Kähler moduli stabilization and the propagation of decidability

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Diophantine equations are in general undecidable, yet appear readily in string theory. We demonstrate that numerous classes of Diophantine equations arising in string theory are decidable and propose that decidability may propagate through networks of string vacua due to additional structure in the theory. Diophantine equations arising in index computations relevant for D3-instanton corrections to the superpotential exhibit propagation of decidability, with new and existing solutions propagating through networks of geometries related by topological transitions. In the geometries we consider, most divisor classes appear in at least one solution, significantly improving prospects for Kähler moduli stabilization across large ensembles of string compactifications.

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I. INTRODUCTION

Many problems arising in string theory are computationally hard, which could be dynamically relevant. While a thorough investigation of the question of how hard they are is still in its infancy, a number of results are known. In [1], which introduced computational complexity into studies of the string landscape, it was shown that a version of the Bousso-Polchinski model [2] for the cosmological constant is NP complete, where NP (Non-deterministic polynomial time) is the set of decision problems for which the problem instances, where the answer is "yes", can be verified in polynomial time. Furthermore, computing the scalar potential in string theory, finding its critical points, and establishing that they are metastable vacua all require solving NP-hard or co-NP-hard problems [3]. Within the last 2 years, machine learning was introduced into the study of string theory problems [4–7] to make progress on computationally hard problems.

Even worse, problems arising in physics may be undecidable. In this paper, we focus on the Diophantine undecidability, which is the solution to Hilbert's tenth problem. In modern language, the problem is to find an algorithm that solves the following decision problem: *HILBERT* Given a polynomial $D(x_1, \ldots, x_s) = 0$ with integer coefficients, does there exist a solution in the integers? $D(x_1, \ldots, x_s) = 0$ is a Diophantine equation. Famously, no such algorithm exists, and therefore HILBERT is undecidable. The proof, which shows that every recursively enumerable set is Diophantine, is due to Matiyasevich [8], building on work [9] of Robinson, Davis, and Putnam; the complete result is known as the MRDP theorem.

The way in which Diophantine equations arise in string compactifications is by the appearance of integer topological data, such as Chern classes, that describe the compactification space, internal gauge fluxes, and the cycles wrapped by branes. Since these data fix some physical observables of the string theory compactification, decision problems involving Diophantine equations abound in studies of the string landscape. Even more broadly, by the MRDP theorem *any* decidable or partially decidable decision problem can be turned into a Diophantine equation, and therefore any yes/no question one might ask about the physics of string theory implicitly introduces another Diophantine equation into its study.

The presence of computationally hard or undecidable problems can affect the dynamics of the underlying system. For instance, folding proteins [10,11] and relaxing spin glasses [12,13] are both associated with NP-hard problems, which correlate with the existence of proteins and spin glasses that relax on timescales exponential in the system size. Such considerations can give an increased understanding of the dynamics and necessitate additional explanations, including why our proteins fold quickly, while general proteins do not. For physics examples, the question can be posed of how the Universe "found" the state it is currently in if this requires solving a hard or undecidable problem. For instance, when applied to finding a small cosmological constant in an eternally inflating universe

[1,3,14–16], this gives a measure that differs significantly from others [17–20].

Given that Diophantine equations arise in string theory in numerous ways and also the size [2,21–25] of the string landscape, one might imagine that the resulting equations can be diverse enough to be up against Diophantine undecidability [26]. Of course, this formal concern is notoriously difficult to quantify, since we do not yet have a handle on all of the Diophantine problems of physical interest in string theory.

The focus of this work is to study cases in which Diophantine equations from string theory are easier to decide than the MRDP theorem might naively suggest. That is, given a class of seemingly intractable Diophantine equations that arise for a physical problem in string theory, does the additional structure of the theory render them decidable? We will see this is sometimes the case.

Since there may be transitions between string vacua/compactifications, for instance, due to topology or flux change, it also natural to consider the landscape as a network where edges represent the transitions; see, e.g., [20]. In this framework, Diophantine equations on one node may be related to Diophantine equations on another, in which case it is natural to study if decidability of one implies decidability of the other. We refer to this as the propagation of decidability through the network.

Though we analyze the decidability of many problems in string theory, our primary focus is on one important example, for the sake of concreteness. We study the stabilization of Kähler moduli in string compactifications. These scalar fields are massless at tree level, but may be stabilized by nonperturbative corrections to the superpotential, specifically world-sheet instantons in heterotic theories or Euclidean D3-instantons (E3) in the type IIB/F-theory compactifications that we study. The existence and structure of the superpotential correction depend crucially on the spectrum of instanton zero modes, of which there are many, but we focus on a constraint [27] on the holomorphic Euler character $\chi(\hat{D}, \mathcal{O}_{\hat{D}}) = 1$, where $\hat{D}$ is a divisor in an elliptically fibered Calabi-Yau fourfold $X \xrightarrow{\pi} B$. This constraint is equivalent [28] to an index related to a divisor $D \subset B$ where $\hat{D} = \pi^{-1}(D)$,

$$\chi_{\mathrm{E3}} = -\frac{1}{2}\int_B c_1(B) \wedge D \wedge D = 1, \qquad (1)$$

with Poincaré duality implied. Expanding D in an integral basis D_i of the second cohomology $H^{1,1}(X, \mathbb{Z})$,

$$D = \sum_{i=1}^{h^{1,1}} n_i D_i, \qquad (2)$$

it may be expressed as a quadratic Diophantine equation in the integers n_i, which is the central object of our study.

Our main result is that networks of base geometries B connected by topological transitions realize the propagation of decidability. Specifically, for a geometry $\tilde{B}$ that is a blowup of another base B, solutions to $\chi_{\mathrm{E3}} = 1$ on B significantly aid in determining solutions to $\tilde{\chi}_{\mathrm{E3}} = 1$ on $\tilde{B}$. This applies to the largest known ensembles of such bases, which exhibit similar physical features and have 2.96×10^{755} and $O(10^{3000})$ geometries [24,25]. Clearly, these extremely large (but finite) networks are intractable by brute force techniques, but propagation of decidability nevertheless allows for concrete statements about instanton solutions. Furthermore, one ensemble [24] may be randomly sampled from a uniform distribution, which is utilized to show that on average 99.2% percent of divisor classes appear as a component in some divisor with $\chi_{\mathrm{E3}} = 1$, even utilizing only the simplest solutions, suggesting that (up to thoroughly discussed caveats) Kähler moduli stabilization across large ensembles of string compactifications may be easier than naively expected.

Our analytic derivations relating solutions on $\tilde{B}$ and B likely give rise to many more solutions in the networks, but some will depend on details of large sequences of blowups that introduce model dependence. We leave a statistical analysis of this type for future work.

This paper is organized as follows. In Sec. II, we review known theorems about Diophantine equations, including decidability of certain cubics and all quadratics. We use them to show that numerous physical Diophantine problems, often of a geometric nature, are decidable. In Sec. III, we study the E3 index on varieties related by blowup and use associated recursion relations that demonstrate the propagation of solutions. In Sec. IV, we derive concrete implications of the results of Sec. III for Kähler moduli stabilization in type IIB/F-theory compactifications. In Sec. V, we conclude.

II. PHYSICS IMPLICATIONS OF KNOWN DIOPHANTINE RESULTS

In this section, we discuss numerous physical applications in string theory of known Diophantine results.

Physically relevant Diophantine equations in string theory are often of relatively low degree. This arises because compactification of the extra dimensions in string, M-, or F-theory often involves a complex algebraic variety X with $\dim_{\mathbb{C}}(X) \leq n$, and the Diophantines can arise from intersection theory on X. Such Diophantines are of degree n or less. It is sometimes the case that the intersection of interest on X can be realized within a fixed subvariety V with $\dim_{\mathbb{C}}(X) \leq d$, in which case the degree is d or below. We will restrict to the case of quadratic and cubic Diophantine equations[1] and then discuss their application in string theory.

[1]Linear Diophantine equations can be solved in polynomial time [29].

In fact, any set of Diophantine equations can be written as a single quartic Diophantine equation [30], albeit in (many) more variables. This is based on the observation that an arbitrary Diophantine equation can be written as a system of quadratic Diophantine equations by introducing auxiliary variables. Then, a set of quadratic Diophantine equations can be turned into a single quartic Diophantine equation by taking the sum of the squares of each individual quadratic equation; since the squares are nonnegative, the resulting equation will have a solution iff the original set of quadratic Diophantine equations had a solution. However, given the MRDP result, this makes it very hard to make general statements about quartic Diophantine equations.

A. Quadratic Diophantines in string theory

A quadratic Diophantine equation is of the form

$$Q(x_1, \ldots, x_s) = 0, \tag{3}$$

where the polynomial $Q(x_1, \ldots, x_s) \in \mathbb{Z}[x_1, \ldots, x_s]$ is of degree 2. The equation may be rewritten as

$$a_{ij}x_i x_j + h_i x_i = n, \tag{4}$$

and $H = \max\{|a_{ij}|, |h_i|, n\}$ is known as the height.

Quadratic Diophantine equations are decidable due to a result of Siegel [31]. One way in which this arises is due to the existence of search bounds. $\Lambda_s(H)$ is a search bound if the existence of an integral solution $(x_1, \ldots, x_s)$ to (4) requires that there is a solution with $|x_i| \leq \Lambda_s(H)$ for $1 \leq i \leq s$. Siegel proved that there is a search bound for any number of variables s, and thus quadratic Diophantine equations are decidable. Though Siegel's search bounds grow exponentially in H, later results [32] demonstrate the existence of search bounds that are polynomial in H for all $s \geq 3$. That is, one can always decide existence of a solution to a quadratic Diophantine equation by searching through a finite set of possibilities that grows polynomially (or exponentially for $s = 2$) in the maximum of the absolute values of its coefficients.

Despite being decidable, determining whether there is a solution to a quadratic Diophantine equation is hard. For instance, even in the two-variable case

$$ax_1^2 + bx_2 + c = 0, \tag{5}$$

determining whether there is a solution $(x, y) \in \mathbb{Z}^2$, it is NP complete via reduction from 3SAT [33].

A few examples of physically relevant quadratic Diophantines, to which these results can be applied, are as follows:

(i) D3 charge: Seven branes with gauge bundles give rise to induced D3-brane charges that appear in consistency conditions for the theory.

Specifically, given a type IIB/F-theory compactification on a Calabi-Yau threefold X with a seven brane on a divisor D and a target D3-brane charge $T \in \mathbb{Z}$, is there a worldvolume flux L (a line bundle) that induces a D3-brane charge T? This may be studied via an integral parametrization of L as

$$c_1(L) = \sum_i n_i \sigma_i, \tag{6}$$

where σ_i is a basis for $H^{1,1}(D, \mathbb{Z})$. Then the answer to the decision problem is yes if and only if

$$\int_D \mathrm{ch}_2(L) = T, \tag{7}$$

where a standard computation expresses the left-handed size as a quadratic Diophantine equation in the variables n_i.

(ii) Chiral three to seven modes: Instanton zero modes crucially affect the structure of nonperturbative corrections to the $4d$ $\mathcal{N} = 1$ superpotential. Some of these zero modes are so-called three to seven strings [34] between a Euclidean D3 instanton and a spacetime filling seven brane, and the correction depends crucially on whether there are chiral modes of this type [35–37].

Specifically, given a type IIB/F-theory compactification on a Calabi-Yau threefold X with a seven brane on a divisor D, is there a divisor $\tilde{D}$ such that a Euclidean D3 brane on $\tilde{D}$ with instanton flux L gives rise to no chiral three to seven modes at the intersection $C = D \cdot \tilde{D}$? Integrally, parametrizing D as a linear combination $D = m_i D_i$ of effective divisors D_i and L in a way similar to before, answering the question is equivalent to solving

$$\int_C \mathrm{Td}(C)\mathrm{ch}(L|_C \otimes K_C^{1/2}) = 0, \tag{8}$$

where the left-hand side may be expressed as a quadratic Diophantine equation in n_i, m_i. Related issues are discussed in [26].

(iii) Bianchi identities: Given a heterotic string compactification on a Calabi-Yau threefold X with a vector bundle V, we need to solve the Bianchi identities for the three-form field H, which can be written as

$$\mathrm{ch}_2(X) - \mathrm{ch}_2(V) = 0. \tag{9}$$

In many cases that have been studied, the bundle V is described as a sum of line bundles, a monad bundle of line bundle sums, or an extension bundle of line bundles. In all cases, we can specify the line bundles via their first Chern classes on the $h^{1,1}(X)$ divisors

D_i of X, $L_i = \mathcal{O}_X(k_i^1, \ldots, k_i^{h^{1,1}})$. The Bianchi identities then become a (set of coupled) quadratic Diophantine equations in the integers k_i^a (note that decidability as discussed above only applies to a single quadratic Diophantine equation, not to a system of equations).

(iv) GLSM (gauged linear sigma model) anomalies: Given a two-dimensional $\mathcal{N} = (0, 2)$ gauged linear sigma model, the U(1) charges q_i^a of all defining fields Φ_i under the a U(1) factors have to be chosen such that the GLSM anomalies vanish. This leads to a (set of coupled) quadratic Diophantine equations in the q_i^a (which can be chosen integral upon changing the U(1) normalization).

Of course, many more physical examples could be produced from index formulae in a similar way. The main one that we will study in this paper has to do with instanton corrections on divisors D in a Kähler threefold B, where in index $\chi_{\mathrm{E3}} = -\frac{1}{2}\int_B c_1 \wedge D \wedge D$ defines a quadratic Diophantine equation in the integers parametrizing D.

B. Cubic Diophantine equations in string theory

There are also a collection of interesting results for cubic Diophantine equations $C(x_1, \ldots, x_s) = 0$. When there is a solution to $C = \partial_i C = 0\ \forall\ i$, we will say that C is singular; if not, it is nonsingular. When the equation is homogeneous of degree 3, we will say that it is a cubic form. Results of interest include as follows:

(i) A cubic form has a nontrivial solution [38] if $s \geq 16$.
(ii) Therefore, cubic forms are decidable if $s \geq 16$.
(iii) A nonsingular (disallowing the trivial solution) cubic form has a solution [39] if $s \geq 10$.

Given these results about cubic forms and the previous ones about quadratic Diophantine equations, it is natural to try to make further progress by decomposing a general cubic Diophantine equation as

$$C(x_1, \ldots, x_s) = F(x_1, \ldots, x_s) + H(x_1, \ldots, x_s), \tag{10}$$

where $F(x_1, \ldots, x_s)$ is a cubic form and $H(x_1, \ldots, x_s)$ is a quadratic polynomial. Then $C(x_1, \ldots, x_s)$ is decidable if any one of the following three conditions holds [40]:

(i) $v(F(x_1, \ldots, x_s)) \geq 17$.
(ii) $s \geq 15$ and $4 \leq v(F(x_1, \ldots, x_s)) \leq s - 3$.
(iii) $s \geq 14$ and we can factor the cubic form as a new variable $\tilde{x}_n$ times a quadratic form in $n-1$ variables $\tilde{x}_i$,

$$C(x_1, \ldots, x_s) = \tilde{x}_n Q(\tilde{x}_1, \ldots, \tilde{x}_{n-1}), \tag{11}$$

subject to some additional conditions on the factorization.

These conditions are all, themselves, decidable.

In these conditions, $v(F)$ is an invariant of the cubic form F defined as follows: given a cubic form, it can be written (nonuniquely) as a sum of products of linear forms L_i and quadratic forms Q_i,

$$F = \sum_{i=1}^{N} L_i Q_i. \tag{12}$$

Now, $v(F)$ is the minimum number of terms N in this sum for which this decomposition is possible.

The simplest results for cubic Diophantine equations are in the case in which they are homogeneous (i.e., cubic forms). While such equations sometimes occur in string theory, one often faces more general cubic Diophantine equations, e.g.,

(i) Existence of elliptic fibrations: It was conjectured by Kollár [41] (which is a proven result by Oguiso [42] and Wilson [43] under some additional assumptions) that a Calabi-Yau threefold X admits an elliptic fibration if $D^3 = 0$, $D^2 \neq 0$, and $D \cdot C \geq 0$ for all algebraic curves $C \subset X$. Deciding the existence of an elliptic fibration using Kollár's criterion thus requires solving a coupled set of a cubic form, a quadratic form inequality, and a linear condition.

(ii) Three generations: In heterotic compactifications on a Calabi-Yau X with a vector bundle V that is given by a sum of line bundles L_i, the condition to obtain three net generations of Standard Model particles is given in terms of the Hirzebruch-Riemann-Roch index theorem and reads

$$\begin{aligned}\sum_i \chi(L_i, X) &= \sum_i \int_X \mathrm{Td}(TX)\mathrm{ch}(L_i) \\ &= \sum_i \frac{1}{6} c_1(L_i)^3 + \frac{1}{12} c_1(L_i) c_2(TX) \\ &\overset{!}{=} 3n,\end{aligned} \tag{13}$$

where n is the order of a freely acting symmetry that is commonly used to obtain the Standard Model gauge group. By parametrizing again the line bundles L_i as $L_i = \mathcal{O}_X(k_i^1, \ldots, k_i^{h^{1,1}})$, this becomes a cubic (inhomogeneous) Diophantine equation. Equation (13) computes the net number of quark doublets, but the net numbers of the other particles are given by similar cubic equations.

Interestingly, decidability of cubic Diophantines also arose recently in quantum field theory, via the derivation [44] of the most general solution to the $U(1)$ anomaly cancellation conditions in four-dimensional $G = U(1)$ theories.

III. PROPAGATION OF INSTANTON SOLUTIONS THROUGH NETWORKS OF STRING GEOMETRIES

We now turn to the central physical problem of this paper: finding Euclidean D3 (E3)-instanton corrections to the superpotential across large networks of compactification manifolds in type IIB string theory and F-theory. In cases where a heterotic dual exists, this will also imply the presence of (world sheet and space-time) instantons in heterotic theories. Such corrections are of great importance for string cosmology and global dynamics on the landscape.

The detailed structure of an instanton correction depends crucially on its spectrum of zero modes, and here we study the simplest case; additional subtleties are discussed in the conclusions. Using various dualities, there are a number of ways to formulate the study of these zero modes. Consider a compactification of M-theory on an elliptically fibered Calabi-Yau fourfold $X \xrightarrow{\pi} B$ with Kähler threefold base B. In [27], Witten showed that if X is smooth, and M5-brane instanton on a vertical divisor $\hat{D}$ in X satisfying $h^i(\hat{D}, \mathcal{O}_{\hat{D}}) = (1, 0, 0, 0)$ contributes to the superpotential. Such instanton divisors have holomorphic Euler character

$$\chi(\hat{D}, \mathcal{O}_{\hat{D}}) = \sum_i (-1)^i h^i(\hat{D}, \mathcal{O}_{\hat{D}}) = 1. \qquad (14)$$

An F-theory compactification may be obtained by taking M-theory on X in the limit of vanishing fiber, in which case the M5-instanton correction becomes and E3-instanton correction, arising from an E3 wrapped on $D \subset B$ where $\hat{D} = \pi^{-1}(D)$. In this duality frame, one would like to rewrite the condition $\chi(\hat{D}, \mathcal{O}_{\hat{D}}) = 1$ in terms of data intrinsic to D and B. This was done via a Leray spectral sequence in work of Kollár [45] that relates cohomology on $\hat{D}$ to cohomology on D. The result matches a detailed IIB instanton zero mode count [28], which defined a new index χ_{E3} related to cohomology on D. If B is $\mathbb{P}^1$ fibered, one can moreover apply heterotic/F-theory duality, and $\hat{D}$ captures world sheet as well as space-time instanton contributions. In order to not rely on a rational ruling of B in the following, we will focus on E3 instantons, but keep in mind implications for heterotic theories if their dual exists.

Witten's instanton condition becomes

$$\chi(\hat{D}, \mathcal{O}_{\hat{D}}) = \chi_{\mathrm{E3}} = \chi(D, \mathcal{O}_D) - \chi(D, K_X) = 1. \qquad (15)$$

We choose to utilize χ_{E3} written in terms of data on B rather than $\chi(\hat{D}, \mathcal{O}_{\hat{D}})$ written in terms of data on X. We will see that this simplifies calculations.

In order to address decidability issues, we would like to formulate the condition as a natural decision problem relevant for determining instanton corrections to the superpotential. *E3-INDEX*: Given a smooth threefold B, is there an effective divisor D with $\chi_{\mathrm{E3}} = 1$? Inserting the expansion of D in Eq. (2) into Eq. (1), the decision problem E3-INDEX becomes a Diophantine equation of degree two in $h^{1,1}(B)$ variables n_i. Given D with $\chi_{\mathrm{E3}} = 1$, we may say that D or the associated set of n_i provides a yes solution to E3-INDEX, i.e., the n_i's solve the Diophantine equation.

Since χ_{E3} is quadratic, E3-INDEX is decidable, a significant improvement from the general case of Diophantine undecidability. Of course, a decision problem being decidable does not mean that it is tractable. For instance, a smooth toric F-theory base is presented in Appendix A, for which the associated χ_{E3} is given in (A4). To the naked eye, this seems intractable, even though it is decidable. To estimate the brute force tractability, we use the search bounds of [32]

$$\Lambda_s(H) = C_4(s) H^{5s+19+74/(s-4)} \quad \text{when } s \geq 5, \qquad (16)$$

where in Appendix A example $s = h^{1,1} = 9$ and therefore $\Lambda_{11}(H) \propto H^{78.8}$. Since (A4) has $H = 44$, taking $C_4(9) \geq 1$ would give a search set S_{search} with

$$|S_{\mathrm{search}}| \geq (2 \times 44^{78.8} + 1)^9 \simeq 10^{1200}. \qquad (17)$$

Using the concrete quadratic decidability result is already intractable in this $h^{1,1} = 9$ case, which is actually quite a low value for $h^{1,1}$. For instance, a generic base in the tree ensemble [24] has $h^{1,1} \simeq 2000$, in which case brute force search using the search bound becomes even worse. One must find another way to solve E3-INDEX.

Our central idea is to utilize additional structure to aid in solving E3-INDEX, in particular the fact that 2 three-dimensional varieties $\tilde{B}$ and B may be related to one another by topological transitions. Specifically, we will study how the index $\tilde{\chi}_{\mathrm{E3}}$ of a divisor $\tilde{D}$ on $\tilde{B}$ is related to χ_{E3} of a divisor D on B, where $\tilde{B}$ and B are related via the blowdown $\pi: \tilde{B} \to B$. The relation is captured by the difference $\Delta\chi = \tilde{\chi}_{\mathrm{E3}} - \chi_{\mathrm{E3}}$, and pullbacks under π will be implicit throughout.

This approach is in the spirit of Mori theory, where a given variety is to be understood from the perspective of simpler models from which it arises via birational transformations. To this end, a related decision problem is *E3-INDEX-PROP* Given a blowdown $\pi: \tilde{B} \to B$ with $\tilde{B}$ and B smooth, is there an effective divisor $\tilde{D}$ on $\tilde{B}$ such that $\tilde{\chi}_{\mathrm{E3}} = 1$? Clearly, any solution to this problem is a solution to E3-INDEX. However, often the blown down variety is simpler to analyze since $h^{1,1}$ and hence the search space become exponentially smaller, so we would study E3-INDEX-PROP on $\tilde{B}$ by its relation to E3-INDEX on B. Specifically, under which circumstances does a solution of E3-INDEX on B persist to $\tilde{B}$? Alternatively, when do properties of B ensure that a new solution exists on $\tilde{B}$?

To aid in answering these questions, we will study the following two cases of E3-INDEX-PROP on $\tilde{B}$:

(i) Case 1: E3-INDEX on B has a solution.

(ii) Case 2: The blowup locus $V \subset B$ has $\chi_V = 1$, where $\chi_V \coloneqq \chi(V, \mathcal{O}_V)$ is the holomorphic Euler character of the blowup locus V.

In both cases, we will show that E3-INDEX-PROP on $\tilde{B}$ has yes solutions. Case 1 is useful if B is simple enough

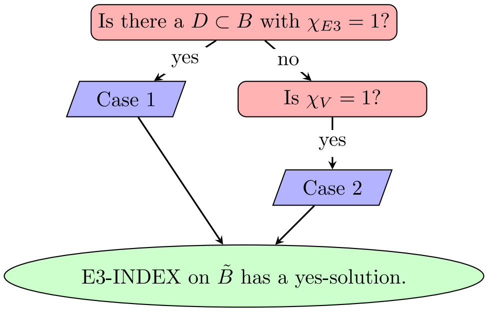

FIG. 1. Flow chart of some cases related to the decision problem E3-INDEX-PROP on $\tilde{B}$, which is a blowup of B. If the answers to the posed questions are both no, there may or may not be a solution; further study is needed, and a few special cases that yield yes solutions are discussed below.

(i.e., the search bounds are low enough) to find solutions to the Diophantine equation (15). Case 2 can be used if E3-INDEX is hard to solve on B, but we have additional information about the nature of the blowup. Moreover, the result can be used in some cases to engineer manifolds for which the answer to E3-INDEX is yes by choosing appropriate blowup loci. We summarize the procedure and propagation of decidability of E3-INDEX in Fig. 1.

In both cases, we will show that E3-INDEX-PROP on $\tilde{B}$ has yes solutions, which will motivate the definition of a third decision problem which is decidable, covers most instances of E3-INDEX and yields yes solutions. *SEQ-E3-INDEX* Given $\tilde{B}$ that has a sequence of blowdowns $\tilde{B} \to \ldots \to \hat{B} \xrightarrow{\pi} B$, such that the blowup locus $V \subset X$ has $\chi_V = 1$, is there an effective divisor $\tilde{D} \subset \tilde{X}$ with $\tilde{\chi}_{\rm E3} = 1$?

Our results and calculations depend critically on whether the blowup is along a curve $C = D_1 \cdot D_2$ or at a single point $P = D_1 \cdot D_2 \cdot D_3$. In the following, we expand a divisor in the blown up variety $\tilde{B}$ as

$$\tilde{D} = \bar{D} + n_e E, \tag{18}$$

where E is the exceptional divisor of the blowup and the divisor $\bar{D}$ can be expanded, analogously to Eq. (2), as

$$\bar{D} = \sum_{i=1}^{h^{11}} \bar{n}_i \mathcal{D}_i. \tag{19}$$

Here, $\mathcal{D}_i$ is the proper transformation of the divisors D_i. When it causes no confusion, we will use the same notation D_i for both the divisor class on X and its pullback under the blowup.

For a blowup at P, we have

$$\tilde{\chi}_{\rm E3} = -\frac{1}{2}\bar{D}^2 c_1 + D_1 D_2 D_3 n_e^2. \tag{20}$$

If there already exists a divisor D on B that solves $\chi_{\rm E3} = 1$, we set $\bar{D} = D$ in Eq. (20) and see that the only solution to $\Delta\chi = 0$ is $n_e = 0$; the solution D to E3-INDEX on B propagates to a solution of E3-INDEX on $\tilde{B}$.

Alternatively, if $\bar{D}^2 c_1 = 0$ and the blowup locus is a single point $D_1 D_2 D_3 = 1$, then $\bar{D} \pm E$ is a solution. For $\bar{D} = 0$, this matches a result in [27], but we note that the solution is much more general. For instance, if $\bar{D}$ is either a K3 surface with trivial normal bundle or if it is a toric divisor associated with a facet interior of a reflexive polytope, the condition $\bar{D}^2 c_1 = 0$ is satisfied and $\bar{D} \pm E$ are solutions; others likely exist.

For a blowup along $C = D_1 \cdot D_2$, we have

$$\begin{aligned}\tilde{\chi}_{\rm E3} &= -\frac{1}{2}\bar{D}^2 c_1 - n_e \bar{D} \cdot C + \chi_C n_e^2 \\ &= \chi_{\rm E3} - n_e \bar{D} \cdot C + \chi_C n_e^2,\end{aligned} \tag{21}$$

where $\chi_C = \frac{1}{2} D_1 D_2 (c_1 - D_1 - D_2)$ is the holomorphic Euler characteristic of the curve C.

Again, if we know a D on B that satisfies $\chi_{\rm E3} = 1$, we can also use it after the blowup, i.e., set $\bar{D} = D$. The condition $\tilde{\chi}_{\rm E3} = 1$ is then equivalent to

$$\Delta\chi = \tilde{\chi}_{E3} - \chi_{\rm E3} = n_e(\chi_C n_e - D \cdot C) = 0. \tag{22}$$

Therefore, we see that a solution (besides the pullback of D, i.e., the $n_e = 0$ case) is $\tilde{D} = D + kE$ where $k = \frac{D \cdot C}{\chi_C}$, where k must be an integer.

Alternatively, if we set all $n_i = 0$ in (19), the condition (21) becomes

$$\tilde{\chi}_{\rm E3} = \chi_C n_e^2 = 1, \tag{23}$$

which is solved only for $\chi_C = 1$ and $n_e = \pm 1$. However, since $-E$ is not effective, only E is a solution. That is, when B is blown up along a genus 0 curve, E is a solution to E3-INDEX on $\tilde{B}$.

Finally, if $\bar{D}^2 c_1 = 0$ but $\bar{D} \neq 0$,

$$\chi_{\rm E3} = n_e(\chi_C n_e - \bar{D} \cdot C) = 1, \tag{24}$$

which requires

$$n_e = \pm 1, \qquad \chi_C = 1 - g(C) = 1 \pm \bar{D} \cdot C. \tag{25}$$

This can be solved in many ways for different choices of $g(C)$ and $\bar{D}$, e.g.,

(i) If $g(C) = 1$, then $\bar{D} \pm E$ is a solution if $\bar{D} \cdot C = \mp 1$.
(ii) If $g(C) = 0$, then $\bar{D} \pm E$ is a solution provided $\bar{D} \cdot C = 0$.

The latter solution can be nontrivial due to self-cancellation in $\bar{D} \cdot C$. For instance, if $c_1 \cdot C = 0$, then $(D_1 + D_2) \cdot C = -2$ since $g(C) = 0$ and $\bar{D} = D_1 + D_2 + D_3 + D_4$ make $\bar{D} \pm E$ a solution provided $(D_3 + D_4) \cdot C = 2$. This sounds contrived, but it arises naturally in toric geometry contexts:

if C is a toric curve meeting at the intersection of two 3 cones with associated divisors (D_3, D_1, D_2) and (D_4, D_1, D_2), and the four divisors $D_{1,2,3,4}$ are facet interiors, then all of the conditions are met.

To summarize, some guaranteed solutions to E3-INDEX-PROP on $\tilde{B}$ are as follows:

(S1) The pullback of a solution from B to $\tilde{B}$.

(S2) The exceptional divisor E. It is always a solution in the point blowup case, and also in the curve blowup case if $g(C) = 0$.

(S3) $\bar{D} + kE$ in the curve blowup case if it is effective and $\bar{D}$ is a solution on B, where $k = \bar{D} \cdot C/\chi_C$.

(S4) $\bar{D} \pm E$, where $\bar{D}$ is not necessarily a solution on B, if certain conditions hold. In the point blowup case, the condition is that $\bar{D}^2 c_1 = 0$. In the curve blowup case, $\bar{D} \pm E$ is a solution provided that $\bar{D}^2 c_1 = 0$ and $\chi_C = 1 - g(C) = 1 \pm \bar{D} \cdot C$.

We have labeled the classes (S1)–(S4) to allow for simplified discussions. Solutions of classes (S1) and (S2) are fairly automatic and correspond to Case 1 and Case 2 above, whereas those of classes (S3) and (S4) require some additional nontrivial checks. There could be other interesting classes of guaranteed solutions, as well.

We wish to also take into account the fact that string geometries arise in large networks. The transition from B to $\tilde{B}$ we discussed, along with its associated solutions, are only two nodes and one edge in that network. One would like to know how solutions propagate through the entire network, rather than just from one node to its neighbor.

This question is addressed by the decision problem SEQ-E3-INDEX defined above. By the results of Case 1 and Case 2, SEQ-E3-INDEX is not only decidable, but has yes solutions. Case 2 guarantees the existence of a solution to E3-INDEX on $\hat{X}$ when blowing up at a curve C with $\chi_C = 1$ or when blowing up at a point, and then Case 1 applied to $\hat{X}$ guarantees a solution on $\tilde{X}$. That is, SEQ-E3-INDEX is always decidable, and yes solutions always exist.

A. Concrete solutions to SEQ-E3-INDEX

We showed above that the exceptional divisor E of a blowup along $V \subset X$ is always a solution to E3-INDEX if $\chi_V = 1$. We now want to see whether there exists a solution of the form $E_1 + kE_2$ after a second blowup where E_1 is the exceptional divisor of the first blowup and E_2 is that of the second blowup.

There four possible combinations, since either of the blowups could be the blowup of a point or of a curve. However, if the second blowup is a point blowup, we must have $k = 0$. Hence, we consider the two cases where the second blowup is along a curve.

First blowup is at a curve: We first consider cases where the first and the second blowups are all blowups along a curve. There are three such cases. The first blowup is along a curve $C = D_1 \cdot D_2$. For the exceptional divisor E_1 of the first blowup to solve E3-INDEX on $\tilde{X}$, we require $\chi_C = 1$.

Blowup 1: The first case is the sequence of blowups,

$$\tilde{X} \xrightarrow{(E_2|E_1,D_3)} \hat{X} \xrightarrow{(E_1|D_1,D_2)} X, \tag{26}$$

where the notation $(E_1|D_1, D_2)$ denotes a blowup along $C = D_1 \cdot D_2$ with exceptional divisor E_1 and similarly for $(E_2|E_1, D_3)$. We have

$$\tilde{\chi}_{\mathrm{E3}} = 1 + D_1 D_2 D_3 (k^2 + k). \tag{27}$$

Since $\chi_C = 1$, we see that $k = 0$ is a solution to $\tilde{\chi}_{\mathrm{E3}} = 1$, as is $k = -1$ provided that $D_1 D_2 D_3 = 1$, which is always the case in toric examples where these three divisors form a three cone. Therefore, we see that besides the pullback of E_1 on $\tilde{X}$, $E_1 - E_2$ is a solution to E3-INDEX under mild assumptions.

Blowup 2: The second case is the sequence of blowups,

$$\tilde{X} \xrightarrow{(E_2|E_1,\tilde{D}_1)} \hat{X} \xrightarrow{(E_1|D_1,D_2)} X. \tag{28}$$

We have

$$\tilde{\chi}_{\mathrm{E3}} = 1 - (D_2 \cdot C)k + k^2. \tag{29}$$

We see that the solutions to $\tilde{\chi}_{\mathrm{E3}} = 1$ are

$$k = 0 \quad \text{or} \quad k = D_2 \cdot C. \tag{30}$$

Therefore, we see that besides the pullback of E_1 on $\tilde{X}$, $E_1 + (D_2 \cdot C)E_2$ is also a solution to E3-INDEX, provided that it is an effective divisor. It is not uncommon that $D_2 \cdot C = -1$, for instance, if D_2 is itself the exceptional divisor of a blowup along a curve that meets D_1 at a point.

Blowup 3: The third case is the sequence of blowups,

$$\tilde{X} \xrightarrow{(E_2|\tilde{D}_1,D_3)} \hat{X} \xrightarrow{(E_1|D_1,D_2)} X. \tag{31}$$

We have

$$\tilde{\chi}_{\mathrm{E3}} = 1 - D_1 D_2 D_3 k + \chi_{C'} k^2, \tag{32}$$

where $C' = D_1 \cdot D_3$; we emphasize that C' is a curve on the original space that is not a blowup locus. We see that the solutions to $\tilde{\chi}_{\mathrm{E3}} = 1$ are

$$k = 0 \quad \text{or} \quad k = \frac{D_1 D_2 D_3}{\chi_{C'}}. \tag{33}$$

Therefore, we see that besides the pullback of E_1 on $\tilde{X}$, $E_1 + \frac{D_1 D_2 D_3}{\chi_{C'}} E_2$ is also a solution to E3-INDEX. First blowup is at a point: There are two cases where the first blowup is at a point and the second is along a curve. In both cases, for the first exceptional divisor to be a solution, the first blowup is at a single point, given by $D_1 D_2 D_3 = 1$.

Blowup 1: The first case is the sequence of blowups,

$$\tilde{X} \xrightarrow{(E_2|\tilde{D}_1,\tilde{D}_2)} \hat{X} \xrightarrow{(E_1|D_1,D_2,D_3)} X. \tag{34}$$

We have

$$\tilde{\chi}_{\mathrm{E3}} = 1 - k + \chi_C k^2, \tag{35}$$

where $C = D_1 \cdot D_2$; we emphasize that C is a curve on the original space that is not a blowup locus. We see that the solutions to $\tilde{\chi}_{\mathrm{E3}} = 1$ are

$$k = 0 \quad \text{or} \quad k = \frac{1}{\chi_C}. \tag{36}$$

Therefore, we see that besides the pullback of E_1 on $\tilde{X}$, $E_1 + \frac{1}{\chi_C} E_2$ is also a solution to E3-INDEX, provided that it is effective, which requires $\chi_C = 1$.

Blowup 2: The second case is the sequence of blowups,

$$\tilde{X} \xrightarrow{(E_2|\tilde{D}_1,E_1)} \hat{X} \xrightarrow{(E_1|D_1,D_2,D_3)} X. \tag{37}$$

We have

$$\tilde{\chi}_{\mathrm{E3}} = 1 + k + k^2. \tag{38}$$

We see that the solutions to $\tilde{\chi}_{\mathrm{E3}} = 1$ are $k = 0, -1$. Therefore, we see that besides the pullback of E_1 on $\tilde{X}$, $E_1 - E_2$ is also a solution to E3-INDEX.

IV. IMPLICATIONS OF PROPAGATION FOR KÄHLER MODULI STABILIZATION

Having formulated various decision problem related to instanton corrections to the superpotential and having found different classes of solutions, we wish to discuss the implications for the stabilization of Kähler moduli.

This is subtle, because as we have emphasized the presence of additional zero modes not captured by the condition $\chi_{\mathrm{E3}} = 1$ can alter the structure of the correction. However, given the importance of Kähler moduli stabilization for realistic string cosmology, it behooves us to proceed with a discussion under the assumption that additional zero modes do not kill the corrections associated with our solutions. This will allow us to discuss how many Kähler moduli appear in the superpotential. A detailed study of the assumption and also the importance of instantons that do not have $\chi_{\mathrm{E3}} = 1$, such as in [28], are interesting directions for future work.

For the purposes of Kähler moduli stabilization, a more relevant question is "How many Kähler moduli appear in at least one instanton correction?" This can be encoded in a counting problem *NAIVE-STABLE*: Given threefold B and all divisors $D \subset B$ with $\chi_{\mathrm{E3}} = 1$, how many Kähler moduli appear in at least one of the instanton corrections? Equivalently, what is the rank of the lattice spanned by the divisors? This is the "naïve" stability count because of the above assumption, and also because a single correction does not guarantee that the modulus is stabilized, for instance, due to a runaway. However, since it can be studied across large networks of geometries, it is a good starting point.

For simplicity, let us study the concrete case of networks of toric varieties that serve as bases of F-theory elliptic fibrations. Consider the so-called tree ensemble [24], which forms a single-component connected network of 2.96×10^{755} toric threefolds. The network is formed by recursively performing blowups of a fixed initial weak-Fano toric threefold B along toric curves, which are $\mathbb{P}^1$'s, and toric points. Either of these blowup loci V have $\chi_V = 1$, and therefore any geometry $\tilde{B}$ in the network (provided it is not the initial threefold B) has a sequence of blowdowns $\tilde{B} \to \ldots \to \hat{B} \to B$. That is, all but one of the 10^{755} geometries in the tree ensemble provide instances of SEQ-E3-INDEX, and therefore have at least one instanton solution.

For Kähler moduli stabilization, one would rather like to know the answer to NAIVE-STABLE on $\tilde{B}$. Since the number of Kähler moduli $h^{1,1}(B)$ on the variety B and $h^{1,1}(\tilde{B})$ on the variety $\tilde{B}$ are fixed, and any single blowup increases $h^{1,1}$ by 1, all directed paths in the network between B and $\tilde{B}$ are necessarily of the same length. However, the toric blowups only have loci V with $\chi_V = 1$, and therefore the exceptional divisor associated to each blowup in the sequence yields a solution to $\tilde{\chi}_{\mathrm{E3}} = 1$ on $\tilde{B}$, i.e., at least

$$\Delta h^{1,1}(\tilde{B}) \coloneqq h^{1,1}(\tilde{B}) - h^{1,1}(B). \tag{39}$$

Kähler moduli satisfy NAIVE-STABLE. The corrections are schematically of the form

$$W = \sum_{i=1}^{h^{11}(\tilde{B})} A_i(\phi) e^{-2\pi T_{E_i}} + \ldots, \tag{40}$$

where

$$T_{E_i} = \int_{E_i} \frac{1}{2} J \wedge J + i C_4 \tag{41}$$

is the complexified Kähler modulus associated to the exceptional divisor E_i, J is the Kähler form, and C_4 is the Ramond-Ramond four form.

The probability that a Kähler modulus on $\tilde{B}$ satisfies NAIVE-STABLE is therefore at least $\Delta h^{11}(\tilde{B})/h^{11}(\tilde{B})$. In the tree ensemble, the expected percentage of Kähler moduli naively stabilized using just exceptional divisors is

$$100\% \times \mathbb{E}_{\tilde{B}\sim U_{\rm tree}}\left[\frac{\Delta h^{11}(\tilde{B})}{h^{11}(\tilde{B})}\right] = 100\% \times \frac{2448}{2483} = 98.6\%, \tag{42}$$

where $U_{\rm tree}$ is the uniform distribution on the tree ensemble. This could be improved even further by the inclusion of divisors pulled back from B in mixing terms, as we will discuss momentarily.

We emphasize that all of these results also apply to the so-called Skeleton ensemble [25], which is less restrictive than the tree ensemble and is estimated to have $O(10^{3000})$ elements. Generally, these results apply to any network of toric varieties constructed by recursively performing toric blowups from an initial toric B, which are automatically along loci with $\chi_V = 1$.

Let us discuss be more details by studying both mixing terms and potential caveats.

One caveat to note is that if E_i intersects a later blowup locus in the sequence, the nature of the blowup could render a generic representative of class E_i a reducible variety, for instance, with normal crossing singularities. In such a case, a careful zero mode analysis is required to determine whether

(1) there are additional zero modes present at the normal crossing locus and
(2) additional physics gives interactions to lift these modes.

This has not been studied in detail in the literature.

However, if an exceptional divisor E is involved in N additional blowups with exceptional divisors E_i, the divisor $E - \sum_{i=1}^N E_i$ can be irreducible and does not suffer from normal crossing singularities. To that end, one would like to understand under which conditions such divisors solve $\tilde{\chi}_{\rm E3} = 1$. For simplicity, suppose that $N = 1$, and consider $\tilde{B} \overset{\pi}{\to} B$ with exceptional divisor $\tilde{E}$ and another divisor E that is the exceptional divisor of a previous blowup along a locus V with $\chi_V = 1$. The latter condition guarantees that E solves $\chi_{\rm E3} = 1$ on B, and therefore it also solves $\tilde{\chi}_{\rm E3} = 1$ on $\tilde{B}$. Since E is a solution, having a new solution of the form $E + k\tilde{E}$ with $k \neq 0$ requires that the blowup locus of π be a curve $\tilde{C} \subset B$, cf. (S3). A solution of the proposed form requires

$$k = \frac{E\cdot\tilde{C}}{\chi_{\tilde{C}}} = -1, \tag{43}$$

where for the sake of irreducibility we took $k = -1$. In the toric case, where $g(\tilde{C}) = 0$, this amounts to requiring that $E\cdot\tilde{C} = -1$.

Corrections of the type that we just discussed induce mixing terms, which could give rise to important competition (such as in racetrack scenarios) and couplings between Kähler moduli in the scalar potential. Any given blowup such as $\tilde{B} \overset{\pi}{\to} B$ in a sequence of blowups allows to study solutions of type (S1)–(S4) for that transition, i.e., whether the exceptional divisor E of the blowup gives rise to a superpotential mixing of that Kähler modulus with some of the others. The pullback solutions (S1) and the exceptional divisor solution (S2) do not, but by their very form, solutions of types (S3) and (S4) give rise to mixing. A detailed study of the prevalence of (S3)- and (S4)-type solutions could give crucial information on Kähler modulus mixing, but is beyond the scope of this paper.

We note, however, that in the previous section we described toric cases in which such solutions exist. For instance, consider a toric divisor D on the original base B in the tree ensemble, which is associated with a fine regular star triangulation of a reflexive polytope. These are not contributing to the simple estimate (42), and we would like to improve the situation. On the biggest three-dimensional polytopes, which dominate the ensemble, 16 of the 38 toric divisors correspond to facet interiors, and therefore satisfy $Dc_1 = 0$. While these do not satisfy $\chi_{\rm E3} = 1$ on B, after any blowup involving D, $\tilde{D} = D - E$ is a solution. Nearly, all geometries in the tree ensemble have such a blowup, and therefore the moduli corresponding to facet interiors raise the expectation for NAIVE-STABLE,

$$100\% \times \mathbb{E}_{\tilde{B}\sim U_{\rm tree}}\left[\frac{\Delta h^{11}(\tilde{B})}{h^{11}(\tilde{B})}\right] = 100\% \times \frac{2464}{2483} = 99.2\%, \tag{44}$$

where the moduli that we have still not taken into account correspond to edge interiors and vertices of the polytope.

Mixing terms are also important from another perspective: they might allow for a correction involving a Kähler modulus T that does not appear on its own in any instanton correction, which is therefore crucial for the stabilization of T. Consider, for instance, the case where B is the result of a blowup along a curve C with $g(C) \neq 0$. Then by (S2), the exceptional divisor E is *not* a solution to $\chi_{\rm E3} = 1$ on B. Working on B, one could try searching for solutions of type (S3) or (S4) which potentially give rise to an instanton contribution involving E. Alternatively, if one passes to $\tilde{B} \overset{\pi}{\to} B$ with exceptional divisor $\tilde{E}$ and blowup locus $\tilde{C}$, then $E - \tilde{E}$ solves $\tilde{\chi}_{\rm E3} = 1$ on $\tilde{B}$ provided that

$$g(\tilde{C}) = 1 - g(C) + E\cdot\tilde{C} \tag{45}$$

and $E - \tilde{E}$ effective. If one wants $\tilde{E}$ to automatically be a solution, $g(\tilde{C}) = 0$ and the condition simplifies to

$$E\cdot\tilde{C} = 1 - g(C), \tag{46}$$

which is a search problem for an appropriate $\tilde{C}$.

V. DISCUSSION AND SUMMARY OF RESULTS

We studied Diophantine equations that arise in string theory with regard to their decidability and physics.

In Sec. II, we reviewed known theorems about Diophantine equations. The central theorems for our applications are that all quadratic Diophantines are decidable, as are cubic form Diophantines in more than 15 variables. We presented simple examples in which quadratic and cubic Diophantine equations appear in string theory. In the quadratic case, we presented examples related to D3-brane charge, chiral three to seven instanton zero modes, Bianchi identities, and GLSM anomalies. In the cubic case, examples included Kollár's condition for the existence of an elliptic fibration in a Calabi-Yau threefold, as well as obtaining three generations of Standard Model chiral multiplets in heterotic line bundle compactifications.

In Sec. III, we focused on a Diophantine equation which is relevant for instanton corrections to the superpotential. More specifically, for an elliptic Calabi-Yau fourfold $X\xrightarrow{\pi}B$, M5-brane instanton corrections may arise from divisors $\hat{D}$ in X satisfying Eq. (14). In this case, the instanton should be thought of in the type IIB/F-theory limit, where it is a Euclidean D3-brane instanton wrapped on D. In cases where a heterotic dual exists, the corresponding instantons are world-sheet and space-time instantons. Since the instanton condition leads to a quadratic Diophantine equation, the question whether there exists a suitable D is decidable. We call this decision problem E3-INDEX.

Our central idea is to make progress on E3-INDEX by considering how the problem behaves under topological transition. That is, given a geometry $\tilde{B}$ that is the blowup of a geometry B, may E3-INDEX on $\tilde{B}$ be understood in terms of the problem on B? From this perspective, one thinks of a network of string geometries related by topological transitions, which are represented by edges, and asks whether decidability of E3-INDEX on one node (geometry) decides E3-INDEX on an adjacent node. This is the propagation of decidability.

We formalize it in terms of a decision problem E3-INDEX-PROP, which introduces the blowup into the problem and the notion that solutions on B may be related to those on $\tilde{B}$, based on the structure of the topology change. We find two rather useful subcases of the problem. In Case 1, E3-INDEX on B itself has a solution D. For both point and curve blowups, that solution pulls back to a solution on $\tilde{B}$, i.e., it has $\tilde{\chi}_{\mathrm{E3}}=1$ on $\tilde{B}$. Additionally, in the case of a blowup along a curve C, an extended solution for E3-INDEX on $\tilde{B}$ exists under mild assumption. It is of the form $D+kE$ where $k=(D\cdot C)/\chi_C$ must be an integer and the divisor must be effective. This is often true in toric varieties. In Case 2, we simply note that the exceptional divisor itself is a solution to E3-INDEX on $\tilde{B}$ if $\chi_V\coloneqq\chi(V,\mathcal{O}_V)=1$, where V is the blowup locus in B. This is the case if V is a genus 0 curve or a point.

The results of the two cases motivate a third decision problem, SEQ-E3-INDEX, which postulates that the variety of interest $\tilde{B}$ may be obtained by a sequence of blowups from a blowup of an initial variety with blowup locus satisfying $\chi_V=1$. Then by the results of Case 1 and Case 2, all instances of SEQ-E3-INDEX have a yes solution, i.e., a divisor with $\tilde{\chi}_{\mathrm{E3}}=1$ on $\tilde{B}$.

We derive numerous classes of solutions to E3-INDEX-PROP on $\tilde{B}$, sometimes with mild assumptions. We categorize them according to whether they are pullbacks of solutions from B [called (S1)], the exceptional divisor of the blowup (S2), of the form $D+kE$ where D is a solution on B (S3), or of the form $D'\pm E$ where D' is not necessarily a solution on B.

In Sec. III A, we derive solutions that depend only on two exceptional divisors in a sequence of blowups, for instance, $E-\tilde{E}$, with E the exceptional divisor of the first blowup. If E itself is a solution, then a necessary condition for $E+n_{\tilde{e}}\tilde{E}$ to be a solution is that the second blowup is a curve blowup, so that interesting solutions involving both E and $\tilde{E}$ arise in the cases of point-then-curve blowup sequences, or curve-then-curve blowup sequences. There are five total possibilities, which we study in detail and present solutions accordingly.

In Sec. IV, we discuss the implications for Kähler moduli stabilization, which are of great importance for string cosmology and global dynamics on the landscape.

We define a problem called NAIVE-STABLE that counts the number of Kähler moduli that appear in some instanton correction. Our results immediately demonstrate that all exceptional divisors in well-studied ensembles of toric geometries generated by blowups are necessarily solutions, which make giving lower bounds on NAIVE-STABLE straightforward. Some mixing terms, associated with facet interiors, are also easy to take into account.

By studying NAIVE-STABLE, we find that 99.2% of Kähler moduli appear in some instanton correction, on average, across the tree ensemble. Additional possibilities for mixing terms are also studied.

This provides ample motivation for future work.

On the formal side, there is additional research that must be done into aspects of instanton physics that are not fully understood. One has to do with instantons wrapped on divisors with normal crossing singularities, which appear in our work and [46]. Open questions include whether the singularities give rise to zero modes and whether additional physics may lift the zero modes. There is also the question of a general formalism for understanding the role of three to seven modes in instanton corrections, though progress has been made in specific contexts, e.g., [47–51]. Lifting deformation modes, for instance, via instanton fluxes [28,52], could also be crucial in stabilizing moduli that do not arise in any divisor with $\chi_{\mathrm{E3}}=1$.

On the statistical/data side, it would be interesting to systematically explore the toric ensembles with respect to the instanton solutions that we have derived. Our analytic analysis and estimates are only the simplest possible cases, and there is likely much more structure to be understood.

More broadly, we have demonstrated a case in which structure in string theory allows for the avoidance of worst-case complexity. In the present case, we showed that geometric structures in string theory allow us to decide Diophantine equations that arise in string theory, despite them being undecidable in general. It would be interesting to find other cases where physical problems in string theory are easier than might naively be expected, both for practical reasons and questions concerning the dynamics of string vacua.

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APPENDIX: AN EXAMPLE OF PROPAGATION

We construct a smooth toric variety X_{P_8} that is the result of eight blowups on $\mathbb{P}^3$. The fan associated with X_{P_8} has the following rays:

$$v_0 = (1,0,0), \quad v_1 = (0,1,0), \quad v_2 = (0,0,1),$$
$$v_3 = (-1,-1,-1), \quad e_1 = (1,1,1), \quad e_2 = (1,1,2),$$
$$e_3 = (1,1,3), \quad e_4 = (1,1,4), \quad e_5 = (1,0,1),$$
$$e_6 = (1,0,2), \quad e_7 = (1,0,3), \quad e_8 = (1,0,4).$$

We will use the same notation for both the rays and their corresponding coordinates. From the linear relation between the rays, we see that X_{P_8} is obtained from $X_{P_0} = \mathbb{P}^3$ via the following sequence of blowups:

$$X_{P_8} \xrightarrow{(e_8|v_2,e_7)} X_{P_7} \xrightarrow{(e_7|v_2,e_6)} X_{P_6} \xrightarrow{(e_6|v_2,e_5)} X_{P_5} \xrightarrow{(e_5|v_0,v_2)} X_{P_4} \xrightarrow{(e_4|v_2,e_3)} X_{P_3} \xrightarrow{(e_3|v_2,e_2)} X_{P_2} \xrightarrow{(e_2|v_2,e_1)} X_{P_1} \xrightarrow{(e_1|v_1,v_2,v_3)} X_{P_0} = \mathbb{P}^3. \quad (A1)$$

We denote the nine generators of the Kähler cone (and by abuse of notation, the Poincaré dual divisors) by G_0, $G_1, \ldots, G_8$. A divisor $D \subset X_{P_8}$ can be expanded as $D = \sum_{i=0}^{8} n_i G_i$.

Using the results of Sec. III, we know that the exceptional divisor E_8 corresponding to the last blowup

$$X_{P_8} \xrightarrow{(e_8|v_2,e_7)} X_{P_7} \quad (A2)$$

is a solution to $\chi_{\mathrm{E3}} = 1$ on X_{P_8}. We can express E_8 as a linear combination of G_i's,

$$E_8 = -G_0 + G_8. \quad (A3)$$

We also compute χ_{E3} on X_{P_8}, which is a quadratic polynomial in the 9 n_i's. We have

$$\begin{aligned}\chi_{\mathrm{E3}} = {} & -18n_0^2 - 22n_1n_0 - 33n_2n_0 - 44n_3n_0 - 42n_4n_0 - 12n_5n_0 - 11n_6n_0 - 40n_7n_0 - 38n_8n_0 - 5n_1^2 \\ & - 12n_2^2 - 22n_3^2 - 21n_4^2 - 2n_5^2 - n_6^2 - 20n_7^2 - 19n_8^2 - 16n_1n_2 - 22n_1n_3 - 33n_2n_3 - 22n_1n_4 - 33n_2n_4 \\ & - 44n_3n_4 - 8n_1n_5 - 12n_2n_5 - 16n_3n_5 - 15n_4n_5 - 5n_1n_6 - 8n_2n_6 - 11n_3n_6 - 11n_4n_6 - 4n_5n_6 \\ & - 22n_1n_7 - 33n_2n_7 - 44n_3n_7 - 42n_4n_7 - 14n_5n_7 - 11n_6n_7 - 22n_1n_8 - 33n_2n_8 - 44n_3n_8 - 42n_4n_8 \\ & - 13n_5n_8 - 11n_6n_8 - 40n_7n_8. \end{aligned} \quad (A4)$$

Using Eq. (A3), it is easy to check that following set of integers solves $\chi_{\mathrm{E3}} = 1$:

$$-n_0 = n_8 = 1 \quad \text{and} \quad n_i = 0, \quad \forall\ i \neq 0, 8. \quad (A5)$$

This agrees with our result that E_8 solves $\chi_{\mathrm{E3}} = 1$ on X_{P_8} via the analytic algebro-geometric method.

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