

Statistics of peaks of weakly non-Gaussian random fields: Effects of bispectrum in two- and three-dimensions

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Analytic expressions for the statistics of peaks of random fields with weak non-Gaussianity are provided. Specifically, the abundance and spatial correlation of peaks are represented by formulas which can be evaluated only by virtually one-dimensional integrals. We assume the non-Gaussianity is weak enough such that it is represented by linear terms of the bispectrum. The formulas are formally given in N -dimensional space, and explicitly given in the case of $N = 1, 2, 3$. Some examples of peak statistics in cosmological fields are calculated for the cosmic density field and weak lensing field, assuming the weak non-Gaussianity is induced by gravity. The formulas of this paper would find a fit in many applications to statistical analyses of cosmological fields.

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I. INTRODUCTION

The statistics of peaks of random fields have been attracting a lot of interest for applications to cosmology. The density peaks are obvious sites for the formation of nonlinear structures [1]. The amplitude of spatial clustering of biased objects is enhanced relative to that of density field [2,3]. This property is naturally expected by statistics of high-density peaks in a Gaussian random field. Mathematical formalism to calculate statistics of peaks in random Gaussian fields is given in seminal papers by Doroshkevich [4] and Bardeen *et al.* [5]. Statistics of peaks, such as abundances, profiles, and correlation functions, in Gaussian random fields have been extensively studied in the literature [6–15]. The clustering of dark matter halos can be modeled by the peaks approach under the assumption that halos form from peaks in the initial Lagrangian density field (for a review, see Ref. [16] and references therein). Lagrangian density field is reasonably assumed to obey Gaussian statistics, as long as the initial condition of the density field in the Universe is Gaussian.

Most of the analytic work on the statistics of peaks assumes the Gaussian statistics of density fields. One of the main reasons for this assumption stems from technical limitations. It is extremely difficult to analytically describe the statistics of peaks in generally non-Gaussian fields, which have infinite degrees of freedom. However, there are several reasons to consider the statistics of peaks in non-Gaussian density fields in cosmology.

For example, the initial density field is not necessarily a Gaussian random field, depending on generation mechanisms of the initial fluctuations (see, e.g., Ref. [17] and references therein). The gravitational evolution induces non-Gaussianity in the density field (see, e.g., Ref. [18] and references therein), and therefore, when the peaks are defined in Eulerian density field, they are not described by the peak theory assuming Gaussian statistics of density fields. The statistics of peaks in the weak lensing fields are also useful in cosmology [19–38]. The weak lensing fields on interested scales are not Gaussian because of the non-linear evolution of the density field which is the source of the weak lensing. The effects of non-Gaussianity are taken into account only numerically in the previous analyses of the weak lensing. Another example of the interest in peaks in non-Gaussian fields is the application to the primordial black holes (PBHs), which is assumed to be formed in the very early Universe [39–42]. The peaks theory of Ref. [5] is applied to the formation of PBHs [43–48].

While deriving analytically complete expressions of the statistics of peaks in generally non-Gaussian fields is difficult, that is possible in some limited cases. For the peak abundance in a special type of non-Gaussian field, chi-square field, an analytic expression can be derived [49,50]. A theory for the abundance of peaks in weakly non-Gaussian fields is pioneered by Refs. [51,52], which generalize the earlier work on the genus statistic and Minkowski functionals in weakly non-Gaussian fields [53,54]. In these papers, the peak abundances in two and three dimensions are expanded in Gram-Charlier series [55–58]. When the non-Gaussianity is weak, and the higher-order cumulants of the distribution do not significantly

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contribute to the statistics of peaks, one obtains an approximate expression for peak abundances by only taking lower-order terms of the series into account. The peak correlations in weakly non-Gaussian fields are derived [59] and are applied to a local-type non-Gaussianity in the primordial density field. Abundances and correlations of peaks in weakly non-Gaussian field in the high-peak limit are also derived [60–63].

In this paper, we follow and extend the methods of those previous papers for peaks in weakly non-Gaussian field, and give explicit formulas with lowest-order non-Gaussianity in two and three dimensions. We consider the abundances and spatial correlations of peaks in a unified formalism, which is developed by Ref. [57]. We first show a formal derivation of the peak statistics in N dimensions, and then find explicit expressions for $N = 1, 2, 3$. In Ref. [52], the formulas for the abundance of peaks are given in a form with multidimensional integrations, which should be evaluated by a semi-Monte-Carlo integration. We find this kind of multidimensional integrations reduces to lower-dimensional integrals, which can be evaluated very fast, extending techniques developed by Refs. [64,65]. This paper contains a set of newly useful formulas for statistics of peaks of weakly non-Gaussian fields, which can be potentially applied to many problems regarding statistics of peaks, such as the peaks in the density field of large-scale structure, and those in weak lensing fields, etc.

This paper is organized as follows. In Sec. II, a formal expression of the number density of peaks in a weakly non-Gaussian field in an N -dimensional space is given, and then analytically explicit expressions for $N = 1, 2, 3$ are derived. In Sec. III, formal expressions of the power spectrum and correlation function of peaks in a weakly non-Gaussian field in an N -dimensional space is given, and then analytically explicit expressions for $N = 2, 3$ are derived. In Sec. IV, three examples of the possible applications to cosmology are presented, i.e., the number density of peaks in a three-dimensional density field, the number density of peaks in a two-dimensional weak lensing field, and three-dimensional correlations of peaks. In these examples, the weak non-Gaussianity is assumed to emerge from weakly nonlinear evolutions by gravitational instability. Finally, conclusions are given in Sec. V.

II. ABUNDANCE OF PEAKS IN WEAKLY NON-GAUSSIAN FIELDS

A. Lowest-order non-Gaussianity

We generally consider a random field $f(\mathbf{x})$ in N -dimensional space, where \mathbf{x} is the N -dimensional coordinates. The field is assumed to have a zero mean,

$$\langle f(\mathbf{x}) \rangle = 0, \quad (1)$$

and the random field is statistically homogeneous and isotropic. We consider expectation values of peak statistics

in non-Gaussian fields. We apply a method of Ref. [57], which provides a general way of evaluating a given expectation value in weakly non-Gaussian fields. The method is based on the expansion by generalized Wiener-Hermite functionals, which is a generalization of the Edgeworth expansion of a single variable in weakly non-Gaussian fields. This basic method is briefly reviewed in the Appendix A.

In this paper, we consider the lowest-order non-Gaussianity, i.e., contributions from the three-point correlation at the lowest order, assuming the higher-order correlations are small enough. In cosmological fields, higher-order correlations frequently obey the so-called hierarchical ordering, in which n -point correlation function $\xi^{(n)}$ is of order $\mathcal{O}(\xi^{n-1})$, where $\xi = \xi^{(2)}$ is the two-point correlation function. In this case, the non-Gaussianity is weak when the two-point correlation ξ is small enough.

Having such a case in our mind, we consider only the linear contribution of the three-point correlation function, or the bispectrum in Fourier space. The expectation value of a functional $\mathcal{F}[f]$ is given by Eq. (A19). When we take into account only the lowest-order non-Gaussianity, we have

$$\begin{aligned} \langle \mathcal{F}[f] \rangle &= \langle \mathcal{F}[f] \rangle_G \\ &+ \frac{1}{6} \int \frac{d^N k_1}{(2\pi)^N} \frac{d^N k_2}{(2\pi)^N} \frac{d^N k_3}{(2\pi)^N} \langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \rangle_c \\ &\times \mathcal{G}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3), \end{aligned} \quad (2)$$

where $\tilde{f}(\mathbf{k})$ is the Fourier transform of $f(\mathbf{x})$, $\langle \cdots \rangle_c$ represents the (three-point) cumulant, and

$$\mathcal{G}_n(\mathbf{k}_1, \dots, \mathbf{k}_n) \equiv (2\pi)^{Nn} \left\langle \frac{\delta^n \mathcal{F}[f]}{\delta \tilde{f}(\mathbf{k}_1) \cdots \delta \tilde{f}(\mathbf{k}_n)} \right\rangle_G \quad (3)$$

represents a Gaussian n -point response function, and the expectation value $\langle \cdots \rangle_G$ is taken for Gaussian distributions with the same power spectrum of the field $f(\mathbf{x})$ (see Appendix A for details).

Due to statistical homogeneity, the three-point cumulant has a form,

$$\langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \rangle_c = (2\pi)^N \delta_D^N(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3), \quad (4)$$

where $\delta_D^N(\mathbf{k})$ is the N -dimensional Dirac's delta function and $B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is the bispectrum. Due to statistical homogeneity and isotropy, the bispectrum is a function of only magnitudes of three wave vectors, k_1 , k_2 , and k_3 . However, we keep the vector notation in the argument of the bispectrum. Thus, Eq. (2) can also be represented by

$$\begin{aligned} \langle \mathcal{F}[f] \rangle &= \mathcal{G}_0 + \frac{1}{6} \int \frac{d^N k_1}{(2\pi)^N} \frac{d^N k_2}{(2\pi)^N} \frac{d^N k_3}{(2\pi)^N} (2\pi)^N \delta_D^N(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \\ &\times B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \mathcal{G}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3). \end{aligned} \quad (5)$$

B. Statistics of field derivatives

Equation (5) is the basic formula of the weakly non-Gaussian expectation values of any kind. In this section, we are interested in the peak abundance of the weakly non-Gaussian field. The peak number density depends on spatial derivatives of the field up to the second order, i.e., f , $\partial_i f$, and $\partial_i \partial_j f$. To evaluate Eq. (3), we need the Gaussian statistics of the peak number density.

The power spectrum $P(k)$ of the random field f is defined by

$$\langle \tilde{f}(\mathbf{k}) \tilde{f}(\mathbf{k}') \rangle_c = (2\pi)^N \delta(\mathbf{k} + \mathbf{k}') P(k), \quad (6)$$

where the appearance of the delta function is a consequence of the statistical homogeneity, and the power spectrum is a function of only the magnitude of the wave vector $k = |\mathbf{k}|$ due to the statistical isotropy. The spectral moment σ_n is defined by

$$\sigma_n^2 = \int \frac{d^N k}{(2\pi)^N} k^{2n} P(k), \quad (7)$$

and the normalized field variables are defined by

$$\alpha \equiv \frac{f}{\sigma_0}, \quad \eta_i \equiv \frac{\partial_i f}{\sigma_1}, \quad \zeta_{ij} \equiv \frac{\partial_i \partial_j f}{\sigma_2}, \quad (8)$$

where $\partial_i = \partial/\partial x_i$ is the spatial derivative.

The Gaussian statistics of the field variables are completely determined by their covariances. They are given by

$$\begin{aligned} \langle \alpha^2 \rangle &= 1, & \langle \alpha \eta_i \rangle &= 0, & \langle \alpha \zeta_{ij} \rangle &= -\frac{\gamma}{N} \delta_{ij}, \\ \langle \eta_i \eta_j \rangle &= \frac{1}{N} \delta_{ij}, & \langle \eta_i \zeta_{jk} \rangle &= 0, \\ \langle \zeta_{ij} \zeta_{kl} \rangle &= \frac{1}{N(N+2)} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \end{aligned} \quad (9)$$

where

$$\gamma \equiv \frac{\sigma_1^2}{\sigma_0 \sigma_2}. \quad (10)$$

Since the set of variables ζ_{ij} is a symmetric tensor, only components with $i \geq j$ are independent.

We denote the set of independent variables as

$$\mathbf{Y} = (\alpha, \eta_1, \dots, \eta_N, \zeta_{11}, \zeta_{12}, \dots, \zeta_{N-1,N}, \zeta_{NN}). \quad (11)$$

The number of components of this vector is $N_0 \equiv 1 + N + N(N+1)/2 = (N+1)(N+2)/2$. The multivariate Gaussian distribution function for these variables at a single point is given by

$$\mathcal{P}_G(\mathbf{Y}) = \frac{1}{\sqrt{(2\pi)^{N_0} \det M}} \exp\left(-\frac{1}{2} \mathbf{Y}^T M^{-1} \mathbf{Y}\right), \quad (12)$$

where $M_{ab} \equiv \langle X_a X_b \rangle$ is a $N_0 \times N_0$ covariance matrix given by Eq. (9). It is useful to define the rotationally invariant quantities,

$$\eta^2 \equiv \boldsymbol{\eta} \cdot \boldsymbol{\eta}, \quad J_1 \equiv -\zeta_{ii}, \quad (13)$$

$$J_2 \equiv \frac{N}{N-1} \tilde{\zeta}_{ij} \tilde{\zeta}_{ji}, \quad (N \geq 2), \quad (14)$$

$$J_3 \equiv \frac{N^2}{(N-1)(N-2)} \tilde{\zeta}_{ij} \tilde{\zeta}_{jk} \tilde{\zeta}_{ki}, \quad (N \geq 3), \quad (15)$$

where repeated indices are summed over and

$$\tilde{\zeta}_{ij} \equiv \zeta_{ij} + \frac{1}{N} \delta_{ij} J_1 \quad (16)$$

is the traceless part of ζ_{ij} . The variable J_2 is considered only for $N \geq 2$, and the variable J_3 is considered only for $N \geq 3$.

In terms of the rotationally invariant variables, the multivariate Gaussian distribution of Eq. (12) is represented by [51,52]

$$\mathcal{P}_G(\mathbf{Y}) \propto \mathcal{N}(\alpha, J_1) \exp\left[-\frac{N}{2} \eta^2 - \frac{(N-1)(N+2)}{4} J_2\right], \quad (17)$$

up to the normalization constant, where

$$\mathcal{N}(\alpha, J_1) = \frac{1}{2\pi\sqrt{1-\gamma^2}} \exp\left[-\frac{\alpha^2 + J_1^2 - 2\gamma\alpha J_1}{2(1-\gamma^2)}\right] \quad (18)$$

is the Gaussian joint distribution function of variables α and J_1 .

C. Number density of peaks in a weakly non-Gaussian field

The number density of peaks above a threshold $f \geq \nu \sigma_0$ is given by [5]

$$n_{\text{pk}}(\nu) = \left(\frac{\sigma_2}{\sigma_1}\right)^N \Theta(\alpha - \nu) \delta^N(\boldsymbol{\eta}) \Theta(\lambda_N) |\det \zeta|, \quad (19)$$

where $\Theta(x)$ is the Heaviside's step function and λ_N is the smallest eigenvalue of the $N \times N$ matrix $(-\zeta_{ij})$. In order to obtain the weakly non-Gaussian corrections of Eq. (5), the Gaussian expectation value of Eq. (3) should be evaluated for $\mathcal{F} = n_{\text{pk}}$. The calculation is straightforward but somehow complicated, and the detailed derivation is given in

Appendix B. The result is usefully represented by using coefficients defined by

$$G_{ijklm}(\nu) \equiv (-1)^k \left\langle n_{\text{pk}}(\nu) H_{ij}(\alpha, J_1) \times L_k^{(N/2-1)} \left(\frac{N}{2} \eta^2 \right) F_{lm}(J_2, J_3) \right\rangle_G, \quad (20)$$

where

$$H_{ij}(\nu, J_1) = \frac{1}{\mathcal{N}(\alpha, J_1)} \left(-\frac{\partial}{\partial \alpha} \right)^i \left(-\frac{\partial}{\partial J_1} \right)^j \mathcal{N}(\alpha, J_1) \quad (21)$$

is the multivariate Hermite polynomials,

$$L_k^{(a)}(x) = \frac{x^{-a} e^x}{k!} \frac{d^k}{dx^k} (x^{k+a} e^{-x}) \quad (22)$$

is the generalized Laguerre polynomials,

$$F_{lm}(J_2, J_3) \equiv (-1)^l J_2^{3m/2} \times L_l^{(3m+(N-2)(N+3)/4)} \left(\frac{(N-1)(N+2)}{4} J_2 \right) \times P_m \left(\frac{J_3}{J_2^{3/2}} \right), \quad (23)$$

and

$$P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2 - 1)^m \quad (24)$$

is the Legendre polynomials.¹ We assume $m = 0$ when $N = 2$ and $l = m = 0$ when $N = 1$. The result of \mathcal{G}_3 is given by Eq. (B21) in Appendix B. In the case of $N = 1$, the terms of G_{ijklm} with $l \neq 0$ or $m \neq 0$ should be omitted. In the case of $N = 2$, the terms with G_{ijklm} with $m \neq 0$ should be omitted. These rules always apply in the following. Substituting Eq. (B21) into Eq. (5), we derive

$$\begin{aligned} \bar{n}_{\text{pk}}(\nu) &\equiv \langle n_{\text{pk}}(\nu) \rangle \\ &= G_{00000} + \frac{\sigma_0}{6} \left[G_{30000} S^{(0)} + 4\gamma G_{21000} S^{(1)} + 3\gamma^2 G_{12000} S_2^{(2)} + \gamma^3 G_{03000} S_1^{(3)} + 4G_{10100} S^{(1)} + \frac{4(N-1)}{N} \gamma G_{01100} S^{(2)} \right. \\ &\quad \left. + 6\gamma^2 G_{10010} (S_2^{(2)} - S^{(2)}) + \frac{6}{N-1} \gamma^3 G_{01010} (N S_2^{(3)} - S_1^{(3)}) + \frac{3(N-2)(N+2)^2}{2(N+4)} \gamma^3 G_{00001} \left(\frac{N+2}{3} S_1^{(3)} - N S_2^{(3)} \right) \right], \end{aligned} \quad (25)$$

where

$$\begin{aligned} S^{(0)} &\equiv \frac{\langle f^3 \rangle_c}{\sigma_0^4}, & S^{(1)} &\equiv -\frac{3 \langle f^2 \Delta f \rangle_c}{4 \sigma_0^2 \sigma_1^2}, \\ S^{(2)} &\equiv -\frac{3N}{2(N-1)} \frac{\langle (\nabla f \cdot \nabla f) \Delta f \rangle_c}{\sigma_1^4}, & S_2^{(2)} &\equiv \frac{\langle f (\Delta f)^2 \rangle_c}{\sigma_1^4}, \\ S_1^{(3)} &\equiv -\frac{\sigma_0^2}{\sigma_1^6} \langle (\Delta f)^3 \rangle_c, & S_2^{(3)} &\equiv -\frac{\sigma_0^2}{\sigma_1^6} \langle f_{ij} f_{ij} \Delta f \rangle_c. \end{aligned} \quad (26)$$

¹The function $F_{lm}(J_2, J_3)$ corresponds to the function $\tilde{F}_{lm}(5J_2, J_3)$ of Ref. [64] and the function $F_{lm}(5J_2, J_3)$ of Ref. [66] in three dimensions, but the normalization is different. Denoting the latter function as F_{lm}^{Diz} , they are related by

$$\tilde{F}_{lm}(5J_2, J_3) = (5/2)^{3m/2} \sqrt{(2m+1)\Gamma(5/2)/\Gamma(l+3m+5/2)} F_{lm}(J_2, J_3)$$

and

$$F_{lm}^{\text{Diz}}(5J_2, J_3) = (5/2)^{3m/2} \sqrt{\Gamma(5/2)/\Gamma(3m+5/2)} F_{lm}(J_2, J_3)$$

when $N = 3$ (a factor $s^{3m/2}$ is missing in Eq. (2.18) of Ref. [66]). Accordingly, the normalizations of bias parameters c_{ijklm} defined later in this paper are different from this literature for $m \neq 0$.

For $N = 1$, we define $S^{(2)} \equiv 0$ because we have the identity $\langle (f')^2 f'' \rangle = 0$. In deriving Eq. (25), we use identities

$$\begin{aligned} \langle f \nabla f \cdot \nabla f \rangle_c &= \frac{2}{3} \sigma_0^2 \sigma_1^2 S^{(1)}, \\ \langle f f_{ij} f_{ij} \rangle_c &= \sigma_1^4 \left(S_2^{(2)} - \frac{N-1}{N} S^{(2)} \right), \\ \langle f_{ij} f_{ij} \rangle_c &= \frac{N-1}{3N} \sigma_1^4 S^{(2)}, \\ \langle f_{ij} f_{jk} f_{ki} \rangle_c &= \frac{\sigma_1^6}{2\sigma_0^2} (S_1^{(3)} - 3S_2^{(3)}), \end{aligned} \quad (27)$$

which are shown by integrations by parts.

D. Calculating coefficients

The remaining task is to calculate the coefficients G_{ijklm} of Eq. (20). Substituting Eqs. (17) and (19) into Eq. (20), we have

$$\begin{aligned} G_{ijklm}(\nu) &= N_1 \left(\frac{\sigma_2}{\sigma_1} \right)^N \left(\frac{N}{2\pi} \right)^{N/2} X_k \int \prod_{i \leq j} d\zeta_{ij} \Theta(\lambda_N) |\det \zeta| \\ &\quad \times H_{i-1,j}(\nu, J_1) F_{lm}(J_2, J_3) \mathcal{N}(\nu, J_1) \\ &\quad \times \exp \left[-\frac{(N-1)(N+2)}{4} J_2 \right], \end{aligned} \quad (28)$$

where

$$X_k \equiv (-1)^k L_k^{(N/2-1)}(0) = \frac{(-1)^k \Gamma(k + N/2)}{\Gamma(k + 1) \Gamma(N/2)} \quad (29)$$

and N_1 is a normalization factor defined by

$$\begin{aligned} N_1^{-1} &\equiv \int d\alpha \prod_{i \leq j} d\zeta_{ij} \mathcal{N}(\alpha, J_1) \exp \left[-\frac{(N-1)(N+2)}{4} J_2 \right] \\ &= \frac{1}{\sqrt{2\pi}} \int \prod_{i \leq j} d\zeta_{ij} \exp \left[-\frac{J_1^2}{2} - \frac{(N-1)(N+2)}{4} J_2 \right]. \end{aligned} \quad (30)$$

For $i = 0$, the functions $H_{-1,j}(\nu, J_1)$ are defined by

$$H_{-1,j}(\nu, J_1) \equiv \frac{1}{\mathcal{N}(\nu, J_1)} \int_{\nu}^{\infty} d\alpha H_{0j}(\alpha, J_1) \mathcal{N}(\alpha, J_1). \quad (31)$$

In deriving Eqs. (28) and (30), we use the property $\int d\alpha \mathcal{N}(\alpha, J_1) = e^{-J_1^2/2}/\sqrt{2\pi}$ and the fact that the Gaussian probability distribution function of $\boldsymbol{\eta}$ is given by $P_G(\boldsymbol{\eta}) d^N \boldsymbol{\eta} = (N/2\pi)^N e^{-N\boldsymbol{\eta}^2/2} d^N \boldsymbol{\eta}$. We change the integration variables as

$$N_1 \prod_{i \leq j} d\zeta_{ij} = \frac{1}{\Omega_N} dx d^{N-1} W d\Omega_N, \quad (32)$$

where $x = J_1 = \lambda_1 + \dots + \lambda_N$, $d^{N-1}W$ represents the volume element of the other (traceless components of) rotationally invariant variables, and $d\Omega_N$ represents the volume element of the rotationally variant (angular) components, and

$$\Omega_N = \int d\Omega_N = \text{Vol}_{\text{SO}(N)} = \frac{2^{N-1} \pi^{(N-1)(N+2)/4}}{\prod_{n=2}^N \Gamma(n/2)}, \quad (N \geq 2), \quad (33)$$

is the volume of N -dimensional rotation group $\text{SO}(N)$. In practice, the volume element $d^{N-1}W$ is obtained by rotating the orthogonal set of coordinates to the principal axes of ζ_{ij} to have the diagonal form $-(\lambda_1, \lambda_2, \dots, \lambda_N)$, ordered by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$.

Because of Eqs. (30) and (32), the normalization condition of variables \mathbf{W} should be

$$\int_D d^{N-1}W \exp \left[-\frac{(N-1)(N+2)}{4} J_2 \right] = 1, \quad (34)$$

where D is the integration domain to satisfy the ordering $\lambda_1 \geq \dots \geq \lambda_N$. Thereby, Eq. (28) can be represented as

$$\begin{aligned} G_{ijklm}(\nu) &= \frac{1}{(2\pi)^{N/2}} \left(\frac{\sigma_2}{\sqrt{N}\sigma_1} \right)^N X_k \\ &\times \int_0^{\infty} dx H_{i-1,j}(\nu, x) \mathcal{N}(\nu, x) f_{lm}(x), \end{aligned} \quad (35)$$

where

$$\begin{aligned} f_{lm}(x) &\equiv N^N \int d^{N-1}W \Theta(\lambda_N) \lambda_1 \dots \lambda_N F_{lm}(J_2, J_3) \\ &\times \exp \left[-\frac{(N-1)(N+2)}{4} J_2 \right]. \end{aligned} \quad (36)$$

In the formula of Eq. (25), a limited number of the coefficients G_{ijklm} is needed. For X_k of Eq. (29), we need only $X_0 = 1$ and $X_1 = -N/2$. For $H_{i-1,j}(\nu, x)$, we need only $H_{-1,0}$, $H_{-1,1}$, $H_{-1,3}$, H_{00} , H_{02} , H_{11} , and H_{20} . These functions are straightforwardly evaluated by Eqs. (21) and (31). For $f_{lm}(x)$, we need only f_{00} , f_{10} , and f_{01} . The necessary functions $f_{lm}(x)$ are evaluated for $N = 1, 2, 3$ in the following subsection.

E. Specific formulas in one-, two-, and three-dimensional spaces

1. One-dimensional case

In one-dimensional space, $N = 1$, only the terms of $G_{ijk00} \equiv G_{ijk}$ should be retained. Equations (25) and (35) in this case reduce to

$$\begin{aligned} \bar{n}_{\text{pk}}(\nu) &= G_{000} + \frac{\sigma_0}{6} [G_{300} S^{(0)} + 4\gamma G_{210} S^{(1)} + 3\gamma^2 G_{120} S_2^{(2)} \\ &\quad + \gamma^3 G_{030} S_1^{(3)} + 4G_{101} S^{(1)}], \end{aligned} \quad (37)$$

and

$$G_{ijk}(\nu) = \frac{1}{\sqrt{2\pi}} \frac{\sigma_2}{\sigma_1} X_k \int_0^{\infty} dx H_{i-1,j}(\nu, x) \mathcal{N}(\nu, x) f(x), \quad (38)$$

where $X_0 = 1$, $X_1 = -1/2$. Putting $N = 1$, $(N-1)J_2 = 0$ and $l = m = 0$ in Eq. (36), we have

$$f(x) \equiv f_{00}(x) = x, \quad (39)$$

for $x \geq 0$. The differential number density $-d\bar{n}_{\text{pk}}/d\nu$ can be evaluated by replacing $H_{i-1,j} \rightarrow H_{ij}$ in Eq. (38), and the resulting expression can be found analytically in this one-dimensional case. Although we do not reproduce the result here, the analytic expression is straightforwardly obtained by using a software package such as *Mathematica*.

2. Two-dimensional case

In two-dimensional space, $N = 2$, only the terms of $G_{ijkl0} \equiv G_{ijkl}$ should be retained. Equations (25) and (35) in this case reduce to

$$\begin{aligned}\bar{n}_{\text{pk}}(\nu) = & G_{0000} + \frac{\sigma_0}{6} [G_{3000}S^{(0)} + 4\gamma G_{2100}S^{(1)} \\ & + 3\gamma^2 G_{1200}S_2^{(2)} + \gamma^3 G_{0300}S_1^{(3)} + 4G_{1010}S^{(1)} \\ & + 2\gamma G_{0110}S^{(2)} + 6\gamma^2 G_{1001}(S_2^{(2)} - S^{(2)}) \\ & + 6\gamma^3 G_{0101}(2S_2^{(3)} - S_1^{(3)})],\end{aligned}\quad (40)$$

and

$$G_{ijkl}(\nu) = \frac{1}{4\pi} \left(\frac{\sigma_2}{\sigma_1}\right)^2 X_k \int_0^\infty dx H_{i-1,j}(\nu, x) \mathcal{N}(\nu, x) f_l(x),\quad (41)$$

where $X_0 = 1$, $X_1 = -1$. To evaluate Eq. (36) in the case of $N = 2$, we introduce a set of variables,

$$x = \lambda_1 + \lambda_2, \quad y = \frac{\lambda_1 - \lambda_2}{2},\quad (42)$$

and we have $J_1 = x$, $J_2 = 4y^2$, and $|\det \zeta| = [x^2 - (2y)^2]/4$. The transformation of the volume element, Eq. (32), in the case of $N = 2$ results in $dW \propto y dy$ [67]. Because of the ordering $\lambda_1 \geq \lambda_2$, the integration domain is given by $y > 0$, and in order to meet the normalization condition, Eq. (34), we have $dW = 8y dy$. Thus, we have

$$f_l(x) \equiv f_{l0}(x) = 8 \int_0^{x/2} dy y e^{-4y^2} (x^2 - 4y^2) (-1)^l L_l(4y^2).\quad (43)$$

For the evaluation of Eq. (40), we need only

$$f_0(x) = e^{-x^2} + x^2 - 1,\quad (44)$$

$$f_1(x) = (1 + x^2)e^{-x^2} - 1.\quad (45)$$

The differential number density $-d\bar{n}_{\text{pk}}/d\nu$ can be evaluated by replacing $H_{i-1,j} \rightarrow H_{ij}$ in Eq. (41), and the resulting expression can be found analytically also in this two-dimensional case. Although the resulting expression is extremely long and we do not reproduce the result here, the analytic expression is straightforwardly obtained by using a software package such as *Mathematica*.

3. Three-dimensional case

In three-dimensional space, $N = 3$, Eqs. (25) and (35) reduce to

$$\begin{aligned}\bar{n}_{\text{pk}}(\nu) = & G_{00000} + \frac{\sigma_0}{6} \left[G_{30000}S^{(0)} + 4\gamma G_{21000}S^{(1)} \right. \\ & + 3\gamma^2 G_{12000}S_2^{(2)} + \gamma^3 G_{03000}S_1^{(3)} + 4G_{10100}S^{(1)} \\ & + \frac{8}{3}\gamma G_{01100}S^{(2)} + 6\gamma^2 G_{10010}(S_2^{(2)} - S^{(2)}) \\ & + 3\gamma^3 G_{01010}(3S_2^{(3)} - S_1^{(3)}) \\ & \left. + \frac{75}{14}\gamma^3 G_{00001} \left(\frac{5}{3}S_1^{(3)} - 3S_2^{(3)} \right) \right]\end{aligned}\quad (46)$$

and

$$\begin{aligned}G_{ijklm}(\nu) = & \frac{1}{(2\pi)^{3/2}} \left(\frac{\sigma_2}{\sqrt{3}\sigma_1} \right)^3 X_k \\ & \times \int_0^\infty dx H_{i-1,j}(\nu, x) \mathcal{N}(\nu, x) f_{lm}(x),\end{aligned}\quad (47)$$

where $X_0 = 1$, $X_1 = -3/2$. To evaluate Eq. (36) in the case of $N = 3$, we introduce a set of variables [5],

$$x = \lambda_1 + \lambda_2 + \lambda_3, \quad y = \frac{\lambda_1 - \lambda_3}{2}, \quad z = \frac{\lambda_1 - 2\lambda_2 + \lambda_3}{2},\quad (48)$$

and we have $J_1 = x$, $J_2 = 3y^2 + z^2$, $J_3 = z^3 - 9y^2z$, and $|\det \zeta| = (x - 2z)[(x + z)^2 - (3y)^2]/27$. The transformation of the volume element, Eq. (32), in the case of $N = 3$ results in $dW \propto y(y^2 - z^2)dy dz$ [5]. Because of the ordering $\lambda_1 \geq \lambda_2 \geq \lambda_3$, the integration domain is given by $-y \leq z \leq y$, and in order to meet the normalization condition, Eq. (34), we have $dW = (2\pi)^{-1/2} 3^2 5^{5/2} y(y^2 - z^2) dy dz$. Thus, we have

$$\begin{aligned}f_{lm}(x) = & \frac{3^2 5^{5/2}}{\sqrt{2\pi}} \left(\int_0^{x/4} dy \int_{-y}^y dz + \int_{x/4}^{x/2} dy \int_{3y-x}^y dz \right) \\ & \times e^{-5(3y^2+z^2)/2} (x - 2z)[(x + z)^2 - (3y)^2] y(y^2 - z^2) \\ & \times F_{lm}(3y^2 + z^2, z^3 - 9y^2z).\end{aligned}\quad (49)$$

For the evaluation of Eq. (40), we need only

$$\begin{aligned}f_{00}(x) = & \frac{x}{2}(x^2 - 3) \left[\text{erf} \left(\frac{1}{2} \sqrt{\frac{5}{2}} x \right) + \text{erf} \left(\sqrt{\frac{5}{2}} x \right) \right] \\ & + \sqrt{\frac{2}{5\pi}} \left[\left(\frac{x^2}{2} - \frac{8}{5} \right) e^{-5x^2/2} \right. \\ & \left. + \left(\frac{31}{4} x^2 + \frac{8}{5} \right) e^{-5x^2/8} \right],\end{aligned}\quad (50)$$

$$f_{10}(x) = -\frac{3x}{2} \left[\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{5}{2}}x\right) + \operatorname{erf}\left(\sqrt{\frac{5}{2}}x\right) \right] - \frac{12}{5} \sqrt{\frac{2}{5\pi}} \left[e^{-5x^2/2} - \left(1 + \frac{15x^2}{8}\right) \times \left(1 + \frac{15x^2}{16}\right) e^{-5x^2/8} \right], \quad (51)$$

$$f_{01}(x) = -\frac{21}{25} \left[\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{5}{2}}x\right) + \operatorname{erf}\left(\sqrt{\frac{5}{2}}x\right) \right] + \frac{27x}{10} \sqrt{\frac{2}{5\pi}} \left[\frac{2}{15} e^{-5x^2/2} + \left(\frac{11}{5} + \frac{x^2}{4} + \frac{5x^4}{16}\right) e^{-5x^2/8} \right]. \quad (52)$$

The differential number density $-d\bar{n}_{\text{pk}}/d\nu$ can be evaluated by replacing $H_{i-1,j} \rightarrow H_{ij}$ in Eq. (47).

The generalized version of multivariate Hermite polynomials, H_{ij} with $i \geq -1$, have analytic expressions: Eq. (31) can be analytically integrated. Therefore, the expression of $G_{ijklm}(\nu)$ of Eq. (47) is just a one-dimensional integration.

III. CORRELATIONS OF PEAKS IN WEAKLY NON-GAUSSIAN FIELDS

A. General formula

The lowest-order non-Gaussian correction to the power spectrum of peaks can be calculated by a method of generalized Wiener-Hermite expansions [57], which is described in Appendix A. The result is given by Eq. (A30). Identifying the biased field \mathcal{F} as the peak number density n_{pk} , we have

$$P_{\text{pk}}(k) = [g_1(\mathbf{k})]^2 P(k) + \frac{1}{2} \int \frac{d^N p}{(2\pi)^N} [g_2(\mathbf{p}, \mathbf{k} - \mathbf{p})]^2 P(p) P(|\mathbf{k} - \mathbf{p}|) + g_1(\mathbf{k}) \int \frac{d^N p}{(2\pi)^N} g_2(\mathbf{p}, \mathbf{k} - \mathbf{p}) B(\mathbf{p}, \mathbf{k} - \mathbf{p}, -\mathbf{k}) + \dots, \quad (53)$$

where $g_n(\mathbf{k}_1, \dots, \mathbf{k}_n) = \mathcal{G}_n(\mathbf{k}_1, \dots, \mathbf{k}_n)/\mathcal{G}_0$. Specifically for peaks, from Eqs. (B18)–(B20), we have

$$g_1(\mathbf{k}) = g_{10000} + g_{01000} k^2, \quad (54)$$

$$g_2(\mathbf{k}_1, \mathbf{k}_2) = g_{20000} + g_{11000}(k_1^2 + k_2^2) + g_{02000} k_1^2 k_2^2 - 2g_{00100} \mathbf{k}_1 \cdot \mathbf{k}_2 + \frac{2Ng_{00010}}{N-1} \left[(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 - \frac{1}{N} k_1^2 k_2^2 \right], \quad (55)$$

where

$$g_{ijklm} \equiv \frac{G_{ijklm}}{\sigma_0^i \sigma_1^{2k} \sigma_2^{j+2l+3m} G_{00000}}. \quad (56)$$

In the case of one dimension, $N = 1$, the last term of Eq. (55) should be omitted. The last coefficient g_{ijklm} is calculated by Eq. (35), or we have

$$g_{ijklm} = \frac{X_k \int_0^\infty dx H_{i-1,j}(\nu, x) \mathcal{N}(\nu, x) f_{lm}(x)}{\sigma_0^i \sigma_1^{2k} \sigma_2^{j+2l+3m} \int_0^\infty dx H_{-1,0}(\nu, x) \mathcal{N}(\nu, x) f_{00}(x)}. \quad (57)$$

The power spectrum of peaks is affected by exclusion effects: the peaks of a smoothed field cannot be too close to each other. Although the exclusion effects affect the small-scale behavior of the correlation function of peaks, the power spectrum of peaks on all scales is largely affected by the effect [15,68–70]. Therefore, the predictions of the perturbative method in this paper are more robust for the correlation function of peaks on large scales [15]. Once the power spectrum of peaks, Eq. (53), is calculated, the correlation function of peaks is given by

$$\xi_{\text{pk}}(r) = \int \frac{d^N k}{(2\pi)^N} e^{i\mathbf{k}\cdot\mathbf{r}} P_{\text{pk}}(k). \quad (58)$$

B. Angular integrations

For fast and accurate evaluations of Eq. (53), one can analytically perform angular integrations, and the resulting expression can be evaluated by one-dimensional fast-Fourier transforms (FFTs). In the case of three dimensions, such a technique is developed in a context of nonlinear perturbation theory [71–74]. We extend the same technique to the two-dimensional case below.

For this purpose, we rewrite the expression of Eq. (53) as

$$P_{\text{pk}}(k) = [g_1(\mathbf{k})]^2 P(k) + \frac{1}{2} \int_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} [g_2(\mathbf{k}_1, \mathbf{k}_2)]^2 P(k_1) P(k_2) + g_1(\mathbf{k}) \int_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} g_2(\mathbf{k}_1, \mathbf{k}_2) B(\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}_1 - \mathbf{k}_2) + \dots, \quad (59)$$

where we use a simplified notation,

$$\int_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \dots \equiv \int \frac{d^N k_1}{(2\pi)^N} \frac{d^N k_2}{(2\pi)^N} (2\pi)^N \delta_{\text{D}}^N(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \dots. \quad (60)$$

Because of the rotational symmetry, the integrands in the second and third terms besides the delta function are functions of only k_1 , k_2 and $\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2$, where $\hat{\mathbf{k}}_i \equiv \mathbf{k}_i/|\mathbf{k}_i|$.

The factor $g_2(\mathbf{k}_1, \mathbf{k}_2)$ and its square are given by a superposition of a form $(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l X(k_1)Y(k_2)$, where l is a non-negative integer. When the bispectrum $B(\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}_1 - \mathbf{k}_2)$ is also given by a superposition of the same form, the integrals in Eq. (59) are given by a superposition of integrals with the following form:

$$\begin{aligned} & \int_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l X(k_1)Y(k_2) \\ &= \int d^N r e^{-i\mathbf{k}\cdot\mathbf{r}} \int \frac{d^N k_1}{(2\pi)^N} \frac{d^N k_2}{(2\pi)^N} e^{i(\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{r}} (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l \\ & \quad \times X(k_1)Y(k_2). \end{aligned} \quad (61)$$

The angular integration of the above integral is analytically possible as follows. First, we notice that the integrals over \mathbf{k}_1 and \mathbf{k}_2 on the right-hand side give a function of r due to rotational symmetry. Therefore, one can replace factors $e^{-i\mathbf{k}\cdot\mathbf{r}}$ and $e^{i(\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{r}}$ by their averages over angle of \mathbf{r} . In two and three dimensions (2D and 3D, respectively), we have

$$e^{-i\mathbf{k}\cdot\mathbf{r}} \rightarrow J_0(kr), \quad e^{i(\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{r}} \rightarrow J_0(|\mathbf{k}_1 + \mathbf{k}_2|r), \quad (2D), \quad (62)$$

$$e^{-i\mathbf{k}\cdot\mathbf{r}} \rightarrow j_0(kr) \quad e^{i(\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{r}} \rightarrow j_0(|\mathbf{k}_1 + \mathbf{k}_2|r), \quad (3D), \quad (63)$$

where $J_n(x)$ and $j_n(x)$ are Bessel functions and spherical Bessel functions, respectively.

1. Two-dimensional case

In two dimensions, the integral of Eq. (61) reduces to

$$\begin{aligned} & \int_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l X(k_1)Y(k_2) \\ &= 2\pi \int r dr J_0(kr) \times \int \frac{d^2 k_1}{(2\pi)^2} \frac{d^2 k_2}{(2\pi)^2} J_0(|\mathbf{k}_1 + \mathbf{k}_2|r) \\ & \quad \times (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l X(k_1)Y(k_2). \end{aligned} \quad (64)$$

We apply an addition theorem of the Bessel function,

$$J_0(|\mathbf{k}_1 + \mathbf{k}_2|r) = \sum_{n=-\infty}^{\infty} (-1)^n J_n(k_1 r) J_n(k_2 r) e^{in\theta_{12}}, \quad (65)$$

where θ_{12} is the angle between \mathbf{k}_1 and \mathbf{k}_2 , i.e., $\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2 = \cos \theta_{12}$. The angular dependence can be written as

$$(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l = \frac{1}{2^l} e^{-il\theta_{12}} \sum_{m=0}^l \binom{l}{m} e^{2im\theta_{12}}. \quad (66)$$

Substituting the above equations into Eq. (64), we have

$$\begin{aligned} & \int_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l X(k_1)Y(k_2) \\ &= 2\pi \int r dr J_0(kr) \times \frac{1}{2^l} \sum_{m=0}^l (-1)^{l-2m} \binom{l}{m} \\ & \quad \times X_{l-2m}(r) Y_{l-2m}(r), \end{aligned} \quad (67)$$

where

$$X_n(r) \equiv \int \frac{k dk}{2\pi} J_n(kr) X(k), \quad (68)$$

$$Y_n(r) \equiv \int \frac{k dk}{2\pi} J_n(kr) Y(k). \quad (69)$$

The last integrals are the one-dimensional Hankel transforms, which can be efficiently evaluated with the one-dimensional FFT using a software package `FFTLog` [75].

Adopting the formula of Eq. (67) in the explicit expression of Eq. (59), the power spectrum of peaks, $P_{\text{pk}}(k)$, can be evaluated by using the one-dimensional (1D) FFT. The correlation function of peaks, Eq. (58), is also evaluated by

$$\xi_{\text{pk}}(r) = \int \frac{k dk}{2\pi} J_0(kr) P_{\text{pk}}(k). \quad (70)$$

2. Three-dimensional case

In three dimensions, the integral of Eq. (61) reduces to

$$\begin{aligned} & \int_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l X(k_1)Y(k_2) \\ &= 4\pi \int r^2 dr j_0(kr) \times \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} j_0(|\mathbf{k}_1 + \mathbf{k}_2|r) \\ & \quad \times (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l X(k_1)Y(k_2). \end{aligned} \quad (71)$$

We apply an addition theorem of the Bessel function,

$$\begin{aligned} & j_0(|\mathbf{k}_1 + \mathbf{k}_2|r) \\ &= \sum_{n=0}^{\infty} (-1)^n (2n+1) j_n(k_1 r) j_n(k_2 r) P_n(\cos \theta_{12}), \end{aligned} \quad (72)$$

where θ_{12} is the angle between \mathbf{k}_1 and \mathbf{k}_2 , i.e., $\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2 = \cos \theta_{12}$, and $P_m(\mu) = (2^m m!)^{-1} (d/dx)^m [(x^2 - 1)^m]$ are Legendre polynomials, which satisfy the orthogonality relation,

$$\frac{1}{2} \int_{-1}^1 d\mu P_n(\mu) P_m(\mu) = \frac{\delta_{nm}}{2n+1}. \quad (73)$$

The angular dependence can be written as

$$(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l = \sum_{m=0}^l (2m+1) \alpha_{lm} P_m(\cos \theta_{12}), \quad (74)$$

where

$$\alpha_{lm} \equiv \frac{1}{2} \int_{-1}^1 d\mu \mu^l P_m(\mu) = \begin{cases} \frac{l!}{2^{(l-m)/2} [(l-m)/2]! (l+m+1)!!} & \left(\begin{array}{l} l \geq m, \\ l+m = \text{even} \end{array} \right), \\ 0 & \text{(otherwise)}. \end{cases} \quad (75)$$

Substituting Eqs. (72) and (74) into Eq. (71), we have

$$\begin{aligned} & \int_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^l X(k_1) Y(k_2) \\ &= 4\pi \int r^2 dr j_0(kr) \sum_{m=0}^l (-1)^m (2m+1) \alpha_{lm} X_m(r) Y_m(r), \end{aligned} \quad (76)$$

where

$$X_m(r) \equiv \int \frac{k^2 dk}{2\pi^2} j_m(kr) X(k), \quad (77)$$

$$Y_m(r) \equiv \int \frac{k^2 dk}{2\pi^2} j_m(kr) Y(k). \quad (78)$$

The last integrals are the one-dimensional Hankel transforms, which can be efficiently evaluated with the one-dimensional FFT.

Adopting the formula of Eq. (76) in the explicit expression of Eq. (59), the power spectrum of peaks, $P_{\text{pk}}(k)$, can be evaluated by using the 1D FFT. The correlation function of peaks, Eq. (58), is also evaluated by

$$\xi_{\text{pk}}(r) = \int \frac{k^2 dk}{2\pi^2} j_0(kr) P_{\text{pk}}(k). \quad (79)$$

IV. WEAK NON-GAUSSIANITY DUE TO NONLINEAR EVOLUTIONS IN THE LARGE-SCALE STRUCTURE

In this section, we numerically calculate the formulas derived in previous sections when the weak non-Gaussianity is evaluated by nonlinear perturbation theory of gravitational instability in the large-scale structure of the

Universe. In the numerical evaluations below, the power spectrum of the three-dimensional density field is calculated by Boltzmann code CLASS [76,77] with a flat Lambda cold dark matter (Λ CDM) model and cosmological parameters $h = 0.6732$, $\Omega_{\text{b}0} h^2 = 0.02238$, $\Omega_{\text{cdm}} h^2 = 0.1201$, $n_s = 0.9660$, and $\sigma_8 = 0.8120$ (Planck 2018 [78]).

A. Number density of peaks in a three-dimensional density field with weak non-Gaussianity induced by gravity

In a three-dimensional space, we consider an example of peaks in the dark matter distribution in three-dimensional space. When the peaks of matter density field are considered, we first smooth the density field with a smoothing kernel $W(kR)$ in Fourier space, where R is the smoothing radius. The field variable $\tilde{f}(\mathbf{k})$ in Fourier space corresponds to

$$\tilde{f}(\mathbf{k}) = W(kR) \delta(\mathbf{k}), \quad (80)$$

where $\delta(\mathbf{k})$ is the density field in Fourier space. In this paper, we adopt a Gaussian smoothing kernel, $W(kR) = e^{-k^2 R^2/2}$. Denoting the linear power spectrum by $P_L(k)$ at an arbitrary redshift, the power spectrum of the smoothed density field at the lowest order is given by

$$P(k) = W^2(kR) P_L(k). \quad (81)$$

Adopting the nonlinear perturbation theory of gravitational instability [18], the bispectrum of smoothed matter density field at the lowest order is given by

$$\begin{aligned} B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= W(k_1 R) W(k_2 R) W(k_3 R) \\ &\times \left[\frac{10}{7} + \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} + \frac{4}{7} \left(\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \right)^2 \right] \\ &\times P_L(k_1) P_L(k_2) + \text{cyc}. \end{aligned} \quad (82)$$

The parameters of Eqs. (26) are given by integrations of the bispectrum in a form,

$$S_j^{(n)} = \frac{\sigma_0^{2n-4}}{\sigma_1^{2n}} \int_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0} s_j^{(n)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3), \quad (83)$$

where

$$\begin{aligned} s_0^{(0)} &= 1, & s_1^{(1)} &= \frac{3}{4} k_3^2, & s_2^{(2)} &= -\frac{9}{4} (\mathbf{k}_1 \cdot \mathbf{k}_2) k_3^2, \\ s_2^{(2)} &= k_1^2 k_2^2, & s_1^{(3)} &= k_1^2 k_2^2 k_3^2, & s_2^{(3)} &= (\mathbf{k}_1 \cdot \mathbf{k}_2)^2 k_3^2. \end{aligned} \quad (84)$$

Symmetrizing the arguments of $s_j^{(n)}$, and using only the first term of Eq. (82), Eq. (83) reduces to an expression of three-dimensional integrals,

$$S_j^{(n)} = \frac{\sigma_0^{2n-4}}{\sigma_1^{2n}} \int_0^\infty \frac{k_1^2 dk_1}{2\pi^2} \frac{k_2^2 dk_2}{2\pi^2} \times \int_{-1}^1 \frac{d\mu}{2} \tilde{s}_j^{(n)}(k_1, k_2, \mu) \tilde{B}(k_1, k_2, \mu), \quad (85)$$

where

$$\begin{aligned} \tilde{B}(k_1, k_2, \mu) &\equiv 3W(k_1 R)W(k_2 R)W[(k_1^2 + k_2^2 + 2k_1 k_2 \mu)^{1/2} R] \\ &\times \left[\frac{10}{7} + \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \mu + \frac{4}{7} \mu^2 \right] P_L(k_1) P_L(k_2), \quad (86) \end{aligned}$$

and

$$\begin{aligned} \tilde{s}^{(0)} &= 1, & \tilde{s}^{(1)} &= \frac{1}{2}(k_1^2 + k_2^2 + k_1 k_2 \mu), \\ \tilde{s}^{(2)} &= \frac{3}{2} k_1^2 k_2^2 (1 - \mu^2), \\ \tilde{s}^{(2)} &= \frac{1}{3} [k_1^4 + k_2^4 + 3k_1^2 k_2^2 + 2k_1 k_2 (k_1^2 + k_2^2) \mu], \\ \tilde{s}_1^{(3)} &= k_1^2 k_2^2 (k_1^2 + k_2^2 + 2k_1 k_2 \mu), \\ s_2^{(3)} &= \frac{1}{3} k_1^2 k_2^2 [(k_1^2 + k_2^2)(2\mu^2 + 1) + 2k_1 k_2 \mu(\mu^2 + 2)]. \end{aligned} \quad (87)$$

The integrals of Eq. (85) with Eqs. (86) and (87) are numerically evaluated. Substituting the results into Eq. (46), the number density of peaks $\bar{n}_{\text{pk}}(\nu)$ in three dimensions can be evaluated.

In Fig. 1, the differential number density of peaks, $-d\bar{n}_{\text{pk}}/d\nu$, is plotted. The Gaussian prediction without the effect of the bispectrum is represented by a dashed line. The gravitational non-Gaussianity increases the number of high-threshold ($\nu \gtrsim 2.4$) peaks, because of the positive skewness in the underlying field.

B. Number density of peaks in a two-dimensional weak lensing field with weak non-Gaussianity induced by gravity

In a two-dimensional space, we consider an example of peaks in the weak lensing field. When the peaks of weak lensing field are considered, we first smooth the lensing field with a smoothing kernel $W(k\vartheta)$, where ϑ is the smoothing angle. The field variable $\tilde{f}(\mathbf{k})$ in Fourier space corresponds to

$$\tilde{f}(\mathbf{k}) = W(k\vartheta)\kappa(\mathbf{k}), \quad (88)$$

where $\kappa(\mathbf{k})$ is the two-dimensional convergence field of weak lensing in Fourier space.

For simplicity, we adopt the flat-sky and Limber's approximations [79] in this paper. Assuming a flat

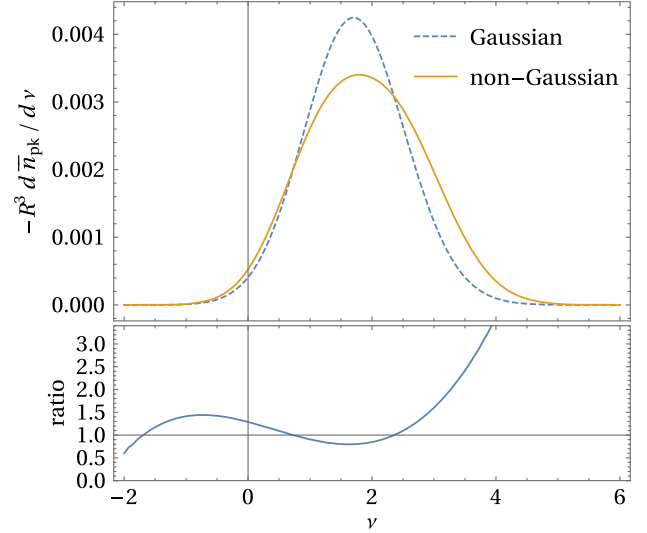


FIG. 1. The differential number density of peaks in three dimensions. In the upper panel, predictions of Gaussian (dashed line) and non-Gaussian (solid line) fields are shown, where the number density is measured in units of the smoothing radius $R = 20 h^{-1}$ Mpc. In the lower panel, the ratio of the non-Gaussian prediction to the Gaussian prediction is plotted.

Universe, the power spectrum and the bispectrum of convergence field are given by [54,80]

$$P_\kappa(k) = \int \chi^2 d\chi q^2(\chi) P_{3\text{D}}\left(\frac{k}{\chi}; \chi\right) \quad (89)$$

and

$$B_\kappa(k_1, k_2, k_3) = \int \chi^2 d\chi q^3(\chi) B_{3\text{D}}\left(\frac{k_1}{\chi}, \frac{k_2}{\chi}, \frac{k_3}{\chi}; \chi\right), \quad (90)$$

where $P_{3\text{D}}(k, \chi)$ and $B_{3\text{D}}(k_1, k_2, k_3, \chi)$ are respectively the power spectrum and bispectrum of a three-dimensional density field at a conformal time, $\tau_0 - \chi$ (τ_0 is the conformal time at the present),

$$q(\chi) \equiv \frac{3H_0^2 \Omega_{\text{m}0} \chi_s - \chi}{2a(\chi) \chi \chi_s} \quad (91)$$

is a weight function of the convergence field, and χ_s is the comoving distance to the source galaxies at a fixed redshift. In reality, the source redshift has a distribution, and the weight function should be replaced by an integral over the source redshift. In this paper, we assume a single redshift for source galaxies just for simplicity.

The two-dimensional power spectrum and bispectrum of the smoothed convergence field are given by $P(k) = W^2(k\vartheta)P_\kappa(k)$ and $B(k_1, k_2, k_3) = W(k_1\vartheta)W(k_2\vartheta)W(k_3\vartheta) \times B_\kappa(k_1, k_2, k_3)$. The three-dimensional power spectrum and bispectrum are given by Eqs. (81) and (82) at the tree level (lowest order) in the perturbation theory. However, one

should apply the nonlinear power spectrum and bispectrum for quantitative predictions for the weak lensing field. For that purpose, we need analytic fitting functions of the nonlinear power spectrum like HaloFit [81,82] and the counterpart of the nonlinear bispectrum [83–85].

The spectral moments of Eq. (7) in the two-dimensional convergence field are given by

$$\sigma_n^2 = \int d\chi \chi^{2n+4} q^2(\chi) \int \frac{kdk}{2\pi} k^{2n} W^2(k\chi\vartheta) P_{3D}(k;\chi). \quad (92)$$

The skewness parameters of Eq. (26) in the two-dimensional convergence field are given by integrations of the bispectrum in a form,

$$S_j^{(n)} = \frac{\sigma_0^{2n-4}}{\sigma_1^{2n}} \int d\chi \chi^{2n+6} q^3(\chi) \int_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0} s_j^{(n)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \times W(k_1\chi\vartheta) W(k_2\chi\vartheta) W(k_3\chi\vartheta) B_{3D}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; \chi), \quad (93)$$

where $s_j^{(n)}$ are given by

$$\begin{aligned} s^{(0)} &= 1, & s^{(1)} &= \frac{3}{4} k_3^2, & s^{(2)} &= -3(\mathbf{k}_1 \cdot \mathbf{k}_2) k_3^2, \\ s_2^{(2)} &= k_1^2 k_2^2, & s_1^{(3)} &= k_1^2 k_2^2 k_3^2, & s_2^{(3)} &= (\mathbf{k}_1 \cdot \mathbf{k}_2)^2 k_3^2. \end{aligned} \quad (94)$$

Although we use the same notation $s_j^{(n)}$ as those in Eq. (84) of the three-dimensional case, the coefficient of $s^{(2)}$ is different in this two-dimensional case, and $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ are two-dimensional vectors. Integrations over these vectors are also two dimensional in Eq. (93).

After symmetrizing the arguments of $s_j^{(n)}$, we can replace the bispectrum B_{3D} by an asymmetric counterpart, B_{3D}^{asym} , which is defined by

$$B_{3D}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; \chi) = \frac{1}{3} [B_{3D}^{\text{asym}}(\mathbf{k}_1, \mathbf{k}_2; \chi) + \text{cyc}]. \quad (95)$$

Since we have $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{0}$, the bispectrum B_{3D} can be always expressible in the form of right-hand side of Eq. (95), even though the choice of functional form of B_{3D}^{asym} is not necessarily unique.

In the case of the tree-level perturbation theory, we have

$$P_{3D}(k; \chi) = D^2(\chi) P_{L0}(k) \quad (96)$$

and

$$B_{3D}^{\text{asym}}(\mathbf{k}_1, \mathbf{k}_2; \chi) = 3D^4(\chi) P_{L0}(k_1) P_{L0}(k_2) \times \left[\frac{10}{7} + \left(\frac{k_1 + k_2}{k_2 + k_1} \right) \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} + \frac{4}{7} \left(\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \right)^2 \right], \quad (97)$$

where $D(\chi)$ is the linear growth factor at a conformal time $\tau_0 - \chi$ and $P_{L0}(k)$ is the linear power spectrum at the present time. Beyond the tree-level perturbation theory, one can apply appropriate nonlinear forms of P_{3D} and B_{3D}^{asym} instead of Eqs. (96) and (97), using, e.g., the HaloFit approaches. For quantitative predictions of the weak lensing field, it is necessary to adopt the nonlinear power spectrum and bispectrum in most of the cases. We use the tree-level perturbation theory in this paper just for simplicity.

The skewness parameters of Eq. (93) reduces to an expression,

$$S_j^{(n)} = \frac{\sigma_0^{2n-4}}{\sigma_1^{2n}} \int d\chi \chi^{2n+6} q^3(\chi) \int \frac{k_1 dk_1}{2\pi} \frac{k_2 dk_2}{2\pi} \times \int_{-1}^1 \frac{d\mu}{\pi \sqrt{1-\mu^2}} \tilde{s}_j^{(n)}(k_1, k_2, \mu) \tilde{B}(k_1, k_2, \mu; \chi), \quad (98)$$

where

$$\begin{aligned} \tilde{B}(k_1, k_2, \mu; \chi) &\equiv W(k_1\chi\vartheta) W(k_2\chi\vartheta) \\ &\times W[(k_1^2 + k_2^2 + 2k_1 k_2 \mu)^{1/2} \chi\vartheta] \\ &\times B_{3D}^{\text{asym}}(\mathbf{k}_1, \mathbf{k}_2; \chi), \end{aligned} \quad (99)$$

and

$$\begin{aligned} \tilde{s}^{(0)} &= 1, & \tilde{s}^{(1)} &= \frac{1}{2} (k_1^2 + k_2^2 + k_1 k_2 \mu), \\ \tilde{s}^{(2)} &= 2k_1^2 k_2^2 (1 - \mu^2), \\ \tilde{s}_2^{(2)} &= \frac{1}{3} [k_1^4 + k_2^4 + 3k_1^2 k_2^2 + 2k_1 k_2 (k_1^2 + k_2^2) \mu], \\ \tilde{s}_1^{(3)} &= k_1^2 k_2^2 (k_1^2 + k_2^2 + 2k_1 k_2 \mu), \\ s_2^{(3)} &= \frac{1}{3} k_1^2 k_2^2 [(k_1^2 + k_2^2)(2\mu^2 + 1) + 2k_1 k_2 \mu(\mu^2 + 2)]. \end{aligned} \quad (100)$$

In the case of the tree-level bispectrum, Eq. (97), the function \tilde{B} of Eq. (99) is equivalent to the one defined in Eq. (86) with replacements $R \rightarrow \chi\vartheta$ and $P_L(k) \rightarrow D^2(\chi) P_{L0}(k)$. The functions $\tilde{s}_j^{(n)}$ in this two-dimensional case are nearly the same as Eq. (87), but the coefficient of $\tilde{s}^{(2)}$ is different from that in the three-dimensional case.

The integrals of Eq. (98) with Eqs. (99) and (100) are numerically evaluated. For efficient evaluations, the results of the three-dimensional integrations for fixed values of χ

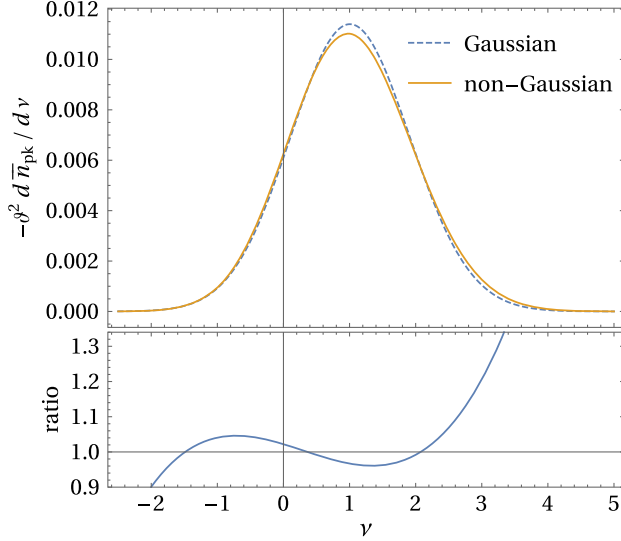


FIG. 2. The differential number density of peaks in two-dimensional weak lensing field. In the upper panel, predictions of Gaussian (dashed line) and non-Gaussian (solid line) fields are shown, where the number density is measured in units of the smoothing angle $\vartheta = 10$ arc min. In the lower panel, the ratio of the non-Gaussian prediction to the Gaussian prediction is plotted. Nonlinearity and noise effects are not included.

are tabulated and interpolated, and finally integrated over χ . Substituting the results into Eq. (40), the number density of peaks $\bar{n}_{\text{pk}}(\nu)$ in two dimensions can be evaluated. In the following example, we simply use the tree-level power spectrum and bispectrum of Eqs. (96) and (97) for illustrative purposes. However, more quantitative evaluations of the weak lensing field require the use of nonlinear power spectrum and bispectrum by, e.g., HaloFit, etc.

In Fig. 2, the differential number density of peaks in the weak lensing field, $-d\bar{n}_{\text{pk}}/d\nu$, is plotted. The Gaussian prediction without the effect of the bispectrum is represented by a dashed line. We apply the Gaussian smoothing function with a smoothing angle $\vartheta = 2$ arc min, and the source redshift is assumed to be fixed at $z_s = 1.5$. In this plot, we simply use the tree-level predictions of the power spectrum and bispectrum by the perturbation theory, Eqs. (96) and (97) as noted above. The gravitational non-Gaussianity increases the number of high-threshold ($\nu \gtrsim 2$) peaks, because of the positive skewness in the underlying field.

The shape of the differential number density of peaks relative to the Gaussian prediction in this plot explains qualitative behavior of the results from the analysis of numerical simulations presented in Refs. [24,38], although the adopted parameters are different. In order to quantitatively compare the prediction with the results of numerical simulations, one needs to use nonlinear fitting functions for the power spectrum and bispectrum, and also needs to take noise effects into account. It is beyond the scope of this paper to make detailed comparison with numerical

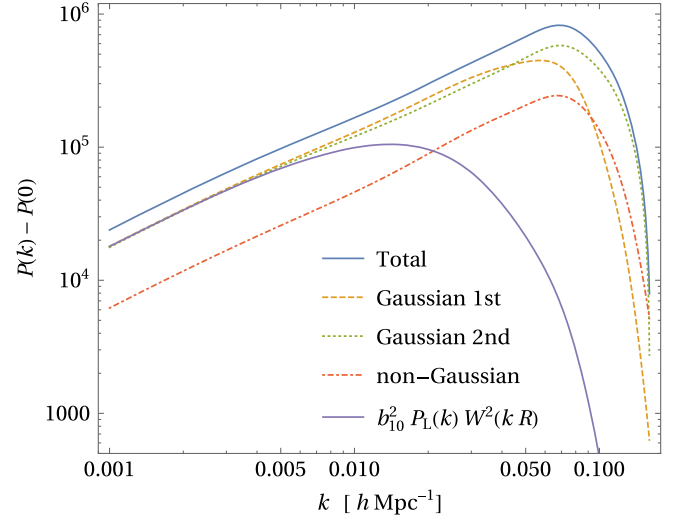


FIG. 3. The power spectrum of peaks in three-dimensional density field with a smoothing radius $R = 20 h^{-1}$ Mpc. Predictions of Gaussian field with first-order and second-order approximations are shown in dashed and dotted lines, respectively (the second-order approximation contains the first-order and second-order contributions). The component of non-Gaussian correction is shown in a dot-dashed line. The total correlation function is shown in a solid line. The scaled power spectrum of the underlying smoothed density field, $b_{10}^2 P_L(k) W^2(kR)$, is also plotted in a lower solid line.

simulations of weak lensing field, which is one of the interesting future applications of this paper.

C. Correlations of peaks with weak non-Gaussianity induced by gravity

For the last example of numerical demonstration, we consider the spatial correlation of peaks with weak non-Gaussianity induced by gravity in three-dimensional space, $N = 3$. Substituting Eqs. (81) and (82) into Eq. (59), we obtain an expression which consists of a superposition of integrals with a form of Eq. (76). Consequently, we need the functions

$$\xi_m^{(n)}(r) \equiv \int \frac{k^2 dk}{2\pi^2} j_m(kr) k^n W^2(kR) P_L(k), \quad (101)$$

$$A_m^{(n)}(r) \equiv \int \frac{k^2 dk}{2\pi^2} j_m(kr) k^n W(kR) P_L(k), \quad (102)$$

$$B_m^{(n)}(r) \equiv \int \frac{k^2 dk}{2\pi^2} j_m(kr) k^n W(kR), \quad (103)$$

to represent the final result. The final expression has the form

$$P_{\text{pk}}(k) = 4\pi \int r^2 dr j_0(kr) \left[\xi_{\text{pk}}^{(1)}(r) + \xi_{\text{pk}}^{(2)}(r) + g_1(k) S_{\text{NG}}(k, r) \right], \quad (104)$$

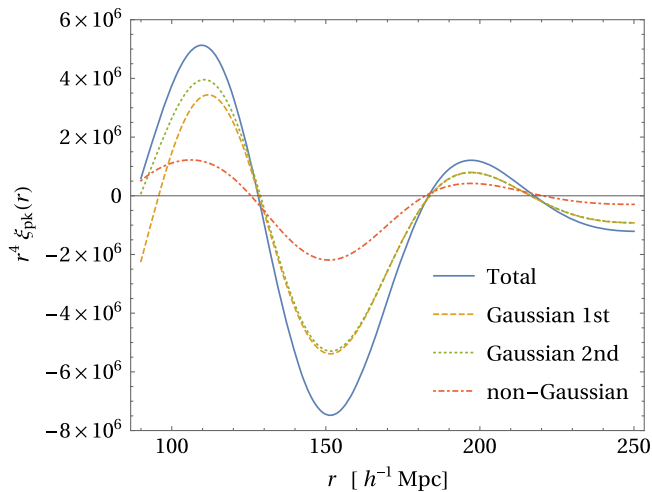


FIG. 4. The correlation function of peaks in three-dimensional density field with a smoothing radius $R = 20 h^{-1}\text{Mpc}$. Predictions of Gaussian field with first-order and second-order approximations are shown in dotted and dashed lines, respectively. The component of non-Gaussian correction is shown in a dot-dashed line. The total correlation function is shown in a solid line.

where $\xi_{\text{pk}}^{(1)}(r)$, $\xi_{\text{pk}}^{(2)}(r)$, and $S_{\text{NG}}(k, r)$ are polynomials of the functions of Eqs. (101)–(103). Their explicit forms are somewhat tedious and given in Appendix C, Eqs. (C2)–(C4).

For the numerical evaluation of Eq. (104), we just need Hankel transforms, which can be efficiently performed by the use of `FFTL`. In Fig. 3, the result of Eq. (104) is plotted, together with partial components of the integral. We subtract off the zero-lag value $P(k \rightarrow 0)$ from the power spectrum for the following reason: As noted in the last paragraph of Sec. III A, it has been suggested that the behavior of the correlation function below the scales of the exclusion zone ($\lesssim R$) nontrivially affects the power spectrum on large scales ($k \rightarrow 0$) [15,68–70]. Accordingly, the second-order approximation of the power spectrum [the contribution of $\xi_{\text{pk}}^{(2)}(r)$ in Eq. (104)] has a nonzero value in the limit of $k \rightarrow 0$, which corresponds to unphysical component in the perturbative expansion. To remove this unphysical effect, we subtract off the zero-lag value $P(k \rightarrow 0)$ from the second-order approximation of the power spectrum. Other components do not have the zero-lag value.

The second-order approximation of the power spectrum with Gaussian components [the first two terms in the integrand of Eq. (104)] is considered to be accurate on scales $\lesssim 0.1h \text{ Mpc}^{-1}$ according to the previous analysis [15]. The shape of the non-Gaussian correction is almost proportional to the Gaussian contribution on most of the scales. Thereby, the total shape of the peak power spectrum does not change much by the effect of non-Gaussianity, but the amplitude does change.

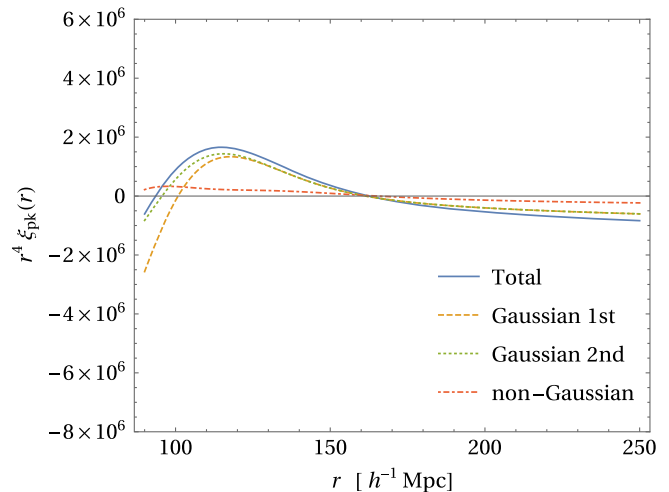


FIG. 5. Same as Fig. 4, but the underlying power spectrum is given by that of a CDM power spectrum without baryons.

Physical implications of the peak clustering are more apparent in configuration space. The corresponding correlation function, Eq. (58), and its components are plotted in Fig. 4. The vertical axis corresponds to r^4 times the correlation function of peaks. Striking features are the existence of peaks at around $100 h^{-1}$ and $200 h^{-1}\text{Mpc}$ and a trough at around $150 h^{-1}\text{Mpc}$. These features are largely due to the effect of baryon acoustic oscillations (BAOs) in the underlying power spectrum. In fact, if the underlying power spectrum is replaced by the those of cold dark matter (CDM) with no baryon, the resulting correlation function is given by Fig. 5. The amplitude of the peak around $100 h^{-1}\text{Mpc}$ is significantly reduced, and the trough and peak on larger scales both vanish. The fact that baryonic features in the peak correlation are significantly enhanced is already pointed out by previous work with Gaussian statistics [14,86]. Here, we see the same property holds with weakly non-Gaussian statistics.

V. CONCLUSIONS

In this paper, analytic formulas for the statistics of peaks of weakly non-Gaussian random field are derived. We consider the lowest-order corrections of non-Gaussianity to the Gaussian predictions, taking the linear terms of the bispectrum into account. First, we generally consider the statistics of peaks in N -dimensional space, and derive formal expressions of number densities, Eq. (25), and the power spectrum, Eq. (53). In order to evaluate the formal expressions, one needs to evaluate G_{ijklm} of Eq. (35). The functions $f_{lm}(x)$ are evaluated in each dimension $N = 1, 2, 3$ as Eqs. (39), (43), and (49). The above equations are our main results of this paper. Useful formulas of angular integrations to evaluate the power spectrum and the correlation functions of peaks for $N = 2, 3$ are given by Eqs. (67) and (76). In order to illustrate possible applications of our results, we

calculate three examples of statistics of peaks for cosmological fields: the number density of peaks in a three-dimensional density field, the number density of peaks in a two-dimensional weak lensing field, and correlations of peaks in a three-dimensional density field. In these examples, the non-Gaussianity is assumed to be induced by nonlinear evolutions of gravitational instability.

The expansion scheme of the peak abundance by the weak non-Gaussianity in this paper is equivalent to the pioneering work of Ref. [52]. In this previous work, the coefficients of the expansion for the three-dimensional peaks involves multidimensional integrations which should be evaluated by semi-Monte-Carlo integration. As for the peak abundance, one of the new developments in this paper is to provide new formulas for the coefficients, all of which can be evaluated by virtually one-dimensional integrations. The new formulas are much easier to evaluate than the previous method, and we believe they can be widely applied to many problems involving peak statistics in cosmology.

For another new development in this paper, we provide new formulas for the peak correlations in the presence of weak non-Gaussianity. The methods of deriving general formulas in two and three dimensions are depicted, and a concrete formula in three dimensions with weak non-Gaussianity induced by gravity is presented [Eqs. (104) and (C2)–(C4)]. Although we do not give the explicit result, the corresponding formula for two-dimensional lensing field can be straightforwardly derived.

An interesting feature of the peak correlations of the matter density field is the enhancement of the effect of BAO in the correlation function of peaks (Fig. 4). Even though the BAO peaks in the correlation function of the density field are smeared by the smoothing, the scale of BAO is still encoded in the correlation function of peaks.

The main purpose of this paper is to provide the analytic formulas for the peak statistics in the presence of the weak non-Gaussianity. There are several directions for applying and extending the results of this paper. First, the peaks of the galaxy number density are obvious sites of the cosmological structures such as the clusters and super-clusters of galaxies. While the analytic formulas for statistics of peaks in Gaussian random fields are only applicable in the Lagrangian density fields, those in weakly non-Gaussian fields are applicable in the Eulerian density fields, which can be directly observable. In the era of large cosmological surveys, the statistics of peaks in the galaxy number density fields would be useful tools beyond the two-point statistics of density fields. Second, analytic formulas of this paper are also useful for the analysis of two-dimensional weak lensing fields. The weak lensing fields on scales of interest are definitely non-Gaussian. In applying the results of this paper, it is necessary to include the effects of noise, which should be rather straightforward. Third, we only take into account the

effect of lowest-order non-Gaussianity characterized by the bispectrum. The next-order contributions include the linear effects of the trispectrum and quadratic effects of the bispectrum. While the next-order contributions of non-Gaussianity are more complicated than those in this paper, it is feasible to extend the results in this direction. Fourth, the analysis of the abundance and correlation of PBHs in the presence of initial non-Gaussianity will be an interesting application of the results of this paper. We hope to address the possibility of the above applications in the near future.

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APPENDIX A: AN EXPANSION METHOD WITH THE GENERALIZED WIENER-HERMITE FUNCTIONALS

In this Appendix, we review a method of Ref. [57] to derive the weakly non-Gaussian corrections to the statistical quantities. The derivation is based on the method of generalized Wiener-Hermite expansion of the biased field [57], and this method is closely related to a method in the integrated perturbation theory [87]. While the derivation of Ref. [57] is mostly presented in configuration space, we present the equivalent method in Fourier space in this Appendix.

We assume the random field f has a zero mean, $\langle f \rangle = 0$, and is statistically homogeneous and isotropic in N -dimensional space. It is convenient to work in Fourier space, and each Fourier coefficient is denoted by $\tilde{f}(\mathbf{k})$. Our convention of the Fourier transform is given by

$$\tilde{f}(\mathbf{k}) = \int d^N x e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}), \quad f(\mathbf{x}) = \int \frac{d^N k}{(2\pi)^N} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{f}(\mathbf{k}). \quad (\text{A1})$$

The statistical properties are specified by the probability distribution functional $\mathcal{P}[\tilde{f}]$, which gives the probability density for a particular form of function $\tilde{f}(\mathbf{k})$.

The partition function is given by a functional integral,

$$\mathcal{Z}[J] = \int \mathcal{D}\tilde{f} \exp \left[i \int \frac{d^N k}{(2\pi)^N} J(\mathbf{k}) \tilde{f}(\mathbf{k}) \right] \mathcal{P}[\tilde{f}], \quad (\text{A2})$$

where $\mathcal{D}\tilde{f}$ is the volume element of the functional integral over the function $\tilde{f}(\mathbf{k})$ with appropriate measures.

According to the cumulant expansion theorem [88], we have

$$\ln \mathcal{Z}[J] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \frac{d^N k_1}{(2\pi)^N} \cdots \frac{d^N k_n}{(2\pi)^N} \langle \tilde{f}(\mathbf{k}_1) \cdots \tilde{f}(\mathbf{k}_n) \rangle_c J(\mathbf{k}_1) \cdots J(\mathbf{k}_n), \quad (\text{A3})$$

where $\langle \cdots \rangle_c$ represents the n -point cumulant. From the above equation, the partition function is represented by

$$\mathcal{Z}[J] = \exp \left[-\frac{1}{2} \int \frac{d^N k}{(2\pi)^N} P(k) J(-\mathbf{k}) J(\mathbf{k}) \right] \exp \left[\sum_{n=3}^{\infty} \frac{i^n}{n!} \int \frac{d^N k_1}{(2\pi)^N} \cdots \frac{d^N k_n}{(2\pi)^N} \langle \tilde{f}(\mathbf{k}_1) \cdots \tilde{f}(\mathbf{k}_n) \rangle_c J(\mathbf{k}_1) \cdots J(\mathbf{k}_n) \right], \quad (\text{A4})$$

where $P(k)$ is the power spectrum defined by

$$\langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \rangle_c = (2\pi)^N \delta^N(\mathbf{k}_1 + \mathbf{k}_2) P(k). \quad (\text{A5})$$

Inverting Eq. (A2), substituting Eq. (A4), and performing Gaussian integration, the probability distribution functional is represented by

$$\begin{aligned} \mathcal{P}[\tilde{f}] &= \int \bar{\mathcal{D}} J \mathcal{Z}[J] \exp \left[-i \int \frac{d^N k}{(2\pi)^N} J(\mathbf{k}) \tilde{f}(\mathbf{k}) \right] \\ &= \exp \left[\sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \int d^N k_1 \cdots d^N k_n \langle \tilde{f}(\mathbf{k}_1) \cdots \tilde{f}(\mathbf{k}_n) \rangle_c \frac{\delta^n}{\delta \tilde{f}(\mathbf{k}_1) \cdots \delta \tilde{f}(\mathbf{k}_n)} \right] \mathcal{P}_G[\tilde{f}], \end{aligned} \quad (\text{A6})$$

where $\bar{\mathcal{D}}[J]$ is the volume element of the functional integral over the function $J(\mathbf{k})$ with appropriate measures, $\delta/\delta \tilde{f}(\mathbf{k})$ is the functional derivative, and

$$\mathcal{P}_G[\tilde{f}] = \int \bar{\mathcal{D}}[J] \exp \left[-\frac{1}{2} \int \frac{d^N k}{(2\pi)^N} P(k) J(-\mathbf{k}) J(\mathbf{k}) - i \int \frac{d^N k}{(2\pi)^N} J(\mathbf{k}) \tilde{f}(\mathbf{k}) \right] \propto \exp \left[-\frac{1}{2} \int \frac{d^N k}{(2\pi)^N} \frac{\tilde{f}(-\mathbf{k}) \tilde{f}(\mathbf{k})}{P(k)} \right] \quad (\text{A7})$$

is the Gaussian probability distribution functional. The last expression is the result of the functional integration up to the normalization constant.

Equation (A6) is a fundamental equation to relate the non-Gaussian statistics to the Gaussian statistics, and the latter is analytically easier to calculate than the former in general. In weakly non-Gaussian cases when the higher-order cumulants are not important, one can expand the exponent and can investigate the effects of lower-order cumulants in the non-Gaussian distributions.

Expanding the exponent of Eq. (A6), we generally have functional derivatives of \mathcal{P}_G , which is straightforwardly calculated by the last expression of Eq. (A7) and results in polynomials of $\tilde{f}(\mathbf{k})$ times \mathcal{P}_G . The Wiener-Hermite functionals are the polynomials of this kind. They are defined by

$$\mathcal{H}_n(\mathbf{k}_1, \dots, \mathbf{k}_n) \equiv \frac{(-1)^n}{\mathcal{P}_G} \frac{\delta^n \mathcal{P}_G}{\delta \tilde{f}(\mathbf{k}_1) \cdots \delta \tilde{f}(\mathbf{k}_n)}, \quad (\text{A8})$$

and $\mathcal{H}_0 = 1$ when $n = 0$. We also define the dual functionals \mathcal{H}_n^* by

$$\mathcal{H}_n^*(\mathbf{k}_1, \dots, \mathbf{k}_n) \equiv (2\pi)^{Nn} P(k_1) \cdots P(k_n) \mathcal{H}_n(-\mathbf{k}_1, \dots, -\mathbf{k}_n). \quad (\text{A9})$$

The first several functionals are given by

$$\mathcal{H}_0^* = 1, \quad (\text{A10})$$

$$\mathcal{H}_1^*(\mathbf{k}) = \tilde{f}(\mathbf{k}), \quad (\text{A11})$$

$$\mathcal{H}_2^*(\mathbf{k}_1, \mathbf{k}_2) = \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) - (2\pi)^N \delta^N(\mathbf{k}_1 + \mathbf{k}_2) P(k_1), \quad (\text{A12})$$

$$\begin{aligned} \mathcal{H}_3^*(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \\ &\quad - [(2\pi)^N \delta^N(\mathbf{k}_1 + \mathbf{k}_2) P(k_1) \tilde{f}(\mathbf{k}_3) + \text{cyc}], \end{aligned} \quad (\text{A13})$$

and so forth. The following orthogonality relation is shown in Ref. [57],

$$\begin{aligned} \langle \mathcal{H}_n^*(\mathbf{k}_1, \dots, \mathbf{k}_n) \mathcal{H}_m(\mathbf{k}'_1, \dots, \mathbf{k}'_m) \rangle_G \\ = \delta_{nm} [\delta^N(\mathbf{k}_1 - \mathbf{k}'_1) \cdots \delta^N(\mathbf{k}_m - \mathbf{k}'_m) + \text{sym}(\mathbf{k}_1, \dots, \mathbf{k}_n)], \end{aligned} \quad (\text{A14})$$

where $\langle \cdots \rangle_G = \int \mathcal{D}\tilde{f} \cdots \mathcal{P}_G$ is the expectation value of the Gaussian statistics with the power spectrum $P(k)$, and $\text{sym}(\mathbf{k}_1, \dots, \mathbf{k}_n)$ indicates $(n! - 1)$ terms to symmetrize the

previous term with respect to permutations of the arguments $\mathbf{k}_1, \dots, \mathbf{k}_n$. Using the generalized Wiener-Hermite functionals, the probability distribution functional of non-Gaussian statistics of Eq. (A6) is represented by

$$\begin{aligned} \mathcal{P} &= \mathcal{H}_0 \mathcal{P}_G + \frac{1}{6} \int d^N k_1 d^N k_2 d^N k_3 \langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \rangle_c \mathcal{H}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \mathcal{P}_G \\ &+ \frac{1}{72} \int d^N k_1 \cdots d^N k_6 \langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \rangle_c \langle \tilde{f}(\mathbf{k}_4) \tilde{f}(\mathbf{k}_5) \tilde{f}(\mathbf{k}_6) \rangle_c \mathcal{H}_6(\mathbf{k}_1, \dots, \mathbf{k}_6) \mathcal{P}_G \\ &+ \frac{1}{24} \int d^N k_1 \cdots d^N k_4 \langle \tilde{f}(\mathbf{k}_1) \cdots \tilde{f}(\mathbf{k}_4) \rangle_c \mathcal{H}_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{P}_G + \cdots \end{aligned} \quad (\text{A15})$$

Assuming the higher-order cumulants are small, weakly non-Gaussian statistics are calculated by the above expansion scheme. This expansion is a generalization of the Edgeworth expansion [56,57,89,90].

Since the generalized Wiener-Hermite functionals are orthogonal functionals with orthogonality given by Eq. (A14), any given functional $\mathcal{F}(\mathbf{x})$ of the random field f can be expanded by the functionals. In Fourier space, the expansion of $\tilde{\mathcal{F}}(\mathbf{k})$ is given by

$$\tilde{\mathcal{F}}(\mathbf{k}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^N k_1}{(2\pi)^N} \cdots \frac{d^N k_n}{(2\pi)^N} (2\pi)^N \delta^N(\mathbf{k}_1 + \cdots + \mathbf{k}_n - \mathbf{k}) \mathcal{G}_n(\mathbf{k}_1, \dots, \mathbf{k}_n) \mathcal{H}_n^*(\mathbf{k}_1, \dots, \mathbf{k}_n), \quad (\text{A16})$$

where the appearance of the delta function in the integrand is a consequence of translational invariance of the space. Due to the orthogonality relation of Eq. (A14), the coefficient functions \mathcal{G}_n are given by

$$(2\pi)^N \delta^N(\mathbf{k}_1 + \cdots + \mathbf{k}_n - \mathbf{k}) \mathcal{G}_n(\mathbf{k}_1, \dots, \mathbf{k}_n) = (2\pi)^{Nn} \langle \tilde{\mathcal{F}}(\mathbf{k}) \mathcal{H}_n(\mathbf{k}_1, \dots, \mathbf{k}_n) \rangle_G = (2\pi)^{Nn} \left\langle \frac{\delta^n \tilde{\mathcal{F}}(\mathbf{k})}{\delta \tilde{f}(\mathbf{k}_1) \cdots \delta \tilde{f}(\mathbf{k}_n)} \right\rangle_G. \quad (\text{A17})$$

Fourier transforming the above equation with respect to \mathbf{k} , we have

$$\mathcal{G}_n(\mathbf{k}_1, \dots, \mathbf{k}_n) = (2\pi)^{Nn} e^{i(\mathbf{k}_1 + \cdots + \mathbf{k}_n) \cdot \mathbf{x}} \langle \mathcal{F}(\mathbf{x}) \mathcal{H}_n(\mathbf{k}_1, \dots, \mathbf{k}_n) \rangle_G = (2\pi)^{Nn} e^{i(\mathbf{k}_1 + \cdots + \mathbf{k}_n) \cdot \mathbf{x}} \left\langle \frac{\delta^n \mathcal{F}(\mathbf{x})}{\delta \tilde{f}(\mathbf{k}_1) \cdots \delta \tilde{f}(\mathbf{k}_n)} \right\rangle_G. \quad (\text{A18})$$

Due to the translational invariance, the second and third expressions are independent of the position \mathbf{x} . Thus, in practice, we can conveniently evaluate the function \mathcal{G}_n by putting $\mathbf{x} = \mathbf{0}$ in the above equation. Using Eqs. (A15) and (A18), the expectation value of any functional \mathcal{F} of f at any position is expanded as

$$\begin{aligned} \langle \mathcal{F} \rangle &= \int \mathcal{D}f \mathcal{F}[f] \mathcal{P} = \mathcal{G}_0 + \frac{1}{6} \int \frac{d^N k_1}{(2\pi)^N} \frac{d^N k_2}{(2\pi)^N} \frac{d^N k_3}{(2\pi)^N} \langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \rangle_c \mathcal{G}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\ &+ \frac{1}{24} \int \frac{d^N k_1}{(2\pi)^N} \cdots \frac{d^N k_4}{(2\pi)^N} \langle \tilde{f}(\mathbf{k}_1) \cdots \tilde{f}(\mathbf{k}_4) \rangle_c \mathcal{G}_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \\ &+ \frac{1}{72} \int \frac{d^N k_1}{(2\pi)^N} \cdots \frac{d^N k_6}{(2\pi)^N} \langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \rangle_c \langle \tilde{f}(\mathbf{k}_4) \tilde{f}(\mathbf{k}_5) \tilde{f}(\mathbf{k}_6) \rangle_c \mathcal{G}_6(\mathbf{k}_1, \dots, \mathbf{k}_6) + \cdots \end{aligned} \quad (\text{A19})$$

This expansion has a diagrammatic interpretation [57]. Higher-order correction terms can be efficiently derived by the diagrammatic rules.

Similarly, we obtain the expansion of the two-point statistics in Fourier space (the power spectrum),

$$\langle \tilde{\mathcal{F}}(\mathbf{k}) \tilde{\mathcal{F}}(\mathbf{k}') \rangle = \int \mathcal{D}\tilde{f} \mathcal{F}(\mathbf{k}) \mathcal{F}(\mathbf{k}') \mathcal{P}. \quad (\text{A20})$$

Substituting Eqs. (A15) and (A16) into the above equation, there appear terms involving factors of the type $\langle \mathcal{H}_n^* \mathcal{H}_m^* \mathcal{H}_l \rangle_G$. These factors can be evaluated by applying

Wick's theorem for Gaussian statistics, and more conveniently evaluated by the diagrammatic method developed in Ref. [57]. In short, contractions of the field which are contained in the same \mathcal{H}_n or \mathcal{H}_n^* should be dropped when applying Wick's theorem. As a result, the factor $\langle \mathcal{H}_n^* \mathcal{H}_m^* \mathcal{H}_l \rangle_G$ is nonzero only when $n + m + l$ is an even number, and we have, e.g.,

$$\langle \mathcal{H}_1^*(\mathbf{k}_1) \mathcal{H}_1^*(\mathbf{k}'_1) \mathcal{H}_0 \rangle_G = (2\pi)^N P(k_1) \delta^N(\mathbf{k}_1 + \mathbf{k}'_1), \quad (\text{A21})$$

$$\langle \mathcal{H}_1^*(\mathbf{k}_1) \mathcal{H}_3^*(\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}'_3) \mathcal{H}_0 \rangle_G = 0, \quad (\text{A22})$$

$$\begin{aligned} \langle \mathcal{H}_2^*(\mathbf{k}_1, \mathbf{k}_2) \mathcal{H}_2^*(\mathbf{k}'_1, \mathbf{k}'_2) \mathcal{H}_0 \rangle_G \\ = (2\pi)^{2N} P(k_1) P(k_2) \delta^N(\mathbf{k}_1 + \mathbf{k}'_1) \delta^N(\mathbf{k}_2 + \mathbf{k}'_2) + \text{sym}, \end{aligned} \quad (\text{A23})$$

$$\langle \mathcal{H}_0^* \mathcal{H}_1^*(\mathbf{k}_1) \mathcal{H}_3(\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}'_3) \rangle_G = 0, \quad (\text{A24})$$

$$\begin{aligned} \langle \mathcal{H}_0^* \mathcal{H}_3^*(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \mathcal{H}_3(\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}'_3) \rangle_G \\ = \delta^N(\mathbf{k}_1 - \mathbf{k}'_1) \delta^N(\mathbf{k}_2 - \mathbf{k}'_2) \delta^N(\mathbf{k}_3 - \mathbf{k}'_3) + \text{sym}, \end{aligned} \quad (\text{A25})$$

$$\begin{aligned} \langle \mathcal{H}_1^*(\mathbf{k}_1) \mathcal{H}_2^*(\mathbf{k}'_1, \mathbf{k}'_2) \mathcal{H}_3(\mathbf{k}''_1, \mathbf{k}''_2, \mathbf{k}''_3) \rangle_G \\ = \delta^N(\mathbf{k}_1 - \mathbf{k}'_1) \delta^N(\mathbf{k}'_1 - \mathbf{k}''_2) \delta^N(\mathbf{k}'_2 - \mathbf{k}''_3) + \text{sym}, \end{aligned} \quad (\text{A26})$$

and so forth, where +sym represents the symmetrization terms which symmetrize the previous term with respect to the arguments of \mathcal{H}_n^* or \mathcal{H}_n . For example, the symmetrization terms of Eqs. (A25) correspond to cyclic permutations with respect to \mathbf{k}_1 , \mathbf{k}_2 , and \mathbf{k}_3 . Substituting Eqs. (A15) and (A16) into Eq. (A20), using Eqs. (A21)–(A24), and subtracting the connected part, we have

$$\begin{aligned} \langle \tilde{\mathcal{F}}(\mathbf{k}) \tilde{\mathcal{F}}(\mathbf{k}') \rangle_c &= (2\pi)^N \delta^N(\mathbf{k} + \mathbf{k}') \mathcal{G}_1(\mathbf{k}) \mathcal{G}_1(-\mathbf{k}) P(k) \\ &+ \frac{1}{2} (2\pi)^N \delta^N(\mathbf{k} + \mathbf{k}') \int \frac{d^N k_1}{(2\pi)^N} \frac{d^N k_2}{(2\pi)^N} (2\pi)^N \delta^N(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \mathcal{G}_2(\mathbf{k}_1, \mathbf{k}_2) \mathcal{G}_2(-\mathbf{k}_1, -\mathbf{k}_2) P(k_1) P(k_2) \\ &+ \frac{1}{2} \left[\mathcal{G}_1(\mathbf{k}) \int \frac{d^N k_1}{(2\pi)^N} \frac{d^N k_2}{(2\pi)^N} (2\pi)^N \delta^N(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}') \mathcal{G}_2(\mathbf{k}_1, \mathbf{k}_2) \langle \tilde{f}(\mathbf{k}) \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \rangle_c + (\mathbf{k} \leftrightarrow \mathbf{k}') \right] + \dots \end{aligned} \quad (\text{A27})$$

This expression also has a diagrammatic interpretation [57]. Higher-order correction terms can be efficiently derived by the diagrammatic rules. In Eqs. (A19) and (A27), the n -point correlations of the Fourier modes $\langle \tilde{f}(\mathbf{k}_1) \dots \tilde{f}(\mathbf{k}_n) \rangle_c$ contain a delta function $\delta^N(\mathbf{k}_1 + \dots + \mathbf{k}_n)$ due to the translational invariance of space, and parts of the integrals are trivially performed. For example, the bispectrum B is defined by

$$\langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \rangle_c = (2\pi)^N \delta^N(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3). \quad (\text{A28})$$

The power spectrum $P_{\mathcal{F}}(k)$ of the biased field \mathcal{F} is given by

$$\frac{\langle \tilde{\mathcal{F}}(\mathbf{k}) \tilde{\mathcal{F}}(\mathbf{k}') \rangle_c}{\langle \mathcal{F} \rangle^2} = (2\pi)^N \delta^N(\mathbf{k} + \mathbf{k}') P_{\mathcal{F}}(k). \quad (\text{A29})$$

Thus, from Eqs. (A19) and (A27) and (A28), we have

$$\begin{aligned} P_{\mathcal{F}}(k) &= [g_1(\mathbf{k})]^2 P(k) \\ &+ \frac{1}{2} \int \frac{d^N p}{(2\pi)^N} [g_2(\mathbf{p}, \mathbf{k} - \mathbf{p})]^2 \\ &\times P(p) P(|\mathbf{k} - \mathbf{p}|) + g_1(\mathbf{k}) \\ &\times \int \frac{d^N p}{(2\pi)^N} g_2(\mathbf{p}, \mathbf{k} - \mathbf{p}) B(\mathbf{p}, \mathbf{k} - \mathbf{p}, -\mathbf{k}) + \dots, \end{aligned} \quad (\text{A30})$$

where

$$g_n(\mathbf{k}_1, \dots, \mathbf{k}_n) \equiv \frac{\mathcal{G}_n(\mathbf{k}_1, \dots, \mathbf{k}_n)}{\mathcal{G}_0}, \quad (\text{A31})$$

and we have used a parity symmetry, $g_n(-\mathbf{k}_1, \dots, -\mathbf{k}_n) = g_n(\mathbf{k}_1, \dots, \mathbf{k}_n)$ and $B(-\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3) = B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$. The non-Gaussian corrections of Eq. (A19) for the denominator of Eq. (A29) do not contribute to the above lowest-order correction, although they contribute to higher-order corrections in general. Equation (A30) has the same form as the result of the integrated perturbation theory [87] if we identify $g_n = c_n$, where c_n is the renormalized bias function. However, this identification is valid only for the lowest-order non-Gaussian approximation, because c_n is defined with non-Gaussian statistics while g_n is defined with Gaussian statistics.

APPENDIX B: GAUSSIAN RESPONSE FUNCTIONS FOR PEAKS

In this Appendix, we calculate the Gaussian n -point response functions of Eq. (3) for the peak number density, $\mathcal{F} = n_{\text{pk}}$, in a generally N -dimensional space. The functions are defined by

$$\mathcal{G}_n(\mathbf{k}_1, \dots, \mathbf{k}_n) \equiv (2\pi)^{Nn} \left\langle \frac{\delta^n n_{\text{pk}}}{\delta \tilde{f}(\mathbf{k}_1) \dots \delta \tilde{f}(\mathbf{k}_n)} \right\rangle_G. \quad (\text{B1})$$

The peak number density $n_{\text{pk}}(\nu)$ given by Eq. (19) is a function of the field derivatives, α , η_i , and ζ_{ij} of Eq. (8). Thereby, the functional derivative $\delta/\delta \tilde{f}(\mathbf{k})$ acting on n_{pk} is replaced by

$$(2\pi)^N \frac{\delta}{\delta \tilde{f}(\mathbf{k})} \rightarrow (2\pi)^N \left[\frac{\delta \alpha}{\delta \tilde{f}(\mathbf{k})} \frac{\partial}{\partial \alpha} + \frac{\delta \eta_i}{\delta \tilde{f}(\mathbf{k})} \frac{\partial}{\partial \eta_i} + \frac{\delta \zeta_{ij}}{\delta \tilde{f}(\mathbf{k})} \frac{\partial}{\partial \zeta_{ij}} \right] \\ = \frac{1}{\sigma_0} \frac{\partial}{\partial \alpha} + \frac{ik_i}{\sigma_1} \frac{\partial}{\partial \eta_i} - \frac{k_i k_j}{\sigma_2} \frac{\partial}{\partial \zeta_{ij}} \equiv \hat{D}(\mathbf{k}). \quad (\text{B2})$$

For the operator $\partial/\partial \zeta_{ij}$, the derivatives are taken as if ζ_{ij} and ζ_{ji} are independent for $i \neq j$ [65]. Thus, Eq. (B1) is rewritten as

$$\mathcal{G}_n(\mathbf{k}_1, \dots, \mathbf{k}_n) = (-1)^n \int d^{N\nu} Y_{n\text{pk}} \hat{D}(\mathbf{k}_1) \cdots \hat{D}(\mathbf{k}_n) \mathcal{P}_G(\mathbf{Y}). \quad (\text{B3})$$

To calculate the differentiations of the above expression, the relations

$$\frac{\partial(\eta^2)}{\partial \eta_i} = 2\eta_i, \quad \frac{\partial J_1}{\partial \zeta_{ij}} = -\delta_{ij}, \\ \frac{\partial J_2}{\partial \zeta_{ij}} = N \tilde{\zeta}_{ji}, \quad \frac{\partial \tilde{\zeta}_{kl}}{\partial \zeta_{ij}} = \delta_{ik} \delta_{jl} - \frac{1}{N} \delta_{ij} \delta_{kl} \quad (\text{B4})$$

are useful. The Gaussian probability distribution function for the field derivatives $\mathcal{P}_G(\mathbf{Y})$ given by Eq. (17) is a

function of only rotationally invariant variables α , η^2 , J_1 , and J_2 . Using the above relations, the first-order derivatives are given by

$$\frac{\partial}{\partial \eta_i} \mathcal{P}_G = 2\eta_i \frac{\partial}{\partial(\eta^2)} \mathcal{P}_G, \\ \frac{\partial}{\partial \zeta_{ij}} \mathcal{P}_G = \left[-\delta_{ij} \frac{\partial}{\partial J_1} + N \tilde{\zeta}_{ji} \frac{\partial}{\partial J_2} \right] \mathcal{P}_G, \quad (\text{B5})$$

the second-order derivatives are given by

$$\frac{\partial^2}{\partial \eta_i \partial \eta_j} \mathcal{P}_G = 2 \left[\delta_{ij} \frac{\partial}{\partial(\eta^2)} + 2\eta_i \eta_j \frac{\partial^2}{\partial(\eta^2)^2} \right] \mathcal{P}_G, \quad (\text{B6})$$

$$\frac{\partial^2}{\partial \zeta_{ij} \partial \zeta_{kl}} \mathcal{P}_G = \left[\delta_{ij} \delta_{kl} \frac{\partial^2}{\partial J_1^2} - \frac{2N}{N-1} (\delta_{ij} \tilde{\zeta}_{lk} + \delta_{kl} \tilde{\zeta}_{ji}) \frac{\partial^2}{\partial J_1 \partial J_2} \right. \\ \left. + \frac{4N^2}{(N-1)^2} \tilde{\zeta}_{ji} \tilde{\zeta}_{lk} \frac{\partial^2}{\partial J_2^2} \right. \\ \left. + \frac{2}{N-1} (N \delta_{il} \delta_{jk} - \delta_{ij} \delta_{kl}) \frac{\partial}{\partial J_2} \right] \mathcal{P}_G, \quad (\text{B7})$$

and the third-order derivatives are given by

$$\frac{\partial^3}{\partial \eta_i \partial \eta_j \partial \eta_k} \mathcal{P}_G = 4 \left[(\delta_{ij} \eta_k + \delta_{jk} \eta_i + \delta_{ki} \eta_j) \frac{\partial^2}{\partial(\eta^2)^2} + 2\eta_i \eta_j \eta_k \frac{\partial^3}{\partial(\eta^2)^3} \right] \mathcal{P}_G, \quad (\text{B8})$$

$$\frac{\partial^3}{\partial \zeta_{ij} \partial \zeta_{kl} \partial \zeta_{mn}} \mathcal{P}_G = \left[-\delta_{ij} \delta_{kl} \delta_{mn} \frac{\partial^3}{\partial J_1^3} + \frac{2N}{N-1} (\delta_{ij} \delta_{kl} \tilde{\zeta}_{nm} + \delta_{ij} \delta_{mn} \tilde{\zeta}_{lk} + \delta_{kl} \delta_{mn} \tilde{\zeta}_{ji}) \frac{\partial^3}{\partial J_1^2 \partial J_2} \right. \\ \left. - \frac{4N^2}{(N-1)^2} (\delta_{ij} \tilde{\zeta}_{lk} \tilde{\zeta}_{nm} + \delta_{kl} \tilde{\zeta}_{ji} \tilde{\zeta}_{nm} + \delta_{mn} \tilde{\zeta}_{ji} \tilde{\zeta}_{lk}) \frac{\partial^3}{\partial J_1 \partial J_2^2} \right. \\ \left. + \frac{8N^3}{(N-1)^3} \tilde{\zeta}_{ji} \tilde{\zeta}_{lk} \tilde{\zeta}_{nm} \frac{\partial^3}{\partial J_2^3} + \frac{2N}{N-1} \left(\frac{3}{N} \delta_{ij} \delta_{kl} \delta_{mn} - \delta_{ij} \delta_{kn} \delta_{lm} - \delta_{kl} \delta_{in} \delta_{jm} - \delta_{il} \delta_{jk} \delta_{mn} \right) \frac{\partial^2}{\partial J_1 \partial J_2} \right. \\ \left. + \frac{4N^2}{(N-1)^2} \left(\delta_{in} \delta_{jm} \tilde{\zeta}_{lk} + \delta_{kn} \delta_{lm} \tilde{\zeta}_{ji} + \delta_{il} \delta_{jk} \tilde{\zeta}_{nm} - \frac{\delta_{ij} \delta_{kl} \tilde{\zeta}_{nm} + \delta_{ij} \delta_{mn} \tilde{\zeta}_{lk} + \delta_{kl} \delta_{mn} \tilde{\zeta}_{ji}}{N} \right) \frac{\partial^2}{\partial J_2^2} \right] \mathcal{P}_G. \quad (\text{B9})$$

The integrand of Eq. (B3) other than the product of operators, $\hat{D}(\mathbf{k}_1) \cdots \hat{D}(\mathbf{k}_n)$, contains only rotationally invariant variables. Thus, we can first average over the angular dependence in the product of operators. Denoting the angular average by $\langle \cdots \rangle_\Omega$, we have

$$\langle \eta_i \rangle_\Omega = 0, \quad \langle \eta_i \eta_j \rangle_\Omega = \frac{1}{N} \delta_{ij} \eta^2, \quad \langle \tilde{\zeta}_{ij} \rangle_\Omega = 0, \quad \langle \tilde{\zeta}_{ij} \tilde{\zeta}_{kl} \rangle_\Omega = \frac{J_2}{N(N+2)} \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{N} \delta_{ij} \delta_{kl} \right), \quad (\text{B10})$$

$$\langle \tilde{\zeta}_{ij} \tilde{\zeta}_{kl} \tilde{\zeta}_{mn} \rangle_\Omega = \frac{J_3}{N^3(N+2)(N+4)} [16\delta_{ij} \delta_{kl} \delta_{mn} - 4N(\delta_{ij} \delta_{km} \delta_{ln} + \delta_{ij} \delta_{kn} \delta_{lm} + \delta_{kl} \delta_{im} \delta_{jn} + \delta_{kl} \delta_{in} \delta_{jm} + \delta_{mn} \delta_{ik} \delta_{jl} + \delta_{mn} \delta_{il} \delta_{jk}) \\ + N^2(\delta_{ik} \delta_{lm} \delta_{jn} + \delta_{jk} \delta_{lm} \delta_{in} + \delta_{il} \delta_{km} \delta_{jn} + \delta_{ik} \delta_{ln} \delta_{jm} + \delta_{jl} \delta_{km} \delta_{in} + \delta_{il} \delta_{kn} \delta_{jm} + \delta_{jk} \delta_{ln} \delta_{im} + \delta_{jl} \delta_{kn} \delta_{im})], \quad (\text{B11})$$

due to rotational symmetry. Using the above equations, we have

$$\langle \mathcal{D}(\mathbf{k}) \mathcal{P}_G \rangle_\Omega = \left(\frac{1}{\sigma_0} \frac{\partial}{\partial \alpha} + \frac{k^2}{\sigma_2} \frac{\partial}{\partial J_1} \right) \mathcal{P}_G, \quad (\text{B12})$$

$$\begin{aligned} \langle \mathcal{D}(\mathbf{k}_1) \mathcal{D}(\mathbf{k}_2) \mathcal{P}_G \rangle_\Omega = & \left\{ \left(\frac{1}{\sigma_0} \frac{\partial}{\partial \alpha} + \frac{k_1^2}{\sigma_2} \frac{\partial}{\partial J_1} \right) \left(\frac{1}{\sigma_0} \frac{\partial}{\partial \alpha} + \frac{k_2^2}{\sigma_2} \frac{\partial}{\partial J_1} \right) - \frac{2(\mathbf{k}_1 \cdot \mathbf{k}_2)}{\sigma_1^2} \left[1 + \frac{2}{N} \eta^2 \frac{\partial}{\partial (\eta^2)} \right] \frac{\partial}{\partial (\eta^2)} \right. \\ & \left. + \frac{2N}{(N-1)\sigma_2^2} \left[(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 - \frac{1}{N} k_1^2 k_2^2 \right] \left[1 + \frac{4J_2}{(N-1)(N+2)} \frac{\partial}{\partial J_2} \right] \frac{\partial}{\partial J_2} \right\} \mathcal{P}_G, \end{aligned} \quad (\text{B13})$$

$$\begin{aligned} \langle \mathcal{D}(\mathbf{k}_1) \mathcal{D}(\mathbf{k}_2) \mathcal{D}(\mathbf{k}_3) \rangle_\Omega \mathcal{P}_G = & \left\{ \left(\frac{1}{\sigma_0} \frac{\partial}{\partial \alpha} + \frac{k_1^2}{\sigma_2} \frac{\partial}{\partial J_1} \right) \left(\frac{1}{\sigma_0} \frac{\partial}{\partial \alpha} + \frac{k_2^2}{\sigma_2} \frac{\partial}{\partial J_1} \right) \left(\frac{1}{\sigma_0} \frac{\partial}{\partial \alpha} + \frac{k_3^2}{\sigma_2} \frac{\partial}{\partial J_1} \right) \right. \\ & - \frac{2(\mathbf{k}_1 \cdot \mathbf{k}_2)}{\sigma_1^2} \left(\frac{1}{\sigma_0} \frac{\partial}{\partial \alpha} + \frac{k_3^2}{\sigma_2} \frac{\partial}{\partial J_1} \right) \left[1 + \frac{2}{N} \eta^2 \frac{\partial}{\partial (\eta^2)} \right] \frac{\partial}{\partial (\eta^2)} + \text{cyc} \\ & + \frac{2N}{(N-1)\sigma_2^2} \left[(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 - \frac{1}{N} k_1^2 k_2^2 \right] \left(\frac{1}{\sigma_0} \frac{\partial}{\partial \alpha} + \frac{k_3^2}{\sigma_2} \frac{\partial}{\partial J_1} \right) \left[1 + \frac{4J_2}{(N-1)(N+2)} \frac{\partial}{\partial J_2} \right] \frac{\partial}{\partial J_2} \\ & + \text{cyc} + \frac{64N^2}{(N-1)^3(N+2)(N+4)\sigma_2^3} \left[(\mathbf{k}_1 \cdot \mathbf{k}_2)(\mathbf{k}_2 \cdot \mathbf{k}_3)(\mathbf{k}_3 \cdot \mathbf{k}_1) \right. \\ & \left. - \frac{k_1^2(\mathbf{k}_2 \cdot \mathbf{k}_3) + \text{cyc}}{N} + \frac{2}{N^2} k_1^2 k_2^2 k_3^2 \right] J_3 \frac{\partial^3}{\partial J_2^3} \left. \right\} \mathcal{P}_G. \end{aligned} \quad (\text{B14})$$

According to the form of \mathcal{P}_G in Eq. (17), we have

$$\left[1 + \frac{2}{N} \eta^2 \frac{\partial}{\partial (\eta^2)} \right] \frac{\partial}{\partial (\eta^2)} \mathcal{P}_G = \frac{N}{2} (\eta^2 - 1) \mathcal{P}_G = -L_1^{(N/2-1)} \left(\frac{N}{2} \eta^2 \right) \mathcal{P}_G, \quad (\text{B15})$$

$$\left[1 + \frac{4J_2}{(N-1)(N+2)} \frac{\partial}{\partial J_2} \right] \frac{\partial}{\partial J_2} \mathcal{P}_G = \frac{(N-1)(N+2)}{4} (J_2 - 1) \mathcal{P}_G = F_{10}(J_2, J_3) \mathcal{P}_G, \quad (\text{B16})$$

$$J_3 \frac{\partial^3}{\partial J_2^3} \mathcal{P}_G = -\frac{(N-1)^3(N+2)^3}{64} J_3 \mathcal{P}_G = -\frac{(N-1)^3(N+2)^3}{64} F_{01}(J_2, J_3) \mathcal{P}_G, \quad (\text{B17})$$

where $F_{lm}(J_2, J_3)$ is defined by Eq. (23). Substituting Eqs. (B15)–(B17) into Eqs. (B12)–(B14), Eq. (B3) is represented in terms of G_{ijklm} of Eq. (20). The results are given by

$$\mathcal{G}_0 = G_{00000}, \quad (\text{B18})$$

$$\mathcal{G}_1(\mathbf{k}) = \frac{G_{10000}}{\sigma_0} + \frac{G_{01000}}{\sigma_2} k^2, \quad (\text{B19})$$

$$\mathcal{G}_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{G_{20000}}{\sigma_0^2} + \frac{G_{11000}}{\sigma_0 \sigma_2} (k_1^2 + k_2^2) + \frac{G_{02000}}{\sigma_2^2} k_1^2 k_2^2 - \frac{2G_{00100}}{\sigma_1^2} \mathbf{k}_1 \cdot \mathbf{k}_2 + \frac{2NG_{00010}}{(N-1)\sigma_2^2} \left[(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 - \frac{1}{N} k_1^2 k_2^2 \right], \quad (\text{B20})$$

$$\begin{aligned} \mathcal{G}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = & \frac{G_{30000}}{\sigma_0^3} + \frac{G_{21000}}{\sigma_0^2 \sigma_2} (k_1^2 + k_2^2 + k_3^2) + \frac{G_{12000}}{\sigma_0 \sigma_2^2} (k_1^2 k_2^2 + \text{cyc}) \\ & + \frac{G_{03000}}{\sigma_2^3} k_1^2 k_2^2 k_3^2 - \frac{2G_{10100}}{\sigma_0 \sigma_1^2} (\mathbf{k}_1 \cdot \mathbf{k}_2 + \text{cyc}) - \frac{2G_{01100}}{\sigma_1^2 \sigma_2} [(\mathbf{k}_1 \cdot \mathbf{k}_2) k_3^2 + \text{cyc}] \\ & + \frac{2NG_{10010}}{(N-1)\sigma_0 \sigma_2^2} \left[(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 - \frac{1}{N} k_1^2 k_2^2 \right] + \frac{2NG_{01010}}{(N-1)\sigma_2^3} \left[(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 k_3^2 + \text{cyc} - \frac{3}{N} k_1^2 k_2^2 k_3^3 \right] \\ & - \frac{N^2(N+2)^2 G_{00001}}{(N+4)\sigma_2^3} \left[(\mathbf{k}_1 \cdot \mathbf{k}_2)(\mathbf{k}_2 \cdot \mathbf{k}_3)(\mathbf{k}_3 \cdot \mathbf{k}_1) - \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 k_3^2 + \text{cyc}}{N} + \frac{2}{N^2} k_1^2 k_2^2 k_3^2 \right]. \end{aligned} \quad (\text{B21})$$

APPENDIX C: RADIAL FUNCTIONS FOR CORRELATIONS OF PEAKS

In this Appendix, radial functions $\xi_{\text{pk}}^{(1)}(r)$, $\xi_{\text{pk}}^{(2)}(r)$, and $S_{\text{NG}}(k, r)$ in Eq. (104) are explicitly given by the functions $\xi_m^{(n)}(r)$, $A_m^{(n)}(r)$, and $B_m^{(n)}(r)$ of Eqs. (101)–(103). The relations are derived from Eqs. (59), (76), (81), and (82). The results for $\xi_{\text{pk}}^{(1)}(r)$ and $\xi_{\text{pk}}^{(2)}(r)$ are already given in Ref. [15]. We reproduce the latter results here for completeness. The result for $S_{\text{NG}}(k, r)$ below is new in this paper.

In the following, we adopt notations

$$b_{ij} \equiv g_{ij000}, \quad \chi_k \equiv g_{00k00}, \quad \omega_{lm} \equiv g_{000lm}. \quad (\text{C1})$$

The final results are given by

$$\xi_{\text{pk}}^{(1)} = b_{10}^2 \xi_0^{(0)} + 2b_{10}b_{01} \xi_0^{(2)} + b_{01}^2 \xi_0^{(4)}, \quad (\text{C2})$$

$$\begin{aligned} \xi_{\text{pk}}^{(2)} = & b_{20}^2 (\xi_0^{(0)})^2 + 4b_{20}b_{11} \xi_0^{(0)} \xi_0^{(2)} + 2b_{11}^2 \xi_0^{(0)} \xi_0^{(4)} + 2 \left(b_{20}b_{02} + b_{11}^2 + \frac{2}{3} \chi_1^2 \right) (\xi_0^{(2)})^2 + 4b_{11}b_{02} \xi_0^{(2)} \xi_0^{(4)} + 4b_{20} \chi_1 (\xi_1^{(1)})^2 \\ & + 8b_{11} \chi_1 \xi_1^{(1)} \xi_1^{(3)} + \left(b_{02}^2 + \frac{4}{5} \omega_{10}^2 \right) (\xi_0^{(4)})^2 + 4 \left(b_{02} + \frac{4}{5} \omega_{10} \right) \chi_1 (\xi_1^{(3)})^2 + 4 \left(b_{20} \omega_{10} + \frac{2}{3} \chi_1^2 \right) (\xi_2^{(2)})^2 \\ & + 8b_{11} \omega_{10} \xi_2^{(2)} \xi_2^{(4)} + 4 \left(b_{02} + \frac{2}{7} \omega_{10} \right) \omega_{10} (\xi_2^{(4)})^2 + \frac{24}{5} \chi_1 \omega_{10} (\xi_3^{(3)})^2 + \frac{72}{35} \omega_{10}^2 (\xi_4^{(4)})^2, \end{aligned} \quad (\text{C3})$$

and

$$\begin{aligned} S_{\text{NG}} = & 2b_{20} \left[\frac{17}{21} (A_0^{(0)})^2 + \frac{4}{21} (A_2^{(0)})^2 - A_1^{(-1)} A_1^{(1)} + \frac{3}{7} A_0^{(0)} B_0^{(0)} + A_1^{(-1)} B_1^{(1)} - \frac{1}{7k^2} A_1^{(1)} B_1^{(1)} + \frac{4}{21k^2} A_0^{(0)} B_0^{(2)} + \frac{8}{21k^2} A_2^{(0)} B_2^{(2)} \right] \\ & + 2b_{11} \left[\frac{34}{21} A_0^{(2)} A_0^{(0)} - (A_1^{(1)})^2 + \frac{8}{21} A_2^{(0)} A_2^{(2)} - A_1^{(-1)} A_1^{(3)} + \frac{3}{7} A_0^{(2)} B_0^{(0)} + A_1^{(1)} B_1^{(1)} + \frac{3}{7} A_0^{(0)} B_0^{(2)} + A_1^{(-1)} B_1^{(3)} \right. \\ & \left. - \frac{3}{7k^2} A_0^{(4)} B_0^{(0)} - \frac{1}{7k^2} A_1^{(3)} B_1^{(1)} - \frac{5}{21k^2} A_0^{(2)} B_0^{(2)} + \frac{8}{21k^2} A_2^{(2)} B_2^{(2)} - \frac{1}{7k^2} A_1^{(1)} B_1^{(3)} + \frac{4}{21k^2} A_0^{(0)} B_0^{(4)} + \frac{8}{21k^2} A_2^{(0)} B_2^{(4)} \right] \\ & + 2b_{02} \left[\frac{17}{21} (A_0^{(2)})^2 + \frac{4}{21} (A_2^{(2)})^2 - A_1^{(1)} A_1^{(3)} + \frac{3}{7} A_0^{(2)} B_0^{(2)} + A_1^{(1)} B_1^{(3)} \right. \\ & \left. - \frac{3}{7k^2} A_0^{(4)} B_0^{(2)} - \frac{1}{7k^2} A_1^{(3)} B_1^{(3)} + \frac{4}{21k^2} A_0^{(2)} B_0^{(4)} + \frac{8}{21k^2} A_2^{(2)} B_2^{(4)} \right] \\ & + 4\chi_1 \left[-\frac{1}{3} A_0^{(2)} A_0^{(0)} + \frac{31}{35} (A_1^{(1)})^2 + \frac{4}{35} (A_3^{(1)})^2 - \frac{2}{3} A_2^{(0)} A_2^{(2)} + \frac{3}{7} A_1^{(1)} B_1^{(1)} + \frac{1}{3} A_0^{(0)} B_0^{(2)} + \frac{2}{3} A_2^{(0)} B_2^{(2)} \right. \\ & \left. - \frac{3}{7k^2} A_1^{(3)} B_1^{(1)} - \frac{1}{21k^2} A_0^{(2)} B_0^{(2)} - \frac{2}{21k^2} A_2^{(2)} B_2^{(2)} + \frac{12}{35k^2} A_1^{(1)} B_1^{(3)} + \frac{8}{35k^2} A_3^{(1)} B_3^{(3)} \right] \\ & + 4\omega_{10} \left[\frac{4}{105} (A_0^{(2)})^2 + \frac{127}{147} (A_2^{(2)})^2 + \frac{24}{245} (A_4^{(2)})^2 - \frac{2}{5} A_1^{(1)} A_1^{(3)} - \frac{3}{5} A_3^{(1)} A_3^{(3)} + \frac{3}{7} A_2^{(2)} B_2^{(2)} + \frac{2}{5} A_1^{(1)} B_1^{(3)} + \frac{3}{5} A_3^{(1)} B_3^{(3)} \right. \\ & \left. - \frac{3}{7k^2} A_2^{(4)} B_2^{(2)} - \frac{2}{35k^2} A_1^{(3)} B_1^{(3)} - \frac{3}{35k^2} A_3^{(3)} B_3^{(3)} + \frac{8}{105k^2} A_0^{(2)} B_0^{(4)} + \frac{44}{147k^2} A_2^{(2)} B_2^{(4)} + \frac{48}{245k^2} A_4^{(2)} B_4^{(4)} \right]. \end{aligned} \quad (\text{C4})$$

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