

Symmetry, symmetry breaking, and pion parton distributions

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Pion valence, glue, and sea distributions are calculated using a continuum approach to the two valence-body bound-state problem. Since the framework is symmetry preserving, physical features of the distributions are properly expressed. The analysis reveals that the emergent phenomenon of dynamical chiral symmetry breaking causes a hardening of the valence-quark distribution function, $q^\pi(x)$. Nevertheless, this distribution exhibits the $x \simeq 1$ behavior predicted by quantum chromodynamics (QCD). At the scale $\zeta_2 := 2$ GeV, the following momentum fractions are predicted: $\langle x_{\text{valence}} \rangle = 0.48(3)$, $\langle x_{\text{glue}} \rangle = 0.41(2)$, $\langle x_{\text{sea}} \rangle = 0.11(2)$. Evolving to $\zeta = 5.2$ GeV, the result for $q^\pi(x)$ agrees with that computed using lattice QCD. These outcomes should both spur improved analyses of existing experiments and stimulate efforts to obtain new data on the pion distribution functions using available and envisioned facilities.

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I. INTRODUCTION

Simply regarding their valence-quark content, pions are Nature's simplest hadrons:

$$\pi^+ \sim u\bar{d}, \quad \pi^- \sim d\bar{u}, \quad \pi^0 \sim u\bar{u} - d\bar{d}; \quad (1)$$

but this appearance is misleading. Despite being hadrons, their physical masses are similar to that of the μ -lepton; and the pion masses vanish in the chiral limit, i.e., in the absence of a Higgs coupling for u - and d -quarks: they are the Nambu-Goldstone (NG) modes generated by dynamical chiral symmetry breaking (DCSB) in the Standard Model. This dichotomous character—simultaneous existence as NG-bosons and bound-states—entails that the challenges of charting and explaining pion structure are of central importance in modern physics. These problems are made more difficult by the crucial role of symmetries and their breaking patterns in determining pion properties, which must be properly incorporated and veraciously expressed in any theoretical treatment.

Given the pions' simple valence-quark content, a basic quantity in any discussion of their structure is the associated distribution function, $q^\pi(x; \zeta)$. This density charts the probability that a valence q -quark in the pion carries a light-front fraction x of the system's total momentum; and one of the earliest predictions of the parton model,

augmented by features of perturbative quantum chromodynamics (pQCD), is [1–3]:

$$q^\pi(x; \zeta = \zeta_H) \sim (1-x)^2, \quad (2)$$

where ζ_H is an energy scale characteristic of nonperturbative dynamics. Moreover, the exponent evolves as ζ increases beyond ζ_H , becoming $2 + \gamma$, where $\gamma \gtrsim 0$ is an anomalous dimension that increases logarithmically with ζ . (In the limit of exact \mathcal{G} -parity symmetry, which is a good approximation in the Standard Model, $u^{\pi^+}(x) = \bar{d}^{\pi^+}(x)$, etc. Hence it is only necessary to discuss one unique distribution.)

Owing to the validity of factorization in QCD, $q^\pi(x)$ is measurable in πN Drell-Yan experiments [4–10]. However, conclusions drawn from analyses of these experiments have proved controversial [11]. For instance, using a leading-order (LO) pQCD analysis of their data, Ref. [9] (the E615 experiment) reported ($\zeta_5 = 5.2$ GeV):

$$q_{\text{E615}}^\pi(x; \zeta_5) \sim (1-x)^1, \quad (3)$$

a marked contradiction of Eq. (2). Subsequent calculations [12] confirmed Eq. (2) and eventually prompted reconsideration of the E615 analysis, with the result that, at next-to-leading order (NLO) and including soft-gluon resummation [13,14], the E615 data can be viewed as being consistent with Eq. (2).

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Notwithstanding these advances, uncertainty over Eq. (2) will remain until other analyses of the E615 data incorporate threshold resummation effects and, crucially, new data are obtained. Prospects for the latter are good because relevant tagged deep-inelastic scattering experiments are approved at the Thomas Jefferson National Accelerator Facility [15–17] and the goal has high priority at other existing and anticipated facilities [18–22].

Meanwhile, progress in theory continues. Novel perspectives and algorithms within lattice-regularized QCD (IQCD) [23–27] are beginning to yield results for the pointwise behavior of the pion’s valence-quark distribution [28–32], offering promise for information beyond the lowest few moments [33–36].

Extensions of the continuum analysis in Ref. [12] are also yielding new insights. For example: a class of corrections to the handbag-diagram representation of the virtual-photon–pion forward Compton scattering amplitude has been identified and shown to restore basic symmetries in calculations of $q^\pi(x; \zeta)$ [37]; and the corrected expression has been used to compute all valence-quark distribution functions in the pion and kaon [38], with the results indicating that the gluon content of the pion is significantly greater than that of the kaon.

This last feature owes to the mechanism responsible for the emergence of mass in the Standard Model. Studies of meson properties [39–42] indicate that the s -quark defines a boundary: emergent mass generation dominates for $\hat{m} < \hat{m}_s$, but the Higgs-mass prevails on $\hat{m} \gtrsim \hat{m}_s$, where \hat{m} is the renormalization group invariant current-mass for a given quark. Hence, comparisons between the properties of systems containing only light quarks and those with one (or more) s -quark(s) are well suited to exposing effects of dynamical mass generation.

Given its potential for validating such observations, there is renewed interest in measuring $u^K(x)/u^\pi(x)$ [15,16,19–22]. Only one data set currently exists [4]. It is old (from 1980) and limited; hence, needs modernizing and expanding in order to be effective in this new role.

The theory predictions also need updating, e.g., the continuum results in Refs. [37,38] were obtained using algebraic models for the elements needed to describe the Compton amplitude, i.e., dressed-quark propagators, meson Bethe-Salpeter amplitudes, and dressed-photon-quark vertex. Therefore, following the recent theory developments, especially concerning the pion, herein we expand upon the calculation of pion parton distributions reported in Ref. [43], providing extensive explanations and including additional material that should, *inter alia*, prove valuable in illuminating the formulation, analyses and results. Notably, this approach to the two-body bound-state problem has successfully unified the treatment of the charged-pion-elastic and neutral-pion-transition form factors [40,42,44–46]. It has also been used to correlate continuum and lattice predictions for the electromagnetic form factors of charged pionlike

mesons, thereby enabling an extrapolation of the IQCD results to the physical pion mass [41].

Our manuscript is arranged as follows. Section II describes the connection between pion Compton scattering and $q^\pi(x)$; and Sec. III recapitulates the analysis of Ref. [37], explaining the flaws of the handbag diagram as a tool for calculating valence-quark distributions and illuminating the corrections that repair its deficiencies and thus produce a symmetry-preserving approximation. Section IV reports our calculation of $q^\pi(x)$ at ζ_H : detailing the kernel used to solve the continuum bound-state problem; computation of the lowest six independent Mellin-moments; and reconstruction of $q^\pi(x)$ therefrom. It also explains that, through a connection between the saturation value of QCD’s process-independent effective charge and the one-loop running coupling, the hadronic scale is determined: $\zeta_H = 0.30$ GeV. Evolution of $q^\pi(x; \zeta_H)$ to $\zeta/\text{GeV} = 2, 5.2$ for comparison with data, phenomenology, and theory is described in Sec. V; and predictions for the glue and sea momentum-distributions are obtained analogously using the singlet evolution equations. Section VI provides a summary and offers perspectives.

II. QUARK DISTRIBUTION FUNCTION

The hadronic tensor relevant to inclusive deep inelastic lepton-pion scattering may be expressed via two invariant structure functions [47]. With the incoming photon possessing momentum q and the target pion, momentum P , then in the deep-inelastic (Bjorken) limit [48], *viz.*

$$q^2 \rightarrow \infty, \quad P \cdot q \rightarrow -\infty, \quad \text{but } x := -q^2/[2P \cdot q] \text{ fixed}, \quad (4)$$

that tensor is ($t_{\mu\nu} = \delta_{\mu\nu} - q_\mu q_\nu / q^2$, $P_\mu^t = t_{\mu\nu} P_\nu$):

$$W_{\mu\nu}(q; P) = F_1(x) t_{\mu\nu} - \frac{F_2(x)}{P \cdot q} P_\mu^t P_\nu^t, \quad (5a)$$

$$F_2(x) = 2xF_1(x). \quad (5b)$$

$F_1(x)$ is the pion structure function, which provides access to the pion’s quark distribution functions:

$$F_1(x) = \sum_{q \in \pi} e_q^2 q^\pi(x), \quad (6)$$

where e_q is the quark’s electric charge. The sum in Eq. (6) runs over all quark flavors; but in the π^+ it is naturally dominated by $u(x)$, $\bar{d}(x)$. Moreover, in the \mathcal{G} -parity symmetric limit, which we employ throughout, $u(x) = \bar{d}(x)$. (Bjorken- x is equivalent to the light-front momentum fraction of the struck parton.) Using the optical theorem, the structure function is given by the imaginary part of the virtual-photon–pion forward Compton scattering amplitude: $\gamma^*(q) + \pi(P) \rightarrow \gamma^*(q) + \pi(P)$.

III. SYMMETRIES AND THE PION VALENCE-QUARK DISTRIBUTION FUNCTION

A. Charged-pion form factor

In order to elucidate the role of symmetries in developing an approximation to the $\gamma^*\pi$ forward Compton amplitude, which is the basis for any computation of $q^\pi(x)$, we first consider the simpler problem of the pion electromagnetic form factor, $F_\pi(Q^2)$. In both cases, one aims to expose structural features of a system characterized by two valence-parton degrees-of-freedom. A useful framework for studying such problems in quantum field theory is provided by the Dyson-Schwinger equations (DSEs) [49], with the one-body gap equation and two-body Bethe-Salpeter equation (BSE) playing leading roles.

The DSEs are a collection of coupled equations and a tractable problem is obtained once a truncation scheme is specified. A weak-coupling expansion reproduces perturbation theory; but although valuable in the analysis of large momentum transfer phenomena in QCD, it cannot be used to obtain nonperturbative information. A symmetry-preserving scheme applicable to hadrons was introduced in Refs. [50,51] and has subsequently been exploited [52–55] and refined [56–62]. The basic point is that the Bethe-Salpeter kernel appropriate to a given two-valence-body problem is computable once that of the one-body gap equation is specified. Following these procedures, one guarantees, *inter alia*, that all Ward-Green-Takahashi (WGT) identities [63–66] are preserved, without fine-tuning, and thereby ensures, e.g., current-conservation and the appearance of NG modes in connection with DCSB. These qualities are essential in connection with studies of electromagnetic interactions involving pions (and other pseudoscalar mesons). (*N.B.* Calculations, such as ours, are typically performed using the Poincaré-covariant Landau gauge because, *inter alia*, this ensures that preservation of symmetries is readily tracked and elucidated via the WGT identities.)

The leading-order term in the procedure of Refs. [50,51] is the rainbow-ladder (RL) truncation. Widely used, it is accurate for an array of systems and properties; in particular, those of ground-state flavor-nonsinglet pseudoscalar mesons because corrections in these channels largely cancel owing to the parameter-free preservation of relevant WGT identities ensured by this scheme.

Even before it was recognized as part of a systematic procedure, RL truncation was used as the basis for a calculation of F_π [67]. As argued therein, to obtain the form factor at this level in the symmetry-preserving truncation, one computes the matrix element depicted in Fig. 1:

$$K_\mu F_\pi(Q^2) = N_c \text{tr} \int_{dk} i\chi_\mu(k+q, k) \times i\Gamma_\pi(k_i; P) S(k-P) i\Gamma_\pi(k_f; -P'), \quad (7)$$

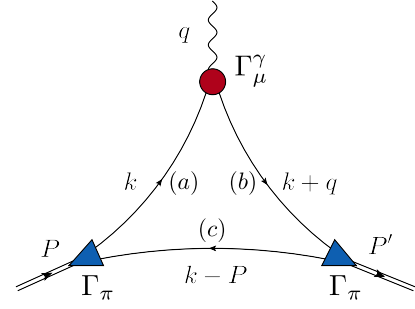


FIG. 1. RL truncation for the charged-pion electromagnetic form factor, Eq. (7): triangles (blue)—pion Bethe-Salpeter amplitudes; circle (red)—amputated dressed-photon-quark vertex; and interior lines—dressed-quark propagators. Poincaré-covariance and electromagnetic current conservation, *inter alia*, are guaranteed so long as each of these elements is computed in RL truncation. For later use, we define line (a) to be that carrying momentum k ; line (b), $k+q$; and line (c), $k-P$.

where $q = P' - P$ is the incoming photon momentum ($Q^2 = q^2$), $2K = P' + P$; $P^2 = -m_\pi^2 = (P')^2$; $k_i = k - P/2$, $k_f = k - P/2 + q/2$; $N_c = 3$; the trace is over spinor indices; and $\int_{dk} := \int \frac{d^4k}{(2\pi)^4}$ is a translationally invariant regularization of the integral.

In Eq. (7), S is the $u = d$ dressed-quark propagator, computed in rainbow truncation; Γ_π is the pion Bethe-Salpeter amplitude, computed with a rainbow-ladder kernel; and $\chi_\mu(k+q, k) = S(k+q)\Gamma_\mu(k+q, k)S(k)$, with Γ_μ the amputated dressed-photon-quark vertex, computed using the same kernel.

In RL truncation, there are no corrections to Eq. (7). To see this, suppose one were to add a gluon emitted from line (a) and reabsorbed by line (a). This would be overcounting because the contribution is already included in the rainbow truncation computation of the dressed-quark propagator. Suppose next that a gluon is emitted by line (a) and absorbed by line (c). That would also be overcounting because such a contribution is already contained in the RL-truncation result for Γ_π . Indeed, no matter which line or lines one chooses to emit and reabsorb a single gluon, the contribution generated is already included in S , Γ_π or Γ_μ . Consequently, Fig. 1 depicts the complete RL result for $F_\pi(Q^2)$. It is the basis for a calculation of this form factor on the entire domain of spacelike Q^2 [40,41,44], which agrees with existing data [68–71] and predicts that QCD scaling violations will be seen in experiments that reach $Q^2 \gtrsim 8 \text{ GeV}^2$.

B. Pion valence-quark distribution function

Viewed simply, RL truncation represents F_π as a three-point function: there are three compound vertices in Fig. 1. Counting in the same way, the photon-pion Compton amplitude is a four-point function; and anyone desiring to supply predictions for $q^\pi(x)$ that are consistent with

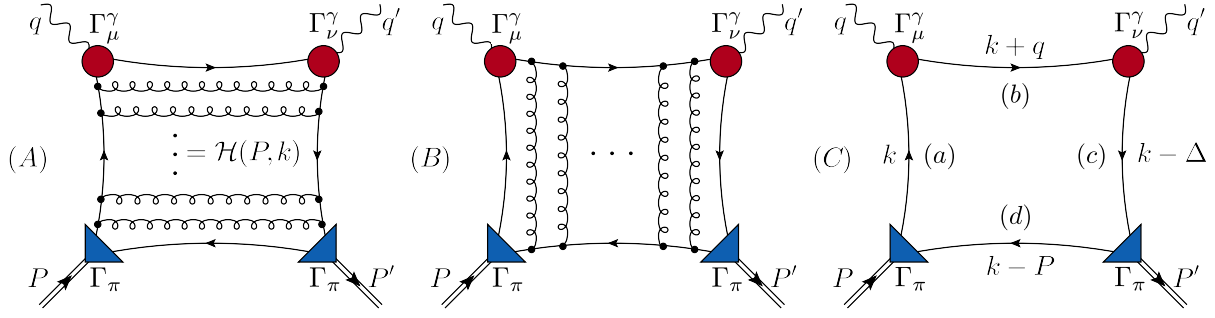


FIG. 2. Collection of diagrams required to complete a symmetry-preserving RL calculation of pion Compton scattering. Amplitude-One ($S1$) = (A) + (B) - (C). The “dots” in (A) and (B) indicate summation of infinitely many ladder-like rungs, beginning with zero rungs. The other two amplitudes are obtained as follows: ($S2$)—switch vertices to which q and q' are attached; and ($S3$)—switch vertex insertions associated with q' and P' . In all panels: triangles (blue)—pion Bethe-Salpeter amplitudes; circles (red)—amputated dressed-photon-quark vertices; and interior lines—dressed-quark propagators. $\Delta = q' - q$. Poincaré-covariance and electromagnetic current conservation, *inter alia*, are guaranteed so long as each of these elements is computed in RL truncation. For later use, using (C), we define line (a) to be that carrying momentum k ; line (b), $k + q$; line (c), $k - \Delta$; and line (d), $k - P$.

those for F_π is immediately presented with the challenge of writing the complete RL truncation for this four-point function. The solution to that problem was presented almost twenty years ago, in connection with the treatment of $\pi\pi \rightarrow \pi\pi$ [72,73]. Translated to the pion Compton amplitude [37], the complete (symmetry-preserving) RL truncation is given by permutations of the three diagrams illustrated in Fig. 2. (An extension to nucleon Compton scattering is described elsewhere [74].)

That the collection of diagrams in Fig. 2 is necessary and sufficient to generate the complete, symmetry-preserving RL treatment of $\gamma\pi \rightarrow \gamma\pi$ is readily made apparent. For example, suppose one were to add a gluon emitted from line (a) and reabsorbed by line (a). This would be over-counting because the contribution is already included in the rainbow truncation dressed-quark propagator. Suppose next that an additional gluon is emitted by line (a) and absorbed by line (d). That would also be over-counting because this diagram is already contained in the RL result for Γ_π . Now imagine that a new gluon is emitted by line (a) and absorbed by line (b). That is over-counting because such contributions are contained in the RL dressed-photon-quark vertex. One must also consider a gluon emitted by line (a) and absorbed by line (c). This is one of the summed diagrams represented by Fig. 2(A); and Fig. 2(B) represents the sum of contributions obtained by laddered gluons between lines (b) and (d) in Fig. 2(C). Allowing only such one-gluon-like exchange effects, which is the basic feature of RL truncation, then there are no distinct additional contributions. On the other hand, if any one of the contributions described and illustrated here is neglected in a given calculation, then that calculation explicitly breaks an array of relevant symmetries.

Consider now the $\gamma^*\pi$ forward pion Compton amplitude in the Bjorken limit, Eq. (4). The ($S3$) permutation of the diagrams in Fig. 2 corresponds to a collection of

so-called *cat's ears* contributions. They are greatly suppressed compared to the other two permutations in the Bjorken limit; hence may be neglected. The ($S2$) permutation corresponds simply to symmetrizing the incoming and outgoing photons and so need not explicitly be considered further. Consequently, in computing $q^\pi(x; \zeta_H)$, one may focus solely on those diagrams drawn explicitly in Fig. 2; namely, in RL truncation [37]:

$$\gamma^*(q) + \pi(P) \rightarrow \gamma^*(q) + \pi(P) \stackrel{\text{Fig. 2}}{=} (A) + (B) - (C). \quad (8)$$

In the forward and Bjorken limits, Fig. 2(C) is the textbook *handbag* contribution to $\gamma^*\pi$ Compton scattering. It has often been used alone to estimate $q^\pi(x; \zeta_H)$. (See, e.g., Refs. [12,75–78] and citations therein and thereof.) If the pion's Bethe-Salpeter amplitude is assumed to be momentum-independent¹ and a Poincaré-invariant regularization of the loop-integral is employed, then Fig. 2(C) yields a result for $q^\pi(x; \zeta_H)$ that preserves both the baryon-number and momentum sum-rules; namely,

$$\int_0^1 dx q^\pi(x; \zeta_H) = 1, \quad (9a)$$

$$\int_0^1 dx x q^\pi(x; \zeta_H) = \frac{1}{2}. \quad (9b)$$

(The right-hand-side of Eq. (9a) remains unity under QCD evolution—DGLAP [81–84].) In fact, one finds [78]

$$q_{\text{CI}}^\pi(x; \zeta_H) \approx \theta(x)\theta(1-x), \quad (10)$$

¹This is the result obtained using an internally-consistent, symmetry-preserving treatment of a vector \otimes vector contact interaction (CI) [79,80].

where $\theta(x)$ is the Heaviside step function. Equation (10) describes a structureless pion, in which a given valence-quark carries all light-front-fractions of the pion's total momentum with equal probability.

If the regularization scheme for the loop in Fig. 2(C) introduces a mass-scale and/or the quark-antiquark interaction is momentum-dependent, then the result obtained violates one or both of the sum rules in Eq. (9) [12,75]. Consequently, Fig. 2(C) alone is a poor approximation when realistic interactions are used.

Consider now Fig. 2(A), which can be written thus:

$$\mathbf{q}_A^\pi(x; \zeta_H) = N_c \text{tr} \int_{dk} \delta_n^x(k_\eta) n \cdot \gamma \mathcal{H}_\pi(P, k), \quad (11)$$

where $\delta_n^x(k_\eta) := \delta(n \cdot k_\eta - xn \cdot P)$; n is a lightlike four-vector, $n^2 = 0$, with $n \cdot P = -m_\pi$ in the pion rest frame; and $k_\eta = k + \eta P$, $k_{\bar{\eta}} = k - (1 - \eta)P$, $\eta \in [0, 1]$. Owing to Poincaré covariance, no observable can legitimately depend on η , i.e., the definition of the relative momentum.

In RL truncation, as illustrated in Fig. 2(A), $\mathcal{H}_\pi(P, k)$ in Eq. (11) is an infinite sum of ladderlike gluon-rungs, beginning with zero rungs. Hence, one may write [85]

$$\mathbf{q}_A^\pi(x; \zeta_H) = N_c \text{tr} \int_{dk} i\Gamma_\pi(k_\eta, -P) S(k_\eta) i\Gamma^n(k; x; \zeta_H) \times S(k_\eta) i\Gamma_\pi(k_{\bar{\eta}}, P) S(k_{\bar{\eta}}), \quad (12)$$

where $\Gamma^n(k; x; \zeta_H)$ is a generalization of the quark-photon vertex, describing a dressed-quark scattering from a zero momentum photon and determined by a RL Bethe-Salpeter equation with inhomogeneity $n \cdot \gamma \delta_n^x(k_\eta)$.

Equation (12) is depicted in Fig. 3(A'); and now a comparison with Fig. 1 makes manifest that the RL treatment of Fig. 2(A) is equivalent to the symmetry preserving analysis of the pion's electromagnetic form factor (at $Q^2 = 0$) [49,86]. Furthermore, Eq. (12) ensures Eq. (9a) because

$$\int_0^1 dx \Gamma^n(k; x; \zeta_H) = n_\mu \Gamma_\mu(k, k) / n \cdot P; \quad (13)$$

thus, using Eq. (7),

$$\int_0^1 dx \mathbf{q}_A^\pi(x; \zeta_H) = F_\pi(Q^2 = 0) = 1. \quad (14)$$

On the other hand, as illustrated by existing calculations, e.g., Refs. [12,85], Eq. (12) violates Eq. (9b). Hence, as explained above, any result for $\mathbf{q}^\pi(x; \zeta_H)$ obtained from Fig. 2(A) alone—equivalently, Fig. 3(A)—is flawed because it violates basic symmetry constraints. Typical consequences include the following: an overestimate of the sea and gluon content of a given bound-state; erroneous

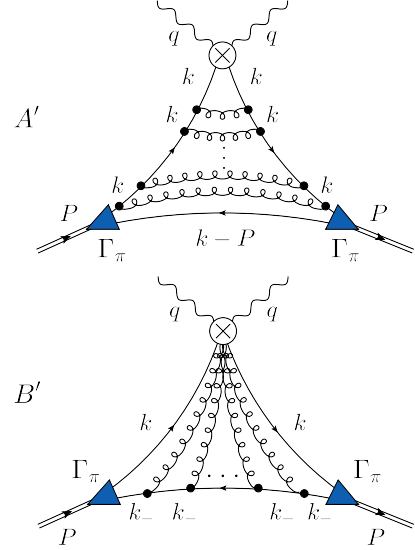


FIG. 3. Employing the optical theorem, the diagrams in Fig. 2 yield these two contributions to $\mathbf{q}^\pi(x)$: upper panel, Eq. (12); and lower panel, Eq. (15). The sum yields the completely symmetry-preserving RL truncation formula for $\mathbf{q}^\pi(x)$.

estimates of the relative size of the valence-quark momentum fractions within different but related bound-states; incorrect identification of ζ_H , if this scale is used as a parameter to fit an empirically-determined distribution [87]; and since these errors are transmitted into the evolved distributions, a lack of credibility in any conclusions and interpretations drawn from the distributions. Furthermore, the symmetry violations and associated errors are amplified by including the $\mathcal{H}(P, k)$ resummation in Fig. 2(A) [Fig. 3(A')] alone because this unbalances the interferences that a fully-consistent RL truncation is guaranteed to preserve. Consequently, less damage is done by working solely with Fig. 2(C).

We turn now to the contribution (B)-(C) in Eq. (8), which has usually been overlooked in calculations of $\mathbf{q}^\pi(x; \zeta_H)$; but whose importance was stressed and illustrated in Ref. [37]. Given that the combination (B)-(C) is crucial if the WGT identities are to be satisfied in a RL analysis of Compton scattering, let us consider their content. A first observation is that $(B_0)-(C) = 0$, i.e., if one omits all terms from the ladderlike sum in Fig. 2(B) then it is completely cancelled by subtracting Fig. 2(C). Hence, $(B_0)-(C)$ is a sum of infinitely many ladderlike rungs, beginning with one rung. This is illustrated in Fig. 3(B'). Studying this figure, the nature of (B)-(C) becomes plain, *viz.* it expresses the impact of the deep-inelastic event as felt by a dressed-quark line embedded *within* the pion bound state. Thinking perturbatively, one might imagine these processes to represent effects associated with initial/final-state interaction corrections to the handbag diagram and thus to be suppressed. However, so long as the gluon exchanges are soft, which is the limit

exposed by the optical theorem analysis, that is not the case because the resummation of ladderlike rungs is resonant. Hence the contribution depicted in Fig. 3(B') is of precisely the same order as that from Fig. 3(A'). In fact, akin to the final state interactions that produce single spin asymmetries [88], the (B)-(C) contribution is leading-twist and its appearance and importance signal failure of the impulse approximation.

These considerations lead to the following form for the (B)-(C) contribution to $q^\pi(x; \zeta_H)$ [37]:

$$q_{BC}^\pi(x; \zeta_H) = N_c \text{tr} \int_{dk} \Gamma_\pi^n(k_\eta, -P; \zeta_H) \times S(k_\eta) \Gamma_\pi(k_{\bar{\eta}}, P) S(k_{\bar{\eta}}), \quad (15)$$

where $\Gamma_\pi^n(k_\eta, -P; \zeta_H)$ is a ‘‘pierced’’ pion Bethe-Salpeter amplitude, computed by summing infinitely many insertions of $[\delta_n^x(k_\eta) n \cdot \partial_{k_\eta} S(k_\eta)]$, between sequentially chosen adjacent gluon-rungs in the diagrammatic expansion of the pion amplitude. Notably, independent of ζ_H , as a consequence of symmetry preservation:

$$\int_0^1 dx q_{BC}^\pi(x; \zeta_H) = 0. \quad (16)$$

We can now write the complete expression for the pion valence-quark distribution function in RL truncation:

$$q^\pi(x; \zeta_H) = q_A^\pi(x; \zeta_H) + q_{BC}^\pi(x; \zeta_H), \quad (17)$$

i.e., one sums the terms in Eqs. (12) and (15).

IV. PREDICTION FOR THE PION VALENCE-QUARK DISTRIBUTION FUNCTION

A. Ward identity approximation for $q^\pi(x)$

As illustrated in Ref. [73], it is challenging to solve for the complete RL u and t channel scattering amplitudes depicted in Figs. 2(A), (B) and needed to describe $\gamma^* \pi \rightarrow \gamma^* \pi$. Herein, we therefore use a simpler approach, employing the approximations introduced in Ref. [37]:

$$i\Gamma^n(k; x; \zeta_H) = \delta_n^x(k_\eta) n \cdot \partial_{k_\eta} S^{-1}(k_\eta), \quad (18a)$$

$$\Gamma_\pi^n(k_\eta, -P; \zeta_H) = n \cdot \partial_{k_\eta} \Gamma_\pi(k_\eta, -P; \zeta_H), \quad (18b)$$

in which case

$$q^\pi(x; \zeta_H) = N_c \text{tr} \int_{dk} \delta_n^x(k_\eta) \times \{n \cdot \partial_{k_\eta} [\Gamma_\pi(k_\eta, -P) S(k_\eta)]\} \Gamma_\pi(k_{\bar{\eta}}, P) S(k_{\bar{\eta}}). \quad (19)$$

Improvement upon Eqs. (18) will be canvassed in future, following Ref. [73]. However, as shown therein, in Ref. [37], and remarked above: the symmetry-preserving nature of our treatment, Fig. 2, ensures cancellations between terms. Hence, corrections in the u channel largely compensate those at equal order in the t channel thereby ensuring accuracy of Eq. (19).

It is straightforward to prove algebraically that the result obtained using Eq. (19) is: independent of η ; ensures

$$q^\pi(x; \zeta_H) = q^\pi(1-x; \zeta_H); \quad (20)$$

satisfies Eqs. (9); and possesses defined subcomponents that comply with Eqs. (14), (16).

B. Computing the inputs for $q^\pi(x)$

In order to calculate $q^\pi(x; \zeta_H)$ from Eq. (19) one must know the dressed light-quark propagator and pion Bethe-Salpeter amplitude. Algebraic *Ansätze* were employed in Ref. [37]. In contrast, herein we follow Ref. [41] and use realistic numerical solutions. Consequently, the result for $q^\pi(x; \zeta_H)$ is completely determined once an interaction kernel is specified for the RL Bethe-Salpeter equation.

We use the interaction explained in Refs. [89,90]:

$$K_{\alpha_1 \alpha'_1 \alpha_2 \alpha'_2} = \mathcal{G}_{\mu\nu}(k) [i\gamma_\mu]_{\alpha_1 \alpha'_1} [i\gamma_\nu]_{\alpha_2 \alpha'_2}, \quad (21a)$$

$$\mathcal{G}_{\mu\nu}(k) = \tilde{\mathcal{G}}(k^2) T_{\mu\nu}(k), \quad (21b)$$

with $k^2 T_{\mu\nu}(k) = k^2 \delta_{\mu\nu} - k_\mu k_\nu$ and $(s = k^2)$

$$\frac{1}{Z_2} \tilde{\mathcal{G}}(s) = \frac{8\pi^2 D}{\omega^4} e^{-s/\omega^2} + \frac{8\pi^2 \gamma_m \mathcal{F}(s)}{\ln[\tau + (1 + s/\Lambda_{\text{QCD}}^2)^2]}, \quad (22)$$

where $\gamma_m = 4/\beta_0$, $\beta_0 = 11 - (2/3)n_f$, $n_f = 4$, $\Lambda_{\text{QCD}} = 0.234 \text{ GeV}$, $\tau = e^2 - 1$ ($\ln e = 1$), and $\mathcal{F}(s) = \{1 - \exp(-s/[4m_t^2])\}/s$, $m_t = 0.5 \text{ GeV}$. The development of Eqs. (21), (22) is summarized in Ref. [89] and their connection with QCD is described in Ref. [59]. Some points may nevertheless bear repeating. Namely, the interaction is deliberately consistent with that determined in studies of QCD's gauge sector and it preserves the one-loop renormalization group behavior of QCD.

Z_2 in Eq. (22) is the dressed-quark wave function renormalization constant. We employ a mass-independent momentum-subtraction renormalization scheme for the gap and inhomogeneous vertex equations, implemented by using the scalar WGT identity and fixing all renormalization constants in the chiral limit [91]. In the first applications of this DSE approach to hadron observables [92,93] (and many that have followed), the renormalization scale was chosen deep in the spacelike region: $\zeta = \zeta_{19} := 19 \text{ GeV}$, primarily to ensure simplicity in the nonperturbative renormalization procedure. This choice entails that

the dressed quasiparticles obtained as solutions to the DSEs remain intact and thus serve as the dominant degrees-of-freedom for all observables. This is adequate for infrared quantities, such as hadron masses: flexibility of model parameters and the bridge with QCD enable valid predictions to be made. However, it generates errors in form factors and parton distributions. With form factors, the correct power-law behavior is obtained, but the scaling violations deriving from anomalous operator dimensions are wrong (see, e.g., Ref. [94]); and for parton distributions, the natural connection between the renormalization scale and the reference scale for evolution equations is lost, again because parton loops are suppressed when renormalizing a RL truncation study at deep spacelike momenta so the computed anomalous dimensions are wrong.

As explained elsewhere [40,42,45], the solution to these problems is to renormalize the DSE solutions at a typical hadronic scale, where the dressed quasiparticles are the correct degrees-of-freedom. This recognizes that a given meson's Poincaré covariant wave function and correlated vertices, too, must evolve with ζ [95–97]. Such evolution enables the dressed-quark and -antiquark degrees-of-freedom, in terms of which the wave function is expressed at a given scale $\zeta^2 = Q^2$, to split into less well-dressed partons via the addition of gluons and sea quarks in the manner prescribed by QCD dynamics. These effects are automatically incorporated in bound-state problems when the complete quark-antiquark scattering kernel is used; but aspects are lost when that kernel is truncated, and so it is with RL truncation. We therefore renormalize our DSEs at the hadronic scale $\zeta = \zeta_H$.

A natural value for the hadronic scale, ζ_H , must now be determined. To that end, recall that QCD possesses a process-independent effective charge [98–100]: $\alpha_{\text{PI}}(k^2)$. This running-coupling saturates in the infrared: $\alpha_{\text{PI}}(0)/\pi \approx 1$, owing to the dynamical generation of a gluon mass-scale [101,102]. These features and a smooth connection with pQCD [and hence Eq. (22)] are expressed in the following algebraic expression:

$$\alpha_{\text{PI}}(k^2) = \frac{\pi\gamma_m}{\ln[(m_\alpha^2 + k^2)/\Lambda_{\text{QCD}}^2]}, \quad (23)$$

$m_\alpha = 0.30 \text{ GeV} \gtrsim \Lambda_{\text{QCD}}$. Evidently, m_α is an essentially nonperturbative scale whose existence ensures that modes with $k^2 \lesssim m_\alpha^2$ are screened from interactions. It therefore serves to define the natural boundary between soft and hard physics; hence, we identify

$$\zeta_H = m_\alpha. \quad (24)$$

Returning to Eqs. (21), (22), computations [41,89,90] reveal that observable properties of light-quark ground-state vector- and flavor-nonsinglet pseudoscalar-mesons are practically insensitive to variations of $\omega \in [0.4, 0.6] \text{ GeV}$, so long as

$$\zeta^3 := D\omega = \text{constant}. \quad (25)$$

This feature extends to numerous properties of the nucleon and Δ -baryon [103–106]. The value of ζ is typically chosen to reproduce the measured value of the pion's leptonic decay constant, f_π . In RL truncation, this requires

$$\zeta = 0.82 \text{ GeV}, \quad (26)$$

with renormalization-group-invariant current-quark mass

$$\hat{m}_u = \hat{m}_d = \hat{m} = 6.7 \text{ MeV}, \quad (27)$$

which corresponds to a one-loop evolved mass of $m^{\zeta^2} = 4.6 \text{ MeV}$. In solving the DSEs relevant to pion physics, we will subsequently employ $\omega = 0.5 \text{ GeV}$, the midpoint of the domain of insensitivity.

The next step on the way to obtaining $\mathbf{q}^\pi(x; \zeta_H)$ is to perform a coupled solution of the dressed-quark gap- and pion Bethe-Salpeter-equations, defined via Eqs. (21), (22), following Ref. [93] and adapting the algorithm improvements from Ref. [107] when necessary.

C. Mellin moments

With S and Γ_π in hand, one can calculate the Mellin-moments:

$$\langle x^m \rangle_{\zeta_H}^\pi = \int_0^1 dx x^m \mathbf{q}^\pi(x; \zeta_H) \quad (28a)$$

$$= \frac{N_c}{n \cdot P} \text{tr} \int_{dk} \left[\frac{n \cdot k_\eta}{n \cdot P} \right]^m \Gamma_\pi(k_\eta, P) S(k_\eta) \\ \times n \cdot \partial_{k_\eta} [\Gamma_\pi(k_\eta, -P) S(k_\eta)]; \quad (28b)$$

and if enough of these moments can be computed, then they can be used to reconstruct the distribution. Usefully, using Eq. (20), one finds that the value of any given odd moment, $\langle x^{m_o} \rangle_{\zeta_H}^\pi$, $m_o = 2\bar{m} + 1$, $\bar{m} \in \mathbb{Z}$, is known once all lower even moments are computed, e.g.:

$$\langle x \rangle_{\zeta_H}^\pi = \frac{1}{2} \langle x^0 \rangle_{\zeta_H}^\pi = \frac{1}{2}, \quad (29a)$$

$$\langle x^3 \rangle_{\zeta_H}^\pi = -\frac{1}{4} \langle x^0 \rangle_{\zeta_H}^\pi + \frac{3}{2} \langle x^2 \rangle_{\zeta_H}^\pi, \quad (29b)$$

$$\langle x^5 \rangle_{\zeta_H}^\pi = \frac{1}{2} \langle x^0 \rangle_{\zeta_H}^\pi - \frac{5}{2} \langle x^2 \rangle_{\zeta_H}^\pi + \frac{5}{2} \langle x^4 \rangle_{\zeta_H}^\pi, \quad (29c)$$

$$\langle x^7 \rangle_{\zeta_H}^\pi = -\frac{17}{8} \langle x^0 \rangle_{\zeta_H}^\pi + \frac{21}{2} \langle x^2 \rangle_{\zeta_H}^\pi \\ - \frac{35}{4} \langle x^4 \rangle_{\zeta_H}^\pi + \frac{7}{2} \langle x^6 \rangle_{\zeta_H}^\pi. \quad (29d)$$

Such identities can be used to validate any numerical method for computing the moments defined by Eq. (28).

Every moment defined by Eq. (28) is finite. However, direct calculation of these moments using numerically determined inputs for the propagator and Bethe-Salpeter amplitude is difficult in practice owing to an amplification of oscillations produced by the $[n \cdot k_\eta]^m$ factor: in any perfect procedure, the oscillations cancel; but that is hard to achieve numerically. We therefore introduce a convergence-factor²

$$C_m(k^2 r^2) = 1/[1 + k^2 r^2]^{m/2}. \quad (30)$$

the moment is computed as a function of r^2 ; and the final value is obtained by extrapolation to $r^2 = 0$. This procedure is reliable for the lowest six moments, $m = 0, 1, \dots, 5$ [108]. According to Eq. (29c), the $m = 5$ moment is not independent; but its direct calculation enables one to ensure that the lower even moments are correct.

One can extend this set of moments by using the Schlessinger point method (SPM) [109–113] to construct an analytic function, $M_S(z)$, whose values at $z = 0, 1, \dots, 5$ agree with the moments computed directly and for which $M_S(7)$ satisfies Eq. (29d) when $M_S(0), M_S(2), M_S(4), M_S(6)$ are used for the even moments. The function $M_S(z)$ then provides an estimate for all moments of the distribution, which is exact for $m \leq 5$.

We illustrate the efficacy of the SPM approach using the nontrivial algebraic model described in Ref. [28] (Eqs. (1), (14), (17) and Sec. IV. A). We computed fifty Mellin moments directly, then used the first six moments and the procedure described above to obtain the following SPM approximation:

$$M_S(z) = \frac{a_0 + a_1 z + a_2 z^2}{a_0 + b_1 z + b_2 z^2 + b_3 z^3}, \quad (31)$$

with the coefficients specified in Table I. Figure 4 compares the moments obtained using the SPM approximation with the true moments: the magnitude of the relative error is $< 0.2\%$ for $m \leq 10$ and $< 1\%$ for $m \leq 15$, i.e., the SPM produces accurate approximations to the first sixteen moments, working with just six. The relative error for the fiftieth moment is -48% .

Having validated the SPM, we computed the moments in Eq. (28) for $m = 0, 1, \dots, 5$ using the numerical results for S and Γ_π obtained with the DSE kernels specified by Eqs. (21), (22).³ Then, to compensate for potential propagation of numerical quadrature error in the moment

²Owing to the nature of the integrand, the convergence factor can be omitted for $m \leq 2$: it only plays a role for $m \geq 3$.

³We used the SPM to assist with extrapolation $r^2 \rightarrow 0$, Eq. (30), and Eqs. (29b), (29c) to check our results.

TABLE I. Computed coefficients for the SPM approximation to the Mellin moments of the pion valence-quark distribution function, Eq. (31). Upper panel—algebraic inputs for the propagator and Bethe-Salpeter amplitude. Lower panel—realistic inputs: column 1, using $m = 0, \dots, 3$; and column 2, $m = 0, \dots, 5$.

Algebraic		
a_0		22.15848512824146
a_1		6.882746278686694
a_2		-0.1087281502480409
b_1		25.18581743280522
b_2		9.520703952313553
b_3		1
Realistic: 0–3		
a_0	6	24.45939048190962
a_1	0.05050505050504085	16.78622673154534
a_2	0	-0.8585917712408488
b_1	5.101010101010094	36.46584243665940
b_2	1	18.84881796585922
b_3	0	1
Realistic: 0–5		
a_0		24.45939048190962
a_1		16.78622673154534
a_2		-0.8585917712408488
b_1		36.46584243665940
b_2		18.84881796585922
b_3		1

computations, we constructed two SPM approximations to the results: one based on the $m = 0, 1, 2, 3$ four-element subset; and another using the complete set of six moments. In each case, the collection of moments is described by a function of the form in Eq. (31) with the coefficients in the lower panel of Table I.

Working with the first eleven SPM-approximant moments in each case, we reconstructed a pion valence-quark distribution; and subsequently defined our result to be the average of these functions:

$$q^\pi(x; \zeta_H) = 213.32x^2(1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911x(1-x) \right]. \quad (32)$$

Notably, the $x \simeq 0, 1$ endpoint behavior of the hadron-scale distribution is completely fixed by algebraic analysis [37]:

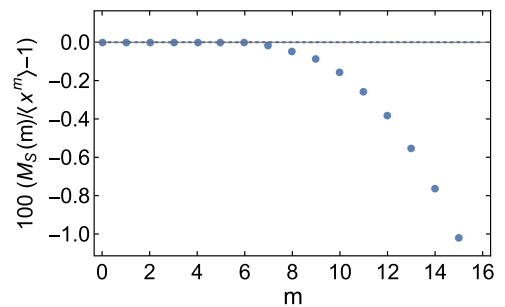


FIG. 4. Comparison between moments evaluated using the SPM approximation in Eq. (31) with those computed directly using the algebraic model in Ref. [28]; the magnitude of the relative error is $< 0.1\%$ for $m \leq 10$.

as shown, the exponent is “2”, reproducing Eq. (2). This feature removes the need to use moments of arbitrarily high order, enabling one to focus instead on the lower-order moments which provide information on the mid- x shape.

One remark may be valuable here. This application of the SPM requires the coefficient of the highest active denominator power in Eq. (31) to be unity. Hence, when one uses Eq. (31) for $m = 0, 1, 2, 3$ moments, $b_2 = 1$ and $a_2 = 0 = b_3$. Referring to the lower panels of Table I, this presents an appearance of sensitivity in the coefficients to the number of moments employed; but that is misleading. The relevant measure is not these coefficients, but the similarity between the curves obtained via reconstruction. Our result, Eq. (32), is depicted in Fig. 5. The mean absolute relative error between its first eleven moments and those of the separate reconstructed distributions is 4(3)%.

Given the remarks in Sec. I, it is worth reiterating that Eq. (32) exhibits the $x \simeq 1$ behavior predicted by the QCD parton model, Eq. (2); and because it is a purely valence distribution, this same behavior is also evident on $x \simeq 0$. However, in contrast to the scale-free valence-quark distribution computed in Ref. [37]:

$$q_{\text{sf}}(x) \approx 30x^2(1-x)^2, \quad (33)$$

obtained using parton-model-like algebraic representations of S, Γ_π , the distribution computed with realistic inputs is a much broader function. A similar effect is observed in the pion’s leading-twist valence-quark distribution amplitude [114] and those of other mesons [108, 115–118]. The cause is the same, *viz.* the valence-quark distribution function is hardened owing to DCSB, which is a realization of the mechanism responsible for the emergence of mass in the Standard Model [119]. Emergent mass is expressed in the momentum-dependence of all QCD Schwinger functions. It is therefore manifest in the point-wise behavior of wave functions, elastic and transition form factors, etc.; and as we have now displayed, also in parton distributions. (This was to be expected, given the connection between light-front wave functions and parton distributions.)

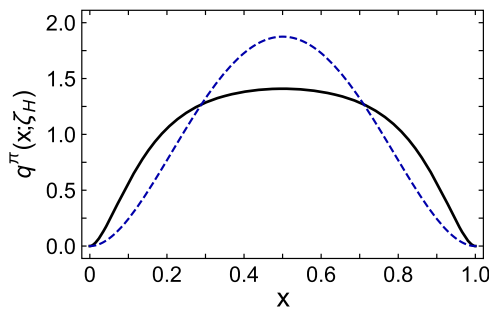


FIG. 5. Solid (black) curve: pion valence-quark distribution function at the hadronic scale, ζ_H , Eq. (32). Dashed (blue) curve: scale-free distribution, Eq. (33).

V. EVOLUTION OF PION DISTRIBUTION FUNCTIONS

The pion valence-quark distribution in Eq. (32) is computed at $\zeta_H = m_\alpha$, Eq. (24). On the other hand, existing IQCD calculations of low-order moments [33–36] and phenomenological fits to pion parton distributions are typically quoted at $\zeta \approx \zeta_2 = 2$ GeV [120–122]; and the scale relevant to the E615 data is $\zeta_5 = 5.2$ GeV [9,13]. We therefore employ the effective charge in Eq. (23) to integrate the one-loop DGLAP equations, therewith evolving $q^\pi(x; \zeta_H = m_\alpha)$ to obtain results for $q^\pi(x; \zeta_2)$ and $q^\pi(x, \zeta_5)$. This procedure ensures that saturation of the effective charge is expressed, e.g., $\alpha_{\text{PI}}(\zeta_H)/(2\pi) = 0.20$, $[\alpha_{\text{PI}}(\zeta_H)/(2\pi)]^2 = 0.04$, stabilizing our evolved results on $\zeta > \zeta_H$. Notably, given that $\zeta_H = m_\alpha$ is fixed by our analysis, all results are predictions. We checked that with fixed ζ_H , varying $m_\alpha \rightarrow (1 \pm 0.1)m_\alpha$ does not measurably affect the evolved distributions. We therefore report results with m_α fixed and an uncertainty determined by varying $\zeta_H \rightarrow (1 \pm 0.1)\zeta_H$.

A. $\zeta_H \rightarrow \zeta_2$

Our prediction for $q^\pi(x; \zeta_2)$ is depicted in Fig. 6. The solid curve and surrounding bands are described by the following function, a generalization of Eq. (32):

$$q^\pi(x) = \mathbf{n}_{q^\pi} x^\alpha (1-x)^\beta \times [1 + \rho x^{\alpha/4} (1-x)^{\beta/4} + \gamma x^{\alpha/2} (1-x)^{\beta/2}], \quad (34)$$

where \mathbf{n}_{q^π} ensures Eq. (9) and the powers and coefficients are listed in Table II. Evidently, the large- x exponent is

$$\beta(\zeta_2) = 2.38(9). \quad (35)$$

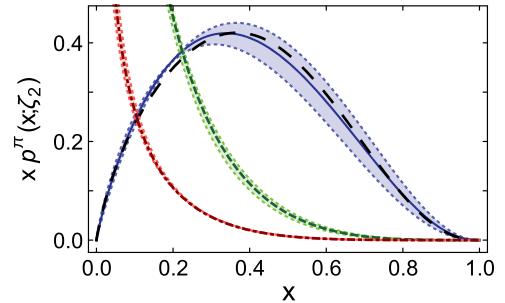


FIG. 6. Pion valence-quark momentum distribution function, $x p^\pi(x; \zeta)$, $p = q$, evolved $\zeta_H \rightarrow \zeta_2 = 2$ GeV—solid (blue) curve embedded in shaded band; and long-dashed (black) curve— ζ_2 result from Ref. [12]. Equations (39), (40): gluon momentum distribution in pion, $x g^\pi(x; \zeta_2)$ —dashed (green) curve within shaded band; and sea-quark momentum distribution, $x S^\pi(x; \zeta_2)$ —dot-dashed (red) curve within shaded band. In all cases, the shaded band indicates the effect of $\zeta_H \rightarrow \zeta_H(1 \pm 0.1)$.

Here it is also worth listing an array of associated, calculated low-order moments in comparison with those obtained in the more recent IQCD simulations:

ζ_2	$\langle x \rangle_u^\pi$	$\langle x^2 \rangle_u^\pi$	$\langle x^3 \rangle_u^\pi$
Ref. [34]	0.24(2)	0.09(3)	0.053(15)
Ref. [35]	0.27(1)	0.13(1)	0.074(10)
Ref. [36]	0.21(1)	0.16(3)	
Ref. [32]	0.254(03)	0.094(12)	0.057(04)
Average	0.24(2)	0.119(18)	0.061(06)
Herein	0.24(2)	0.098(10)	0.049(07)

Both continuum and IQCD results agree on the light-front momentum fraction carried by valence-quarks in the pion at $\zeta = \zeta_2$:

$$\langle 2x \rangle_q^\pi = 0.48(3), \quad (37)$$

i.e., roughly one-half. This is consistent with a recent phenomenological analysis of data on π -nucleus Drell-Yan and leading neutron electroproduction [122]: $\langle 2x \rangle_q^\pi = 0.48(1)$ at $\zeta = 2.24$ GeV.

As explained above, the pion is purely a bound-state of a dressed-quark and dressed-antiquark at the hadronic scale, ζ_H . Sea and glue distributions are zero at ζ_H . They are generated by QCD evolution on $\zeta > \zeta_H$. Using LO evolution with the coupling in Eq. (23) we obtain the sea and glue distributions in Fig. 6, from which one computes the following momentum fractions ($\zeta = \zeta_2$):

$$\langle x \rangle_g^\pi = 0.41(2), \quad \langle x \rangle_{\text{sea}}^\pi = 0.11(2). \quad (38)$$

The ordering of these values agrees with that in [122], but our gluon momentum-fraction is $\sim 20\%$ larger and that of the sea is commensurately smaller.

Our computed glue and sea momentum distributions are fairly approximated using the following simple functional form:

$$xp^\pi(x; \zeta) = \mathcal{A}x^\alpha(1-x)^\beta, \quad (39)$$

TABLE II. Coefficients and powers that reproduce the computed pion valence-quark distribution functions, depicted in Figs. 6, 7, when used in Eq. (34).

	n_{q^r}	α	β	ρ	γ
ζ_2	9.83	-0.080	2.29	-1.27	0.511
	8.31	-0.127	2.37	-1.19	0.469
	7.01	-0.162	2.47	-1.12	0.453
ζ_5	7.81	-0.153	2.54	-1.20	0.505
	7.28	-0.169	2.66	-1.21	0.531
	6.48	-0.188	2.78	-1.19	0.555

with the coefficient and powers listed here ($p = g = \text{glue}$, $p = S = \text{sea}$):

	p	\mathcal{A}	α	β
ζ_2	g	0.40 ∓ 0.03	-0.55 ∓ 0.03	3.47 ± 0.13
	S	0.13 ∓ 0.01	-0.53 ∓ 0.05	4.51 ± 0.03
ζ_5	g	0.34 ∓ 0.04	-0.62 ∓ 0.04	3.75 ± 0.12
	S	0.12 ± 0.02	-0.61 ∓ 0.07	4.77 ± 0.03

B. $\zeta_H \rightarrow \zeta_5$

Our predictions for the pion parton distributions at a scale relevant to the E615 experiment, i.e., $\zeta_5 = 5.2$ GeV [9,13], are depicted in Fig. 7. The solid curve and surrounding bands are described by the function in Eq. (34) with the powers and coefficients listed in Table II. Evidently, the large- x exponent is

$$\beta(\zeta_5) = 2.66(12). \quad (41)$$

Working with results obtained in an exploratory IQCD calculation [31], one finds $\beta_{\text{IQCD}}(\zeta_5) = 2.45(58)$; and also the following comparison between low-order moments:

ζ_5	$\langle x \rangle_u^\pi$	$\langle x^2 \rangle_u^\pi$	$\langle x^3 \rangle_u^\pi$
Ref. [31]	0.17(1)	0.060(9)	0.028(7)
Herein	0.21(2)	0.076(9)	0.036(5)

The data in Fig. 7 is that reported in Ref. [9], rescaled according to the analysis in Ref. [14]. Our prediction agrees with the rescaled data. Importantly, no parameters were varied in order to achieve this outcome.

As above, the predictions for the glue and sea distributions in Fig. 7 were obtained using LO evolution from $\zeta_H = m_\alpha \rightarrow \zeta_5$ with the coupling in Eq. (23); and from these distributions one obtains the following momentum fractions ($\zeta = \zeta_5$):

$$\langle x \rangle_g^\pi = 0.45(1), \quad \langle x \rangle_{\text{sea}}^\pi = 0.14(2). \quad (43)$$

The glue and sea momentum distributions are fairly described by the function in Eq. (39) evaluated using the coefficient and powers in the lower rows of Eq. (40).⁴

Figure 7 also displays the IQCD result for the pion valence-quark distribution function [31] evolved to the E615 scale: dot-dot-dashed (grey) curve within bands. As could be anticipated from the comparisons listed in

⁴Recall that in the neighborhood $\Lambda_{\text{QCD}}^2/\zeta^2 \simeq 0$, for any hadron [123]: $\langle x \rangle_q = 0$, $\langle x \rangle_g = 4/7 \approx 0.57$, $\langle x \rangle_S = 3/7 \approx 0.43$.

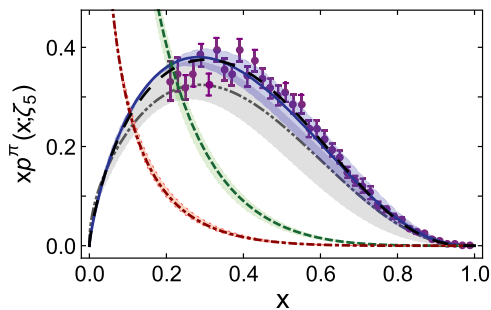


FIG. 7. Pion valence-quark momentum distribution function, $xq^\pi(x; \zeta)$, evolved $\zeta_H \rightarrow \zeta_5 = 5.2$ GeV—solid (blue) curve embedded in shaded band; and long-dashed (black) curve— ζ_5 result from Ref. [12]. Gluon momentum distribution in pion, $xg^\pi(x; \zeta_2)$ —dashed (green) curve within shaded band; and sea-quark momentum distribution, $xS^\pi(x; \zeta_2)$ —dot-dashed (red) curve within shaded band. See Eqs. (39), (40). In all the above cases, the shaded band indicates the effect of $\zeta_H \rightarrow \zeta_H(1 \pm 0.1)$. Dot-dot-dashed (grey) curve within shaded band—IQCD result [31]. Data (purple) from Ref. [9], rescaled according to the analysis in Ref. [14].

connection with Eq. (42), the pointwise form of the IQCD prediction agrees with our result (within errors). This is significant: two disparate treatments of the pion bound-state problem have now arrived at the same prediction for the pion’s valence-quark distribution function.

VI. SUMMARY AND PERSPECTIVE

Using a continuum approach to the two valence-body bound-state problem in quantum field theory, we presented a symmetry-preserving calculation of the pion’s valence-quark distribution function, $q^\pi(x; \zeta_H)$, ζ_H is the hadronic scale [Sec. III]; and thereby unified the result with kindred predictions for the electromagnetic pion elastic and transition form factors [40–42, 44–46] and numerous other observables (e.g., Refs. [106, 124]). Within this framework, the pion is purely a bound-state of a dressed-quark and dressed-antiquark at ζ_H ; consequently,

$$q^\pi(x; \zeta_H) = q^\pi(1-x; \zeta_H). \quad (44)$$

Capitalizing on this, we directly computed the first three independent Mellin moments of $q^\pi(x; \zeta_H)$ and therefrom developed analytic approximations that delivered estimates for the next three. Our prediction for $q^\pi(x; \zeta_H)$ was reconstructed from this information on the first six independent moments and the algebraically computed $x \simeq 0$, 1 endpoint behavior [Sec. IV].

In continuum studies, the value of the hadronic scale, ζ_H , has typically been a parameter; usually chosen to obtain agreement with the value of some Mellin moments determined in phenomenological analyses of data [87]. That is not the case herein. Instead, the value $\zeta_H = m_\alpha$, Eq. (23), is determined at the outset by connecting the one-loop

running coupling with QCD’s process-independent effective charge [98, 99].

Our result for $q^\pi(x; \zeta_H)$ [Eq. (32)] exhibits the $x \simeq 1$ behavior predicted by the QCD parton model, Eq. (2). Moreover, $q^\pi(x; \zeta_H)$ is a broad function. As with meson distribution amplitudes, this hardening is a consequence of dynamical chiral symmetry breaking (DCSB), itself a realization of the mechanism responsible for the emergence of mass in the Standard Model.

With the hadronic scale fixed, we used the process-independent effective charge to integrate the evolution equations and obtain $q^\pi(x; \zeta_2 = 2$ GeV) and $q^\pi(x, \zeta_5 = 5.2$ GeV) [Sec. V]; and, simultaneously, predictions for the associated glue and sea quark distribution functions within the pion. At ζ_2 , the scale typical of both lattice-QCD studies and phenomenological analyses of data, we determined the following momentum budget for the pion:

$$\langle x_{\text{valence}} \rangle = 0.48(3), \quad (45a)$$

$$\langle x_{\text{glue}} \rangle = 0.41(2), \quad (45b)$$

$$\langle x_{\text{sea}} \rangle = 0.11(2), \quad (45c)$$

confirming the large gluon momentum-fraction found in earlier continuum analyses [12, 38]. Furthermore, our prediction for $q^\pi(x, \zeta_5)$ [Fig. 7] agrees with πN Drell-Yan data [9] rescaled as suggested by the complete next-to-leading-order (NLO) reanalysis in Ref. [14].

Of particular importance is the agreement between our parameter-free result for $q^\pi(x, \zeta_5)$ and that obtained in a recent, exploratory IQCD calculation [31]. With this confluence, two disparate treatments of the pion bound-state problem have arrived at the *same* prediction for the pion’s valence-quark distribution function. This should stimulate a reconsideration of extant phenomenological analyses so that the next attempts involve a complete NLO analysis of data, including the threshold resummation effects which seem so crucial to obtaining a sound extraction of $q^\pi(x)$. The results presented herein also support efforts to obtain new data on pion distribution functions, such as those approved at the Thomas Jefferson National Accelerator Facility [15–17] and identified as high priority at other facilities [18–22].

A worthwhile extension of the analysis described herein is the calculation of analogous kaon distribution functions. This will enable a sophisticated reevaluation of predictions from an earlier algebraic analysis [38], which indicated that the gluon content of the kaon is significantly smaller than that of the pion and identified the origin of this effect to be DCSB and its role in forming the almost-massless pion [119].

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