$$\sigma_{ii} = \frac{4\pi}{|F_i(z)|^2} = \frac{4\pi}{|F_i'(0)|^2 |z - z_0|^2}, \tag{35}$$

which exhibits a typical resonance behavior. Here, it is understood that z takes real or pure imaginary values, depending on whether it is above or below the third threshold. To be more explicit, if we take only the first approximation for z_0, we have the following cross sections for above and below the third threshold,

$$\sigma_{ii} = \begin{cases} \frac{4\pi}{|F_i'(0)|^2[(k_3+\varepsilon_2)^2+\varepsilon_1^2]}, \\ \frac{4\pi}{|F_i'(0)|^2[(\kappa_3+\varepsilon_1)^2+\varepsilon_2^2]}, \end{cases} \tag{36}$$

where the first/second line is for above/below the threshold, with $k_3 = z = i\kappa_3$, $\kappa_3 > 0$ in the second case. Since we have $\varepsilon_2 > 0$, the peak above the third threshold must be in the tail region. If $\varepsilon_1 < 0$, then we could see a full peak just below the threshold. If $\varepsilon_1 > 0$, however, a cusp will show up exactly at the threshold.

B. Resonance scenario in Ross-Shaw theory: General case

Here we would like to go beyond the approximation of $M_{12} = 0$. We will show below that, the above results in fact hold in the most general case of three-channel scattering.

For this purpose, we investigate Eq. (13) and Eq. (14) and realize that, in order to have a resonant behavior, the matrix $w = (M - ik)^{-1}$ needs to be singular. This implies that the determinant D defined in Eq. (16) must vanish. Therefore, when viewed as a complex function of $k_3 = z$, we may define

$$D(z) = \begin{vmatrix} M_{11} - ik_1(z) & M_{12} & M_{13} \\ M_{12} & M_{22} - ik_2(z) & M_{23} \\ M_{13} & M_{23} & M_{33} - iz \end{vmatrix}, \tag{37}$$

the complex resonance pole z_0 should be solved for under the condition of $D(z_0) = 0$, in the neighborhood of the origin. In the above equation, functions $k_1(z)$ and $k_2(z)$ should be obtained by using the energy condition equation (19). Expanding both k_1 and k_2 around the origin we see that $k_{1,2}(z) = k_{1,2}^{(0)} + \gamma_{1,2} z^2$. Therefore, close to the origin, equation $D(z_0) = 0$ yields a quintic equation for z_0. Since $|z_0| \ll 1$, we may expand the determinant $D(z_0)$ into a Taylor expansion. To first order, we get

$$D(z_0) \approx D(0) - (iz_0)\Delta_{33}(0) + \cdots, \tag{38}$$

where $\cdots$ designates terms of higher orders in z_0 and with $\Delta_{33}(0)$ being the cofactor for the matrix element $(M_{33} - ik_3)$ in the 3×3 matrix $(M - ik)$, i.e.,

$$\Delta_{33}(0) = (M_{11} - ik_1^{(0)})(M_{22} - ik_2^{(0)}) - M_{12}^2. \tag{39}$$

Therefore, to this order, the solution is

$$iz_0^{(1)} = \varepsilon \equiv \frac{D(0)}{\Delta_{33}(0)} = M_{33} + M_{13} \frac{\begin{vmatrix} M_{12} & M_{22} - ik_2^{(0)} \\ M_{13} & M_{23} \end{vmatrix}}{\begin{vmatrix} M_{11} - ik_1^{(0)} & M_{12} \\ M_{12} & M_{22} - ik_2^{(0)} \end{vmatrix}} - M_{23} \frac{\begin{vmatrix} M_{11} - ik_1^{(0)} & M_{12} \\ M_{13} & M_{23} \end{vmatrix}}{\begin{vmatrix} M_{11} - ik_1^{(0)} & M_{12} \\ M_{12} & M_{22} - ik_2^{(0)} \end{vmatrix}}. \tag{40}$$

It is easy to verify that, in the limit of $M_{12} = 0$, this reproduces the previous result in Eq. (24). The discussions about the elastic scattering cross section remains unchanged. The only thing that needs to be modified is the explicit expression for the solution z_0 to various orders, which, to the first order, is now shown in Eq. (40) instead of

Eq. (24). Again, higher order expressions can be obtained easily if necessary.

IV. MULTICHANNEL LÜSCHER FORMULA AND THE STRATEGY FOR LATTICE COMPUTATIONS

In this section, we briefly outline the strategies for a lattice calculation within the multichannel Lüscher approach for three channels. As we have mentioned in Sec. I, in the case of three channels, one first needs a parametrization for the S matrix in terms of functions, and furthermore in terms of the Ross-Shaw parameters.

The most general form of the S matrix for three channels, assuming time reversal symmetry, was first given by Waldenstrøm in 1974 and it looks like the following [27]:

$$S = \begin{bmatrix} \eta_1 e^{2i\delta_1} & iX_{12}e^{i(\delta_{12})} & iX_{13}e^{i(\delta_{13})} \\ iX_{12}e^{i(\delta_{12})} & \eta_2 e^{2i\delta_2} & iX_{23}e^{i(\delta_{23})} \\ iX_{13}e^{i(\delta_{13})} & iX_{23}e^{i(\delta_{23})} & \eta_3 e^{2i\delta_3} \end{bmatrix}, \tag{41}$$

where δ_1, δ_2, and δ_3 are scattering phases in channel 1, 2, and 3, respectively, and $\eta_i \in [0, 1]$, $i = 1, 2, 3$ are called the inelasticity parameters for each channel, all of which are functions of the energy. The other parameters X_{ij} and δ_{12}, δ_{23}, and δ_{23} are related to the δ_i and η_i in a complicated manner; hence, the are also functions of the energy. Interested readers can consult Ref. [27] for details. These six functions of energy are then parametrized within the Ross-Shaw theory in terms of 9 real parameters: 6 for the scattering length matrix M, 3 for the effective ranges. Note that S matrix is related to the T matrix via $S = 1 + 2iT$ while the latter is further related to the Ross-Shaw M matrix via Eq. (13).

The multichannel Lüscher formula has many forms. The most convenient one is the one that is directly related to the Ross-Shaw M matrix,

$$\det\left[M - B^{(\mathbf{P})}\right] = 0, \tag{42}$$

where the matrix $B^{(\mathbf{P})}$, called the box function by Colin Morningstar *et al.* [28], is a complicated but computable function involving modified zeta functions that can be obtained from the energy eigenvalues in a finite box. The label $\mathbf{P}$ designates the total three-momentum of the two-particle system so that it applies also to moving frames. The corresponding constraint equations that are derived in the previous section needs to be boosted accordingly using an appropriate Lorentz transformation. The explicit expression for the box function reads

$$\begin{aligned} &\langle J' m_{J'} L' S' a' | B^{(\mathbf{P})} | J m_J L S a \rangle \\ &= -i\delta_{aa'}\delta_{SS'}(u_a)^{L+L'+1} W^{(\mathbf{P}a)}_{L' m_{L'}; L m_L}(k_i^2) \langle J' m_{J'} | L' m_{L'}, S m_S \rangle \\ &\quad \times \langle L m_L, S m_S | J m_J \rangle. \end{aligned} \tag{43}$$

Here, J, m_J, L, and S correspond to the total angular momentum quantum number, the third component of total angular momentum, the orbital angular momentum, and the spin quantum number of the two-particle state. The index a designates other quantum numbers, e.g., channel or isospin, etc. The function $W^{(\mathbf{P}a)}_{L' m_{L'}; L m_L}(k_i^2)$ involves zeta functions and the arguments k_i^2 with $i = 1, 2, 3$ represent the momenta in the corresponding channels which are related to the energy via Eq. (19). Therefore, Eq. (42) shows that the matrices M and B, which are defined explicitly in Ref. [28], share the same threshold behavior, which is also indicated in Ref. [29].

For a given set of parameters in the Ross-Shaw matrix M, the multichannel Lüscher formula (42) can be viewed as an equation for the energy eigenvalues that enter the equation via the box function $B^{(\mathbf{P})}$. Therefore, when solved numerically it yields a set of energy eigenvalues in the finite box. These energy levels can be compared with the real energy levels obtained from the lattice simulations. This comparison in turn yields an estimate for various M_{ij}'s in the Ross-Shaw matrix, as illustrated in Ref. [21]. On the other hand, as we have obtained the conditions that need to be satisfied by these parameters in order to have a resonance peak close to the threshold of the third channel, cf. Eq. (28), one can directly check if the lattice extracted parameters really support such a scenario or not, as was already done in the two-channel case in Ref. [21].

It is interesting to note that, in the general Ross-Shaw theory, Eq. (6) can be utilized to any energy region. In particular, if we investigate only the region close to the third threshold, it is good enough to use the zero-range expansion. This sets all the effective ranges to zero, leaving us with only six parameters. In other words, if we focus on the energy region very close to the threshold, the zero-range approximation is always valid. Of course, by utilizing the multichannel Lüscher formula, other energy levels that are somewhat distant from the threshold enter the game (via fitting of M_{ij}'s); therefore, there could be some deviations from the zero-range approximation. Still, extraction of the M_{ij}'s and a check of whether they satisfy the constraints as outlined in Eq. (28) offer a crucial test. This comparison will hopefully clarify, or at least shed some light on, the differences from the two different approaches so far: the HALQCD approach and the conventional Lüscher approach. In fact, one could try to arrange a situation where as many as possible energy levels are close to the third threshold. In such a case, one could utilize the zero-range approximation without any problem as long as one drops the energy levels that are too distant from the threshold.

V. CONCLUSIONS

To shed more light on the nature of the resonancelike structure $Z_c(3900)$, lattice studies have been performed over the years. However, some puzzles still remain. The existing lattice studies fall into two categories: the ones using Lüscher's approach and the ones using the HALQCD approach. The results from these two types of approaches are not consistent with each other as they should be. This discrepancy needs to be clarified.

In this paper, we study the problem using the three-channel Ross-Shaw theory, which is the generalization of the effective range expansion. We have obtained the constraint conditions that need to be satisfied by various parameters of the theory in order to have a narrow resonance close to the threshold of the third channel, a scenario that $Z_c(3900)$ realizes. We have pointed out that, combined with the multichannel Lüscher formula, a real lattice computation could be performed which will yield the results for these parameters and furthermore, one could check if these constraint relations are supported by the lattice results or not. We have also outlined the strategies of such lattice simulations on how to extract these parameters in a more reliable fashion. Currently, we are working on the simulation details along the lines that are described here and we hope to report the results soon.

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