Isotropization of slowly expanding spacetimes

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We show that the homogeneous, massless Einstein-Vlasov system with toroidal spatial topology and diagonal Bianchi type I symmetry for initial data close to isotropic data isotropizes towards the future and in particular asymptotes to a radiative Einstein-de Sitter model. We use an energy method to obtain quantitative estimates on the rate of isotropization in this class of models.

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I. INTRODUCTION

Determining the asymptotic behavior for cosmological models is a fundamental objective of mathematical cosmology. A lot of effort has been made to investigate the stability of the isotropy of the Universe (cf., e.g., [1–6], and references therein). For the Einstein-Vlasov system, which models universes containing ensembles of self-gravitating collisionless particles [7–9], this program is quite advanced for the class of spatially homogeneous (SH) spacetimes.

Much of the respective literature is based on an approach by Rendall [10] as well as Rendall and Tod [11] in which certain symmetries are imposed on the Vlasov matter distribution, as a consequence of which, together with spatial homogeneity, the Einstein-Vlasov system reduces to a system of autonomous (time-invariant) ODEs. Hence, dynamical systems theory can be applied to analyze these systems. The task to utilize this approach for all those types of SH cosmologies to which it is applicable has been accomplished in [12–17] and recently been completed in [18]. To the latter source we also refer to for a recent and more detailed summary of this approach.

The dynamical systems approach is powerful in particular in that it is capable of yielding global stability results. It however has some detriments: Firstly, its results are not of full generality since it relies on the above mentioned symmetry assumptions on the matter distribution. Secondly, it cannot be applied to all types of SH cosmologies, since not all of these concrete symmetry assumptions are known, or can be found in principle. Finally, its applicability is limited to the spatially homogenous context since for cosmologies with less spatial symmetry the Einstein-Vlasov system does not reduce to ODEs.

Hence, there has been an interest in adopting other techniques as well. Nungesser [19–21] and Nungesser et. al. [22] performed a small data future stability analysis for several types of spatially homogenous Einstein-Vlasov cosmologies (cf. also the result [23] on the Einstein-Boltzmann system). While the stability results obtained by this approach concern generally the small data regime, it is fit to overcome the above detriments of the dynamical systems approach. The former thus complements the latter in the SH context, and leaves open the possibility of generalization to spacetimes with less spatial symmetry. The focus of this literature has been on the future stability in the case of massive particles. The dynamics in the massless case is of interest to be analyzed separately since it has been shown to behave substantially differently from the massive case for various spatial topologies; cf. e.g., [15].

In the present paper we prove the isotropization of small perturbations of the Einstein-de Sitter (EdS) model within the class of diagonal Bianchi type I solutions to the massless Einstein-Vlasov system. Though the Bianchi I Einstein-Vlasov system has already been investigated thoroughly in [10,15], and despite the fact that we use the same symmetry assumptions as these sources, our results are novel and complement the latter two in the following points: Firstly, the result of [10] is limited to the massive case with regards to the future asymptotics, while we treat the massless case. Secondly, we use an energy method by which we not only recover the result of [15] with regards to future asymptotics, but which also captures the decay rates for the perturbations away from isotropy—the monotone function techniques used in [15] did not. Finally, we expect the energy method to also be applicable to the nondiagonal case, and that it is sufficiently robust to allow for a generalization to the class of inhomogeneous solutions.After this work was completed we learned that a related problem was considered independently in [24].

We start out in Sec. II with some background on the radiative EdS model tailored to the present context. In Sec. III we lay out the setup for our analysis on the diagonal Bianchi I Einstein-Vlasov system, and we formulate our result in theorem III.7. Section IV is then devoted to the

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proof of the latter by a small data stability analysis using an energy method. Two of the calculations for the proof can be found in the Appendix.

II. THE RADIATIVE EINSTEIN-DE SITTER MODEL

The radiative EdS model (cf. e.g., [25])

$$((0,\infty)\times\mathbb{T}^3, -dt^2 + t\cdot\gamma),$$

where $(\mathbb{T}^3,\gamma)$ is a flat torus, is a solution to the Einstein equations coupled to a radiation fluid, i.e., a perfect fluid with pressures equal to $1/3$ times the energy density. With a scale factor $a(t)=\sqrt{t}$ it expands significantly slower than the related FLRW vacuum solution on hyperbolic spatial topologies (the Milne model) with $a(t)=t$ and also slower than the corresponding solution on $\mathbb{T}^3$ for dust (i.e., a pressureless perfect fluid), the EdS model, with $a(t)=t^{2/3}$; cf. [9]. The EdS models pose interesting examples of *matter-dominated* cosmological spacetimes, i.e., spacetimes whose asymptotic behavior is altered by the presence of matter.

While for initial data close to the Milne geometry on hyperbolic spatial manifolds vacuum and nonvacuum future asymptotics are similar [26], on toroidal spatial topologies vacuum asymptotics deviate drastically from the matter dominated regime of the EdS models [9]. It is of essential interest to investigate the stability properties of those model solutions in order to understand whether their behavior is representative for generic spacetimes with similar initial data. The stability properties of EdS models are unknown except in the homogeneous context, i.e., for Bianchi type I models. In that case it has been shown by Nungesser that the massive EdS model is a future attractor of the Einstein-Vlasov system with massive particles [19]. The analogous problem for the radiative EdS model, which concerns massless particles (or radiation) is addressed for the massless Einstein-Vlasov system in the present paper for the restricted class of diagonal Bianchi type I models. We show that initial data sufficiently close to an isotropic state for the massless Einstein-Vlasov system isotropizes towards the future and asymptotes towards a member of the family of radiative EdS models with suitable decay rates for the perturbations.

The slower expansion rate for the radiative case makes it *a priori* more difficult to establish sufficiently strong decay estimates. We point out that nonlinear stability results are established for exponential scale factors [27] or polynomial scale factors with significantly higher exponents [26]. Indeed, our analysis requires a more careful treatment of the evolution equation for the shear tensor and metric perturbations. We use a fine-tuned corrected energy, which controls shear tensor and metric perturbation simultaneously, to obtain the crucial decay estimates.

Our theorem assures that the radiative EdS model is an attractor in the restricted class of diagonal Bianchi I symmetric solutions to the massless Einstein-Vlasov system. To what extent stability holds in less restricted sets of solutions as for instance the set of surface-symmetric or $\mathbb{T}^2$-symmetric solutions is an open problem that can be addressed using the framework of previous works as for instance [28] and will be the subject of future studies.

III. THE DIAGONAL BIANCHI TYPE I EINSTEIN-VLASOV SYSTEM

In much of this section we closely follow Sec. 2 of [15]. Section III A gives some background on spatially homogenous cosmologies of Bianchi type. For a deeper background on the Bianchi classification and on the choice of basis we refer to [29]. In Sec. III B we discuss Vlasov matter in the context of Bianchi I. For a thorough background on the Einstein-Vlasov system we refer to [7–9]. After that we specialize to reflection symmetric (or diagonal) models in Sec. III C. In Sec. III D we then formulate a reflection symmetric Bianchi I Einstein-Vlasov system for massless particles. Finally, we state our main result in the form of theorem III.7 in Sec. III E and emphasize the relation to the result of [15].

A. Bianchi cosmologies

Bianchi spacetimes admit a Lie algebra of Killing vector fields K_1, K_2, and K_3 which are tangent to the orbits of the group which is identified with the universal covering space of Bianchi models. These orbits are called surface of homogeneity. Moreover, the Killing vector fields satisfy the commutation relation $[K_i,K_j]=C^k_{ij}K_k$ where C^k_{ij} are structure constants. Bianchi I models are characterized by $C^k_{ij}=0$. Choosing a unit vector field n normal to the group orbits, one has a natural choice for the time coordinate. One can choose a basis $\{E_i\}$ of the surfaces of homogeneity such that they commute with the Killing vector fields. In this way, one can construct the so-called left-invariant frame $\{n,E_i\}$ which is generated by the right-invariant Killing vector fields; cf. [29] [Sec 1.5.2]. We now consider general Bianchi I spacetimes of the form $\bar{g}=-dt^2+\mathbf{g}$, where $\mathbf{g}=g_{ij}(t)W^i\otimes W^j$, where W^i denotes the dual one-forms to the left-invariant basis E_i. We denote by k_{ij} the second fundamental form and decompose via $k_{ij}=\sigma_{ij}-Hg_{ij}$, where $H=-\frac{1}{3}\mathrm{tr}_g k$ and σ_{ij} is the trace-free part of k_{ij}. Moreover, we define the rescaled trace-free part and its square by

$$\Sigma_i^j \coloneqq H^{-1}\sigma_i^j \quad \text{and} \quad F\coloneqq \Sigma_i^j\Sigma_j^i, \tag{3.1}$$

respectively. The physical interpretation of the defined quantities is as follows: H is the Hubble scalar and represents a measure of the overall, isotropic, rate of spatial expansion. σ_{ij} is the shear tensor and represents a measure of the anisotropic rate of spatial expansion. Respectively Σ_i^j is the Hubble normalized shear tensor, and consequently F represents an overall measure of anisotropy. In particular, $F = 0$ marks an isotropic state.

As matter model we consider a collisionless kinetic gas of massless particles, i.e., massless Vlaosv matter. The respective energy-momentum tensor is determined by a distribution function f which solves a transport equation, the Vlasov equation. $f(t, x^i, p^i)$ represents the density of particles at time t and position x^i with momentum p^i.

B. Vlasov matter

In the case of Bianchi I symmetry the Vlasov equation reduces to

$$\partial_t f + 2k^i_j p^j \partial_{p^i} f = 0. \tag{3.2}$$

In particular, compatibility with spatial homogeneity forces f to be independent of the spatial coordinates, i.e., $f = f(t, p^i)$, and hence (3.2) does not contain any spatial derivatives. The Vlasov equation (3.2) has the general spatially homogeneous solution in Bianchi type I symmetry of the form (cf. [30] [Sec. 4])

$$f(t, p^i) = f_0(p_i).$$

We decompose the components of the energy-momentum tensor into the spatial part S_{ij}, the energy density ρ and the momentum density j_i, given by

$$\rho := \int_{\mathbb{R}^3\setminus\{0\}} f_0(p_k)|p|_g(\det g)^{-\frac{1}{2}} d^3p,$$

$$S_{ij} := \int_{\mathbb{R}^3\setminus\{0\}} f_0(p_k)\frac{p_i p_j}{|p|_g}(\det g)^{-\frac{1}{2}} d^3p,$$

$$j_i := \int_{\mathbb{R}^3\setminus\{0\}} f_0(p_k)p_i(\det g)^{-\frac{1}{2}} d^3p,$$

where $|p|_g := (g_{ij}p^i p^j)^{1/2}$ and $d^3p := dp_1 dp_2 dp_3$. In Bianchi type I the momentum constraint implies $j_k = 0$.

C. Reflection symmetry; diagonal models

In the present paper we restrict ourselves to the subclass of a Bianchi type I Einstein-Vlasov system which admits reflection symmetry (or diagonality) in the following sense; cf. [10,15]. On the initial data the following conditions are imposed:

$$\begin{aligned} f_0(p_1, p_2, p_3) &= f_0(p_1, -p_2, -p_3) = f_0(-p_1, -p_2, p_3) \\ &= f_0(-p_1, p_2, -p_3), \\ g(t_0) &= \mathrm{diag}(g_{11}(t_0), g_{22}(t_0), g_{33}(t_0)), \\ k(t_0) &= \mathrm{diag}(k_{11}(t_0), k_{22}(t_0), k_{33}(t_0)). \end{aligned}$$

Under these conditions $j_k = 0$ is satisfied and $S_{ij}(t_0)$ is diagonal. The evolution equations preserve the diagonality for all time, i.e., g_{ij}, k_{ij}, and S_{ij} are diagonal for all time.

D. Formulation of the system

Next, we define the dimensionless variables following [15]. We define (no summation over repeated indices, unless explicitly mentioned)

$$\Omega := \frac{8\pi\rho}{3H^2}, \qquad s_i := \frac{g^{ii}}{\sum_k g^{kk}}, \qquad \Sigma_i := -\Sigma^i_i = -\frac{k^i_i}{H} - 1,$$

with

$$s_1 + s_2 + s_3 = 1 \quad \text{and} \quad \Sigma_1 + \Sigma_2 + \Sigma_3 = 0. \tag{3.3}$$

For simplicity we also collect the three s_i and Σ_i components in tuples $\vec{s}$ and $\vec{\Sigma}$, respectively. Further, we define

$$w_i := \frac{S^i_i}{\rho}, \qquad w := \frac{1}{3}\sum_i w_i. \tag{3.4}$$

In the case at hand, i.e., in the massless case, we have $w = 1/3$, which follows from (3.4) and the fact that $\mathrm{tr}S = \rho$ for massless particles. It is important to note that the w_i can be written as functions of $\vec{s}$ (cf. (9) in [15]):

$$w_i(\vec{s}) = \frac{s_i \int f_0 p_i^2 (\sum_k s_k p_k^2)^{-\frac{1}{2}} d^3\tilde{p}}{\int f_0 (\sum_k s_k p_k^2)^{\frac{1}{2}} d^3\tilde{p}} =: s_i Y_i(\vec{s}). \tag{3.5}$$

The reflection-symmetric Bianchi I Einstein-Vlasov system for massless particles finally takes the form

$$\Omega = 1 - \frac{1}{6}F, \tag{3.6a}$$

$$H' = -H(3 - \Omega), \tag{3.6b}$$

$$\Sigma_i' = -\Omega(\Sigma_i + 1 - 3w_i), \tag{3.6c}$$

$$s_i' = -2s_i\left(\Sigma_i - \sum_k s_k \Sigma_k\right), \tag{3.6d}$$

where the prime denotes $\partial_\tau = H^{-1}\partial_t$ (cf. [15]). We see that (3.6b) is decoupled from (3.6c) and (3.6d). Hence, the latter two define a six-dimensional reduced system.

E. Main result

For the system introduced above we prove the following theorem.

Theorem III.7. Consider C^∞ initial data for the massless Einstein-Vlasov system with diagonal Bianchi I symmetry, (g_0, H_0, F_0) at $t_0 = (2H(t_0))^{-1}$ with f_0 sufficiently close to an isotropic distribution function. There exists an $\varepsilon > 0$ such that $F_0 < \varepsilon$ and $|g_{ij} - \delta_{ij}| < \varepsilon$ imply the following future asymptotics for a constant $C > 0$,

$$F(t) \lesssim \varepsilon t^{-1.16/2+\varepsilon}, \quad 2t \leq H^{-1}(t) \leq 2t(1 + C\varepsilon t^{-1/2}), \tag{3.7}$$

and

$$tg^{ij} \to g^{ij}_{\infty} \text{ as } t \to \infty \text{ with } |g^{ij}_{\infty} - tg^{ij}(t)| \lesssim \varepsilon t^{-1.16/4+\varepsilon}. \tag{3.8}$$

In particular, the rescaled square of the shear tensor, $F = |\Sigma|_g^2$ vanishes asymptotically, i.e., the spacetime isotropizes. Moreover, the rescaled spatial metric $t^{-1}g(t)$ converges to a limit metric g_∞, which remains ε-close to the initial metric g_0. As a result, the radiative EdS model is orbitally stable in the set of solutions to the massless Einstein-Vlasov system with diagonal Bianchi I symmetry.

Remark III.10.—The choice of initial time $t_0 = (2H(t_0))^{-1}$ is made for technical reasons, and does not restrict the generality.

Remark III.11.—The value 1.16 of the decay rate is approximated. We obtain this value numerically from a solution of an algebraic equation [cf. (4.12)]. We do not claim this decay rate is sharp, but it is the optimal value in the scope of the method we use.

Remark III.12.—The orbital stability in the previous theorem was already proven by Heinzle and Uggla in [15]. However, their proof did not provide decay rates for the perturbations. Moreover, the dynamical systems method utilized in [15] does not provide a natural extension to the inhomogeneous case, while the energy methods are flexible in their application and in principle extend to less symmetric scenarios.

IV. STABILITY ANALYSIS

In the remainder of this paper we prove theorem III.7. The basic idea of the proof is to use the fact that for the attractor of the system, which represents the isotropic state, $w_i(\vec{s}^*) = 1/3$ holds for all $i = 1, 2, 3$, as will be shown. We then linearize the system around this state in Sec. IV A, and prove by a small data stability analysis that it is indeed a local attractor of the system in the remainder of this section. For this we define an energy function in Sec. IV C, find the optimal estimates for its decay to zero in Sec. IV D, and finally translate these rates to the decay rates of the quantities of theorem III.7 in Sec. IV E. For two calculations of this section we refer to the Appendix.

The type of energy method we apply here has been used for the Einstein equations in different contexts (cf. [26] and references therein). Here it is used in a context where the behavior of the matter variables has a strong effect on the geometry (matter dominated regime). We expect that it can be used in different classes of Bianchi models containing Vlasov matter. The case of Bianchi type II symmetry is currently work in progress [31].

A. Linearization around a state of isotropic geometry

The first observation concerns the attractor geometry, which we eventually show to be isotropic. Let us identify this state. Firstly, from the definition and interpretation of Σ_i in Sec. III D, we know that an isotropically expanding state is characterized by $\vec{\Sigma} = 0$ and $w_i = w \ \forall \ i$. Secondly, it has been shown in [15] that to each initial matter distribution f_0 there is a unique $\vec{s} = \vec{s}^*$ which represents the rescaled 3-metric for which $w_i(\vec{s}^*) = 1/3 \ \forall \ i$. Hence, from (3.4) we see that $\vec{s}^*$ corresponds to a state in which the matter attains isotropic pressures. Therefore, assuming an isotropic attractor, it must be uniquely characterized by $(\vec{s}, \vec{\Sigma}) = (\vec{s}^*, 0)$, which marks an equilibrium point of the reduced system (3.6c)–(3.6d).

We now linearize (3.6a)–(3.6d) around the isotropic state. To shift the corresponding equilibrium point to the origin we define $\bar{s}_i \coloneqq s_i - s_i^*$ and express the reduced system as

$$\Sigma_i' = -\Omega(\Sigma_i + 1 - 3w_i),$$
$$\bar{s}_i' = -2(\bar{s}_i + s_i^*)\left(\Sigma_i - \sum_k (\bar{s}_k + s_k^*)\Sigma_k\right),$$

with w_i given by (3.5). The corresponding linearized system at $(\vec{\bar{s}}, \vec{\Sigma}) = (0, 0)$ then reads

$$\Sigma_i' = -\Sigma_i + 3\bar{s}_i Y_i(s_k^*) + 3s_i^* \sum_j \frac{\partial Y_i}{\partial s_j}\bigg|_{s_k^*} \bar{s}_j + O(|\vec{\Sigma}|_\delta^2) + O(|\vec{\bar{s}}|_\delta^2), \tag{4.1a}$$

$$\bar{s}_i' = -2s_i^*\Sigma_i + 2s_i^* \sum_k s_k^* \Sigma_k + O(|\vec{\bar{s}}|_\delta |\vec{\Sigma}|_\delta), \tag{4.1b}$$

where $|.|_\delta$ denotes the Euclidean norm. Note that $|\vec{\Sigma}|_\delta^2 = F$; cf. (3.1).

In order to perform a stability analysis of the origin of (4.1a)–(4.1b) we impose a smallness assumption on $\vec{\Sigma}$ and $\vec{\bar{s}}$. Consequently, the higher order terms of the system can be treated as error terms, which can be absorbed in the final energy estimate [see (4.13) below]. It is the linear terms that determine the decay rates for small initial data.

Remark IV.2.—When calculating the eigenvalues and eigenvectors of the linearization at our assumed attractor point, one finds that it is a degenerate equilibrium point with one zero eigenvalue. One could then try to pursue a center manifold analysis to obtain the local stability of this point, and decay estimates; cf. for instance [32]. In the present work we chose an alternative path, using the energy method laid out below, because of its potential to be generalized to spatially inhomogenous cases.

B. Closeness to an isotropic initial matter state

The linear system (4.1a)–(4.1b) contains factors of $Y_i(\vec{s}^*)$ and its derivatives. These depend on the initial particle distribution, as a consequence of $\vec{s}^*$ depending on f_0; cf. Sec. IV A. In the following we adopt the notation $\vec{s}^*[f_0]$ when we want to emphasize this dependence.

Let us consider for now an isotropic initial particle distribution $f_0^{\rm iso}(p_k) = f_0^{\rm iso}(|p|_\delta)$. From (3.3) the corresponding pressures are isotropic as well and thus $w_i = 1/3 \ \forall\, i$. Furthermore, we then know from the discussion in Sec. IV A that $f_0^{\rm iso}$ is associated with a unique $\vec{s}^*[f_0^{\rm iso}]$. Thus, from (3.3) and (3.5) we have $s_i^*[f_0^{\rm iso}] = 1/3 \ \forall\, i$. Gathering the information and plugging it into (3.5) we find

$$Y_i(\vec{s}^*[f_0^{\rm iso}]) = 1.$$

Next, we calculate the derivatives of Y_i for our isotropic matter state. The result is

$$\left.\frac{\partial Y_i}{\partial s_j}\right|_{\vec{s}^*[f_0^{\rm iso}]} = \begin{cases} -\frac{7}{5}; & \text{if } i = j \\ -\frac{4}{5}; & \text{if } i \neq j \end{cases}, \qquad (4.2)$$

and we refer to Appendix A 1 for the calculation. Now, since all functionals are continuous in f_0, by continuity we can conclude that the values of Y_i and $\partial Y_i/\partial s_j$ for f_0 close to isotropic are ε-close to (4.2).

Motivated by the discussion of this subsection, we introduce the following parametrization for matter states which deviate from the isotropic one:

$$f_0 = f_0^{\rm iso} + \xi \tilde{f}_0, \qquad s_i^*[f_0] = 1/3 + c_i/3, \quad (4.3)$$

$$Y_i(\vec{s}^*[f_0]) = 1 + a_i, \qquad \left.\frac{\partial Y_i}{\partial s_j}\right|_{\vec{s}^*[f_0]} = \begin{cases} -\frac{7}{5} + \epsilon_1; & \text{if } i = j, \\ -\frac{4}{5} + \epsilon_2; & \text{if } i \neq j, \end{cases} \quad (4.4)$$

where $\tilde{f}_0$ denotes some perturbation. The parameters $c_i, \xi, a_i, \epsilon_1, \epsilon_2$ thereby control the deviation, and we impose on them the smallness assumption

$$|c_i|, |\xi|, |a_i|, |\epsilon_1|, |\epsilon_2| \ll 1, \qquad \forall\, i. \qquad (4.5)$$

In the following we drop again the functional notation and if we write $\vec{s}^*$, then what we mean is a close to isotropic $\vec{s}^*$ in the sense of (4.3).

C. The energy function

Next, we define an energy of the system to eventually control the deviation from $\sum_i \bar{s}_i^2 + \Sigma_i^2$, i.e., we define

$$E \coloneqq E_1 + E_2 + E_3 \quad \text{with} \quad E_i \coloneqq \alpha \bar{s}_i^2 + \beta \bar{s}_i \Sigma_i + (\Sigma_i)^2, \quad (4.6)$$

and where $\alpha > \beta^2/4$ and $\beta \in \mathbb{R}$, in order to have $E \geq 0$ and $E = 0$ only at the isotropic state. For small initial data (cf. Secs. IV A and IV B) we anticipate E to be strictly monotonically decreasing and ultimately going to zero, and we seek to obtain estimates on its decay rate which we can then translate to the rates of the geometric quantities in theorem III.7.

In order to achieve this goal we derive the evolution equation for E by building the prime of (4.6) and using (4.1a)–(4.1b). The calculation is straightforward but lengthy. We quote here the result and refer to Appendix A 2 for more details. It is important to note however, that in the course of this calculation we plug in the parametrizations (4.3)–(4.4). Introducing the notation $\bar{s}^2 \coloneqq \sum_i \bar{s}_i^2$ and $\vec{\bar{s}} \cdot \vec{\Sigma} \coloneqq \sum_i \bar{s}_i \Sigma_i$ the result reads

$$\begin{aligned} E' = {}& \frac{8}{5}\beta \bar{s}^2 - 2\left(1 + \frac{1}{3}\beta\right) F + \left(\frac{24}{5} - \beta - \frac{4}{3}\alpha\right) \vec{\bar{s}} \cdot \vec{\Sigma} \\ & - \frac{8}{5}\beta(\bar{s}_1\bar{s}_2 + \bar{s}_1\bar{s}_3 + \bar{s}_2\bar{s}_3) + \mathcal{G}(\epsilon_1, \epsilon_2) \\ & + \sum_i \mathcal{H}_i(a_i, c_i) + \mathcal{O}^3(|\vec{\Sigma}|_\delta, |\vec{\bar{s}}|_\delta), \end{aligned} \qquad (4.7)$$

with

$$\begin{aligned} \mathcal{G}(\epsilon_1, \epsilon_2) \coloneqq {}& \epsilon_1(\beta \bar{s}^2 + 2\vec{\bar{s}} \cdot \vec{\Sigma}) \\ & + 2\epsilon_2(\beta(\bar{s}_1\bar{s}_2 + \bar{s}_1\bar{s}_3 + \bar{s}_2\bar{s}_3) - \vec{\bar{s}} \cdot \vec{\Sigma}), \\ \mathcal{H}_i(a_i, c_i) \coloneqq {}& 3\beta a_i \bar{s}_i^2 + 2\left(3a_i - \frac{2}{3}\alpha c_i\right)\bar{s}_i \Sigma_i - \frac{2}{3}\beta c_i \Sigma_i^2 \\ & + \frac{2}{3}(2\alpha \bar{s}_i + \beta \Sigma_i) s_i^* \sum_k c_k \Sigma_k \\ & + (2\Sigma_i + \beta \bar{s}_i) c_i \sum_j \left.\frac{\partial Y_i}{\partial s_j}\right|_{s_k^*} \bar{s}_j, \end{aligned} \qquad (4.8)$$

$$\begin{aligned} \mathcal{O}^3(|\vec{\Sigma}|_\delta, |\vec{\bar{s}}|_\delta) \coloneqq {}& O(|\vec{\Sigma}|_\delta^3) + O(|\vec{\Sigma}|_\delta^2 |\vec{\bar{s}}|_\delta) \\ & + O(|\vec{\Sigma}|_\delta |\vec{\bar{s}}|_\delta^2) + O(|\vec{\bar{s}}|_\delta^3), \end{aligned} \qquad (4.9)$$

and where $\partial Y_i/\partial s_j|_{s_k^*}$ is understood to be substituted by the parametrization (4.4).

D. Optimal decay estimate for the energy

In order to obtain a decay estimate on E we start by using the binomial inequality on (4.7) and introducing a decay inducing coefficient $\lambda > 0$, which yields

$$\begin{aligned} E' \leq {}& -\lambda E + q(\vec{\bar{s}}, \vec{\Sigma}) + \mathcal{G}(\epsilon_1, \epsilon_2) \\ & + \sum_i \mathcal{H}_i(a_i, c_i) + \mathcal{O}^3(|\vec{\Sigma}|_\delta, |\vec{\bar{s}}|_\delta), \end{aligned} \qquad (4.10)$$

with the quadratic form

$$q(\vec{\bar{s}},\vec{\Sigma}) \coloneqq \left(\lambda\alpha+\frac{16}{5}\beta\right)\bar{s}^2+\left(\beta(\lambda-1)+\frac{24}{5}-\frac{4}{3}\alpha\right)\vec{\bar{s}}\cdot\vec{\Sigma} + \left(\lambda-2-\frac{2}{3}\beta\right)F. \tag{4.11}$$

The estimate (4.10) contains three parameters, α, β, and λ. We can vary these within their bounds in order to optimize the decay rate of E.

When introducing α and β in (4.6) we put on them the constraint $\alpha > \beta^2/4$ such that E is non-negative. Further, we just introduced the constant $\lambda > 0$. In fact we want to maximize the value of λ in order to obtain the optimal decay rate. At the same time however we have to demand that q stays negative semidefinite, such that it does not counter the decay.

In summary, we wish to find a triple (α,β,λ) that maximizes λ under the following constraints:

(i) $\alpha > \frac{\beta^2}{4}$, $\beta\in\mathbb{R}$, and $\lambda>0$,

(ii) q is negative semidefinite.

After inspecting the conditions (i) and (ii), it turns out that the optimal value of λ is achieved when $q=0$ (this can be verified for instance using a computer algebra system such as *Mathematica*). In this way we maximize the value of λ while making the quadratic form q vanish. In other words, if e_1 and e_2 are the two eigenvalues of the quadratic form (4.11), then the system which corresponds to the conditions (i) and $q=0$, i.e.,

$$\{\alpha>\beta^2/4, \lambda>0, e_1=0, e_2=0\},$$

is satisfied for the value of λ by solving the equation

$$15\lambda^3-45\lambda^2+142\lambda-128=0. \tag{4.12}$$

One finds the numerical result $\lambda\approx 1.16426$. Accordingly, $\alpha\approx 3.4455$ and $\beta\approx -1.2536$. Hence, for this choice of constants we have

$$E' \le -\lambda E + C\varepsilon E, \tag{4.13}$$

where we estimated $\mathcal{G}$, $\sum_i \mathcal{H}_i$, and $\mathcal{O}^3(|\vec{\Sigma}|_\delta, |\vec{\bar{s}}|_\delta)$ using suitable constant $C>0$ and $0<\varepsilon\ll 1$, because of the smallness of the initial data and (4.5). This decay estimate is optimal within the framework of our method.

E. Decay estimates for the geometric quantities

From (4.13) it follows

$$\bar{s}_i \le e^{(-\lambda/2+\varepsilon)\tau}, \qquad |\Sigma_i| \le e^{(-\lambda/2+\varepsilon)\tau}. \tag{4.14}$$

What is left to do is to express these rates in terms of metric time t. Using $\partial_\tau = H^{-1}\partial_t$, we can rewrite (3.6b) as

$$\partial_t(H^{-1}) = 3-\Omega \le 3.$$

Using $d\tau/dt = H$ and $t_0 = (2H(t_0))^{-1}$, one finds

$$e^{-\tau} \le Ct^{-\frac{1}{3}},$$

for some constant $C>0$. From (4.14) it is readily seen that $F\le Kt^{-\lambda/3+\varepsilon/3} \eqqcolon Kt^{-\kappa}$ for some positive constant K and $0<\kappa<\lambda/3$. We may rewrite (3.6b) again as

$$\partial_t(H^{-1}) = 2+\frac{1}{6}F,$$

which implies

$$2\le \partial_t(H^{-1}) \le 2+Kt^{-\kappa}.$$

Integrating this inequality and keeping in mind that $t_0=(2H(t_0))^{-1}$, we arrive at

$$2t\le H^{-1}\le 2t+\frac{K}{1-\kappa}t^{1-\kappa} \Leftrightarrow H=\frac{1}{2}t^{-1}[1+O(t^{-\kappa})]. \tag{4.15}$$

Now, using again $d\tau/dt=H$ and integrating (4.15) we finally find

$$t^{1/2} = t_0^{1/2}e^{\tau-\tau_0+\zeta}, \tag{4.16}$$

where $\zeta \coloneqq O(\varepsilon(t^{-\kappa}+t_0^{-\kappa}))$ which is a small number. Therefore, from (4.14) and (4.16) we finally get

$$|\Sigma_i| = O(t^{-\lambda/4+\varepsilon}),$$
$$|s_i - s_i^*| = O(t^{-\lambda/4+\varepsilon}), \qquad \forall\, i\in\{1,2,3\}, \tag{4.17}$$

where λ is the solution of (4.12).

It only remains to obtain the decay estimate of the metric. From $\dot{g}^{ij}=k^{ij}$, we get (no summation on i in the following assumed)

$$\frac{d}{dt}(tg^{ii}) = -[2H\Sigma_i + (2H-t^{-1})](tg^{ii}). \tag{4.18}$$

Using (4.15) and (4.17), and after integrating (4.18), one finds

$$tg^{ii} \le Ct_0 g^{ii}(t_0), \tag{4.19}$$

where C is positive constant. On the other hand, integrating (4.18) on the interval $[t,\infty)$ with $t\ge t_0$, yields

$$|g^{ii}_\infty - tg^{ii}(t)| \le C\varepsilon t^{-\lambda/4+\varepsilon}, \tag{4.20}$$

where $g^{ij}_\infty \coloneqq \lim_{t\to\infty} tg^{ii}(t)$ and C is some positive constant. This completes the proof of theorem III.7.

V. CONCLUSION

In this work we considered a class of homogeneous spacetimes, i.e., the diagonal Bianchi type I spacetimes, with toroidal topology as solutions to the massless Einstein-Vlasov system. We showed that those systems with initial data close to the radiative Einstein-de Sitter model described in Sec. IV B, isotropize towards the future. In particular, the radiative Einstein-de Sitter model is an orbital attractor of such systems.

The novelty of this work lies in the use of the energy method in this context. This method could be applied to other Bianchi classes, e.g., Bianchi type II [31], or even to more general cases such as inhomogeneous spacetimes, which could in principle be the subject of future works.

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APPENDIX: CALCULATIONS

1. Calculation of $\partial Y_i/\partial s_j$

Calculating the derivatives of Y_i from the definition (3.5) we have

$$\frac{\partial Y_i}{\partial s_j} = -\frac{\int f_0 p_i^2 p_j^2 (\sum_k s_k p_k^2)^{-\frac{3}{2}} d^3p}{2\int f_0 (\sum_k s_k p_k^2)^{\frac{1}{2}} d^3p} - \frac{\int f_0 p_i^2 (\sum_k s_k p_k^2)^{-\frac{1}{2}} d^3p \int f_0 p_j^2 (\sum_k s_k p_k^2)^{-\frac{1}{2}} d^3p}{2(\int f_0 (\sum_k s_k p_k^2)^{\frac{1}{2}} d^3p)^2},$$

which is manifestly negative. Choosing $s_k = 1/3$ this becomes

$$\left.\frac{\partial Y_i}{\partial s_j}\right|_{s_k=1/3} = -\frac{9}{2}\frac{I_2^{ij}[f_0]}{I_0[f_0]} - \frac{9}{2}\frac{I_1^i[f_0]I_1^j[f_0]}{I_0[f_0]^2} \tag{A1}$$

with the integrals

$$I_0[f_0] \coloneqq \int f_0 |p|_\delta d^3p, \qquad I_1^k[f_0] \coloneqq \int f_0 p_k^2 |p|_\delta^{-1} d^3p,$$

$$I_2^{ij}[f_0] \coloneqq \int f_0 p_i^2 p_j^2 |p|_\delta^{-3} d^3p.$$

Specializing now to an isotropic initial particle distribution $f_0(p_k) = f_0^{\text{iso}}(|p|_\delta)$ and adopting spherical coordinates $(|p|_\delta, \theta, \varphi)$ in momentum space, such that $d^3p = |p|_\delta^2 \sin\theta d|p|_\delta d\theta d\varphi$, one can verify that

$$I_0[f_0^{\text{iso}}] = 4\pi \int f_0^{\text{iso}}(|p|_\delta)|p|_\delta^3 dp \eqqcolon 4\pi \tilde{I}_0,$$

$$I_1^i[f_0^{\text{iso}}] = \frac{4\pi}{3}\tilde{I}_0; \qquad \forall\, i \in \{1,2,3\},$$

$$I_2^{ij}[f_0^{\text{iso}}] = \begin{cases} \frac{4\pi}{5}\tilde{I}_0; & \text{if } i = j, \\ \frac{4\pi}{15}\tilde{I}_0; & \text{if } i \neq j. \end{cases}$$

Substituting this into (A1) yields the result (4.2).

2. Calculation of E'

From the definition of the energy (4.6) we see that in order to calculate E' we require the terms $(E_i^2)'$, $(\bar{s}_i^2)'$ and $(\bar{s}_i\Sigma_i)'$. Using the product rule and (4.1a)–(4.1b) a direct calculation up to second order in $\Sigma_i, \bar{s}_i$ gives

$$(\Sigma_i^2)' = -2\Sigma_i^2 + 6Y_i(\vec{s}^*)\bar{s}_i\Sigma_i + 6s_i^*\Sigma_i \sum_j \left.\frac{\partial Y_i}{\partial s_j}\right|_{s_k^*} \bar{s}_j + O(|\vec{\Sigma}|_\delta^3) + O(|\vec{\Sigma}|_\delta |\vec{s}|_\delta^2),$$

$$(\bar{s}_i^2)' = -4s_i^*\bar{s}_i\Sigma_i + 4s_i^*\bar{s}_i \sum_k s_k^*\Sigma_k + O(|\vec{\Sigma}|_\delta |\vec{s}|_\delta^2),$$

$$(\bar{s}_i\Sigma_i)' = -\bar{s}_i\Sigma_i + 3Y_i(\vec{s}^*)\bar{s}_i^2 - 2s_i^*\Sigma_i^2 + 2s_i^*\Sigma_i\sum_k s_k^*\Sigma_k + 3s_i^*\bar{s}_i\sum_j \left.\frac{\partial Y_i}{\partial s_j}\right|_{s_k^*}\bar{s}_j + O(|\vec{s}|_\delta^3) + O(|\vec{\Sigma}|_\delta^2|\vec{s}|_\delta).$$

(4.3)–(4.4) for s_i^* and $Y_i(\vec{s}^*)$. For E_i' we then have

$$E_i' = 3\beta\bar{s}_i^2 - 2\left(1+\frac{1}{3}\beta\right)\Sigma_i^2 + \left(6-\beta-\frac{4}{3}\alpha\right)\bar{s}_i\Sigma_i + (\beta\bar{s}_i + 2\Sigma_i)\sum_j \left.\frac{\partial Y_i}{\partial s_j}\right|_{s_k^*}\bar{s}_j + \mathcal{H}_i(a_i,c_i) + \mathcal{O}^3(|\vec{\Sigma}|_\delta, |\vec{s}|_\delta),$$

where $\mathcal{H}_i(a_i,c_i)$ and $\mathcal{O}^3(|\vec{\Sigma}|_\delta, |\vec{s}|_\delta)$ are given by (4.8) and (4.9), respectively. What is left to do is to sum up the E_i'. Using the notation $\bar{s}^2 \coloneqq \sum_i \bar{s}_i^2$ and $\vec{s}\cdot\vec{\Sigma} \coloneqq \sum_i \bar{s}_i\Sigma_i$ we get

$$\begin{aligned} E' &= 3\beta\bar{s}^2 - 2\left(1+\frac{1}{3}\beta\right)F + \left(6-\beta-\frac{4}{3}\alpha\right)\vec{s}\cdot\vec{\Sigma} + \sum_{i,j}(\beta\bar{s}_i + 2\Sigma_i)\left.\frac{\partial Y_i}{\partial s_j}\right|_{s_k^*}\bar{s}_j + \sum_i \mathcal{H}_i(a_i,c_i) + \mathcal{O}^3(|\vec{\Sigma}|_\delta, |\vec{s}|_\delta) \\ &\underset{\text{(IV.9)}}{\overset{\text{(IV.5)}}{=}} 3\beta\bar{s}^2 - 2\left(1+\frac{1}{3}\beta\right)F + \left(6-\beta-\frac{4}{3}\alpha\right)\vec{s}\cdot\vec{\Sigma} - \beta\left(\frac{7}{5}\bar{s}^2 + \frac{8}{5}(\bar{s}_1\bar{s}_2 + \bar{s}_1\bar{s}_3 + \bar{s}_2\bar{s}_3)\right) \\ &\quad - 2\left[\frac{7}{5}\vec{s}\cdot\vec{\Sigma} + \frac{4}{5}(\Sigma_1(\bar{s}_2+\bar{s}_3) + \Sigma_2(\bar{s}_1+\bar{s}_3) + \Sigma_3(\bar{s}_1+\bar{s}_2))\right] + G(\epsilon_1,\epsilon_2) + \sum_i \mathcal{H}_i(a_i,c_i) + \mathcal{O}^3(|\vec{\Sigma}|_\delta, |\vec{s}|_\delta). \end{aligned}$$

Finally, the term in square brackets can be simplified by using the trace-freeness of the shear, i.e., $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$, to show that

$$\Sigma_1(\bar{s}_2 + \bar{s}_3) + \Sigma_2(\bar{s}_1 + \bar{s}_3) + \Sigma_3(\bar{s}_1 + \bar{s}_2) = -\vec{\bar{s}} \cdot \vec{\Sigma}.$$

With this we obtain (4.7).

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