Geometry and integrability in $\mathcal{N}=8$ supersymmetric mechanics

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We construct the $\mathcal{N}=8$ supersymmetric mechanics with a potential term whose configuration space is the special Kähler manifold of rigid type and show that it can be viewed as the Kähler counterpart of $\mathcal{N}=4$ mechanics related to "curved Witten-Dijkgraaf-Verlinde-Verlinde equations." Then, we consider the special case of the supersymmetric mechanics with a nonzero potential term defined on the family of $U(1)$-invariant one-(complex)dimensional special Kähler metrics. The bosonic parts of these systems include superintegrable deformations of perturbed two-dimensional oscillators and Coulomb systems.

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I. INTRODUCTION

The construction of $\mathcal{N}$-extended supersymmetric mechanics has remained one of the main research goals of the supersymmetric community since the introduction of the concept of supersymmetry. Nevertheless, until now there has been no regular way to find the $\mathcal{N}>2$ supersymmetric extensions of the given mechanical systems. The traditional way to increase the number of supersymmetries (without exceeding the number of fermionic degrees of freedom (d.o.f.)) is to provide the configuration space with complex structure(s) (with an appropriate specification of the potential term), i.e., to restrict the configuration space to Kähler, hyper-Kähler, or quaternionic manifolds. For example, on a generic N-dimensional Riemann manifold one can always construct the $\mathcal{N}=2$ supersymmetric mechanics with $(N|2N)$ (i.e., N bosonic and $2N$ fermionic) d.o.f.; requiring that the configuration space be a generic N-(complex)dimensional Kähler manifold and properly specifying the potential, we can construct the $\mathcal{N}=4$ supersymmetric mechanics with $(N|2N)$ (complex) d.o.f. The bosonic part of these Hamiltonians reads

$$H_{(2)}=\frac{1}{2}g^{ij}(x)(p_ip_j+\partial_iW(x)\partial_jW(x)) \quad\longrightarrow\quad H_{(4)}=\frac{1}{2}g^{\bar{a}b}(z,\bar{z})(\bar{\pi}_a\pi_b+\bar{\partial}_a\bar{U}(\bar{z})\partial_bU(z)), \tag{1.1}$$

with (p_i,x^i) and (π_a,z^a) being canonically conjugate pairs and $g^{ij}(x)$ and $g^{\bar{a}b}$ being the inverse Riemann and Kähler metrics, respectively.

Another way to increase the number of supersymmetries (above $\mathcal{N}=2$ supersymmetry) is to double the number of fermionic d.o.f., which introduces additional geometric objects. For example, to construct the $\mathcal{N}=4$ supersymmetric extension of a free-particle system in generic configuration space, we have to double the number of fermionic d.o.f. from $2N$ to $4N$ and introduce the third-rank symmetric tensor $F_{ijk}(x)dx^idx^jdx^k$, which satisfies the *curved Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations* [1],

$$F_{kmj;i}=F_{kmi;j},\qquad F_{jkp}g^{pq}F_{imq}-F_{ikp}g^{pq}F_{jmq}=R_{ijkm}, \tag{1.2}$$

where R_{ijkl} are the components of the Riemann tensor of $(M_0,g_{ij}dx^idx^j)$, and the subscript ";" denotes a covariant derivative with the Levi-Civita connection.

Similarly, to construct the $\mathcal{N}=8$ supersymmetric extension of a free-particle system on a Kähler manifold we have to increase the number of (real) fermionic variables from $4N$ to $8N$ and introduce the third-rank holomorphic symmetric tensor $f_{abc}(z)dz^adz^bdz^c$, which satisfies the equations [2]

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$$f_{abc;d} = f_{abd;c}, \qquad R_{a\bar{b}c\bar{d}} = -f_{ace}g^{\bar{e}'e}\bar{f}_{\bar{e}'\bar{b}\bar{d}}, \qquad (1.3)$$

where $f_{abc;d} = f_{abc,d} - \Gamma^e_{ad}f_{ebc} - \Gamma^e_{bd}f_{eac} - \Gamma^e_{cd}f_{eab}$, and $R_{a\bar{b}c\bar{d}}$ and Γ^a_{bc} are the nonzero components of the Riemann tensor and Levi-Civita connection, which are defined as

$$\Gamma^a_{bc} = g^{a\bar{d}}g_{b\bar{d},c}, \qquad R_{a\bar{b}c\bar{d}} = g_{n\bar{b}}(\Gamma^n_{ac})_{,\bar{d}}. \qquad (1.4)$$

These manifolds are known as special Kähler manifold of rigid type [3] and they have been extensively studied since their introduction within the context of Seiberg-Witten duality [4]. The similarity between these systems has not been noticed before.

In this paper we show that this similarity holds for the supersymmetric mechanics with a potential term as well. Namely, after reviewing the main properties of $\mathcal{N}=4$ supersymmetric mechanics connected with the solution of the modified WDVV equations [1,5,6] (Sec. II), we construct on the special Kähler manifold of rigid type the $\mathcal{N}=8$ supersymmetric mechanics with potential term (Sec. III). We find that when we double the supersymmetries the prepotentials $W(x)$ and $U(z)$ in the bosonic Hamiltonians (1.1) should satisfy the following equations:

$$W_{i;j} + F_{ijk}g^{km}W_m = 0, \qquad U_{a;b} - f_{abc}g^{\bar{d}c}\bar{U}_{\bar{d}} = 0. \qquad (1.5)$$

Finally, in Sec. IV we present the general solution of the one-(complex)dimensional $U(1)$-symmetric special Kähler manifold and find the admissible set of potentials for $\mathcal{N}=8$ supersymmetric mechanics. The bosonic parts of these supersymmetric mechanics include the superintegrable perturbations of a deformed two-dimensional oscillator and the Coulomb system suggested in Refs. [7,8].

II. $\mathcal{N}=4$ MECHANICS ON RIEMANN MANIFOLDS

In order to construct the $\mathcal{N}=4$ supersymmetric mechanics on an N-dimensional Riemannian manifold $(M_0, g_{ij}(x)dx^i dx^j)$ we extend the cotangent bundle $(T^*M_0, dp_i \wedge dx^i)$ by $4N$ fermionic variables $\psi^{i\alpha}$, $\bar{\psi}^j_\beta = (\psi^\beta_j)^\dagger$, with $su(2)$ indices $\alpha, \beta = 1,2$ which are raised and lowered as follows: $A_\alpha = \epsilon_{\alpha\beta}A^\beta$, $A^\alpha = \epsilon^{\alpha\beta}A_\beta$, with $\epsilon_{12} = \epsilon^{21} = 1$. We then define the following transition maps from one chart to the other:

$$\tilde{x}^i = \tilde{x}^i(x), \qquad \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i}p_j, \qquad \tilde{\psi}^{ia} = \frac{\partial \tilde{x}^i(x)}{\partial x^j}\psi^{ja}. \qquad (2.1)$$

Then, we equip this supermanifold with a supersymplectic structure, which is manifestly invariant with respect to Eq. (2.1):

$$\Omega = dp_i \wedge dx^i + \mathrm{i}d(\psi^{i\alpha}g_{ij}D\bar{\psi}^j_\alpha - \bar{\psi}^{i\alpha}g_{ij}D\psi^j_\alpha) = dp_i \wedge dx^i + \mathrm{i}R_{ijkl}\psi^{i\alpha}\bar{\psi}^j_\alpha dx^k \wedge dx^l + 2\mathrm{i}g_{ij}D\psi^{i\alpha} \wedge D\bar{\psi}^j_\alpha, \qquad (2.2)$$

where $D\psi^{i\alpha} \equiv d\psi^{i\alpha} + \Gamma^i_{jk}\psi^{j\alpha}dx^k$, $\alpha = 1,2$, and Γ^i_{jk} and R_{ijkl} are the components of the connection and curvature of the metric $g_{ij}(x)dx^i dx^j$.

The Poisson brackets corresponding to this (super)symplectic structure are defined by the following nonzero relations:

$$\{p_j, x^i\} = \delta^i_j, \qquad \{p_i, \psi^{j\alpha}\} = -\Gamma^j_{ik}\psi^{k\alpha}, \qquad \{p_i, p_j\} = 2\mathrm{i}R_{ijkm}\psi^{k\alpha}\bar{\psi}^m_\alpha, \qquad \{\psi^{i\alpha}, \bar{\psi}^j_\beta\} = -\frac{\mathrm{i}}{2}\delta^\alpha_\beta g^{ij}. \qquad (2.3)$$

Our goal is to construct the supercharges Q^α and $\bar{Q}_\beta$ and the Hamiltonian $\mathcal{H}$, which obey the $\mathcal{N}=4$, $d=1$ Poincaré superalgebra

$$\{Q^\alpha, \bar{Q}_\beta\} = -\frac{\mathrm{i}}{2}\delta^\alpha_\beta\mathcal{H},, \qquad \{Q^\alpha, Q^\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0. \qquad (2.4)$$

To this end, following Ref. [1], we first equip the Riemann manifold $(M_0, g_{ij}(x)dx^i dx^j)$ with the third-rank symmetric tensor $F_{ijk}(x)dx^i dx^j dx^k$, which obeys Eq. (1.2).

The first equation in Eq. (1.2) defines the well-known Codazzi tensor, while the second equation could be viewed as a generalization of the Witten-Dijkgraaf-Verlinde-Verlinde equation [9] to Riemann manifolds, and was referred to the as *curved WDVV equation* in Refs. [1,5,6].

To construct the supersymmetric mechanics with a nontrivial potential we can define a closed one-form on $(M_0, g_{ij}dx^i dx^j, F_{ijk}(x)dx^i dx^j dx^k)$ that obeys the following compatibility condition:

$$W^{(1)} = W_i(x)dx^i, \quad dW^{(1)} = 0, \quad W_{i;j} + F_{ijk}g^{km}W_m = 0. \qquad (2.5)$$

Clearly, it can be locally presented as an exact one-form $W^{(1)} = dW(x)$, where the locally defined function $W(x)$ is called the "prepotential."

With these objects in hand, we can construct the $\mathcal{N}=4$ supersymmetric mechanics defined by the following supercharges and Hamiltonian [5]:

$$Q^\alpha = p_i\psi^{i\alpha} + \mathrm{i}W_i\psi^{i\alpha} + \mathrm{i}F_{ijk}\psi^{i\beta}\psi^j_\beta\bar\psi^{k\alpha}, \qquad \bar Q_\alpha = p_i\bar\psi^i_\alpha - \mathrm{i}W_i\bar\psi^i_\alpha + \mathrm{i}F_{ijk}\bar\psi^i_\beta\bar\psi^{j\beta}\psi^k_\alpha, \tag{2.6}$$

$$\mathcal{H} = g^{ij}p_ip_j + g^{ij}W_iW_j + 4W_{i;j}\psi^{i\alpha}\bar\psi^j_\alpha - 4[F_{kmj;i} + R_{imjk}]\psi^{i\alpha}\bar\psi^m_\alpha\psi^{j\beta}\bar\psi^k_\beta. \tag{2.7}$$

It follows from Eq. (1.2) that there exists a special coordinate frame where the metrics (and, respectively, Christoffel symbols and Riemann tensor) takes the form [6]

$$g_{ij} = \frac{\partial^2\mathcal{A}}{\partial x^i\partial x^j}, \qquad \Gamma_{ijk} = \frac{1}{2}\frac{\partial^3 A(x)}{\partial x^i\partial x^j\partial x^k}, \qquad R_{ijkm} = \Gamma_{imp}g^{pq}\Gamma_{qjk} - \Gamma_{ikp}g^{pq}\Gamma_{qjm}. \tag{2.8}$$

From the last equation it becomes clear that the choice $F_{ijk} = \pm\Gamma_{ijk}$ solves the curved WDVV equations (1.2). Then, by solving Eq. (2.5) we get the two sets of solutions

$$\left(F_{ijk} = -\Gamma_{ijk}, W = \sum_i c_i\frac{\partial\mathcal{A}}{\partial x^i}\right), \qquad \left(F_{ijk} = \Gamma_{ijk}, W = \sum_i c_i x^i\right), \quad \text{with} \quad c_i = \text{const}. \tag{2.9}$$

The first solution is that obtained in Ref. [10]. The second solution can be transformed to the first one by a Legendre transformation

$$x^i \to u_i = \partial_i\mathcal{A}(x), \qquad \mathcal{A}(x) \to \tilde{\mathcal{A}}(u) = (u_ix^i - \mathcal{A}(x))|_{u_i=\partial_i\mathcal{A}(x)}. \tag{2.10}$$

In this coordinate frame the system (2.6)–(2.7) coincides with the N-dimensional $\mathcal{N}=4$ supersymmetric mechanics constructed by using the N scalar supermultiplets [10] (the respective system with a single supermultiplet was investigated in Ref. [11]).

However, in many cases it is more convenient to solve Eqs. (1.2) and (2.5) in other frames. Below, we will exemplify this by presenting their solutions on $so(N)$-invariant special Riemann manifolds.

A. $SO(N)$-invariant Riemann manifolds

Let us consider the curved WDVV and potential equations (1.2) and (2.5) in isotropic [$so(N)$-invariant] spaces with the metric represented in a conformally flat form,

$$g_{ij}dx^idx^j = \sum_{i=1}^N g(r)dx^idx^i, \quad \text{where } r^2 = \sum_{i=1}^N x^ix^i. \tag{2.11}$$

Let us show that these manifolds always admit nontrivial solutions.

Indeed, let $F_{(0)ijk}(x)$ and $W_{(0)}(x)$ be the solutions of the WDVV and potential equations in Euclidian space, which obey some additional condition

$$F_{(0)ikp}F_{(0)pjm} = F_{(0)jkp}F_{(0)pim},$$
$$\partial_i\partial_jW_{(0)} + F_{(0)ijk}\partial_kW_{(0)} = 0, \tag{2.12}$$

with

$$\sum_{i=1}^N x^iF_{(0)ijk} = \delta_{jk}, \qquad \sum_{i=1}^N x^i\partial_iW_{(0)} = \alpha_0, \tag{2.13}$$

where $F_{(0)ijk} = \frac{\partial^3 F_{(0)}}{\partial x^i\partial x^j\partial x^k}$ and α_0 is some constant. The variety of pairs $(F_{(0)}, W_{(0)})$ that obey these equations was presented in Ref. [12].

These flat solutions can be lifted to the solutions of the *curved WDVV and potential equations* in isotropic spaces as follows (here we use a slightly different notation than that in Refs. [1,5]):

$$F^\sigma_{\kappa|ijk} = g(r)\left(F_{(0)ijk} + \Gamma(r)\frac{\delta_{ij}x^k + \delta_{jk}x^i + \delta_{ik}x^j}{r^2} - A(r)\frac{x^ix^jx^k}{r^4}\right), \tag{2.14}$$

where

$$\Gamma(r) = \frac{r}{2}\frac{d\log g}{dr}, \qquad A(r) = 2\Gamma - \frac{r\Gamma'/2}{\Gamma+1}. \tag{2.15}$$

The corresponding solutions of the curved potential equation and the respective Hamiltonian are given by

$$W = W_{(0)} - \alpha_0\int\frac{\Gamma}{1+\Gamma}\frac{dr}{r} \Rightarrow H = g^{-1}(r)\sum_{i=1}^N(p_ip_i + W_{(0)i}W_{(0)i}) - \frac{2\alpha_0^2}{rg(r)}\left(1 - \frac{1}{(1+\Gamma)^2}\right). \tag{2.16}$$

Note that the "curved" counterpart of the initial Hamiltonian yields an additional potential term with coupling constant α_0^2. In the particular case of a sphere and two-sheet hyperboloid (pseudosphere), when $g = (1 + \epsilon r^2)^{-2}$ (with $\epsilon = 1$ corresponding to the sphere and $\epsilon = -1$ to the pseudosphere), it coincides with the potential of the superintegrable (pseudo)spherical generalization of a harmonic oscillator known as a Higgs oscillator [13].

Thus, with a specific choice of the initial prepotential $W_{(0)}(x)$ we can construct $\mathcal{N} = 4$ supersymmetric superintegrable deformations of a Higgs oscillator. For example, the choice $W_{(0)} = \sum_{i=1}^N \alpha_i \log x^i$, $F_{(0)} = \frac{1}{2}\sum_{i=1}^N (x^i)^2 \log x^i$ yields superintegrable (pseudo)spherical deformations of an N-dimensional oscillator with extra centrifugal terms (which is also known as a Rosochatius system) [14], with an additional restriction on the oscillator frequency [5],

$$H_{\mathrm{Ros}} = (1+\epsilon r^2)^2\left(\sum_{i=1}^N p_i^2 + \sum_i \frac{\alpha_i^2}{x^{i2}} + \epsilon\frac{4(\sum_i \alpha_i)^2}{(1-\epsilon r^2)^2}\right). \tag{2.17}$$

Taking the solutions of Eq. (2.12) corresponding to the three-particle rational Calogero model [15], we get the following (pseudo)spherical Hamiltonian:

$$H_{3\mathrm{Calogero}} = (1+\epsilon r^2)^2\left(\sum_{i=1}^3 p_i^2 + \sum_{i>j=1}^3 \frac{2g^2}{(x_i - x_j)^2} + \epsilon\frac{36 g^2}{(1-\epsilon r^2)^2}\right). \tag{2.18}$$

This is a particular case of a superintegrable (pseudo) spherical Calogero-Higgs oscillator [16].

III. $\mathcal{N} = 8$ MECHANICS ON SPECIAL KÄHLER MANIFOLDS

In this section we generalize the system presented in Ref. [2] and construct, on the special Kähler manifolds of the rigid type, the $(N|4N)$-(complex)dimensional mechanics with a potential term, which possesses the $\mathcal{N} = 8$ supersymmetry

$$\{Q_{i\alpha}, Q_{j\beta}\} = \{\bar{Q}_{i\alpha}, \bar{Q}_{j\beta}\} = 0, \qquad \{Q_{i\alpha}, \bar{Q}_{j\beta}\} = -\mathrm{i}\epsilon_{ij}\epsilon_{\alpha\beta}\mathcal{H}. \tag{3.1}$$

For this purpose we define the $(2N|4N)_{\mathbb{C}}$-dimensional phase superspace equipped with the supersymplectic structure

$$\Omega = d\pi_a \wedge dz^a + d\bar{\pi}_{\bar{a}} \wedge d\bar{z}^a - R_{a\bar{b}c\bar{d}}\eta^c_{i\alpha}\bar{\eta}^{d|i\alpha} dz^a \wedge d\bar{z}^b + g_{a\bar{b}} D\eta^a_{i\alpha} \wedge D\bar{\eta}^{b|i\alpha}, \qquad D\eta^a_{i\alpha} = d\eta^a_{i\alpha} + \Gamma^a_{bc}\eta^b_{i\alpha} dz^c, \tag{3.2}$$

where the fermionic variables $\eta^a_{i\alpha}$ and $\bar{\eta}^{\bar{a}}_{i\alpha}$ related as $(\eta^a_{i\alpha})^\dagger = \bar{\eta}^{\bar{a}i\alpha}$. Here $\alpha, i = 1, 2$ are $su(2)$ indices which are raised and lowered as follows: $A_\alpha = \epsilon_{\alpha\beta}A^\beta$, $A^\alpha = \epsilon^{\alpha\beta}A_\beta$, $A_i = \epsilon_{ij}A^j$, $A^i = \epsilon^{ij}A_j$, with $\epsilon_{12} = \epsilon^{21} = 1$.

This supersymplectic structure is manifestly invariant under the coordinate transformation

$$\tilde{z}^a = \tilde{z}^a(z), \qquad \tilde{\pi}_a = \frac{\partial z^b}{\partial \tilde{z}^a}\pi_b, \qquad \tilde{\eta}^a_{i\alpha} = \frac{\partial \tilde{z}^a}{\partial z^b}\eta^b_{i\alpha}, \tag{3.3}$$

i.e., $\eta^a_{i\alpha}$ transforms as dz^a.

The Poisson brackets corresponding to Eq. (3.2) are defined by the relations

$$\{\pi_a, z^b\} = \delta^b_a, \qquad \{\pi_a, \eta^b_{i\alpha}\} = -\Gamma^b_{ac}\eta^c_{i\alpha},$$
$$\{\pi_a, \bar{\pi}_{\bar{b}}\} = \mathrm{i}R_{a\bar{b}c\bar{d}}\eta^c_{i\alpha}\bar{\eta}^{\bar{d}i\alpha}, \qquad \{\eta^a_{i\alpha}, \bar{\eta}^{bj\beta}\} = -\mathrm{i}g^{a\bar{b}}\delta^j_i\delta^\beta_\alpha. \tag{3.4}$$

To construct the supersymmetric mechanics with a nonzero potential we have to equip the Kähler manifold with the closed holomorphic one-form

$$U^{(1)} = U_a(z) dz^a, \qquad U_a = \frac{\partial U(z)}{\partial z^a}, \tag{3.5}$$

where $U(z)$ is a locally defined holomorphic function called the "prepotential."

With these ingredients in hand we can construct the $\mathcal{N} = 8$ supersymmetric mechanics *with* a potential term. Having in mind the structure of supercharges of the $\mathcal{N} = 4$ supersymmetric mechanics on a generic Kähler manifold [17], and that of the $\mathcal{N} = 8$ supersymmetric mechanics (*without* a potential term) on special Kähler manifolds [2], we choose the following *Ansätze* for supercharges:

$$Q_{i\alpha} = \pi_a\eta^a_{i\alpha} + \bar{U}_{\bar{a}}T^\beta_\alpha\bar{\eta}^{\bar{a}}_{i\beta} + \frac{\mathrm{i}}{3}\bar{f}_{\bar{a}\bar{b}\bar{c}}\bar{\eta}^{\bar{a}}_{i\beta}\bar{\eta}^{\bar{b}j\beta}\bar{\eta}^{\bar{c}}_{j\alpha},$$
$$\bar{Q}_{i\alpha} = \bar{\pi}_{\bar{a}}\bar{\eta}^{\bar{a}}_{i\alpha} - U_a T^\gamma_\alpha \eta^a_{i\gamma} + \frac{\mathrm{i}}{3}f_{abc}\eta^a_{i\beta}\eta^{bj\beta}\eta^c_{j\alpha}, \tag{3.6}$$

where the matrix T^β_α collects the parameters that control the explicit breaking of the $su(2)$ symmetry realized on the greek indices, and, without loss of generality, is parametrized by the two angle-like parameters α_0 and β_0,

$$T^\beta_\alpha = \begin{pmatrix} \cos\alpha_0 & \mathrm{e}^{\mathrm{i}\beta_0}\sin\alpha_0 \\ \mathrm{e}^{-\mathrm{i}\beta_0}\sin\alpha_0 & -\cos\alpha_0 \end{pmatrix}. \tag{3.7}$$

We should stress that it is impossible to introduce an interaction that preserves both $su(2)$ symmetries [realized on the greek and latin indices from the middle of alphabet (i,j,k)]. However, the simultaneous breaking of both of these symmetries results in the appearance of the central charges in the super Poincaré algebra [18].

The components of the (anti)holomorphic symmetric tensors $f_{abc}(z)$ and $\bar{f}_{\bar{a}\bar{b}\bar{c}}(\bar{z})$ have to obey the constraints (1.3), and U_a and $\bar{U}_{\bar{a}}$ were defined in Eq. (3.5).

Then, by taking their Poisson brackets we find that these supercharges span the $\mathcal{N}=8$ Poincaré superalgebra (3.1) if U_a and $\bar{U}_{\bar{a}}$ obey the equations

$$U_{a;b} - f_{abc}g^{\bar{d}c}\bar{U}_{\bar{d}} = 0, \tag{3.8}$$

with $U_{a;b} = U_{a,b} - \Gamma^c_{ab}U_c$. In such a case, the Hamiltonian reads

$$\begin{aligned}\mathcal{H} = {}& g^{a\bar{b}}(\pi_a\bar{\pi}_{\bar{b}} + U_a\bar{U}_{\bar{b}}) - \frac{\mathrm{i}}{2}U_a g^{a\bar{e}}\bar{f}_{\bar{e}\bar{b}\bar{c}}\bar{\eta}^{\bar{b}\alpha}_i T^\beta_\alpha \bar{\eta}^{\bar{c}i}_\beta - \frac{\mathrm{i}}{2}\bar{U}_{\bar{a}}g^{\bar{a}e}f_{ebc}\eta^{b\alpha}_i T^\beta_\alpha \eta^{ci}_\beta \\ & - \frac{1}{12}f_{abc;d}\eta^{ai\rho}\eta^b_{i\gamma}\eta^{cj\gamma}\eta^d_{j\rho} - \frac{1}{12}\bar{f}_{\bar{a}\bar{b}\bar{c};\bar{d}}\bar{\eta}^{\bar{a}i\rho}\bar{\eta}^{\bar{b}}_{i\gamma}\bar{\eta}^{\bar{c}j\gamma}\bar{\eta}^{\bar{d}}_{j\rho} - \frac{1}{4}f_{abe}g^{\bar{e}'e}\bar{f}_{\bar{e}'\bar{c}\bar{d}}(\eta^{ai}_\alpha\eta^b_{i\beta}\bar{\eta}^{\bar{c}j\alpha}\bar{\eta}^{\bar{d}\beta}_j + \eta^{a\alpha}_i\eta^b_{j\alpha}\bar{\eta}^{\bar{c}j\beta}\bar{\eta}^{\bar{d}i}_\beta).\end{aligned} \tag{3.9}$$

Equation (1.3) can be expressed in the distinguished coordinate frame via a single holomorphic function $\mathcal{F}(z)$ (the "Seiberg-Witten potential") (see, e.g., Ref. [3]),

$$g_{a\bar{b}} = \mathbb{R}\mathrm{e}\,\partial_a\partial_b\mathcal{F}(z), \quad \Gamma_{\bar{a}bc} = \partial_a\partial_b\partial_c\mathcal{F}(z), \Rightarrow f_{abc} = \Gamma_{\bar{a}bc}. \tag{3.10}$$

In this coordinate frame Eq. (3.8) becomes

$$\partial_a\partial_b U - (\partial_d U + \partial_{\bar{d}}\bar{U})g^{\bar{d}c}\partial_a\partial_b\partial_c\mathcal{F} = 0. \tag{3.11}$$

From this equation we immediately get the following solution:

$$U(z) = \sum_{a=1}^N (m^a\partial_a\mathcal{F}(z) + \mathrm{i}n_a z^a), \tag{3.12}$$

where m^a and n_a are real constants.

The bosonic part of the constructed $\mathcal{N}=8$ supersymmetric mechanics respects the "T-duality" transformation, which is the complex counterpart of the Legendre transformation (2.10),

$$z^a \to u_a = \partial_a\mathcal{F}, \quad \mathcal{F}(z) \to \tilde{\mathcal{F}}(u) = (z^a u_a - \mathcal{F}(z))|_{u_a=\partial_a\mathcal{F}}. \tag{3.13}$$

For the potential it reads

$$\begin{aligned}U(z) &= \sum_{a=1}^N m^a\partial_a\mathcal{F}(z) + \mathrm{i}n_a z^a \to U(u) \\ &= \sum_{a=1}^N m^a u_a + \mathrm{i}n_a\partial^a\tilde{\mathcal{F}}(u).\end{aligned} \tag{3.14}$$

The extension of the duality transformation to the whole phase superspace is as follows:

$$(z^a,\pi_a,\eta^a_{i\alpha}) \to (u_a,p^a,\xi^a_{i\alpha}), \quad \text{where } u_a = \partial_a\mathcal{F}(z), \qquad p^a\frac{\partial^2\mathcal{F}}{\partial z^a\partial z^b} = -\pi_b, \qquad \theta_{ai\alpha} = \frac{\partial^2\mathcal{F}}{\partial z^a\partial z^b}\xi^b_{i\alpha}. \tag{3.15}$$

Looking back at the presented model of $\mathcal{N}=8$ supersymmetric mechanics we can observe many similarities with the $\mathcal{N}=4$ supersymmetric mechanics described in the previous section, which prompts us to consider it as a complex counterpart of the latter. In particular, the notion of a "special Kähler manifold of rigid type" [Eq. (1.3)] can be viewed as the complex analog of the curved WDVV equations (1.2), and the restriction on the prepotential $U(z)$ can be viewed as he complex counterpart of those on the real one (1.5). In both cases, there exist special coordinate frames where the metrics and the respective third-rank tensors are expressed via a single function, cf. Eqs. (2.8) and (3.10). Further similarities can be noticed by comparing Eqs. (3.12) and (2.9).

However, the requirement of "special Kähleriality" (1.3) is more restrictive than Eq. (1.2). For example, a special Kähler manifold of rigid type necessarily has a negative curvature, while the "curved WDVV equations" do not yield such a restriction; the "curved WDVV equations" admit nontrivial solutions on generic $so(N)$-invariant Riemann manifolds (including N-dimensional spheres and hyperboloids). In contrast to this, complex projective spaces (and their noncompact counterparts) cannot be equipped with the structure of a special Kähler manifold. Moreover, it seems that special Kähler metrics could possess $U(N)$ isometry only in the simplest case where $N=1$, which we consider in the next section.

IV. TWO-DIMENSIONAL SYSTEMS

In this section we construct the one-(complex)dimensional special Kähler manifolds which are invariant under the $U(1)$ transformation $z \to e^{i\lambda} z$, and then find the potentials that admit the $\mathcal{N} = 8$ supersymmetric extension.

Choosing the metric g to be a function of $z\bar{z}$ only, i.e., setting $g = g(z\bar{z})dzd\bar{z}$, one may explicitly solve the second equation in Eq. (1.3) as

$$g(z\bar{z})dzd\bar{z} = (c_1(z\bar{z})^{n_1} + c_2(z\bar{z})^{n_2})dzd\bar{z}, \qquad f(z)[dz]^3 = \sqrt{-c_1c_2}(n_1 - n_2)z^{n_1+n_2-1}[dz]^3, \qquad c_1c_2 < 0. \quad (4.1)$$

The corresponding Kähler potential reads

$$K(z,\bar{z}) = \frac{c_1(z\bar{z})^{n_1+1}}{(n_1+1)^2} + \frac{c_2(z\bar{z})^{n_2+1}}{(n_2+1)^2}. \quad (4.2)$$

Then, performing the transformation $\frac{\sqrt{c_1}}{n_1+1} z^{n_1+1} \to z$, we can simplify these structures as follows:

$$ds^2 = (1 - \kappa^2(z\bar{z})^m)dzd\bar{z},$$
$$f(z)[dz]^3 = \kappa m z^{m-1}[dz]^3, \quad \text{with} \quad |z| \in [0, \kappa^{-1/m}). \quad (4.3)$$

The Christoffel symbol and the Riemann curvature are

$$\Gamma^1_{11} = -\frac{\kappa^2 m z^{m-1}\bar{z}^m}{1 - \kappa^2(z\bar{z})^m}, \qquad R_{1\bar{1}1\bar{1}} = -\frac{\kappa^2 m^2 (z\bar{z})^{m-1}}{1-\kappa^2(z\bar{z})^m}. \quad (4.4)$$

For this special case the potential equation (3.8) takes the form

$$U'' + \frac{\kappa^2 m z^{m-1}\bar{z}^m}{1-\kappa^2(z\bar{z})^m}U' - \frac{\kappa m z^{m-1}}{1-\kappa^2(z\bar{z})^m}\bar{U}' = 0. \quad (4.5)$$

Then, we obtain

$$\frac{d\bar{U}(\bar{z})}{dz} = \frac{d}{dz}\left(\frac{1-\kappa^2(z\bar{z})^m}{\kappa m z^{m-1}}U''(z) + \kappa\bar{z}^m U'(z)\right) = 0. \quad (4.6)$$

From this equation we immediately get the solution

$$U'(z) = \kappa a z^m + \bar{a}, \quad (4.7)$$

with a being an arbitrary complex constant.

Thus, the one-(complex)dimensional $\mathcal{N} = 8$ supersymmetric mechanics is defined by the following bosonic Hamiltonian:

$$H_{\kappa,m,a} = \frac{\pi\bar{\pi} + |\kappa a z^m + \bar{a}|^2}{1-\kappa^2(z\bar{z})^m}, \quad \text{with}$$
$$\{\pi, z\}_0 = \{\bar{\pi}, \bar{z}\}_0 = 1, \qquad \{\pi, \bar{\pi}\}_0 = \{z, \bar{z}\}_0 = 0. \quad (4.8)$$

The presence of a nonzero potential breaks the kinematical $U(1)$ symmetry, $z \to e^{i\lambda}z$, $\pi \to e^{i\lambda}\pi$, but in the free-particle case $a = 0$ the Hamiltonian becomes manifestly invariant under this transformation and thus defines the integrable system

$$H_{\kappa,m,0} = \frac{\pi\bar{\pi}}{1-\kappa^2(z\bar{z})^m} \qquad J = i(z\pi - \bar{z}\bar{\pi}), \qquad \{H_0, J\}_0 = 0, \quad (4.9)$$

where J is the generator of $U(1)$ symmetry.

However, for specific values of m the system can have hidden symmetries. The simplest example corresponds to the $m = -2$ case.

(i) $m = -2$:
In this case the Hamiltonian (4.8) admits a separation of variables in the polar coordinates,

$$z = re^{i\varphi}, \qquad \pi = \frac{e^{-i\varphi}}{2}\left(p_r - \frac{ip_\varphi}{r}\right), \qquad H_{\kappa,-2,a} = \frac{p_r^2 + |a|^2(1 + \frac{\kappa^2}{r^4})}{4(1-\frac{\kappa^2}{r^2})} + \kappa\frac{p_\varphi^2 + \kappa|a|^2\cos(\varphi + \arg a)}{4(r^2-\kappa^2)}, \quad (4.10)$$

which allows us to immediately find the quadratic constant of motion

$$H_{\kappa,-2,a} = \frac{\pi\bar{\pi} + |\kappa a z^{-2} + \bar{a}|^2}{1-\frac{\kappa^2}{|z|^2}}, \qquad I = p_\varphi^2 + 2\kappa|a|^2\cos(\varphi + \arg a) = (z\pi - \bar{z}\bar{\pi})^2 - 4\kappa\frac{\bar{a}^2z^2 + a^2\bar{z}^2}{z\bar{z}}. \quad (4.11)$$

To find additional values of the parameter m leading to (super)integrable systems, one has to do the following. Fixing the energy surface of the Hamiltonian (4.8), one may rewrite it as

$$\pi\bar{\pi} + \kappa^2(|a|^2 + E_{\kappa,m,a})|z|^{2m} + \kappa a^2 z^m + \kappa\bar{a}^2\bar{z}^m = E_{\kappa,m,a} - |a|^2. \quad (4.12)$$

From this expression we immediately deduce that for $m=1$ it coincides with the energy surface of a two-dimensional oscillator interacting with a linear electric field, which could be absorbed by the trivial shift of the complex coordinate z, while for $m=-1/2$ it can be easily transformed to the $m=1$ case using the Bohlin–Levi-Civita transformation $z=\tilde{z}^2$, which relates the energy surfaces of a two-dimensional oscillator and the Coulomb problem [19]. Hence, for the particular values of $m=1,\,-1/2$ the Hamiltonian (4.8) possesses two functionally independent constants of motion and hence becomes superintegrable. Let us consider these cases in full detail.

(ii) $m=1$:

In this particular case, the Hamiltonian (4.8) takes the form

$$H_{\kappa,1,a}=\frac{\pi\bar{\pi}+|\kappa a z+\bar{a}|^2}{1-\kappa^2|z|^2}. \tag{4.13}$$

It possesses a hidden symmetry given by the deformed $U(1)$ generator J presented in Eq. (4.9),

$$J_{\kappa,1}=\mathrm{i}\left[\left(z+\frac{\bar{a}^2}{\kappa(|a|^2+H_{\kappa,1,a})}\right)\pi-\left(\bar{z}+\frac{a^2}{\kappa(|a|^2+H_{\kappa,1,a})}\right)\bar{\pi}\right], \tag{4.14}$$

and by the complex constant of motion

$$F_\kappa=\pi^2+\kappa^2(|a|^2+H_{\kappa,1,a})\left(\bar{z}+\frac{a^2}{\kappa(|a|^2+H_{\kappa,1,a})}\right)^2, \tag{4.15}$$

which can be interpreted as a deformation of the so-called Fradkin tensor written in terms of complex coordinates $z=(x_1+\mathrm{i}x_2)/\sqrt{2}$ and conjugate momentum.

They form the nonlinear algebra

$$\{J_{\kappa,1},F_\kappa\}=2\mathrm{i}F,\qquad \{J_{\kappa,1},\bar{F}_\kappa\}=-2\mathrm{i}\bar{F}_\kappa,\qquad \{F_\kappa,\bar{F}_\kappa\}=4\mathrm{i}\kappa^2(|a|^2+H_{\kappa,1,a})J_{\kappa,1}. \tag{4.16}$$

To emphasize the relationship between this system and an oscillator, let us rewrite the Hamiltonian (4.13) as follows:

$$H_{\kappa,1,a}=H^\kappa_{\mathrm{osc}}+\frac{|\omega|^2}{2\kappa^2},\qquad H^\kappa_{\mathrm{osc}}=\frac{\pi\bar{\pi}+|\omega|^2 z\bar{z}+\bar{E}z+E\bar{z}}{1-\kappa^2(z\bar{z})},\qquad \omega\coloneqq\sqrt{2}\kappa|a|,\qquad E\coloneqq\kappa\bar{a}^2. \tag{4.17}$$

The function in the numerator can be interpreted as a two-dimensional isotropic oscillator with frequency $|\omega|$ interacting with an electric field $\mathbf{E}=(E_1,E_2)$, with $E=(E_1+\mathrm{i}E_2)/2$. The parameters κ and a can be expressed using ω and E as follows:

$$\kappa=\frac{1}{2}\frac{|\omega|^2}{|E|},\qquad a=\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{\arg E}{2}}\frac{|E|}{|\omega|}. \tag{4.18}$$

(iii) $m=-1/2$:

In this case, the Hamiltonian (4.8) acquires the form

$$H_{\kappa,-1/2,a}=\frac{\pi\bar{\pi}+|\kappa a\frac{1}{\sqrt{z}}+\bar{a}|^2}{1-\frac{\kappa^2}{|z|}}. \tag{4.19}$$

It possesses a hidden symmetry given by the deformed $U(1)$ generator $J_{\kappa,-1/2}$,

$$J_{\kappa,-1/2}=2\mathrm{i}\left[\left(z-\frac{\kappa a^2\sqrt{z}}{H_{\kappa,-1/2,a}-|a|^2}\right)\pi-\left(\bar{z}-\frac{\kappa\bar{a}^2\sqrt{\bar{z}}}{H_{\kappa,-1/2,a}-|a|^2}\right)\bar{\pi}\right], \tag{4.20}$$

and by the fact that the complex constant of motion is a deformation of the two-dimensional Runge-Lenz vector $\mathbf{A}=(A_1,A_2)$, with $A_\kappa=(A_1+\mathrm{i}A_2)/2$:

$$A_\kappa = z\pi^2 - (H_{\kappa,-1/2,a} - |a|^2)\left(\sqrt{\bar z} - \frac{\kappa\bar a^2}{H_{\kappa,-1/2,a} - |a|^2}\right)^2. \tag{4.21}$$

They form the nonlinear algebra

$$\{J_{\kappa,-1/2}, A_\kappa\} = 2\mathrm{i}A_\kappa, \qquad \{J_{\kappa,-1/2}, \bar A_\kappa\} = -2\mathrm{i}\bar A_\kappa, \qquad \{A_\kappa, \bar A_\kappa\} = -\mathrm{i}(H_{\kappa,-1/2,a} - |a|^2)J_{\kappa,-1/2}. \tag{4.22}$$

The Hamiltonian (4.19) can be interpreted as a deformation of the two-dimensional Coulomb problem perturbed by the potential $\delta V = k(\frac{a^2}{\sqrt{z}} + \frac{\bar a^2}{\sqrt{\bar z}})$.

The bosonic Hamiltonian H (4.8) possesses the duality transformation

$$H_{\kappa,m,a} = \frac{\pi\bar\pi + |\kappa a z^m + \bar a|^2}{1-\kappa^2|z|^m} = -\frac{\tilde\pi\,\bar{\tilde\pi} + |\tilde\kappa\,\tilde a]\tilde z^{\tilde m} + \bar{\tilde a}|^2}{1-\tilde\kappa^2|\tilde z|^{\tilde m}} = -H_{\tilde\kappa,\tilde m,\tilde a}, \tag{4.23}$$

where the variables are related as (cf. [19])

$$z = \frac{\tilde\kappa\tilde z^{\tilde m+1}}{\tilde m+1}, \qquad \pi = \frac{\tilde\pi}{\tilde\kappa\tilde z^{\tilde m}}, \tag{4.24}$$

and the following constraints on the parameters are imposed:

$$(m+1)(\tilde m+1) = 1, \qquad \kappa\tilde\kappa^{m+1} = |\tilde m+1|^m, \qquad \tilde a = \bar a. \tag{4.25}$$

To be self-consistent, in the transformations (4.24) we should change the admitted values of the coordinates

$$\text{from } |z| \in [0, \kappa^{-1/m}) \quad \text{to} \quad |\tilde z| \in [\tilde\kappa^{-1/\tilde m}, \infty). \tag{4.26}$$

Explicitly, the supercharges of the $\mathcal{N}=8$ supersymmetric extensions of the presented bosonic systems read

$$Q_{i\alpha} = \pi\eta_{i\alpha} + (\kappa\bar a\bar z^m + a)T^\beta_\alpha\bar\eta_{i\beta} + \frac{\mathrm{i}}{3}\kappa m\bar z^{m-1}\bar\eta_{i\beta}\bar\eta^{j\beta}\bar\eta_{j\alpha},$$

while the Hamiltonian has the form

$$\begin{aligned}\mathcal{H}_{\kappa,m,a} = {} & \frac{\pi\bar\pi + |\kappa a z^m + \bar a|^2}{1-\kappa^2(z\bar z)^m} + \mathrm{i}\frac{m\kappa z^{m-1}(a+\kappa\bar a\bar z^m)}{2(1-\kappa^2(z\bar z)^m)}\eta^\alpha_i T^\beta_\alpha\eta^i_\beta - \mathrm{i}\frac{m\kappa\bar z^{m-1}(\bar a+\kappa a z^m)}{2(1-\kappa^2(z\bar z)^m)}\bar\eta^\alpha_i T^\beta_\alpha\bar\eta^i_\beta \\ & - \kappa m\frac{m-1+\kappa^2(1+2m)(z\bar z)^m}{12(1-\kappa^2(z\bar z)^m)}(z^{m-2}\eta_{i\alpha}\eta^{i\beta}\eta_{j\beta}\eta^{j\alpha} + \bar z^{m-2}\bar\eta_{i\alpha}\bar\eta^{i\beta}\bar\eta_{j\beta}\bar\eta^{j\alpha}) \\ & - \frac{\kappa^2 m^2(z\bar z)^{m-1}}{4(1-\kappa^2(z\bar z)^m)}(\eta^i_\alpha\eta_{i\beta}\bar\eta^{j\alpha}\bar\eta^\beta_j + \eta^\alpha_i\eta_{j\alpha}\bar\eta^{j\beta}\bar\eta^i_\beta).\end{aligned} \tag{4.27}$$

The $U(1)$ transformation $z\to \mathrm{e}^{\mathrm{i}\lambda}z$ extended to the supersymplectic structure (3.2) is

$$z \to \mathrm{e}^{\mathrm{i}\lambda}z, \qquad \pi \to \mathrm{e}^{-\mathrm{i}\lambda}\pi, \qquad \eta_{i\alpha} \to \mathrm{e}^{\mathrm{i}\lambda}\eta_{i\alpha}. \tag{4.28}$$

It is defined by the generator

$$\mathcal{J} = \mathrm{i}(z\pi - \bar z\,\bar\pi) - \frac{\partial^2 h(z\bar z)}{\partial z\partial\bar z}\eta^{i\alpha}\bar\eta_{i\alpha}, \qquad h(z,\bar z) = z\bar z - \frac{\kappa^2(z\bar z)^{m+1}}{m+1}, \tag{4.29}$$

where $h(z,\bar z)$ is the Killing potential for $U(1)$ isometry.

It is seen that the supercharges and the Hamiltonian in Eq. (4.27) are not invariant under this transformation for generic m, even for $a=0$. This is not surprising, since the third-order tensor $f(a)[dz]^3$ in Eq. (4.3) is invariant under the $U(1)$ transformation $z\to\mathrm{e}^{\mathrm{i}\nu}$ only for $m=-2$, while the one-form (4.7) is not $U(1)$ invariant at all. Since $\eta_{i\alpha}$ transforms as dz, we conclude that the supercharges and supersymmetric Hamiltonian fail to be $U(1)$ invariant in the generic case. Hence, it is only for the case $a=0, m=-2$ [corresponding to the bosonic Hamiltonian (4.11)] that we can construct the $\mathcal{N}=8$ supersymmetric extension with the supercharges and Hamiltonian that are invariant under the transformation (4.28).

On the other hand, the Hamiltonian $\mathcal{H}_{\kappa,m,0}$, in contrast to the supercharges $Q_{(\kappa,m,0)i\alpha}$, is invariant under the transformation

$$z \to e^{i\lambda} z, \qquad \pi \to e^{-i\lambda}\pi, \qquad \eta_{i\alpha} \to e^{i\frac{2-m}{4}\lambda}\eta_{i\alpha}. \tag{4.30}$$

Hence, it commutes with the generator

$$\tilde{\mathcal{J}} = i(z\pi - \bar{z}\bar{\pi}) - \frac{\partial^2 h(z\bar{z})}{\partial z \partial \bar{z}}\eta^{i\alpha}\bar{\eta}_{i\alpha} + \frac{m+2}{4}g(z\bar{z})\eta_{i\alpha}\bar{\eta}^{i\alpha}, \qquad \{\tilde{\mathcal{J}}, \mathcal{H}_{m,0}\} = 0, \tag{4.31}$$

where $h(z\bar{z})$ is the Killing potential (4.29) and $g(z\bar{z}) = 1-\kappa^2(z\bar{z})^m$ is the component of the special Kähler metrics (4.1).

V. CONCLUDING REMARKS

In this paper we have constructed the $\mathcal{N}=8$ supersymmetric mechanics with a potential term, whose configuration space is a special Kähler manifold of rigid type. We observed that it can be viewed as a complex counterpart of the recently suggested $\mathcal{N}=4$ supersymmetric mechanics [1,5]. Then, we constructed the $U(1)$-invariant one-dimensional special Kähler manifold and corresponding $\mathcal{N}=8$ supersymmetric mechanics, including $\mathcal{N}=8$ supersymmetric extensions of superintegrable perturbations of a deformed two-dimensional oscillator and Coulomb systems considered in Ref. [8] as particular cases. It is an open question whether $\mathcal{N}=8$ supersymmetric counterparts of the hidden symmetries of these superintegrable systems exist.

A less straightforward and more interesting task is the construction of the nonlinear $\mathcal{N}=8$ supersymmetric mechanics generalizing the nonlinear supersymmetric systems considered by Plyushchay *et al.* (see, e.g., Ref. [20] and references therein). We believe that a nonlinear supersymmetric mechanics on the two-dimensional manifolds considered in Ref. [21] could be directly extended to the examples of the superintegrable models presented in Sec. IV of this paper. This topic definitely deserves a separate consideration, in a separate study.

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