Conductivity bound of the strongly interacting and disordered graphene from gauge/gravity duality

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The carriers in graphene tuned close to the Dirac point envisage signatures of the strongly interacting fluid and are subject to hydrodynamic description. The important question is whether strong disorder induces the metal-insulator transition in this two-dimensional material. The bound on the conductivity tensor found earlier within the single current description implies that the system does not feature metal-insulator transition. The linear spectrum of the graphene imposes the phase-space constraints and calls for the two-current description of interacting electron and hole liquids. Based on the gauge/gravity correspondence, using the linear response of the black brane with broken translation symmetry in Einstein-Maxwell gravity with the auxiliary $U(1)$-gauge field, responsible for the second current, we have calculated the lower bound of the DC conductivity in the holographic model of graphene. The calculations show that the bound on the conductivity depends on the coupling between both $U(1)$ fields and for a physically justified range of parameters it departs only weakly from the value found for a model with the single $U(1)$ field.

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I. INTRODUCTION

Disorder and interactions inside solids are responsible for finite values of the transport coefficients and play a very important role in establishing their detailed behavior. Importantly, the role of both disorder and interactions depends on the spatial dimensionality of the condensed matter system. Doping of the intrinsic semiconductors, being the key ingredient of numerous electronic applications, is an important example illustrating the role of disorder in the weakly interacting three-dimensional materials.

It has been predicted and verified experimentally that in three-dimensional systems both strong disorder or strong electron-electron interactions can induce metal to insulator transition. In the noninteracting systems this phenomenon is called the Anderson transition [1]; while in the presence of electron-electron interaction, the transition is known as Mott [2] or if interactions and disorder play a significant role, the Anderson-Hubbard one [3].

On the other hand, the two-dimensional systems are far more complicated from the experimental [4–6] as well as the theoretical points of view. The theoretical description of the interacting [7] systems in question does not give unique results. The recent application of the gauge/gravity analogy to study the strongly interacting two-dimensional disordered materials has revealed the absence of the disorder driven metal-insulator transition in the system [8]. The result is valid in the hydrodynamic limit for the electron mean free path much smaller than the typical scale of the spatial inhomogeneities [9].

The hydrodynamic limit of the electron flow has been identified experimentally in very clean systems, as predicted a long time ago [10]. In fact, the signatures of the hydrodynamic behavior have been observed over recent years in many materials including the high mobility (Al,Ga)As wires [11,12]. More recent measurements have envisaged the hydrodynamic signatures in many other materials. One should mention the shear viscosity measurements in the ultracold Fermi gases [13], strongly correlated oxides [14], and graphene [15,16]. The comprehensive discussion of this novel set of experiments is given in [17].

The special interest is devoted to graphene, the two-dimensional system that envisages a hydrodynamic behavior of the carriers, observed in a number of recent experiments [18–21], especially when the material is tuned close to the particle-hole symmetry point. Because of the strong scattering of charge carriers, nearby the charge neutrality point, the thermoelectric power of graphene is strongly enhanced [18] and approaches the hydrodynamic limit. The departures from the Wiedemann-Franz ratio due to the increase of thermal conductivity [16] have been interpreted as the indication of hydrodynamic behavior in the material in question. On the other hand, the

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hydrodynamic viscosity of electrons has been measured [15] in high mobility graphene samples. The viscous effects were observed [19] and shown to facilitate high mobility transport at temperatures below 150 K. The recent theoretical and experimental studies of hydrodynamic effect in graphene have been reviewed in [22,23].

Even though the hydrodynamic flow is expected to be observed in a very clean system, the disorder seems to be an important factor that sometimes even facilitates the hydrodynamic behavior [24]. The signatures of the Stokes nonlinear flow with the low Reynolds number [25] have been detected in graphene [21], as the appearance of vortices leading to the negative resistance of the material.

At the Fermi level graphene exhibits a massless relativistic spectrum with Dirac cone. As was mentioned above, close to the charge neutrality point, it sustains a strongly interacting material, ideal system for studies by means of gauge/gravity duality methods. In this system, the thermoelectric transport coefficients have been found using the hydrodynamic approach [26–28], with a fairly good agreement with the experimental data.

Recently, this attitude has been generalized to the model with two distinct $U(1)$-gauge currents, which is solved by the AdS/CFT analogy [29]. The model in question allowed the successful quantitative comparison between theory and experimental data. The paper [29] gives a number of arguments behind the introduction of two gauge fields and associated currents. One reason for the appearance of two currents in graphene is the charge imbalance between electrons and holes in the system with linear spectrum. It has been found that the two-current model allows for a quantitatively correct description of the thermal conductance of graphene. The paper [30] presented the further generalization, taking into account the possible coupling between both currents. In Ref. [30] the transport properties of graphene using the model in question were elaborated. Moreover the perpendicular magnetic field to the graphene sheet was taken into account. It was assumed that the charges bounded with the two gauge fields are proportional to each other with the factor g, which is responsible for the possible imbalance of the positive and negative charges in graphene close to the charge neutrality point. It was found that the kinetic and transport coefficients were influenced by α-coupling constant and factor g. The increase of α leads to the increase of the width of normalized thermal conductivity, while in the case when $g = 0$, the effect has been quite the opposite (we have the decrease of the width). On the other hand, the α-coupling constant affects the Wiedeman-Franz ratio (WFR), changing the width and heights of the curves. The general tendency is that the WFR diminishes while the value of the α-coupling constant grows.

Moreover, the coupling constant in question impels the charge dependence of the diagonal resistivity and the WFR; i.e., the increase of α causes the decrease of both ρ^{xx} and W^{xx}. The Seebeck and Nernst coefficients were affected by magnetic field and α. The influence in question, for large value of S^{xx}, changes the shape of the curve from two minima and a maximum curve to the one with a minimum (for $B = 0$) and two small maxima for larger absolute values. The Hall angle was also influenced by the coupling constant. In the studied case the density dependence of the thermoelectric coefficient α_{ij} and Seebeck coefficient S^{xx} agree with experimental data.

The generalizations of these studies were given in [31], where the holographic calculation of magnetotransport coefficients in the $3+1$-dimensional system with Dirac-like spectrum was presented. The calculations envisage the influence of g and α on the coefficients. Namely, the magnetic field dependence on resistivity ρ^{xx} and ρ^{xy} depicts that the bigger values of α one takes the smaller resistivity we achieve.

In general one expects the presence of additional gauge fields in graphene due to geometric and other reasons [32,33]. The use of gauge/gravity duality allows for the exact solution of the strongly coupled field theoretical models. We use this approach to elaborate the effect of interactions and disorder on the hydrodynamic transport of graphene modeled by the $3+1$-dimensional anti–de Sitter (AdS) space time with the black brane background that breaks translational symmetry [34,35].

The studies of electrical transport in a strongly coupled system include the case of strange metals in two spatial dimensions at finite temperature and charge density, holographically dual to Einstein-Maxwell theory with a potential in asymptotically four-dimensional AdS manifold. One finds that the electrical conductivity is bounded from below by a universal minimum conductance. The inspection of Stokes-like equations in the spacetime in question shows that it cannot exhibit metal-insulator transitions [8]. The bound on the incoherent thermal conductivity obtained by analyzing the linear perturbations of black brane with broken translation symmetry in AdS Einstein-Maxwell dilaton gravity was performed in [36]. It turns out that the thermal conductivity has nonzero value (at finite temperature), as far as the dilaton potential being bounded from below.

In [37] the analytical lower bound on the conductivity in holographic model AdS Einstein-Maxwell dilaton theory, in terms of black horizon data, using the Stokes equations on black object event horizon was provided. In the considered model the metal-insulator transition is not driven by disorder, but it is caused by coupling the scalar field to the Maxwell one. Studies in the rotational and translational symmetries breaking system reveal that the ratio of the determinant of the electrical conductivities along any spatial directions, to black brane area density, having the zero charge limit in account, tends to the universal value [38]. The conductivity bounds were also elaborated in the case of probe brane models [39], massive gravity [40], as well as effective holographic theories [41].

It was shown in the two latter cases that there were no bounds on conductivities.

In [42] the Navier-Stokes equations of the model with two $U(1)$-gauge fields were derived. The paper elaborates the black brane response to the electric fields and temperature gradient. The DC transport coefficients for the holographic Dirac semimetals are found. Here we analyze a similar model with a goal to establish the bounds on the conductivity of the Dirac fluid in graphene subject to the influence of the α-coupling constant between the two $U(1)$-gauge fields. The main result is that the coupling between the currents, in the following quantified by the parameter α, only slightly modifies the bounds [see Eqs. (63) and (85)] on the conductivity for $\alpha \leq 1$. Larger values of α lead to strong decrease of the bound and finally to metal-insulator transition at $\alpha = 2$, when the conductivity bound vanishes.

The organization of the paper is as follows. In Sec. II we introduce the gravitational background and the action used to describe two interacting currents in graphene. Section III is devoted to the description of the perturbations of the event horizon allowing the derivation of the appropriate hydrodynamic description. We calculate the conductivity of the system in the background of the uncharged black brane in Sec. IV. In Sec. V the case of the charged background black brane is discussed, where we also derive the lower bounds on the conductivity. The variational approach has been applied in Sec. VI to study the conductivity bounds. In Sec. VII we end up with the summary and conclusions.

II. BACKGROUND HOLOGRAPHIC MODEL

In our paper we deal with the generalization of the previously studied models [34,35], by adding two interacting $U(1)$-gauge fields. The aim is to find the influence of them on DC thermoelectric transport coefficients and to compare with the existing results. In our model the gravitational action in $(3+1)$ dimensions is taken in the form

$$S = \int \sqrt{-g} d^4x \left(R + \frac{6}{L^2} - \frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{\alpha}{4} F_{\mu\nu} B^{\mu\nu} \right), \tag{1}$$

where R is the scalar curvature of the spacetime, and ϕ stands for the scalar field, which as we see later on contributes a viscositylike term to the hydrodynamic equations. $F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}$ are the ordinary Maxwell field strength tensor, while the second $U(1)$-gauge field $B_{\mu\nu}$ is given by $B_{\mu\nu} = 2\nabla_{[\mu} B_{\nu]}$. α is the coupling constant between both gauge fields. L is the radius of AdS spacetime.

The presence of an additional gauge field is motivated by the desire of describing carrier flow in graphene, near the particle-hole symmetry point. These two currents may be interpreted as connected with electrons and holes. The approach in question provides quantitatively correct description of the thermal conductivity of graphene close to the Dirac point [29]. Allowing for the interaction between the two $U(1)$-gauge currents, the coupling α provides additional degree of freedom and *inter alia* affects [30] the magnetic field dependence of the nondiagonal transport coefficients, especially for the low values of the aforementioned field. The important novel aspect of the two-current model is the tensor structure [29,30] of the transport coefficients with the general entries, e.g., for the conductivity σ^{ij}_{ab}, where a, b refer to two fields denoted above as F and B and i, j refer to the spatial directions [cf. Eqs. (53) and (54)]. The identifications of the charges $Q_F = -e n_e$ and $Q_B = +e n_h$ with electrons and holes and the total electric current $J^j = J^j_F + J^j_B$ as well as assuming that the electric fields $E^i_F = E^i_B = E^i$ lead to the value of the total conductivity elements $\sigma^{ij} = \sum_{a,b} \sigma^{ij}_{ab}$. The presence of the coupling α between the fields leads to nonzero values of σ^{ij}_{FB}. Moreover, independently of whether the coupling vanishes or not it is important to keep the tensor structure of the kinetic and transport coefficients [29,30]. The analogous studies of the magnetotransport coefficients of Dirac semimetals [31] being the three-dimensional analogues of graphene require similar treatment of the conductivity. In both cases, in order to define the other transport coefficients, like thermoelectric tensor or Hall angle, one needs to take the full tensorial character of the conductivity into consideration.

In the studied action (1) we have to do with the second gauge field coupled to the ordinary Maxwell one. The justifications of such a kind of gravity with electromagnetism coupled to the other gauge field follow from the top-down perspective [43]. Namely, starting from the string/M theory the reduction to the lower dimensional gravity is performed. It is relevant in the holographic correspondence attitude, because the theory in question is a fully consistent quantum theory (string/M theory) and this fact guarantees that any predicted phenomenon by the top-down theory will be physical. This point has been discussed in [42].

Variation of the action S with respect to the metric, the scalar, and gauge fields yields the following equations of motion:

$$G_{\mu\nu} - g_{\mu\nu}\frac{3}{L^2} = T_{\mu\nu}(\phi) + T_{\mu\nu}(F) + T_{\mu\nu}(B) + \alpha T_{\mu\nu}(F, B), \tag{2}$$

$$\nabla_\mu F^{\mu\nu} + \frac{\alpha}{2}\nabla_\mu B^{\mu\nu} = 0, \tag{3}$$

$$\nabla_\mu B^{\mu\nu} + \frac{\alpha}{2}\nabla_\mu F^{\mu\nu} = 0, \tag{4}$$

$$\nabla_\mu \nabla^\mu \phi = 0, \tag{5}$$

where we have denoted by $G_{\mu\nu}$ the Einstein tensor, while the energy momentum tensors for the fields in the theory are given by

$$T_{\mu\nu}(\phi) = \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{4}g_{\mu\nu}\nabla_\delta\phi\nabla^\delta\phi, \tag{6}$$

$$T_{\mu\nu}(F) = \frac{1}{2}F_{\mu\delta}F_\nu{}^\delta - \frac{1}{8}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{7}$$

$$T_{\mu\nu}(B) = \frac{1}{2}B_{\mu\delta}B_\nu{}^\delta - \frac{1}{8}g_{\mu\nu}B_{\alpha\beta}B^{\alpha\beta}, \tag{8}$$

$$T_{\mu\nu}(F,B) = \frac{1}{2}F_{\mu\delta}B_\nu{}^\delta - \frac{1}{8}g_{\mu\nu}F_{\alpha\beta}B^{\alpha\beta}. \tag{9}$$

For the gauge fields in the considered theory we assume the following components:

$$A_\mu dx^\mu = a_t dt, \qquad B_\mu dx^\mu = b_t dt. \tag{10}$$

III. PERTURBATIONS OF BACKGROUND BLACK BRANE

In the following analysis we consider the line element provided by

$$ds^2 = -U(r)G(r,x_i)dt^2 + \frac{F(r,x_i)dr^2}{U(r)} + ds^2(\Sigma_2), \tag{11}$$

where Σ_2 stands for the two-dimensional hypersurface at chosen radial r coordinate. The dependence of function G and F on the x_i coordinates takes care of their spatial variations. We also take the following components of the fields:

$$A = a_t dt, \qquad B = b_t dt. \tag{12}$$

As in [34], the line element at $r \to \infty$ approaches the AdS boundary with the following conditions:

$$U \to r^2, \qquad F \to 1, \qquad G \to G(x), \qquad g_{ij} \to r^2\bar{g}_{ij} \tag{13}$$

$$a_t(r,x_i) \to \mu(x), \qquad b_t(r,x_i) \to \mu_d(x),$$
$$\phi(r,x_i) \to r^{\Delta-3}\bar{\phi}(x_i), \tag{14}$$

where $\mu(x)$ and $\mu_d(x)$ are the spatially dependent chemical potentials (at the boundary) connected with the adequate $U(1)$-gauge field. We also assume the periodic boundary conditions with period L_i in the ith direction $f(x_i + L_i) = f(x_i)$, and if required work with quantities averaged over the volume of periodicity $\mathbb{E}[f] = \frac{1}{L_{x_1}L_{x_2}}\int dx_1 dx_2 f$. $\bar{\phi}(x_i)$ above serves as a boundary source of the field $\phi(r,x_i)$ and Δ is the scaling dimension of it.

The black brane event horizon that has Σ_2 topology is situated at $r = 0$. Having in mind the Edington-Finkelstein ingoing coordinates, the near-horizon expansions of the metric tensor components and fields are given by [35]

$$U(r) = r(4\pi T + U^{(1)}r + \ldots), \tag{15}$$

$$G(r,x_i) = G^{(0)}(x) + G^{(1)}(x)r + \ldots, \tag{16}$$

$$F(r,x_i) = F^{(0)}(x) + F^{(1)}(x)r + \ldots, \tag{17}$$

$$g_{ij} = g_{ij}^{(0)} + g_{ij}^{(1)}r + \ldots, \tag{18}$$

$$a_t(r,x_i) = r(a_t^{(0)}G^{(0)}(x) + a_t^{(1)}(x)r + \ldots), \tag{19}$$

$$b_t(r,x_i) = r(b_t^{(0)}G^{(0)}(x) + b_t^{(1)}(x)r + \ldots), \tag{20}$$

$$\phi(r,x_i) = \phi^{(0)}(x) + \phi^{(1)}(x)r + \ldots, \tag{21}$$

with the auxiliary condition written as $G^{(0)}(x) = F^{(0)}(x)$.

If one implements the $U(1)$-gauge and temperature gradient in the black brane spacetime, at fixed r coordinate, then the black object responds. In our considerations we have to take into account linear perturbations described by [35]

$$\delta(ds^2) = \delta g_{\alpha\beta}\,dx^\alpha dx^\beta - 2tM\xi_a dt\,dx^a, \tag{22}$$

$$\delta A = \delta a_\beta\,dx^\beta - t\,E_a dx^a + t\,N\,\xi_b\,dx^b, \tag{23}$$

$$\delta B = \delta b_\beta\,dx^\beta - t\,B_a dx^a + t\,N_d\,\xi_b\,dx^b, \tag{24}$$

as well as the perturbation of scalar field, $\delta\phi$. In what follows we consider $\delta g_{\mu\nu}$, δa_μ, δb_μ, and $\delta\phi$ as functions of (r, x_m). On the other hand E_a, B_a, ξ_i, depend on x_i coordinates and are closed forms on Σ_2. Moreover, the regularity at the black brane event horizon implies the following:

$$\delta g_{tt} = U(r)(\delta g_{tt}^{(0)}(x_i) + \mathcal{O}(r)), \quad \delta g_{tr} = \delta g_{tr}^{(0)}(x_i) + \mathcal{O}(r), \tag{25}$$

$$\delta g_{rr} = \frac{1}{U(r)}(\delta g_{rr}^{(0)}(x_i) + \mathcal{O}(r)), \quad \delta g_{ij} = \delta g_{ij}^{(0)}(x_i) + \mathcal{O}(r), \tag{26}$$

$$\delta g_{ti} = \delta g_{ti}^{(0)}(x_i) - GU\xi_i\frac{\ln r}{4\pi T} + \mathcal{O}(r),$$
$$\delta g_{ri} = \frac{1}{U(r)}(\delta g_{ri}^{(0)}(x_i) + \mathcal{O}(r)), \tag{27}$$

$$\delta a_t = \delta a_t^{(0)}(x_i) + \mathcal{O}(r), \quad \delta a_i = \frac{\ln r}{4\pi T}(-E_i + N\xi_i) + \mathcal{O}(r), \tag{28}$$

$$\delta a_r = \frac{1}{U(r)}(\delta a_r^{(0)}(x_i) + \mathcal{O}(r)), \quad (29)$$

$$\delta b_t = \delta b_t^{(0)}(x_i) + \mathcal{O}(r), \quad \delta b_i = \frac{\ln r}{4\pi T}(-B_i + N_d \xi_i) + \mathcal{O}(r), \quad (30)$$

$$\delta b_r = \frac{1}{U(r)}(\delta b_r^{(0)}(x_i) + \mathcal{O}(r)). \quad (31)$$

It turns out that the constraint on the leading order has to be imposed,

$$\delta g_{tt}^{(0)} + \delta g_{rr}^{(0)} - 2\delta g_{rt}^{(0)} = 0, \qquad \delta g_{ri}^{(0)} = \delta g_{ti}^{(0)},$$
$$\delta a_r^{(0)} = \delta a_t^{(0)}, \qquad \delta b_r^{(0)} = \delta b_t^{(0)}. \quad (32)$$

A. Equations for perturbations at the event horizon

One imposes on a subset of the linearized black brane perturbations, i.e., $\delta g_{it}^{(0)}$, $\delta g_{rt}^{(0)}$, $\delta a_t^{(0)}$, $\delta b_t^{(0)}$, the relations as follows [42]:

$$\nabla_i \nabla^i w + \nabla_i E^i + \nabla_i (a_t^{(0)} v^i) + \frac{\alpha}{2}[\nabla_m \nabla^m w_d + \nabla_m B^m + \nabla_m (b_t^{(0)} v^m)] = 0, \quad (33)$$

$$\nabla_i \nabla^i w_d + \nabla_i B^i + \nabla_i (b_t^{(0)} v^i) + \frac{\alpha}{2}[\nabla_m \nabla^m w + \nabla_m E^m + \nabla_m (a_t^{(0)} v^m)] = 0, \quad (34)$$

$$b_t^{(0)}[\nabla_i w_d + B_i + \frac{\alpha}{2}(\nabla_i w + E_i)] + a_t^{(0)}[\nabla_i w + E_i + \frac{\alpha}{2}(\nabla_i w_d + B_i)] - \nabla_i \phi^{(0)} \nabla_m \phi^{(0)} v^m + 2\nabla^m \nabla_{(m} v_{i)} + 4\pi T \xi_i - \nabla_i p = 0, \quad (35)$$

$$\nabla_i v^i = 0, \quad (36)$$

where we have denoted

$$w = \delta a_t^{(0)}, \qquad w_d = \delta b_t^{(0)},$$
$$p = -4\pi T \frac{\delta g_{rt}^{(0)}}{G^{(0)}} - \delta g_{it}^{(0)} \nabla^i \ln G^{(0)}, \qquad v_i = -\delta g_{it}^{(0)}. \quad (37)$$

The above equations result from the conservation of charge and heat currents in the unperturbed system. The variables introduced in (37) are a subset of all perturbations. They are found to fulfil at the horizon the generalized Navier-Stokes equations (33)–(36). As discussed earlier for the single current model [35] the scalar field contributes the viscosity-like term as also does the curvature of the horizon. The latter is best visible by writing

$$2\nabla^m \nabla_{(m} v_{j)} = \nabla^2 v_j + R_{ji} v^i. \quad (38)$$

IV. CONDUCTIVITY FOR THE UNCHARGED BLACK BRANE

In this section we consider the case without ϕ field, responsible for dissipation. Further, for the connectedness with [8] we define the quantities

$$Q = a_t^{(0)}, \qquad Q_d = b_t^{(0)},$$
$$\nabla_j w = -\nabla_j \mu, \qquad \nabla_j w_d = -\nabla_j \mu_d. \quad (39)$$

Let us first study the conductivities for Dirac semimetals in the uncharged black object case, i.e., $Q = 0$, $Q_d = 0$. For the considered situation equation (33) and (34) decoupled to the relations

$$\nabla_i \left[\sqrt{g^{(0)}}(E^i - \nabla^i \mu)\right] = 0, \quad (40)$$

$$\nabla_i \left[\sqrt{g^{(0)}}(B^i - \nabla^i \mu_d)\right] = 0. \quad (41)$$

Because of the fact that they constitute the linear equations, we may set

$$\mu = \mu_a E^a, \qquad \mu_d = \mu_{da} B^a, \quad (42)$$

in Eqs. (40) and (41). This substitution reveals that

$$\nabla_i \left[\sqrt{g^{(0)}}(\delta^i{}_k - \nabla^i \mu_k) E^k\right] = 0, \quad (43)$$

$$\nabla_i \left[\sqrt{g^{(0)}}(\delta^i{}_k - \nabla^i \mu_{dk}) B^k\right] = 0. \quad (44)$$

On the other hand, it follows that for some constants $\psi^i{}_k$, $(\psi_d)^i_k$ and functions Ψ^i, $\Psi_d{}^i$, one obtains

$$\sqrt{g^{(0)}} g_{(0)}^{ij} (\delta^k{}_j - \nabla_j \mu^k) = \epsilon^{ij}(\psi^i{}_k - \nabla_j \Psi^k), \quad (45)$$

$$\sqrt{g^{(0)}} g_{(0)}^{ij} (\delta^k{}_j - \nabla_j \mu_d^k) = \epsilon^{ij}((\psi_d)^i_k - \nabla_j \Psi_d^k). \quad (46)$$

Using the properties of the antisymmetric two-dimensional tensor ϵ^{ij}, it can be proved that the above equations are equivalent to

$$-\epsilon^{am}(\delta^k{}_m - \nabla_m \mu^k) = \sqrt{g^{(0)}} g_{(0)}^{aj} (\psi^k{}_j - \nabla_j \Psi^k), \quad (47)$$

$$-\epsilon^{am}(\delta^k{}_m - \nabla_m \mu_d^k) = \sqrt{g^{(0)}} g_{(0)}^{aj} ((\psi_d)^k{}_j - \nabla_j \Psi_d^k). \quad (48)$$

Taking the spatial derivatives of the relations (47) and (48), using the uniqueness and linearity arguments, one obtains that

$$\nabla_i \Psi^k = \psi^k{}_j \nabla_i \mu^j, \tag{49}$$

$$\nabla_i \Psi^k_d = (\psi_d)^k{}_j \nabla_i \mu^j_d. \tag{50}$$

Combining relations (47) and (48) and (49) and (50), we arrive at the following:

$$\sqrt{g^{(0)}} g^{ij}_{(0)} (\delta^k{}_j - \nabla_j \mu^k) = -\epsilon^{ij} (\delta^r{}_j - \nabla_j \mu^r)(\psi^{-1})^k_r, \tag{51}$$

$$\sqrt{g^{(0)}} g^{ij}_{(0)} (\delta^k{}_j - \nabla_j \mu_d{}^k) = -\epsilon^{ij} (\delta^r{}_j - \nabla_j \mu^r)(\psi_d^{-1})^k_r. \tag{52}$$

Multiplying Eq. (51) by E_k and relation (52) by B_k, we arrive at the component of the gauge currents $J^i_{(F)}$, $J^i_{(B)}$, respectively. Having in mind that neglecting heat transport the gauge currents can be written as

$$J^i_{(F)} = \sigma^{ij}_{FF} E_j + \sigma^{ij}_{FB} B_j, \tag{53}$$

$$J^i_{(B)} = \sigma^{ij}_{BF} E_j + \sigma^{ij}_{BB} B_j, \tag{54}$$

we arrive at the conclusion that the DC conductivities constitute relations as follows:

$$\sigma^{ij}_{FF} = \epsilon^{im} \psi^j{}_m, \qquad \sigma^{ij}_{FB} = \frac{\alpha}{2} \epsilon^{im} \psi^j{}_m, \tag{55}$$

$$\sigma^{ij}_{BB} = \epsilon^{im} (\psi_d)^j{}_m, \qquad \sigma^{ij}_{BF} = \frac{\alpha}{2} \epsilon^{im} (\psi_d)^j{}_m. \tag{56}$$

With the inspection of Eqs. (45) and (46) and (51) and (52), we draw a conclusion that their consistency is ensured if $\psi_a{}^k \psi_k{}^b = -\delta_a{}^b$ and similarly $(\psi_d)_a{}^k (\psi_d)_k{}^b = -\delta_a{}^b$. On the other hand, the relations (55) and (56) enable us to write

$$\det \sigma_{FF} = \frac{1}{2!} \epsilon^{ab} \epsilon_{mk} (\sigma_{FF})_a{}^m (\sigma_{FF})_b{}^k = \frac{1}{2!} \epsilon^{ab} \epsilon_{mk} \psi_a{}^m \psi_b{}^k, \tag{57}$$

$$\begin{aligned}\det \sigma_{BB} &= \frac{1}{2!} \epsilon^{ab} \epsilon_{mk} (\sigma_{BB})_a{}^m (\sigma_{BB})_b{}^k \\ &= \frac{1}{2!} \epsilon^{ab} \epsilon_{mk} (\psi_d)_a{}^m (\psi_d)_b{}^k,\end{aligned} \tag{58}$$

$$\begin{aligned}\det \sigma_{FB} &= \frac{1}{2!} \epsilon^{ab} \epsilon_{mk} (\sigma_{FF})_a{}^m (\sigma_{FF})_b{}^k \\ &= \frac{1}{2!} \frac{\alpha^2}{4} \epsilon^{ab} \epsilon_{mk} \psi_a{}^m \psi_b{}^k,\end{aligned} \tag{59}$$

$$\begin{aligned}\det \sigma_{BF} &= \frac{1}{2!} \epsilon^{ab} \epsilon_{mk} (\sigma_{BB})_a{}^m (\sigma_{BB})_b{}^k \\ &= \frac{1}{2!} \frac{\alpha^2}{4} \epsilon^{ab} \epsilon_{mk} (\psi_d)_a{}^m (\psi_d)_b{}^k.\end{aligned} \tag{60}$$

Having in mind the consistency condition, mentioned above, one arrives at the following:

$$\det \sigma_{FF} = 1, \qquad \det \sigma_{BB} = 1, \tag{61}$$

which implies that

$$\det \sigma_{FB} = \frac{\alpha^2}{4}, \qquad \det \sigma_{BF} = \frac{\alpha^2}{4}. \tag{62}$$

Consequently, for the determinant of the conductivity in the theory under consideration, we obtain

$$\begin{aligned}\det \sigma &= \left(\frac{1}{2!}\right)^2 \beta \epsilon_{bj_1} \epsilon_{mk} \epsilon^{dj_2} \epsilon_{sz} \psi^m{}_b \psi^k{}_{j_1} (\psi_d)^s{}_d (\psi_d)^z{}_{j_2} \\ &= \beta \det \sigma_{FF} \det \sigma_{BB} = \beta,\end{aligned} \tag{63}$$

where we set

$$\beta = \tilde{\alpha} \left(1 + \frac{\alpha^2}{4}\right), \tag{64}$$

and $\tilde{\alpha} = 1 - \frac{\alpha^2}{4}$.

Let us assume that the conductivities σ_{ab}, a, $b = F$, B do not depend on the spatial directions. Under these circumstances they can be considered as scalars. However, the full conductivity σ of the system is the 2×2 matrix with entries $\{\sigma_{FF}, \sigma_{FB}; \sigma_{BF}, \sigma_{BB}\}$ and thus on the basis of Eqs. (61) and (62) one has that $\det \sigma = \sigma_{FF} \sigma_{BB} - \sigma_{FB} \sigma_{BF} = 1 - (\alpha/2)^4 = \beta$. It can be seen that the considered bound is also valid for $\alpha = 0$, when the matrix is diagonal, $\sigma = \mathrm{diag}\{\sigma_{FF}, \sigma_{BB}\}$.

V. CHARGED BLACK BRANE CASE

In order to study the charged case, let us define for the adequate $U(1)$-gauge field, bulk dual tensors [44]

$$\mathbb{F}^{rj} = -\frac{1}{2} \frac{\epsilon^{rjab}}{\sqrt{-g}} F_{ab}, \qquad \mathbb{B}^{rj} = -\frac{1}{2} \frac{\epsilon^{rjab}}{\sqrt{-g}} B_{ab}, \tag{65}$$

with the property $\epsilon^{rtxy} = \frac{1}{\sqrt{-g}}$, as $r \to \infty$. It implies that one can find constant dual currents densities connected with Maxwell and auxiliary gauge fields, in the boundary theory, which imply

$$I^i_{(F)} = \epsilon^{ij} \left(E_j + \frac{\alpha}{2} B_j\right), \qquad I^i_{(B)} = \epsilon^{ij} \left(B_j + \frac{\alpha}{2} E_j\right). \tag{66}$$

The equations of motion $\partial_i J^i_{(F)} = 0, \qquad \partial_i J^i_{(B)} = 0$ yield that

$$\partial_r \mathbb{E} \left[\mathbb{F}^{ir} + \frac{\alpha}{2} \mathbb{B}^{ir}\right] = 0, \tag{67}$$

$$\partial_r \mathbb{E} \left[\mathbb{B}^{ir} + \frac{\alpha}{2} \mathbb{F}^{ir}\right] = 0, \tag{68}$$

where we have denoted the spatial average by $\mathbb{E}[M] = \frac{1}{L^2}\int dx^2 M$ for the coordinates that satisfy the periodic boundary conditions $x_i \to x_i + L$.

Just using the duals, at $r \to 0$, we get

$$\mathbb{E}\left[\mathbb{F}^{it} + \frac{\alpha}{2}\mathbb{B}^{it}\right] = \epsilon^{im} J_{(F)m}, \tag{69}$$

$$\mathbb{E}\left[\mathbb{B}^{it} + \frac{\alpha}{2}\mathbb{F}^{it}\right] = \epsilon^{im} J_{(B)m}. \tag{70}$$

The spatial averaged dual electric currents connected with Maxwell and auxiliary fields are independent of the radius of the bulk. It means that they can be defined on the black object event horizon

$$\mathbb{E}\left[\mathbb{F}^{it} + \frac{\alpha}{2}\mathbb{B}^{it}\right] = \mathcal{E}_j + \frac{\alpha}{2}\mathcal{B}_j = \tau^{jk}_{(F)} I_{(F)k}, \tag{71}$$

$$\mathbb{E}\left[\mathbb{B}^{it} + \frac{\alpha}{2}\mathbb{F}^{it}\right] = \mathcal{B}_j + \frac{\alpha}{2}\mathcal{E}_j = \tau^{jk}_{(B)} I_{(B)k}, \tag{72}$$

where $\tau^{jk}_{(F)}$ and $\tau^{jk}_{(B)}$ are the dual resistivity tensors bounded with Maxwell and hidden sector gauge fields. One can relate $\mathcal{E}_i$, $\mathcal{B}_i$, $I^i_{(F)}$, $I^i_{(B)}$, $J^k_{(F)}$, $J^k_{(B)}$ and obtain

$$J_{(F)a} = \epsilon_{ia}\epsilon_{jk}\tau^{ij}_{(F)}\left(E^k + \frac{\alpha}{2}B^k\right), \tag{73}$$

$$J_{(B)a} = \epsilon_{ia}\epsilon_{jk}\tau^{ij}_{(B)}\left(B^k + \frac{\alpha}{2}E^k\right). \tag{74}$$

It leads to the following relations:

$$\sigma^{ij}_{FF} = \epsilon^{mi}\epsilon^{nj}\tau^{(F)}_{mn}, \tag{75}$$

$$\sigma^{ij}_{FB} = \epsilon^{mi}\epsilon^{nj}\tau^{(F)}_{mn}\frac{\alpha}{2}, \tag{76}$$

$$\sigma^{ij}_{BB} = \epsilon^{mi}\epsilon^{nj}\tau^{(B)}_{mn}, \tag{77}$$

$$\sigma^{ij}_{BF} = \epsilon^{mi}\epsilon^{nj}\tau^{(B)}_{mn}\frac{\alpha}{2}. \tag{78}$$

Consequently, it can be found that

$$\det\sigma_{FF} = \frac{1}{2!}\epsilon^{mi}\epsilon^{nj}\tau^{(F)}_{mn}, \tag{79}$$

$$\det\sigma_{FB} = \frac{1}{2!}\frac{\alpha^2}{4}\epsilon^{mi}\epsilon^{nj}\tau^{(F)}_{mn}, \tag{80}$$

$$\det\sigma_{BB} = \frac{1}{2!}\epsilon^{mi}\epsilon^{nj}\tau^{(B)}_{mn}, \tag{81}$$

$$\det\sigma_{BF} = \frac{1}{2!}\frac{\alpha^2}{4}\epsilon^{mi}\epsilon^{nj}\tau^{(B)}_{mn}. \tag{82}$$

As in [8] we assume that the boundary theory constitutes the particle vortex dual, which leads to the conjecture that

$$\det\sigma_{FF} = \frac{1}{\det\tau_{(F)}}, \qquad \det\sigma_{BB} = \frac{1}{\det\tau_{(B)}}, \tag{83}$$

which in turn precedes the conditions

$$(\det\tau_{(F)})^2 = 1, \qquad (\det\tau_{(B)})^2 = 1. \tag{84}$$

By virtue of the above relations, in the charge case the determinant of the conductivity is given by

$$\begin{aligned}\det\sigma &= \left(\frac{1}{2!}\right)^2 \beta_1 \epsilon_{i_1k}\epsilon_{j_1l}\epsilon_{i_2a}\epsilon_{j_2b}\tau^{i_1j_1}_{(F)}\tau^{kl}_{(F)}\tau^{i_2j_2}_{(B)}\tau^{ab}_{(B)}\\ &= \beta_1\ \det\tau_{(F)}\ \det\tau_{(B)} = \beta_1,\end{aligned} \tag{85}$$

where β_1 is given as follows:

$$\beta_1 = \tilde{\alpha}\left(1 + \frac{\alpha^2}{4}\right). \tag{86}$$

The bound we have obtained in the charged case is the same as in the uncharged model found earlier. The parameter β_1 is a monotonously diminishing function of the coupling α from its canonical value 1, when $\alpha \to 0$ and to the value 0 for $\alpha \to 2$. On physical grounds one expects $\alpha \leq 1$. The theory predicts the lowering of the bound from its value 1 towards ≈ 0.94 at $\alpha = 1$.

VI. VARIATIONAL ATTITUDE

This section is devoted to the variational techniques implemented in order to establish the lower bounds on DC conductivities. The bounds are achieved in an analogous way as the upper bounds of resistance of a disordered resistor network, based on Thomson's principle [45,46]. It states that if one runs a set of "trial" currents through a resistor network, being subject to certain boundary conditions, the upper bound of the inverse conductivity can be computed by applying the variational principle to the power dissipated by the trial currents in question. It happens that the power dissipated by trial currents is minimal for the true distribution of the aforementioned currents.

To proceed further, let us recall that the Stokes equation on the black brane event horizon can be recast in the form as derived in Ref. [42],

$$\begin{aligned}&\int \sqrt{g^{(0)}}d^2x[2\nabla^{(i}v^{j)}\nabla_{(i}v_{j)} + (\nabla_i w + E_i)(\nabla^i w + E^i)\\ &\quad + (\nabla_i w_d + B_i)(\nabla^i w_d + B^i)\\ &\alpha(\nabla_i w + E_i)(\nabla^i w_d + B^i) + v^m\nabla_m\phi^{(0)}\nabla_j\phi^{(0)}v^j]\\ &\quad = \int d^2x[Q^{i(0)}\xi_i + J^{i(0)}_{(F)}E_i + J^{i(0)}_{(B)}B_i],\end{aligned} \tag{87}$$

with the adequate definitions of the currents, given by

$$J^{i(0)}_{(F)} = J^I_{(F)}|_{\mathcal{H}} = \sqrt{g^{(0)}}g^{ij}_{(0)}[(\nabla_j(\delta a_t^{(0)}) + E_j - a_t^{(0)}\delta g_{tj}^{(0)}) + \frac{\alpha}{2}(\nabla_j(\delta b_t^{(0)}) + B_j - b_t^{(0)}\delta g_{tj}^{(0)})], \tag{88}$$

$$J^{i(0)}_{(B)} = J^I_{(B)}|_{\mathcal{H}} = \sqrt{g^{(0)}}g^{ij}_{(0)}[(\nabla_j(\delta b_t^{(0)}) + B_j - b_t^{(0)}\delta g_{tj}^{(0)}) + \frac{\alpha}{2}(\nabla_j(\delta a_t^{(0)}) + E_j - a_t^{(0)}\delta g_{tj}^{(0)})], \tag{89}$$

$$Q^{i(0)} = Q^i|_{\mathcal{H}} = -4\pi T\sqrt{g^{(0)}}g^{ij}_{(0)}\delta g^{(0)}_{tj}. \tag{90}$$

Additionally one has that the following conservation relations are fulfilled:

$$\nabla_i J^{i(0)}_{(F)} = 0, \qquad \nabla_i J^{i(0)}_{(B)} = 0, \qquad \nabla_i Q^{i(0)} = 0. \tag{91}$$

Further, let us define

$$J^{i(0)}_{(F)} = \sqrt{g^{(0)}}\mathcal{J}^i_F, \qquad J^{i(0)}_{(F)} = \sqrt{g^{(0)}}\mathcal{J}^i_F, \qquad Q^{i(0)} = 4\pi T\sqrt{g^{(0)}}v^i. \tag{92}$$

Consequently the relation (87) may be rewritten in the form

$$\begin{aligned}\int d^2x[Q^{i(0)}\xi_i + J^{i(0)}_{(F)}E_i + J^{i(0)}_{(B)}B_i] = \int \sqrt{g^{(0)}}d^2x\Big[&2\nabla^{(i}v^{j)}\nabla_{(i}v_{j)} + v^m\nabla_m\phi^{(0)}\nabla_j\phi^{(0)}v^j \\ &+ \left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}^i_F - \frac{\alpha}{2}\mathcal{J}^i_B\right) - a_t^{(0)}v^i\right]\left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}_{iF} - \frac{\alpha}{2}\mathcal{J}_{iB}\right) - a_t^{(0)}v_i\right] \\ &+ \left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}^i_B - \frac{\alpha}{2}\mathcal{J}^i_F\right) - b_t^{(0)}v^i\right]\left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}_{iB} - \frac{\alpha}{2}\mathcal{J}_{iF}\right) - b_t^{(0)}v_i\right] \\ &+ \alpha\left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}^i_F - \frac{\alpha}{2}\mathcal{J}^i_B\right) - a_t^{(0)}v^i\right]\left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}_{iB} - \frac{\alpha}{2}\mathcal{J}_{iF}\right) - b_t^{(0)}v_i\right]\Big],\end{aligned} \tag{93}$$

which is the subject of the following analysis.

A. Bound on conductivities

In order to establish the bounds on the conductivities in the holographic model of graphene, we analyze the left-hand side of Eq. (93), which includes the definition of the dissipated power. The dissipated power is provided by the following expression:

$$P = J^i_F E_i + J^i_B B_i + Q^i\xi_i, \tag{94}$$

where the above quantities are normalized by averaging them spatially over the black brane event horizon, i.e.,

$$J^i_F = \mathbb{E}[J^{i(0)}_{(F)}], \qquad J^i_B = \mathbb{E}[J^{i(0)}_{(B)}], \qquad Q^i = \mathbb{E}[Q^{i(0)}]. \tag{95}$$

In what follows, we consider compact and flat spatial dimensions of the dual theory.

Using Eq. (93) the dissipative power (94) implies

$$\begin{aligned}P = \mathbb{E}\Big[&2\nabla^{(i}v^{j)}\nabla_{(i}v_{j)} + v^m\nabla_m\phi^{(0)}\nabla_j\phi^{(0)}v^j + \left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}^i_F - \frac{\alpha}{2}\mathcal{J}^i_B\right) - a_t^{(0)}v^i\right]\left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}_{iF} - \frac{\alpha}{2}\mathcal{J}_{iB}\right) - a_t^{(0)}v_i\right] \\ &+ \left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}^i_B - \frac{\alpha}{2}\mathcal{J}^i_F\right) - b_t^{(0)}v^i\right]\left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}_{iB} - \frac{\alpha}{2}\mathcal{J}_{iF}\right) - b_t^{(0)}v_i\right] \\ &+ \alpha\left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}^i_F - \frac{\alpha}{2}\mathcal{J}^i_B\right) - a_t^{(0)}v^i\right]\left[\frac{1}{\tilde{\alpha}}\left(\mathcal{J}_{iB} - \frac{\alpha}{2}\mathcal{J}_{iF}\right) - b_t^{(0)}v_i\right]\Big].\end{aligned} \tag{96}$$

One can consider P as a functional of v^i and the $U(1)$-gauge currents. It means that for an arbitrary conserved periodic set of charge and heat currents directed along v^i one has that

$$\mathcal{J}^i_F = \tilde{\mathcal{J}}^i_F + \tilde{\tilde{\mathcal{J}}}{}^i_F, \qquad \mathcal{J}^i_B = \tilde{\mathcal{J}}^i_B + \tilde{\tilde{\mathcal{J}}}{}^i_B, \qquad v^i = \tilde{v}^i + \tilde{\tilde{v}}{}^i, \tag{97}$$

where $(\tilde{v}^i, \tilde{\mathcal{J}}{}^i_F, \tilde{\mathcal{J}}{}^i_B)$ stands for the exact solution of the underlying system of hydrodynamical equations, being subject to the adequate boundary conditions. On the other hand, $(\tilde{\tilde{v}}{}^i, \tilde{\tilde{\mathcal{J}}}{}^i_F, \tilde{\tilde{\mathcal{J}}}{}^i_B)$ denote the deviations from the exact solution. Expansion of P reveals that we get

$$P[v^i, \mathcal{J}^i_F, \mathcal{J}^i_B] = P[\tilde{v}^i + \tilde{\tilde{v}}{}^i, \tilde{\mathcal{J}}^i_F + \tilde{\tilde{\mathcal{J}}}{}^i_F, \tilde{\mathcal{J}}^i_B + \tilde{\tilde{\mathcal{J}}}{}^i_B] = P[\tilde{v}^i, \tilde{\mathcal{J}}^i_F, \tilde{\mathcal{J}}^i_B] + P[\tilde{\tilde{v}}{}^i, \tilde{\tilde{\mathcal{J}}}{}^i_F, \tilde{\tilde{\mathcal{J}}}{}^i_B] + 2K, \tag{98}$$

where the quantity $2K$ implies

$$\begin{aligned} 2K =& 2\left[\frac{1}{\tilde{\alpha}}\left(\tilde{\mathcal{J}}^i_F - \frac{\alpha}{2}\mathcal{J}^i_B\right) - a_t^{(0)}\tilde{v}^i\right]\left[\frac{1}{\tilde{\alpha}}\left(\tilde{\tilde{\mathcal{J}}}_{iF} - \frac{\alpha}{2}\tilde{\tilde{\mathcal{J}}}_{iB}\right) - a_t^{(0)}\tilde{\tilde{v}}_i\right] + 2\left[\frac{1}{\tilde{\alpha}}\left(\tilde{\mathcal{J}}^i_B - \frac{\alpha}{2}\mathcal{J}^i_F\right) - a_t^{(0)}\tilde{v}^i\right]\left[\frac{1}{\tilde{\alpha}}\left(\tilde{\tilde{\mathcal{J}}}_{iB} - \frac{\alpha}{2}\tilde{\tilde{\mathcal{J}}}_{iF}\right) - a_t^{(0)}\tilde{\tilde{v}}_i\right] \\ &+ \left[\frac{1}{\tilde{\alpha}}\left(\tilde{\mathcal{J}}^i_F - \frac{\alpha}{2}\mathcal{J}^i_B\right) - a_t^{(0)}\tilde{v}^i\right]\left[\frac{1}{\tilde{\alpha}}\left(\tilde{\tilde{\mathcal{J}}}_{iB} - \frac{\alpha}{2}\tilde{\tilde{\mathcal{J}}}_{iF}\right) - a_t^{(0)}\tilde{\tilde{v}}_i\right] + \left[\frac{1}{\tilde{\alpha}}\left(\tilde{\mathcal{J}}^i_B - \frac{\alpha}{2}\mathcal{J}^i_F\right) - a_t^{(0)}\tilde{v}^i\right]\left[\frac{1}{\tilde{\alpha}}\left(\tilde{\tilde{\mathcal{J}}}_{iB} - \frac{\alpha}{2}\tilde{\tilde{\mathcal{J}}}_{iF}\right) - a_t^{(0)}\tilde{\tilde{v}}_i\right] \\ &+ 4\nabla^{(i}\tilde{v}^{j)}\nabla_{(i}\tilde{\tilde{v}}_{j)} + 2\tilde{\tilde{v}}^m\nabla_m\phi^{(0)}\nabla_i\phi^{(0)}\tilde{v}^i. \end{aligned} \tag{99}$$

It can be shown using the current equations and integration by parts that $K = 0$. Consequently, it reveals that

$$P[v^i, \mathcal{J}^i_F, \mathcal{J}^i_B] \geq P[\tilde{v}^i, \tilde{\mathcal{J}}^i_F, \tilde{\mathcal{J}}^i_B]. \tag{100}$$

As was explained in [8], for the charged black brane one may set $v_i = 0$, which trivially fulfils the constraints equations. Then we arrive at

$$P[0, \mathcal{J}^i_F, \mathcal{J}^i_B] = \int \sqrt{g^{(0)}} d^2x \frac{\epsilon^{i_1}{}_a \epsilon^{j_1}{}_b}{\det\sigma}[\sigma^{BB}_{i_1 j_1}\mathcal{J}^a_F\mathcal{J}^b_F + \sigma^{FF}_{i_1 j_1}\mathcal{J}^a_B\mathcal{J}^b_B - (\sigma^{FB}_{i_1 j_1} + \sigma^{BF}_{i_1 j_1})\mathcal{J}^a_F\mathcal{J}^b_B]. \tag{101}$$

On the other hand, using Eq. (96), one arrives at the following:

$$P[0, \mathcal{J}^i_F, \mathcal{J}^i_B] = \int \sqrt{g^{(0)}} d^2x \left[\frac{1}{\tilde{\alpha}}(\mathcal{J}^i_F)^2 + \frac{1}{\tilde{\alpha}}(\mathcal{J}^i_B)^2 - \frac{\alpha}{\tilde{\alpha}}\mathcal{J}^i_F\mathcal{J}_{iB}\right]. \tag{102}$$

Comparison of Eqs. (101) and (102) gives us the conditions imposed on the electrical conductivities, in the general case.

To commence with, let us analyze limits of the obtained relations. First one supposes that in the absence of the heat current, we consider only the single current case. In this case $B_i = 0$, $\alpha = 0$ and the relations (53) and (54) reveal that

$$\sigma^{ij}_{BF} = 0, \qquad \sigma^{ij}_{FB} = 0, \qquad \sigma^{ij}_{BB} = 0, \tag{103}$$

and $E^i = \frac{\mathcal{J}^i_F}{\sigma_{FF}}$. Taking into account (53) and calculating the dissipative power we get

$$P = \int \sqrt{g^{(0)}} d^2x \frac{\mathcal{J}^i_F\mathcal{J}_{iF}}{\sigma_{FF}} = \int \sqrt{g^{(0)}} d^2x \mathcal{J}^i_F\mathcal{J}_{iF}. \tag{104}$$

It implies that the following relation takes place:

$$\sigma_{FF} \geq 1. \tag{105}$$

Consequently, for the model with only auxiliary $U(1)$-gauge field, one has that

$$E_i = 0, \qquad \sigma^{ij}_{FB} = 0, \tag{106}$$

and $B_i = \frac{\mathcal{J}^i_B}{\sigma_{BB}}$. The same reasoning as above leads to the relation

$$P = \int \sqrt{g^{(0)}} d^2x \frac{\mathcal{J}^i_B\mathcal{J}_{iB}}{\sigma_{BB}} = \frac{1}{\tilde{\alpha}}\int \sqrt{g^{(0)}} d^2x \mathcal{J}^i_B\mathcal{J}_{iB}, \tag{107}$$

and it yields that

$$\sigma_{BB} \geq \tilde{\alpha}. \tag{108}$$

In the next step, because of the complexity of the exact relations, let us suppose the existence only of $\mathcal{J}^x_F$ and $\mathcal{J}^x_B$ currents. By straightforward calculations it can be envisaged that $P[0, \mathcal{J}^i_F, \mathcal{J}^i_B]$ reduces to

$$P[0,\mathcal{J}_F^i,\mathcal{J}_B^i] = \int \sqrt{g^{(0)}} d^2x \frac{1}{\det\sigma} [\sigma_{yy}^{BB} (\mathcal{J}_F^x)^2 + \sigma_{yy}^{FF} (\mathcal{J}_B^x)^2 - (\sigma_{yy}^{FB} + \sigma_{yy}^{BF}) \mathcal{J}_F^x \mathcal{J}_B^x]. \quad (109)$$

Comparing relations (109) and (102), the estimations for the adequate components of $\sigma_{\alpha\beta}^{ij}$ tensor can be achieved,

$$\frac{\sigma_{yy}^{BB}}{\det\sigma} = \frac{1}{\tilde{\alpha}}, \qquad \frac{\sigma_{yy}^{FF}}{\det\sigma} = \frac{1}{\tilde{\alpha}}, \qquad \frac{\sigma_{yy}^{FB} + \sigma_{yy}^{BF}}{\det\sigma} = \frac{\alpha}{\tilde{\alpha}}. \quad (110)$$

VII. SUMMARY AND CONCLUSIONS

In our paper we have studied the lower bounds of the electrical conductivities in the holographic model of the strongly interacting two-dimensional graphene sheet with disorder by means of the gauge-gravity duality. It happens that graphene close to the particle-hole symmetry point is a laboratory system fulfilling the strong coupling requirements. On the gravity side we elaborate the Einstein-Maxwell theory supplemented by the auxiliary $U(1)$-gauge field. The ordinary Maxwell and the auxiliary fields are coupled by the *kinetic mixing* term, with a coupling constant α. In the studies we pay attention to the linear response of the black brane to the electric fields of the aforementioned gauge fields. On the field theory side, the situation coincides with the existence of two transport currents, which in graphene may correspond to electron and hole currents. The mixing parameter α may be responsible for the phase space constraints of scattering events in the system with Dirac spectrum.

We have found the modifications of the bounds due to the coupling between the currents. For the physically expected values of the α-coupling constant that is smaller than 1, the obtained bound β for the conductivity tensor σ, $\det\sigma = \beta$ is only slightly below 1.

It would be of interest to analyze the existence of the similar bound in Dirac or Weyl semimetals, being the three-dimensional analogues of graphene.

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