T-model inflation and bouncing cosmology

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We examine the dynamics of a closed cosmology of which the matter source is that of a conformally coupled scalar field with a broken $SO(1, 1)$ symmetry, which corresponds to the α attractors proposed by Linde and Kallosh. Following a field redefinition, such models give rise to "T-model" inflationary potentials, the dynamics of which provide both an inflationary phase and a classical bounce. We show that the Universe can undergo bounces far from the regime of quantum gravity (i.e., at energy densities lower than the Planck density). We analyze perturbations on this background with particular attention given to the effects of a double-bouncing scenario (with rapid recollapse between bounces) on the long wavelength modes. We demonstrate that the predictions of such models agree well with observations and might explain the suppression of power in the low multiples of the cosmic microwave background.

I. INTRODUCTION

The inflationary paradigm has become a central feature of modern cosmology since it both addresses conceptual issues such as the flatness, horizon, and entropy problems and also provides a natural mechanism by which primordial fluctuations can seed structure formation [1–4] (for a review, see Ref. [5]). This is achieved by the *nearly* de Sitter geometry of the expanding early Universe, which generates an almost scale-invariant spectrum of perturbations. To achieve this, the dominant form of matter must have an equation of state $p \approx -\rho$, i.e., $w = -1$, that of a cosmological constant. For inflation to end, w must increase in order to allow other forms of matter to contribute significantly to the evolution of the Universe. Typically, models invoke scalar fields to realize such an evolution; when the field is potential dominated, it behaves as a cosmological constant. As it descends down the potential and becomes kinetically dominated, however, it acts more like a stiff fluid ($w = 1$). In a closed cosmology, the $k > 0$ Friedmann-Lemâître-Robertson-Walker (FLRW) model wherein the spatial manifold is a 3-sphere, the same accelerated expansion can lead to a bounce from a collapsing universe to an expanding one. It is therefore natural to ask whether the same mechanism can give rise to both phenomena. Qualitatively, the collapse and the expansion

impose requirements that are "mirror images" of one another on the field. In a contracting branch, the field must gain energy faster than other energy sources, yet become almost constant under evolution at small length scales. In the expanding branch, the opposite is true; the energy density should dissipate slowly only at small scales and faster as the Universe expands. Typically, a scalar field with a suitable potential is invoked to play the role of the inflaton (the field responsible for inflation), as the dynamics of such a system allow for this dual role.

Observational data from the Planck satellite have shown both excellent agreement with the FLRW cosmological paradigm and place strict bounds on cosmological parameters [6,7]. Taken together with the baryon acoustic oscillations, the size of the curvature constant, Ω_k, is quite tightly constrained around zero—the best-fit parameter value gives $\Omega_k = -0.005^{+0.016}_{-0.017}$ under the parametrization $\sum \Omega_i = 1$. These bounds are likely to be improved by future observations. However, there is a limit which will be reached if the observations $|\Omega_k| < 10^{-5}$ become degenerate with the effects of a curvature perturbation of which the wavelength is greater than the Hubble horizon. Therefore, in the absence of a positive measurement of Ω_k, the closed Universe is impossible to rule out as a physical system. On the inflationary side, recent progress has stringently constrained models. The ratio of amplitudes of tensor perturbations to those of scalar perturbations, denoted r, together with the scalar spectral tilt n_s, strongly disfavor most monomial potentials, particularly the minimal quadratic

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and quartic models. Concave models which exhibit plateaus, however, appear to be in good correspondence with the observational measurements, with a large class giving rise to $n_s \approx 0.968$ and small r (dependent on specific field configurations) [8–10].

Since inflation is invoked to solve issues of fine-tuning of the initial parameters, it behooves us to use inflationary models which are in some sense "natural" or "generic"—a solution to the horizon and flatness problems that requires a very specific choice of potential simply passes on the burden of explaining away fine-tuning [11,12]. As the simplest of models are observationally excluded, we will consider inflationary models that arise naturally in the context of extended theories of gravity, and in particular from the conformal coupling of a scalar field to the gravitational action. In the case of the scalar field with a broken $SO(1,1)$ symmetry, such theories give rise to "T-models" of inflation [13], which (predominantly) asymptotically become plateaus for large field values. Although the quantitative results we present will be closely tied to this choice, the qualitative behavior is not; the vast majority of inflationary systems (i.e., those which allow for accelerated expansion) also allow for a bounce in a closed universe. In passing, we mention that the α-attractors proposal may also unify inflation with dark energy [14–16], but we will not pursue this direction further in this paper.

The model we discuss differs from other bouncing models which introduce exotic matter, higher derivatives, or quantum effects to overcome the gravitational collapse. Rather than considering a universe which must begin at infinite extent, within this model, the universe could begin from an initial singularity in the prebounce phase. Such a universe would expand to finite extent before recollapsing into a bounce at an energy density smaller than the Planck density. The bounce would be then followed by a long period of inflation which gives rise to the Universe we observe. This would avoid the problems of having runaway growth of inhomogeneities which can spoil the bounce. This provides an entirely different scenario from the conventional collapsing universe of infinite size and is also explored in Ref. [17].

This paper is laid out as follows. In Sec. II, we describe the particular realization of the gravitational action that gives rise to this scenario and establish how the symmetry gives rise to the plateaus of our inflationary potential. This is followed by Secs. III and IV, in which we establish the necessary conditions for the Universe to undergo a bounce and describe the space of bouncing solutions. In Sec. V, we exhibit a simple model that features an analytic bouncing solution. Numerical simulations of the full dynamics are given in Sec. VI. In Sec. VII, we discuss the role of growing perturbations in a homogeneous and isotropic background as the Universe collapses. We discuss observational implications in Sec. VIII. Finally, we give some concluding remarks in Sec. IX.

Unless otherwise noted, we will be working in reduced Planck units throughout this paper, in which $M_P^{-2} = 8\pi G = 1$ and $c = 1$.

II. CONFORMAL ACTION

We begin by considering theories of conformally invariant gravity induced by the evolution of a dynamical gravitational constant, as first proposed by Brans and Dicke [18]. To give rise to a more interesting inflationary scenario, we will consider a class of matter couplings featuring two fields which initially exhibit a $SO(1,1)$ symmetry. Such models are of pressing interest, as it has recently been found that the apparent fifth force they would exhibit is suppressed due to Weyl invariance, and therefore such models are cosmologically viable [19]. Closely following the seminal analysis of Ref. [8], let us consider a conformally invariant Lagrangian for gravity (nonminimally) coupled to two scalar fields, χ and ψ, given by

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi + \frac{1}{12}(\chi^2 - \psi^2)R \right. \\ \left. - \frac{1}{36}F\left(\frac{\psi}{\chi}\right)(\psi^2 - \chi^2)^2\right]. \tag{2.1}$$

In the above, F is an arbitrary function which breaks the $SO(1,1)$ symmetry globally (if F were constant instead, the action would exhibit a global de Sitter symmetry). Although the sign of the ψ^2 terms may appear to make our Lagrangian pathological under the production of unphysical states (such as negative-energy ghosts), we note that this is in fact just a choice of gauge [20]. To make contact with the usual formulation of minimally coupled gravity, we impose the gauge-fixing condition $\chi^2 - \psi^2 = 6\alpha$ (the "rapidity" gauge) and hence parametrize the remaining degree of freedom by $\tanh(\frac{\phi}{\sqrt{6\alpha}}) = \frac{\psi}{\chi}$ in terms of a single scalar field.[1] Noting that $d\psi = \chi d\phi$ and $d\chi = \psi d\phi$, we find that the Lagrangian can be recast in the following more familiar form:

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - F\left(\tanh\frac{\phi}{\sqrt{6\alpha}}\right)\right]. \tag{2.2}$$

It is worthwhile to note at this point the role played by F in this construction. The function tanh has a range of $(-1, 1)$ for real values of ϕ and is a monotonic function on the entire real line. Therefore, we consider functions F which are differentiable on this domain. Such functions will asymptote to constants for large values of ϕ when

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[1]Note that this parametrization does not fully capture all possible values of ϕ. We may also write $\frac{\psi}{\chi} = \coth(\frac{\phi}{\sqrt{6\alpha}})$. Depending on the form of F, the two parametrizations may not overlap, giving rise to distinct models [21].

convoluted with tanh, since $d(F \circ \tanh) = (dF \circ \tanh)\operatorname{sech}^2$. Thus, if F is differentiable on the domain, the convolution asymptotes to a constant, and the $SO(1,1)$ symmetry will be restored asymptotically. However, if we consider functions F which are not differentiable everywhere, and in particular functions which have a pole in $(-1, 1)$, the domain of the convolution will no longer be finite, and the symmetry cannot be restored at large values of ϕ. Such a system is investigated in Ref. [22], in which it is shown that placing simple poles at the end points of the domain allows for the reconstruction of familiar inflationary potentials. It is easy to see that setting $F = G \circ \operatorname{arc} \tanh$ allows the reconstruction of any function G via such a process. This is possible exactly because arctanh is monotonic on the interval $[-1, 1]$ with poles at the end points. We further note that our choice of gauge is qualitatively unimportant except that it imposes the restriction $\chi^2 > \psi^2$ (otherwise, our system exhibits the wrong sign for gravitational couplings and will be unstable, leading to the production of ghosts).

The inflationary models captured by Eq. (2.2) are known as "attractor models," in reference to their convergent predictions [8]. If F is chosen to be an even monomial function, the resulting models are often termed T-models due to the shape of their potential, which features plateaus away from the origin. Here, we must note that these plateaus arise as a result of the noncompact topology of the symmetry group we have employed. To show this, consider our scalar field ϕ to live upon $\mathbf{R}^n$. Upon gauge fixing such that the action reproduces the standard minimally coupled Einstein-Hilbert gravitational action, there remains a noncompact direction in the symmetry group of the matter field, since the gauge fixing must map the invariant function of the matter variables onto a positive, real constant $F(\phi) = C$. Surfaces of constant C are non-compact (as they form the symmetry group), yet are parametrized by members of $\mathbf{R}^n/\mathbf{R}_+ = SO(n)$, which is a compact group. Therefore, in order to cover the entire symmetry group, we require an invertible map from a noncompact space onto a compact space, which must be a monotonic function with finite range yet infinite domain. Such functions must approach constants as their arguments tend to infinity and hence necessarily form plateaus when taking on the role of a potential under any finite convolutions. As a contrasting case, if we had used $SO(2)$ as our symmetry group, the map would have been tan, and hence our potentials would not be bounded from above. Thus, T-models are qualitatively generic for noncompact symmetry groups.

We now consider a homogeneous and isotropic cosmology on the manifold $\mathcal{M} = \mathbf{R} \times S^3$. Spatial slices have constant positive curvature and correspond to $k > 0$ Robertson-Walker geometries. In such a system, there is only one dynamical geometric degree of freedom, the scale factor $a = a(t)$. The line element in this case becomes

$$ds^2 = -dt^2 + a^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 d\theta^2) \right]. \quad (2.3)$$

Thus, once we restrict ourselves to matter compatible with the space-time symmetries (i.e., homogeneity and isotropy), we are left with a familiar inflationary Lagrangian,

$$a^{-3}\mathcal{L} = 3\left(\frac{\dot{a}^2}{a^2} - \frac{k}{a^2} \right) - \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right), \quad (2.4)$$

where $V(\phi) = F \circ \tanh \frac{\phi}{\sqrt{6}}$. Throughout the rest of this work, we will perform our analysis for even monomials, i.e., $F(x) \propto x^{2n}$, e.g., $V(\phi) = \Lambda \tanh^2(\frac{\phi}{\sqrt{6\alpha}})$. A complete analysis of these models in the context of dynamical systems was recently completed [23], and in it, the extended behavior of the entire system was found to have unique end points under evolution, and the universality of such evolutions was established.

III. BACKGROUND DYNAMICS

In this section, we will follow the analysis of Ref. [24], in which a scalar field is used to give rise to bouncing cosmologies as a counterpart to a decaying cosmological constant. We will focus on the simplest potentials that can arise as a consequence of the broken $SO(1,1)$ symmetry. From our Lagrangian in Eq. (2.4), we obtain the familiar Friedmann equation, in which $H = \dot{a}/a$:

$$H^2 = \frac{1}{3}\rho - \frac{k}{a^2}. \quad (3.1)$$

It is immediately apparent that one cannot have $H = 0$ (or equivalently $\dot{a} = 0$) simultaneously with positive energy density and nonpositive curvature $k \leq 0$. There exist a number of models which exhibit classical bounces through the use of ghost matter ($\rho < 0$) or phantom ($P < -\rho$) matter. Such matter may be introduced by fiat in order to remove the singularity [25] or as a result of a more fundamental physical process (such as the evolution of the fine-structure constant [26]). However, in this work, our goal is to examine bounces in a classical context with matter which obeys the usual (weak) energy condition. To achieve this, we require $\Omega_k < 0$. Thus, a bounce can only occur when $\rho = 3k/a_b^2$, where a_b is the scale factor at the bounce (with orientation chosen such that $a > 0$ at all times). At this point, we can expect either a bounce or a recollapse, depending on the sign of the second derivative of the scale factor (positive or negative, respectively).

We now examine how the conditions for a bounce are reflected in Raychaudhuri's equation, which is

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p). \quad (3.2)$$

Hence, a bounce occurs if $p < -\frac{1}{3}\rho$. In terms of a barotropic perfect fluid, we require $w < -1/3$. The matter

during a bounce will consist of a homogeneous, isotropic scalar field with a suitable potential. Since the equation of state depends upon the relationship between kinetic energy and potential energy of the scalar field, the same field can play multiple roles; indeed, this is one of the motivations for invoking fields such as inflatons. For a homogeneous, isotropic scalar field, the energy density is given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{3.3}$$

and the pressure is

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{3.4}$$

As a result, w is given by

$$w = \frac{p}{\rho} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \tag{3.5}$$

We see that $w = 1$ corresponds to kinetic domination and $w = -1$ corresponds to potential domination. Therefore, we require $V(\phi) > \dot{\phi}^2$ in order to have $w < -1/3$.

We now consider a general model of a potential featuring a plateau, given by $V(\phi) = \Lambda u(\phi)$ where the maximum value of the function $u(\phi)$ is 1 as $\phi \to \pm\infty$ (without loss of generality). This potential generalizes the usual T-models and can arise from functions F which do not necessarily have poles. As a result, potential domination means that any bounce must occur when

$$\rho \leq \frac{\Lambda}{2}. \tag{3.6}$$

Note that if our field is to both provide the dynamics of the bounce and play the role of inflaton this constraint indicates that the bounce must occur far from the quantum gravity scale. This is because the energy scale of inflation is of the order of 10^{16} GeV, whereas the Planck scale (where we expect quantum gravity effects to become important) is around 10^{19} GeV. In such cases, our Universe will bounce from contracting to expanding without ever approaching the quantum gravity scale. In turn, we find that the minimal radius of curvature at which a bounce can occur is $l_m = \sqrt{\frac{2}{\Lambda}}$. From the Planck data, we find that this is far from the Planck length for $|\Omega_k|$ below the measured upper bound as well as above the lower bound of being indistinguishable from a superhorizon fluctuation. Hence, the Universe can indeed undergo a bounce wherein both the matter and geometrical sectors are far from the quantum gravity regime.

We can use the condition (3.6) together with the Friedmann equation (3.1) to show that any value of w between -1 and 1 can be realized by imposing the following condition at the bounce point ($a = a_b$):

$$V(\phi) = \frac{3k}{a_b^2}(1 - w) \qquad \dot{\phi}^2 = \frac{6k}{a_b^2}(1 + w). \tag{3.7}$$

Recall that if $w > -1/3$ this is a point of recollapse rather than a bounce. Thus, all we must strictly require is that $V(\phi)$ has a range which contains this value. In terms of the T-model potentials described by $V(\phi) = \Lambda \tanh^{2n}(\frac{\phi}{\sqrt{6\alpha}})$, this is achieved by

$$\phi > \sqrt{6\alpha}\,\text{arc}\tanh\left[\left(\frac{2k}{a_b^2}\right)^{\frac{1}{2n}}\right]. \tag{3.8}$$

Note that this a necessary but not sufficient condition for a bounce. For the bounce to occur, we also require that the potential energy is larger than twice the kinetic energy. Furthermore, we must stress that we are considering the dynamics of a universe which is contracting; as such, the usual Hubble friction term $3H\dot{\phi}$ gives rise to *antifriction* when $H < 0$, serving to increase the kinetic energy instead. Therefore, when the inflaton reaches the plateau in a contracting branch of the Friedmann equation, it is near a minimum of kinetic energy density on the solution (and hence a maximum of w). If this is not sufficient to cause a bounce (i.e., if we have $w > -1/3$), then the inflaton will accelerate, the kinetic component of energy density will dominate the evolution, and the system will reach a singularity. We were very recently made aware of a similar discussion in Ref. [17], which parallels much of the background dynamics.

An important issue arises when considering models beyond the homogeneous and isotropic background dynamics. The de Sitter expansion is an attractor in the space of inhomogeneous and isotropic cosmologies [27,28]. As such, we should be unsurprised to see after inflation a universe that is homogeneous and isotropic; this was one of the key motivations behind inflationary models in the first instance. However during a collapsing phase, the opposite is true. Inhomogeneities grow to a point at which their effects may overwhelm the background dynamics and thus avoid the bounce. Therefore, we must address the issue of whether such a bounce can be physically realized from initial conditions set far before the bounce, such as at the onset of collapse.

In the model we present, we have restricted the matter content to be that of the scalar field alone, as this is the dominant contribution to the stress-energy tensor at high energy scales. If we examine the prebounce phase of such a solution, we find that as the universe expands the scalar field finds the minimum of its potential and in oscillating around it appears to have an effective equation of state similar to that of dust, with $w = 0$ or even faster falloff, with averaged $w > 0$. See the Appendix for more details of this. Hence, at a large scale factor, this model becomes dominated by the curvature term and reaches a finite extent. Further to the past, this model is again in an expanding

phase and becoming homogeneous. Therefore, setting initial conditions far in the past would not necessarily be an impediment to the universe undergoing a bounce. A universe which begins at a singularity with some inhomogeneity, expands, and recollapses can have smaller inhomogeneity at the bounce point. See Sec. VI and Fig. 5 therein for a numerical simulation demonstrating this possibility.

In a more physically motivated model, we should also consider the introduction of a (small) cosmological constant to make contact with late time observations in our own Universe and consider the case in which the inflaton decays into other matter. The background dynamics of such a model can begin from a universe of infinite size at $t \to -\infty$; however, this is also not necessarily the case. Such universes can also begin at singularities, or have repeated bounce and recollapses. As an example, consider a universe at a bounce point which has the scalar field low enough on its potential, and rolling down the potential, such that there is not a lot of inflation. Such a universe may never become dominated by the cosmological constant, but it would recollapse to a singularity. The time reverse of such a model is a universe that begins at a singularity and recollapses to a bounce then expands through inflation to a de Sitter universe.

Finally, we note that in order to match with observations of our Universe it is not necessary that the entire Universe bounces but that at least a single patch of size approximately corresponding to the Hubble radius during inflation bounces. As such, we can consider the scenario discussed in Ref. [24] in which initial conditions are set far in the past for an inhomogeneous universe. If the inhomogeneities are randomly distributed in a large universe, the majority of any section of a spatial neighborhood may be dominated by their inhomogeneities during collapse. This will lead to these regions becoming singular. However, some sections will have sufficiently small inhomogeneities that do not spoil the bounce, and thus expand again, and undergo inflation. This inflationary period will push any singular regions far beyond the Hubble horizon of the expanding universe after the bounce, possibly beyond the asymptotic de Sitter horizon also.[2] Thus, as observers, we should not be surprised to find ourselves in the region which underwent a bounce any more than we should be surprised to find ourselves in a Universe where the cosmological constant is compatible with our existence.

IV. PHASE SPACE

The Hamiltonian treatment for our model follows directly from the Lagrangian in Eq. (2.4), with canonical

structure inherited from the conformal theory in Eq. (2.1). For clarity of exposition, we will work with the volume $\nu = a^3$ instead of the scale factor throughout this section. The primary reason for this is to make our algebra considerably more transparent, since the canonical conjugate to the ν is the Hubble parameter H, i.e., the Poisson bracket is given by $\{\nu, H\} = \frac{1}{2}$. Our phase space is thus parametrized by $\{\nu, H; \phi, P_\phi\}$, where P_ϕ is the conjugate momentum. Our solutions are further determined by the value of the curvature $\Omega_k \equiv -k/\rho_c$, defined at the bounce (as opposed to today, as usual).

A priori, we have a five-dimensional space to consider. We then wish to find the space Γ consisting of physically indistinguishable solutions. We find that the Hamiltonian $\mathcal{H}$ is (using the symbol $\equiv$ to note the constraint):

$$\mathcal{H} = -3H^2\nu + \frac{P_\phi^2}{2\nu} + \nu V + \Omega_k \nu^{1/3} \equiv 0. \quad (4.1)$$

Here, it is useful to note that the Hamiltonian constraint in Eq. (4.1) and the Friedmann equation (3.1) are equivalent. We therefore have a four-dimensional phase space with a single constraint (the diffeomorphism constraint that usually arises in the $3 + 1$ decomposition of general relativity is automatically satisfied by the homogeneity of our system). Thus, we should expect to see a two-dimensional space of solutions to our systems generated by the Hamiltonian flow. The symplectic structure is given by the symplectic form

$$\omega = 2d\nu \wedge dH + d\phi \wedge dP_\phi. \quad (4.2)$$

From this, we find that our familiar Hamiltonian vector field on phase space now includes a term induced by the presence of curvature:

$$X_\mathcal{H} = 6\nu H \frac{\partial}{\partial \nu} - 6\left(\frac{P_\phi^2}{\nu^2} + \frac{2\Omega_k}{3\nu^{2/3}}\right)\frac{\partial}{\partial H} + \frac{P_\phi}{\nu}\frac{\partial}{\partial \phi} - \nu V' \frac{\partial}{\partial P_\phi}. \quad (4.3)$$

Since we are in a closed Universe ($\Omega_k < 0$), the coefficient of $\frac{\partial}{\partial H}$ is not necessarily positive. In other terms, the Hubble parameter is not monotonically nonincreasing on all solutions. This means that we cannot choose a fixed Hubble slice to parametrize the space of solutions to our model since there will be solutions which cross a given value of the Hubble multiple times (e.g., in both expanding and recollapsing phases). Moreover, we cannot necessarily count solutions at the bounce/recollapse point. This is because Λ-dominated late-time solutions will not recollapse, and small-scale kinetically dominated solutions will not bounce. However, if we restrict ourselves to subspaces of the set of physical solutions, we can often choose our subsets to have well-defined properties. As an example,

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[2]If the universe undergoes eternal inflation during this period, this one patch can seed a large, even potentially infinite, number of patches which later may recollapse with their own inhomogeneities. Many of these patches will again become singular, but with a large number, some of them will by chance be sufficiently homogeneous that they can again bounce and the cycle continues.

consider the space of solutions which undergo a bounce. These necessarily satisfy $H = 0$ and $w < -1/3$ simultaneously. Thus, we can count such solutions by constructing a measure on a subset of the full phase space that intersects these conditions. These subspaces may yet be noncompact; in this particular instance, the intersection of the Hamiltonian constraint with the $H = 0$ surface is noncompact, since any choice $\nu \in \mathbf{R}_+$ can always satisfy the constraint for a fixed Ω_k, simply by scaling P_ϕ accordingly.

The solution to this issue is familiar in the flat case [29] and involves the restriction of the analysis to physical degrees of freedom rather than phase-space variables. Solutions related by rescaling of both P_ϕ and ν such that their ratio remains fixed are dynamically related. Indeed, it is this ratio which constitutes the physical parameter $\dot{\phi}$ (both the equations of motion for H and ϕ are written only in terms of $\dot{\phi}$, not ν nor P_ϕ alone, in the flat case). When introducing curvature, it would appear that this is no longer the case because ν contributes directly to $\dot{H}$ through the curvature. However, we note again that Ω_k is only determined up to an overall scaling in an intrinsic description of physics; a simultaneous rescaling of Ω_k, ν, and P_ϕ can be performed such that there is no physical distinction to a cosmologist working within such a universe.

It may seem strange to the reader that we are allowing a change in the constants of our system under the action of this rescaling symmetry. We note that any observer within this system would only be able to infer Ω_k through observations of the physical degrees of freedom of their universe (such as, for example, $\dot{H}$), which are unaffected by this change. As such, a cosmologist living in this universe could only determine physics up to a choice of fixing this symmetry. In fact, we have taken this into account by fixing $\sum \Omega_i = 1$. Such a change is generated by the action on the extended space of both phase space and the constants of motion by

$$\mathcal{G} = P_\phi \frac{\partial}{\partial P_\phi} + \nu \frac{\partial}{\partial \nu} + \Omega_k^{2/3} \frac{\partial}{\partial \Omega_k}. \quad (4.4)$$

In the above expression, $\mathcal{G}$ is the adaptation of the minimal coupling symmetry explored in Ref. [30] to a curved cosmology. It commutes with the Hamiltonian constraint and preserves the Hamiltonian flow and thus does not affect the dynamics of the intrinsic system. Note that $\mathcal{G}$ is not a symplectomorphism, but rather a "scaled" symplectomorphism, $\mathbf{L}_G \omega \propto \omega$, where $\mathbf{L}$ denotes the Lie derivative. We can employ the generator $\mathcal{G}$ in order to project our space of solutions, essentially gauge fixing Ω_k to any given value. Equivalently, our space of bouncing solutions Γ_b is determined at the point of bounce by the triple $\{\phi_b, \nu_b, \Omega_k\}$, modulo the action of $\mathcal{G}$ which acts as an equivalence relation between solutions

$$\{\phi_b, \nu_b, \Omega_k\} \sim \{\phi_b, \beta\nu_b, \beta^{2/3}\Omega_k\} \quad \forall \ \beta \in \mathbf{R}_+. \quad (4.5)$$

Thus, we can choose a member of each equivalence class by setting $\Omega_k = -1$, for example. To evaluate our space of solutions, however, it is more convenient to work with the invariants of $\mathcal{G}$; the first is the ratio $P_\phi/\nu = \dot{\phi}$, and we choose the second to be $\xi = \Omega_k^{2/3}/\nu$. In these terms, the symplectic structure pulled back to the $H = 0$ surface (denoted by $\bar{\omega}$) becomes

$$\bar{\omega} = \frac{2\xi - 3V}{2\sqrt{\xi - V}} \frac{\Omega_k^{3/2}}{\xi^{5/2}} \, d\phi \wedge d\xi. \quad (4.6)$$

We note that the orientation of $\bar{\omega}$ can flip depending on the sign of $2\xi - 3V$. When positive, this measures solutions which are at the point of recollapse. Conversely, when negative, it measures solutions that bounce. Although Ω_k appears in the expression of $\bar{\omega}$, this is merely an overall choice of scale (due to the fact that $\mathcal{G}$ is a scaled symplectomorphism). It will not contribute to any calculation of the fraction of solutions that have a certain property.

We have thus identified the space of bouncing solutions, Γ_b, to be the subspace defined by the intersection $\mathcal{H}^{-1}(0) \cap (V < F < 3V/2)$. Its symplectic structure is nondegenerate (due to being a 2-form on a two-dimensional space of bouncing solutions Γ_b) and hence can act as a natural measure on the space, derived from the Liouville measure. Any other measure can be obtained by taking $\bar{\omega}$ and multiplying by some function of ξ and ϕ. This space is still noncompact for potentials which feature a plateau; this is unsurprising as it reflects the facts that the plateau of the potential in the infinite limit restores the symmetry of the system and the physical differences between solutions become indistinguishable for large $|\phi_b|$. Ratios of the measure on the space of bouncing solutions can be calculated by introducing a cutoff in ϕ beyond which we identify solutions due to the stochastic effects overwhelming the classical dynamics. There is a pole of order ξ^2 about the second symmetric point where $V = 0$, which can also be regularized either by identifying solutions with $|\phi| < \epsilon$ for some $\epsilon > 0$ or equivalently assuming V has a small nonzero minimum. Evaluating this measure, we find that our system is dominated by solutions that bounce with the particle on the plateau, giving the right conditions for a long period of slow-roll inflation to follow. This result holds qualitatively for all plateau models, since the set of ϕ that satisfy $V(\phi) < B$ for some B becomes the entire real line when B exceeds the height of the plateau. In a manner similar to the choice of F overcoming the existence of the plateau, the choice of function used with the volume form could overturn this result. However, such a function would have to disfavor values of V which allow the particle to reach the plateau, and thus would also require poles.

V. APPROXIMATE ANALYTIC SOLUTIONS

To provide a clear explanation of why we observe bounces in such models, let us consider the cosmological system in two limiting cases of energy in a collapsing universe. In the low energy limit $\rho \ll \Lambda$, the inflaton is low on the potential. The dynamics of the field are those of a harmonic oscillator with a small antifriction term (a greatly underdamped system). As such, w oscillates between the extreme values of ± 1, and so on average (across a cycle), we can model the system with an approximately constant equation of state with barotropic parameter $\bar{w} = 0$. As the energy density increases, this approximation becomes invalid, and w decreases; see the Appendix for details. Thus, our dynamics is akin to that of a closed universe with dust as the dominant matter component, and the Friedmann equation becomes

$$H^2 = \frac{1}{3}\left(\frac{\Omega_d}{a^3} + \frac{\Omega_k}{a^2}\right), \qquad (5.1)$$

where Ω_d is a constant. We reiterate at this point that, since we are dealing with a closed universe, Ω_k is negative. Such models have simple parametric solutions in conformal time $dt = ad\eta$, given by

$$a \propto 1 - \cos\eta, \qquad (5.2)$$

$$t \propto \eta - \sin\eta. \qquad (5.3)$$

Since we are in the regime in which the scalar field oscillations are small (and hence both the harmonic oscillator approximation and the dustlike behavior are valid), these are valid in our system around the points of recollapse ($\eta \approx \pi$). There is a well-known redundancy in our description of these models: rescaling the constants Ω_i by an appropriate power law relation yields equivalent dynamics, since each constant now carries a dimensionful scale. The general form of this symmetry gives rise to the attractor solutions in cosmology by imposing that the Liouville measure is preserved (see Refs. [27–29]).

In this section, we will deviate from the convention of fixing the sum of the Ω_i to unity. Our goal here is to discuss an approximate analytic solution to the equations of motion, and the exact equations of motion contain the evolution of the inflaton, for which the equation of state is dynamic. In essence, we shall show that the dual role played by the inflaton (which sometimes acts as cosmological constant and other times acts as a stiff fluid) can be mimicked by transfer between the different Ω. Because of the distinct scaling of these (as required in order to preserve the dynamics), a transfer between two components will change their sum. Thus, to obey the convention, each Ω would have to change. Instead, we opt to keep Ω_k fixed during evolution, since there is no transfer between it and the matter components. This is because Ω_k arises from the geometry of the spatial slice, which we do not couple to matter.

We are using the harmonic oscillator approximation, which is valid because we are considering the simplest form of the function, given by $F(x) = x^2$. For higher powers, and appropriate rescaling of the width, the potential approaches that of a square well, with potential given by

$$V = \begin{cases} \Lambda & |\phi| > \theta/2 \\ 0 & |\phi| < \theta/2 \end{cases}, \qquad (5.4)$$

where $\pm\theta/2$ are the approximate end points of the plateau. We first consider the behavior when the particle is in the well ($\rho < \Lambda$). In this case, our dynamics is given by

$$H^2 = \frac{1}{3}\left(\frac{\Omega_o}{a^6} + \frac{\Omega_k}{a^2}\right). \qquad (5.5)$$

Again, this equation can be solved exactly in parametric form by considering conformal time wherein we find

$$a^2 = \sqrt{-\frac{\Omega_o}{\Omega_k}}\cos\left(\sqrt{-\frac{8\Omega_k}{3}}(\eta - \eta_r)\right), \qquad (5.6)$$

where η_r is the conformal time at which the universe begins to recollapse, i.e., the scale factor reaches its maximal value. We can then recover t through an elliptic integral.

When we consider the case in which $\rho > \Lambda$, the particle will not oscillate but will behave as if it is on a flat potential. This can be further split into the constant term (the height of the potential), which behaves like a cosmological constant, and the kinetic term, which evolves like a massless scalar field. Thus, the Friedmann equation is well approximated by

$$H^2 = \frac{1}{3}\left(\Lambda + \frac{\Omega_s}{a^6} + \frac{\Omega_k}{a^2}\right). \qquad (5.7)$$

If $\Omega_s = 0$ in such models, we recover the closed de Sitter solution given by

$$a = \sqrt{-\frac{\Omega_k}{\Lambda}}\cosh\left[\sqrt{\frac{\Lambda}{3}}(t - t_o)\right]. \qquad (5.8)$$

In the general case in which the kinetic component scalar field is nonzero, we find that a bounce occurs if Ω_s is less than a critical value Ω_c. To determine Ω_c, consider that the condition for a bounce in Eq. (5.7) becomes a cubic equation in $b = a^2$,

$$b^3 + \frac{\Omega_k}{\Lambda}b^2 + \frac{\Omega_s}{\Lambda} = 0, \qquad (5.9)$$

and the critical condition is that this occurs at a local minimum of H over a. To see why this is true, consider the

roots of our cubic equation for some value of Ω_s. Since both Λ and Ω_s are positive, one root is negative and thus not physical. Thus, we consider either of the two positive roots. If this is not a local minimum of the cubic function, we can increase Ω_s, which increases H, and continue to do this until the cubic no longer has a root. This happens when the root is at the minimum of the cubic (and hence a repeated root):

$$3b^2 + 2\frac{\Omega_c}{\Lambda}b = 0. \tag{5.10}$$

We solve this for b and reinsert into the cubic equation to find the critical value Ω_c, which is

$$\Omega_c = -\frac{4\Omega_k^3}{27\Lambda^2}. \tag{5.11}$$

These solutions, well away from the bounce point, asymptote to the de Sitter solutions above. However, as the solution evolves from the bounce point, Hubble friction will eventually slow the scalar field to a halt.[3] Subsequently, it will turn around[4] and, after a period of slow-roll inflation, fall down the potential removing the effective cosmological constant term, allowing the expanding universe to recollapse. We note that if a period of constant slow roll or ultraslow roll occurs due to the flatness of the potential it is going to be necessarily transient [31–33].

To establish when general solutions will bounce, we consider the motion of the inflaton field itself. Consider the point at which the inflaton hits the wall. If the kinetic energy ($\frac{\dot\phi^2}{2} = \frac{\Omega_o}{a^6}$) is less than the height of the potential Λ, the field will reflect and continue, and we can repeat our analysis at the next interaction. If the kinetic energy is greater than the height of the wall, it will be reduced by Λ as it traverses the wall, and hence $\Omega_s = \Omega_o - \Lambda a^6$. To obtain an analytic solution for such a scenario, we will neglect the contributions of the kinetic terms of the scalar field when on the plateau and the potential terms when in the valley. We also note that conservation of the scalar field energy density during the transition ensures the continuity of the Hubble parameter.

Between the reflections, we can find the behavior of the inflaton. Using $\dot\phi^2/2 = \Omega_o/a^6$, we can write the equation of motion as follows in terms of the scale factor:

$$\frac{d\phi}{da} = \frac{\dot\phi}{\dot a} = \frac{\dot\phi}{Ha} = \frac{1}{\Omega}\sqrt{\frac{6\Omega_o}{\Omega_o + a^4\Omega_k}}. \tag{5.12}$$

---

[3]For the moment, we ignore the effect of quantum fluctuations. We discuss this later on.
[4]A slight tilt in the plateau is assumed.

This can be exactly integrated and inverted to give the scale factor a in terms of ϕ:

$$a^2 = \sqrt{-\frac{\Omega_o}{\Omega_k}}\mathrm{sech}\left[\sqrt{\frac{2}{3}}(\phi - \phi_o)\right]. \tag{5.13}$$

From this, we can express the kinetic energy in terms of ϕ as

$$\frac{1}{2}\dot\phi^2 = \sqrt{-\frac{\Omega_k^3}{\Omega_o}}\cosh^3\left[\sqrt{\frac{2}{3}}(\phi - \phi_o)\right]. \tag{5.14}$$

We can therefore consider the following scenario: if the well is sufficiently narrow, the kinetic energy between reflections cannot increase by Ω_c or more, and hence the universe must bounce. The limiting case for a reflection happens when the kinetic energy is equal to the height of the potential. Therefore, we consider a reflection at $\phi = -\theta/2$ with kinetic energy Λ. For this to occur, we require that $\Lambda > -\Omega_k^{3/2}/\sqrt{\Omega_o}$ (otherwise, the kinetic energy is always sufficient to escape the well, and our cosmology is always described by Eq. (5.7) and asymptotes to the $k > 0$ de Sitter expansion). This fixes

$$\phi_o = -\frac{\theta}{2} - \sqrt{\frac{3}{2}}\mathrm{arccosh}\left(\frac{\Omega_o^{1/6}\Lambda^{1/3}}{\sqrt{-\Omega_k}}\right). \tag{5.15}$$

Note that the term inside the inverse hyperbolic cosine function is composed of a dimensionless combination of our constants. The particle next encounters the wall of the potential at $\phi = \theta/2$. If by this stage the kinetic energy has increased by less than Ω_c, the remaining kinetic energy after the particle leaves the potential well will be sufficiently small enough that the universe will bounce. The amount of kinetic energy gained is a function of the width θ of the potential. Therefore, we can choose θ to ensure that the amount of kinetic energy gained is always less than this. This is achieved by

$$\theta = \sqrt{\frac{3}{2}}\left[\mathrm{arccosh}\left(\sqrt{-\frac{\Omega_o}{\Omega_k^3}}\Lambda + 4\frac{\sqrt{-\Omega_k^3\Omega_o}}{27\Lambda^2}\right)^{1/3}\right. $$
$$\left. - \mathrm{arccosh}\left(\sqrt{-\frac{\Omega_o}{\Omega_k^3}}\Lambda\right)^{1/3}\right]. \tag{5.16}$$

Thus, we can, in principle, ensure that a given branch of the homogeneous, isotropic, and closed cosmology will always bounce, provided we choose the correct potential. The exact solution given here is derived in the case of the square well, which is the most analytically tractable situation. However, we see from Eq. (5.11) that once the potential is well approximated by a plateau a bounce will occur as long

as the kinetic energy of the inflaton is sufficiently small. Therefore, for any of the generic *T*-model potentials, there will always exist in phase space a set of conditions leading to a bouncing universe.[5]

Beyond the bounce, our analytic approximation will break down. As the Hubble parameter is now positive, the system will encounter friction, and the scalar field will slow. Since the true potential is never actually flat, the combination of this friction and the gradient of the potential will be sufficient to halt the inflaton and reverse its motion, beginning a long period of slow roll back down the potential. To make this apparent, consider the behavior of the scalar field on the plateau: for low kinetic energies, the dominant contribution to the Hubble parameter will be the potential, and hence we find that the velocity of the field follows

$$\dot{\phi} \propto \exp(-3\Lambda t). \tag{5.17}$$

As a result, the field will come to a halt at a finite value of ϕ after an infinite interval in time. However, since the true potential has a gradient, this behavior cannot happen. In fact, the true dynamics will diverge from that of the square well when the Hubble friction term becomes of the same order as the gradient of the potential. There is always a point at which the field turns around, and the square well cannot capture this behavior.

One way in which we can alleviate this problem is to compactify the scalar field on an S^1 or set a reflection at some ϕ_r to represent the point at which the scalar field would turn around. We will focus our discussion on the latter case; this can recreate a phenomenologically interesting double-bouncing solution. This behavior is quantitatively distinct from that which we would expect from a scalar field on a *T*-model potential. However, it encompasses the relevant qualitative features, namely that the field encounters the valley again after the bounce. To facilitate the analytic calculation of perturbations on this background, we express the evolution of the scale factor in terms of conformal time. We can explicitly construct a double-bouncing cosmological model using Eq. (5.7) during the bounces and Eq. (5.5) for recollapse. Such solutions are continuous in a and H, and this condition determines the values η_1 and η_2 of conformal time at which we connect the descriptions; *ab initio*, these are free parameters of our double-bouncing model but will be constrained by the cosmological parameters. We find that the evolution of the scalar factor is
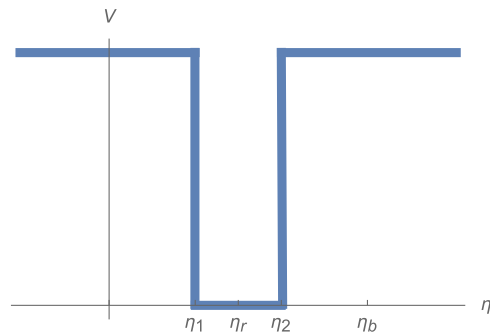

FIG. 1. The double-bouncing cosmological model under discussion in this section. The evolution of the potential is shown with respect to conformal time. The first bounce occurs at $\eta = 0$, and the recollapse occurs at $\eta = \eta_r$.

$$a = \begin{cases} \sqrt{-\frac{\Omega_k}{\Lambda}} \sec(\omega\eta) & \eta < \eta_1, \\[2mm] \left(\sqrt{-\frac{\Omega_s}{\Omega_k}} \cos[2\omega(\eta - \eta_r)]\right)^{1/2} & \eta_1 < \eta < \eta_2, \\[2mm] \sqrt{-\frac{\Omega_k}{\Lambda}} \sec[\omega(\eta - \eta_b)] & \eta > \eta_2, \end{cases} \tag{5.18}$$

where $\omega = \sqrt{-\frac{\Omega_k}{3}}$.

Without loss of generality, we choose the first bounce to occur at $\eta = 0$, recollapse to occur at $\eta = \eta_r$, and second bounce to occur at $\eta = \eta_b$. The transitions between the distinct descriptions occur at $\eta = \eta_1, \eta_2$. From this result, we immediately identify the value of the scale factor at the bounce as $a_b = \sqrt{-\frac{\Omega_k}{\Lambda}}$ and at recollapse as $a_r = (-\frac{\Omega_s}{\Omega_k})^{1/4}$. Conservation of energy at transitions between the plateau and valley sets the scale factor here as $\Omega_s = \Lambda a_t^6$. Thus, we find that these values of the scale factor at these times are related to one another. Indeed, we can express the scale factor at recollapse in terms of that at the bounce and that of the transitions:

$$a_r = \left(-\frac{\Omega_s}{\Omega_k}\right)^{1/4} = \left(-\frac{\Lambda a_t^6}{\Omega_k}\right)^{1/4} = \sqrt{\frac{a_t^3}{a_b}}. \tag{5.19}$$

Thus, if we parametrize our space of double-bouncing cosmologies by the ratio of scale factor at recollapse to that at the bounce, $\gamma = a_r/a_b$, then our system is completely specified. Our approximations hold most closely in the symmetric solution in which the universe recollapses with the inflaton at the center of the square well (where all the energy is kinetic, and thus $w = 1$). As a result, all energy lost to Hubble friction in the expanding phase is regained during the collapsing phase, and the inflaton has sufficient energy to reach the plateau when it encounters the wall. Thus, we can fix the parameters of (5.18): $\eta_1 = \frac{1}{\sqrt{k}} \arccos(\gamma^{-\frac{2}{3}})$, $\eta_r = 3\eta_1/2$, $\eta_2 = 2\eta_1$, and $\eta_b = 3\eta_1$. The behavior of this system is shown in Fig. 1.

We end this section by noting that the conditions which relate the width of the potential to the parameters Ω_k, Ω_o,
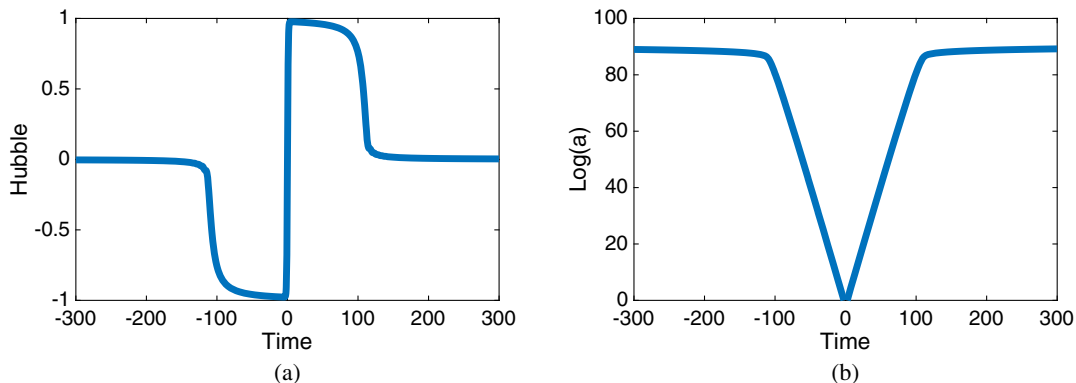
FIG. 2. Hubble (a) and log of scale factor (b) against time for a solution with a bounce at $t = 0$. We see a typical evolution with singularities in both the future and past with a bounce between them. This displays the behavior of a simple bounce with a large number of e-folds of inflation/deflation on either side. This is a toy model with $\alpha = 1$, $\mu = 1$, $a = 0.224$, $\Omega_k = -0.05$, and $w = -0.9$ at the bounce.

and Λ guarantee that the cosmology will undergo a bounce. However, Ω_o is dynamically generated; unlike Ω_k and Λ, it changes when the particle encounters the well. It will not necessarily take the same value each time the particle falls into (or leaves) the well, since it depends on the amount of Hubble friction (or antifriction in contracting cases) that the scalar field encounters during its evolution. In the particular example of the square well, we find that $\Omega_o = a^6(\frac{1}{2}\dot{\phi}^2 + \Lambda)$ at the point at which the particle reaches the well if we allow the inflaton to reach it after a bounce has occurred. Therefore, a fixed potential will not guarantee that the resulting cosmology is never singular, although it may guarantee a bounce for a given combination of Ω_k, Λ, and Ω_s.

VI. NUMERICAL EVOLUTIONS

In Fig. 2, we show a typical bouncing solution chosen with $w = -0.9$ at the bounce point. Such solutions can undergo a long period of inflation while the inflaton is on the plateau of the potential and produce a spectral tilt and scalar-to-tensor ratio which agree with observations. In such solutions, the scalar field is high on the potential at the bounce, with the inflationary/deflationary periods happening with the scalar field on the same plateau. There is an asymmetry between the expanding and contracting phases due to the velocity of the scalar field at the bounce; as the inflaton approaches the plateau, its kinetic energy drops, pushing w toward -1. However, if there is sufficient energy to reach the plateau, the antifriction will again reaccelerate the inflaton.

In Fig. 3, we see a more complex evolution in which the universe expands, recollapses, and then undergoes multiple bounce-recollapse cycles before entering into a de Sitter expansion phase. We can also see the evolution of the scale factor on long timescales in which the inflationary expansion is evident. The de Sitter phase in this evolution
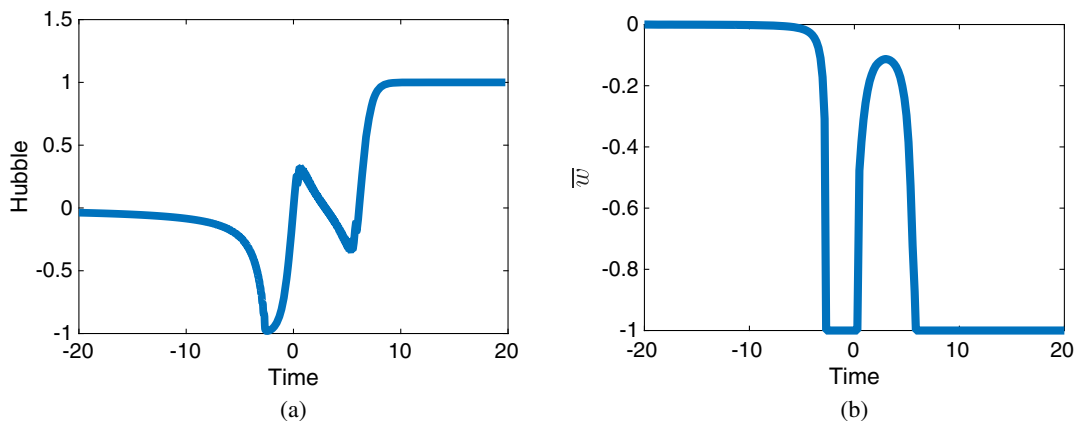

FIG. 3. On the left, the evolution of a large universe that undergoes a double-bounce. The parameters of the model were chosen such that at the bounce point $w \approx -1$, $\Omega_k = -1$, $a = 1$ with a much narrower potential ($\alpha \approx 10^{-5}$). On the right, we show the cycle-averaged behavior of the equation-of-state parameter, $\bar{w}$, with averaging done across short cycles during fast oscillations, and over short time periods outside of this. In particular, we note that between bounces there is a period of slow roll from the particle perspective, with $w \approx -1$; however, due to the effects of curvature, the Hubble parameter evolves rather quickly.

continues for a large number of *e*-folds, beyond the range of our numerical calculation. We also see a close-up of the behavior of the Hubble parameter about the primary bounce. Here, we see that the system actually undergoes a very short recollapsing phase before bouncing for a second time. Such behavior is typical for systems in which the potential is shallow, since the inflaton can quickly exit the inflationary regime, allowing the curvature to overcome the scalar field and causing a recollapse. During this phase, the negative Hubble parameter acts as antifriction, which accelerates the scalar field, allowing it to climb higher up the opposite side of the potential. As long as this effect does not endow the scalar field with more kinetic energy than the height of the potential, the field approaches the plateau in the right phase for a second bounce to occur, followed by a period of inflation. After the bounce, the Hubble parameter is positive and thus acts as friction on the system, which allows for a long period of slow-roll inflation.

A double-bounce typically occurs as follows: the universe first bounces as the inflaton moving down the potential on the shallow slope of the plateau. After the bounce, there is a short period in which the Hubble parameter grows yet the frictional effect is low. If the phase of the system is such that the inflaton then enters the steeper region of the potential, it can gain sufficient momentum to cross the minimum close to the point of recollapse. The antifriction drives it up the opposite side of the potential, and the universe bounces for a second time. Usually, this second bounce will occur with the inflaton rolling up the potential, and hence it will be followed by a long period of slow-roll expansion.

The double-bouncing scenario (and multiple bounces in general) are of particular importance when considering the amplification of perturbative modes during evolution. This is a phenomenologically interesting feature which runs parallel to that discussed in Ref. [34] in which it was shown that a closed inflationary universe can suppress low multipoles. As we shall see in the following section, we expect to see the longest wavelength modes amplified the most through a bounce. However, the typical behavior of a double-bouncing solution is that one bounce is particularly short lived when compared to the frequency of oscillations of the longest wavelengths. Therefore, the phase of recollapse and the second bounce will be effectively smoothed over the longest wavelength modes. However, for a faster oscillating mode, the relative phase of the mode will be important; the significant friction encountered during recollapse while in a kinetic phase will lead to suppression of the amplitude, whereas friction during a potential dominated phase will not.

Before concluding this section, we examine the possibility that as the field rises onto the plateau but before turnaround its variation becomes dominated by its quantum fluctuations. If this were to happen, the field would become oblivious to the scalar potential, and the inflaton condensate would spread stochastically, leading to eternal inflation. This is because there would be regions where the quantum "kicks" would conspire to keep the field on the plateau forever. If this were the case, a double-bounce would not be possible.

Quantum fluctuations can dominate the motion when the classical velocity of the inflaton in field space $|\dot{\phi}|$ is smaller that the typical quantum fluctuation $\delta\phi = H/2\pi$ per Hubble time $\delta t = 1/H$. Thus, the criterion in order for the quantum fluctuations to be subdominant is

$$|\dot{\phi}| > \frac{\delta\phi}{\delta t} = \frac{H^2}{2\pi} \Rightarrow |V'| > \frac{3}{2\pi}H^3, \qquad (6.1)$$

where we used the slow-roll equation $3H\dot{\phi} = -V'$, with the prime denoting a derivative with respect to the field. Thus, if the slope of the potential is too small, quantum fluctuations take over, and eternal inflation occurs. One might think that the above criterion is certainly violated in our case, because at the turnaround point the inflaton stops momentarily and $\dot{\phi} = 0$. However, it was shown in Ref. [35] that, if the slope is substantial, the duration of the period when $|\dot{\phi}| < \delta\phi/\delta t$ is less than a Hubble time, which means turnaround occurs before a single quantum kick. In this case, eternal inflation is avoided. The condition for this to happen is none other than Eq. (6.1), provided slow roll is assumed before turnaround.

How far the inflaton can advance along the potential plateau depends on its initial kinetic density and on the particular scalar potential envisaged. Assuming slow roll determines the kinetic density in terms of the potential, because of the inflationary attractor. To investigate if the danger of eternal inflation can be avoided in a double-bounce scenario, we consider the *T*-model potential

$$V(\phi) = \Lambda \tanh^2(\phi/\sqrt{6\alpha}). \qquad (6.2)$$

Using that $H = \sqrt{V/3}$, it is straightforward to find that Eq. (6.1) is equivalent to

$$\sinh^2(\phi/\sqrt{6\alpha}) < \frac{4\pi}{\sqrt{2\alpha\Lambda}}. \qquad (6.3)$$

The above condition can reduce to

$$\phi/\sqrt{6\alpha} < \begin{cases} \frac{1}{2}\ln(\frac{16\pi}{\sqrt{2\alpha\Lambda}}) & \text{when } \phi > \sqrt{6\alpha}, \\[2mm] \frac{2\sqrt{\pi}}{(2\alpha\Lambda)^{1/4}} & \text{when } \phi < \sqrt{6\alpha}. \end{cases} \qquad (6.4)$$

The inflationary energy scale is roughly $\Lambda^{1/4} \sim 10^{-2}$. This means that the above criterion for $\phi < \sqrt{6\alpha}$ is always satisfied, when $\alpha < 100$ or so. This is expected, for such inflaton values are not on the plateau. In the opposite case, when $\phi > \sqrt{6\alpha}$ (on the plateau now), the criterion in Eq. (6.4) becomes $\phi < \sqrt{6\alpha}(6.4 - \frac{1}{4}\ln\alpha)$. Choosing
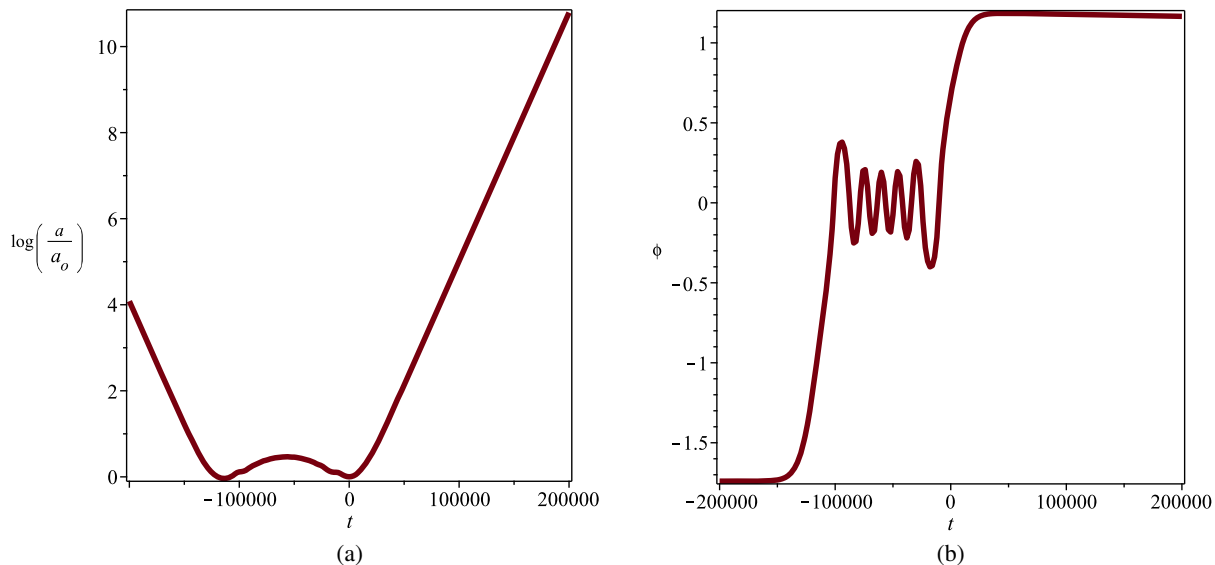
(a)

(b)

FIG. 4. Log of scale factor (a) and scalar field (b) against time for a solution for the double-bouncing scenario described above. A large universe reduces to a minimal size before undergoing a pair of bounces separated by a short-lived expanding then recollapsing phase. We see that the scalar field avoids the eternal inflation region during the expanding phase that follows the second bounce. This solution continues to inflate for around 90 e-folds.

$\alpha = 0.01$, we find the bound $\phi < 1.8$. Thus, turnaround has to occur at a smaller value of ϕ if eternal inflation is to be avoided.

Through numerical investigation, using the parameters $\Lambda = 10^{-8}$, $\alpha = 0.01$, and $\Omega_k = -0.005$, we find initial conditions at one bounce that has a double bounce with a large universe at either end. The maximum value of ϕ is 1.2 and so is below the eternal inflation value. The system collapses, bounces, recollapses, and then has about 90 e-folds of inflation. So, it is indeed possible to get the double bounce without hitting the eternal inflation problem. The scalar field begins on the plateau of the potential during the collapsing phase and reaches a turning point at the first bounce. This is followed by an oscillating phase during the short-lived expansion and recollapse, and then returning to the plateau for the inflationary phase following the second bounce. This evolution is shown in Fig. 4.

Another interesting possibility to consider is that our Universe may begin in the far past at a singularity. From this, the Universe would expand, then recollapse to a bounce before undergoing inflation. This was recently discussed in Ref. [17], and we direct the reader to the analysis therein for a discussion of the motivation and interesting features of such a model. Such an evolution is indeed possible within the model which we consider, and an example numerical evolution is shown in Fig. 5. In such an evolution, the inflaton may be found on the plateau at early times, and its kinetic energy tends to infinity as $t \to -\infty$. However, during the initial expansion, the inflaton falls down the potential and oscillates about the minimum. During this oscillatory phase, the effective averaged equation of state has $\bar{w} = 0$, and thus the universe becomes curvature dominated and

recollapses. Following this recollapse from a finite maximal scale factor, the inflaton can find itself in the right circumstances (low kinetic energy, high potential) for the universe to bounce, which can be followed by a prolonged period of inflation. Thus (as discussed in Ref. [17]), a universe can "bounce from nothing."

VII. PERTURBATIONS

The background model we have thus far described is qualitatively symmetric; for each solution to the equations of motion, there is a counterpart with corresponding dynamics under reversal of time about the bounce point. However, when we consider the evolution of the inhomogeneous modes that evolve on this background, we should not expect to find the same symmetry. This has been explored well in the case of matter bounce scenarios, an alternative theory to inflation which posits the exit of superhorizon modes during a contracting phase. In particular, nonsymmetric phenomena include the generation of black holes, which will persist across any bounces, and the entropic amplification of superhorizon scales as the universe collapses.

Let us consider the space of scalar perturbations on this background geometry. Recall that we are dealing with a closed FLRW universe, and hence our spatial manifold is S^3. We must therefore consider perturbations that consist of hyperspherical harmonics. Following Refs. [36,37], we write the eigenfunctions of the Laplace operator on S^3 as

$$\nabla^2 Q_{lmn} = \frac{n^2 - 1}{r_0^2} Q_{lmn}, \qquad (7.1)$$

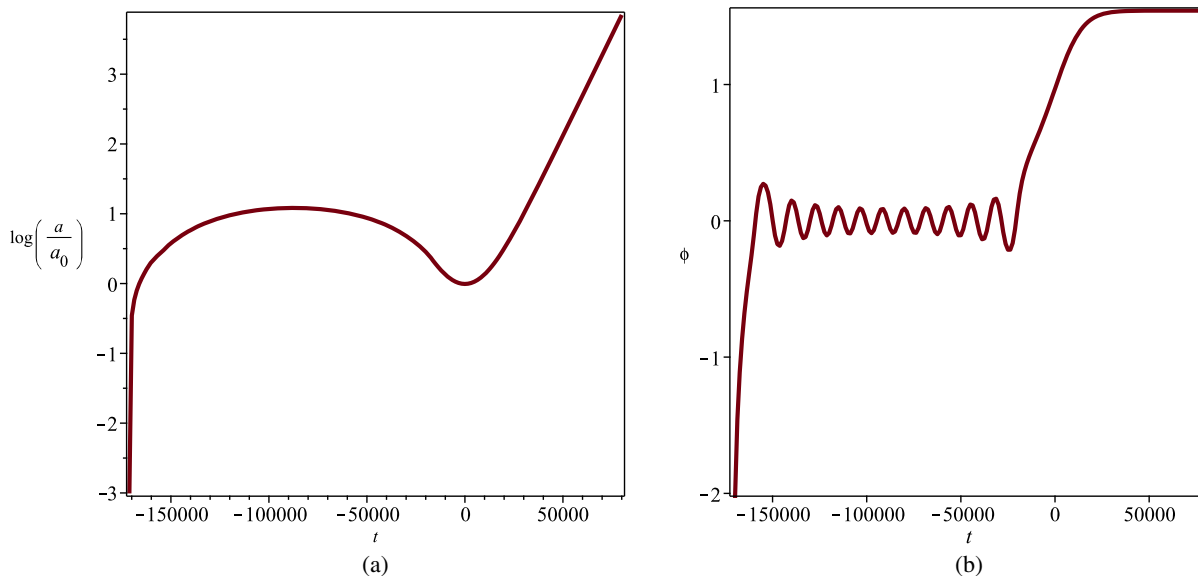where r_0 is the comoving radius of curvature.

FIG. 5. Log of scale factor (a) and scalar field (b) against time for an evolution in which the universe begins at a singularity in the past. The universe in this case expands from an initial singularity to a local maximum size, then collapses back to a bounce. Following this bounce, the universe then undergoes an extended period of slow-roll inflation (not shown in the figure due to the focus on singularity and bounce).

These constitute the S^3 equivalents of Fourier modes. As we take the limit in which the volume of the 3-sphere becomes large compared to the wavelength of mode, these modes are well approximated by the usual Euclidean Fourier modes with wave number $k^2 = (n^2 - 1)/r_0^2$. However, we are particularly interested in the effects of a bounce in which the physical volume of the 3-sphere is at its smallest. Furthermore, at the bounce point, the Hubble parameter vanishes exactly. Therefore, the usual frictional term will be absent. This is true regardless of the mode under consideration. An extensive study of the behavior of perturbations in bouncing cosmologies was performed in Refs. [38,39]. Although the analysis was performed in the context of loop quantum cosmology, the results are fairly generic. We will deviate from this type of analysis in two ways: we consider closed cosmologies, and, furthermore, we do not set the initial conditions for the semiclassical perturbations at the bounce point. In a quantum context, setting the initial conditions at the bounce point may be a logical choice, since that it is the point of highest energy density, and therefore the point at which quantum effects should be strongest both in terms of matter and geometry. However, we will adopt a different approach here which is better suited to addressing issues of the classical bounce. Since significant portions of phase space do not bounce even in the unperturbed theory, we consider the effects of perturbations to a

bouncing background that begin on a collapsing branch rather than setting conditions at the bounce itself. To do otherwise would be to make an undue assumption, namely that the system always bounces.

Let us first consider scalar curvature perturbations on our background space-time. We take Eq. (2.17) from Ref. [36] as the starting point for our analysis, in which scalar perturbations are expressed in terms of the variable $q = (\dot{\phi}/H)\zeta$, with ζ denoting the usual comoving curvature perturbation. Although we will follow this analysis closely, we must deviate in our choice of variables, as q is ill defined at the bounce. We therefore use $x = \dot{\phi}\zeta = Hq$ as our basic variable, which remains well defined throughout the evolution of the perturbation. Away from the bounce point, q is easily recovered, and we can continue the analysis in the standard inflationary case. Thus, we find our equation of motion for x is given by

$$a_x \ddot{x} + b_x \dot{x} + c_x x = 0. \tag{7.2}$$

To simplify our analysis, we will write the coefficients a_x, b_x, and c_x in terms of the Hubble parameter to make it clear which terms become important near the bounce and which terms dominate during slow-roll expansion:

$$a_x = 2H^2(n^2 - 4) + 3\dot{\phi}^2,$$

$$b_x = 10H^3(n^2 - 4) + H[(4n^2 + 11)\dot{\phi}^2 - 4(n^2 - 4)V + 6V'\dot{\phi}],$$

$$c_x = -8H^4(n^2 - 7)(n^2 - 4) - 2(\dot{\phi}^2 + 2V)[(n^2 - 1)\dot{\phi}^2 - 4(n^2 - 4)V] + 24H(n^2 + 2)V'\dot{\phi}$$

$$+ 24V'^2 + 12V''\dot{\phi}^2 + 4H^2[(n^4 + 8n^2 + 9)\dot{\phi}^2 + 2(n^2 - 4)(n^2 - 9)V + V'']. \tag{7.3}$$

Although these coefficients are complicated, it is immediately apparent that our system has a well-defined limit when H vanishes. We are therefore able to numerically solve the equations of motion for x through a bounce.

The evolution of tensor perturbations is simpler [37]. The dynamical evolution of the amplitude of the transverse, traceless tensor perturbation h of wave number n is given by

$$\ddot{h} + 3H\dot{h} + \left(V + \frac{1}{2}\dot{\phi}^2 - H^2\right)(n^2 - 1)h = 0. \quad (7.4)$$

In both cases, we expect to see only negligible deviations from the usual power spectra generated during the inflationary phase postbounce, and as such, we expect that, even in the presence of a bounce, T-models maintain their attractor nature. We will quantify this statement more carefully in the following section, allowing us to make contact with observations for this class of models. For now, we will primarily concern ourselves with the behavior of perturbations before the bounce occurs and the backreaction they can have upon our space-time.

To illustrate the effects of the bounce upon perturbation modes, let us first consider the case in which the bounce occurs when the inflaton can be found on the plateau of the potential. In this case, we neglect the effect of the kinetic energy upon the evolution of the field, and we arrive at the closed de Sitter model of which the dynamics are described by

$$a = a_b \cosh(\lambda t),$$
$$H = \lambda \tanh(\lambda t), \quad (7.5)$$

where $\lambda^2 = V/3$ as set by the value of the potential on the plateau. To simplify the discussion and showcase the qualitative evolution of the different modes, we have set $\Lambda = 1 = -\Omega_k$. With these assumptions, the evolution of both the tensor and scalar modes becomes identical. The solution for the perturbation variable x can be found analytically as follows, where $P^{\frac{3}{2}}_{n-1/2}$ is the Legendre polynomial of the first kind and $Q^{\frac{3}{2}}_{n-1/2}$ is the associated Legendre function of the second kind:

$$h = (-\text{sech}^2\lambda t)^{3/4}[c_1 P^{\frac{3}{2}}_{n-\frac{1}{2}}(\tanh \lambda t) + c_2 Q^{\frac{3}{2}}_{n-\frac{1}{2}}(\tanh \lambda t)]. \quad (7.6)$$

This solution can be split into two parts, which can broadly be thought of as the components of which the amplitude stays fixed and that which changes as the scale factor evolves. As the universe collapses, the amplitude of the evolving mode increases, and conversely, when the universe expands, its amplitude reduces. In a perfectly symmetric bounce such as the one described above, there

is no mixing between these modes, and thus the amplitudes of oscillations asymptote toward symmetry. However, when we introduce an asymmetric bounce, the modes mix; a part of the increasing amplitude of the growing mode is transferred into the constant mode (and vice versa). Therefore, across an asymmetric bounce, we can expect to see that the fluctuations are amplified (see Fig. 6). Usually, modes outside of the Hubble radius $|H|^{-1}$ freeze out. However, in a contracting universe, Eq. (7.6) informs us that the amplitude of such modes increases instead. Once the mode is shorter than the Hubble radius, the oscillations dominate over this expansion, and so the amplification slows significantly. Thus, we expect that all modes are amplified, and the modes with lowest wave number (i.e., the longest wavelength) are amplified the most [40,41].

VIII. CONTACT WITH OBSERVATIONS

We now turn to examine how the phenomenology of a double-bounce scenario fares in the context of inflation. Recall that if the background dynamics allow for a double bounce we expect that, soon after the second bounce, the Hubble parameter will reach an approximately constant value, and the universe will be described well by a de Sitter approximation. As noted in the previous section, perturbations will be enhanced as the universe evolves through the bounce. However, as we will show in this section, we expect that the power spectrum for smaller modes (which inflate for a larger number of e-folds) will not will not deviate from the usual results postbounce. As a result, we will demonstrate that it is possible to realize a double-bounce model with predictions in good agreement with current observations. For the rest of this section, we will restore units of $M_P^{-1} = \sqrt{8\pi G}$.

We first note that postbounce, de Sitter space is achieved very quickly as long as the field starts reasonably high on the plateau, as we can see in Fig. 3. Since the number of e-folds is defined as $N = -Hdt$ and $H = 0$ at the bounce, we expect the number of e-folds until (quasi-)de Sitter is achieved to be almost negligible. The deviation from de Sitter is encoded in ϵ, with $\epsilon = 0$ corresponding to exact de Sitter. Using the solutions through the bounce (7.5), we find that $|\epsilon|$ becomes smaller than $|\epsilon|_{\text{dS}}$ at a number of e-folds given by

$$N_{\text{dS}} = \ln \cosh \text{csch}^{-1}|\epsilon|_{\text{dS}}. \quad (8.1)$$

We can set $|\epsilon|_{\text{dS}} = 1$ as a hard limit for quasi-de Sitter. This returns $N_{\text{dS}} = 0.367$, whereas $|\epsilon|_{\text{dS}} = 0.1$ returns $N_{\text{dS}} = 2.31$. Therefore, generically after the second bounce, the universe will rapidly approach the flat de Sitter evolution within $O(1)$ e-folds; only scales that leave the horizon before that are going to see anything other than a quasi-de Sitter universe.

For large scales, we expect two distinct behaviors: first, we expect finite-size effects to become important as the
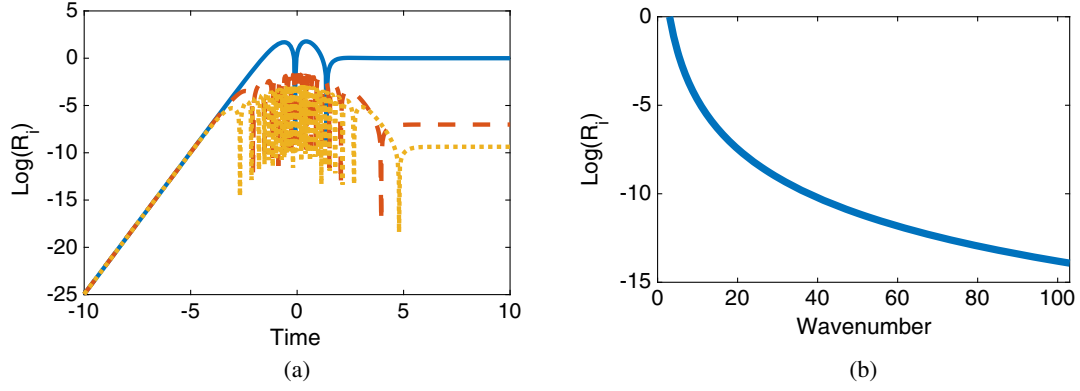
FIG. 6. On the left, the logs of the ratios of amplitudes against time, normalized such that the lowest mode $n = 1$ has unit amplitude upon exiting the horizon. Plotted are the behaviors of the $n = 3$, 18 and 33 (solid blue, dashed red, and dotted yellow, respectively). On the right, the logs of ratios of the amplitude increases of the wave modes of tensor and scalar perturbations across a bounce to that of the lowest mode ($n = 3$). We see that the largest increases are found at the low end of the spectrum, with the distribution becoming smaller and flatter as the wave number increases. We should expect to see a red tilt to the spectrum of perturbations, but the amplitude of this will strongly depend on the total number of e-folds following the bounce; large expansions will suppress this effect significantly.

scales approach the radius of the universe, and, second, we expect the effects of the bounce to cause a deviation from the usual near scale invariance, since large scales leave the horizon before de Sitter is achieved. Finite-size effects are known to become important at large scales, affecting the power spectrum. However, the presence of the bounce further modifies the usual treatment, regardless of the spatial curvature of the universe. As noted above, the gauge-independent perturbation $q = \delta\phi + (\dot{\phi}/H)\zeta$ is ill defined in a bounce. However, we may instead work in the uniform curvature gauge, where $\zeta = 0$. In this case, we consider the variable $v = a\delta\phi$, which remains well defined through the bounce and follows the equation of motion [42]

$$v_n'' + \left(\frac{n^2 - 1}{r_0^2} - \frac{a''}{a}\right)u_n = 0, \qquad (8.2)$$

where the prime indicates differentiation with respect to the conformal time $\eta = \int dt/a$. This is nothing more than collection of a simple harmonic oscillators with time-dependent masses, which means that the definition of a vacuum is time dependent as well. In the de Sitter case, $a''/a \approx 6/\eta^2$ close to the singularity, which means that a Euclidean vacuum (the well-known Bunch-Davies initial condition) can be imposed. However, close to the bounce where $a \propto \cosh \lambda t$ and $t = 2\lambda^{-1} \tanh^{-1}[\tan(\lambda\eta/2)]$, we find that in the early limit

$$v_n'' + \left(\frac{n^2 - 1}{r_0^2} - \frac{1}{\lambda^{-2} + \frac{\eta^2}{6}}\right)u_n = 0. \qquad (8.3)$$

Therefore, we can apply the Bunch-Davies vacuum condition at the bounce if all modes become nearly massless at the bounce, which is the case for small λ (we show this below). The physical meaning of this condition can be thought of as

follows: the bounce "smooths out" the perturbations as the horizon becomes infinitely large, since there is no typical length scale. This is akin to assuming a (nearly) maximally symmetric space for observationally relevant scales.

In the domain where $\eta\lambda \gg 1$ (which occurs soon after the bounce but before smaller modes leave the horizon), the solution to this equation is the usual superposition of the Bessel functions

$$v_n = c_1\sqrt{n\eta}J_{\frac{\sqrt{3}}{2}}\left[\frac{(n^2 - 1)\eta}{r_0}\right] + c_2\sqrt{n\eta}Y_{\frac{\sqrt{3}}{2}}\left[\frac{(n^2 - 1)\eta}{r_0}\right]. \quad (8.4)$$

We impose the usual normalization condition $u'u^* - u^{*'}u = i$ and the requirement that at some early time just after the bounce the oscillations were free $v \propto e^{\pm ik\eta}$. Then, identifying $\delta\phi$ as the comoving curvature perturbation $\mathcal{R}$ in the zero-curvature gauge, and since $\langle\delta\phi|\delta\phi\rangle \sim H^2$ along with $\mathcal{R} = H\delta\phi/\dot{\phi}$, we arrive at the well-known expression for the dimensionless spectrum of scalar perturbations,

$$\mathcal{P}_\zeta = \frac{1}{8\pi^2 M_P^2}\frac{H^4}{\dot{\phi}^2} \approx \frac{1}{8\pi^2 M_P^2}\frac{H^2}{\epsilon}, \qquad (8.5)$$

where the normalization comes from the usual cosmic microwave background (CMB) transfer functions.

As noted above, we expect a deviation from the nearly flat spectrum for scales that exit the horizon roughly early enough after the bounce, before quasi-de Sitter is reached but after the initial conditions can be assumed to be Euclidean. First, we note that the Cosmic Background Explorer (COBE) normalization may be used to estimate the value of $\lambda M_P = \sqrt{V_0/3}$ of the potential. Using

$$\frac{H^2}{M_P^2\epsilon} = 1.59 \times 10^{-6}, \qquad (8.6)$$

and $\epsilon \sim 10^{-2}$ at the pivot scale, we may estimate $\lambda = 4.21 \times 10^{-5} M_P$. Note that for this value of λ the scales in which there is departure from de Sitter but the initial conditions are valid are set by the scale $a_b/\lambda \approx 10^{30}$ Mpc for $a_b \sim e^{-N_*}$, where the ΛCDM evolution of the scale factor is dwarfed by the early exponential expansion (compare with the radius of the observable Universe 14.3 Gpc). Therefore, we are certainly safe in assuming time-independent vacua as initial conditions.

Using the closed de Sitter expression for the scale factor, the dimensionless spectrum looks like

$$\mathcal{P}_\zeta = -\frac{\lambda^2}{M_P^2} \sinh^2(\lambda t) \tanh^2(\lambda t). \tag{8.7}$$

Horizon crossing occurs at $k = aH$ as usual, which gives

$$k = a_b \lambda \sinh(\lambda t). \tag{8.8}$$

Therefore, we find

$$\mathcal{P}_\zeta = \frac{k^2}{M_P^2 a_b^2} \tanh^2\left[\operatorname{arcsinh}\left(\frac{k}{a_b\lambda}\right)\right]$$
$$= \frac{\lambda^2}{M_P^2} - \frac{k^2}{M_P^2 a_b^2} + \mathcal{O}(k^4/a_b^4), \tag{8.9}$$

where we expand for large wavelengths for which the above assumption of time-independent vacua holds.

If we assume that outside of largest observable scales the de Sitter approximation quickly broke down (i.e., assuming the bare minimum e-folds of growth), then the scales in which we expect to see deviation from near scale invariance correspond to $e^{-N_{dS}} r_{CMB}$, where $r_{CMB} = 14.3$ Gpc. This in turn corresponds to an angular scale of $\theta = e^{-N_{dS}}/(2\pi)$, varying from 6° to 0.9° for $N_{dS} \in [0.367, 2.31]$ as estimated above. This corresponds to multipoles of $\ell \lesssim 180$ to $\ell \lesssim 30$; in particular, a suppression to the latter, corresponding to a slightly slower relaxation to de Sitter, agrees with observations. Note that the relative correction to the spectrum corresponding to the lowest multipoles is of the order

$$\frac{\Delta \mathcal{P}_\zeta}{\mathcal{P}_\zeta} = -\frac{k^2}{a_b^2 \lambda^2} = -\frac{\ell^2/r_{CMB}^2}{e^{-2N_{tot}} \lambda^2}. \tag{8.10}$$

By varying the total number of e-foldings N_{tot}, it is possible to arrive at a percentage-level modification of the order of 0.1% to 1% to the spectrum. Such an effect is largely model independent, insofar as it only requires a bounce followed by a relaxation to de Sitter, and may explain current observations. In fact, matching the suppression with observations can lead to limits on the number of e-folds; for instance, in the above example, $N_{tot} \gtrsim 125$ is specifically

ruled out, as it predicts a much stronger suppression than observed at low multipoles.

Note that this effect is complementary to the finite-curvature effects that can cause an enhancement of the power spectra at low n, dominating over the enhancement effect. Scales with $n \gg 3$ quickly find themselves deep in the Hubble horizon. The pivot scales $k = 0.05$ Mpc^{-1} and $k = 0.002$ Mpc^{-1} correspond to $n = 3152$ and $n = 126$, respectively, for the smallest Universe compatible with observations [37]. Therefore, we expect that the dependence of the vacuum on the wave number is negligible, and the power spectra are only modified in an observationally relevant way due to the bounce, not finite-curvature effects. Moreover, the suppression is not contrary to the enhancement of perturbations through the bounce found in the previous section, since we have set the initial conditions close the bounce and not at the horizon exit. It is true that perturbations are amplified through a bounce, but as the comoving horizon becomes infinite and all modes can talk to each other, we may reasonably assume that gauge-invariant perturbations approach a maximally symmetric set of initial conditions.

Having discussed the implications of the bounce on the low multipole modes, we now turn our attention to the predictions of our α-attractor potential for scales which experience de Sitter for their entire lifetime. We remind the reader that our model is governed by the Lagrangian

$$\sqrt{-g}\mathcal{L} = \frac{1}{2}M_P^2 R - \frac{1}{2}(\partial\phi)^2 - 3\alpha m^2 M_P^2 \tanh^2 \frac{\phi}{\sqrt{6\alpha}M_P}, \tag{8.11}$$

where we have reinstated α in the potential for dimensional consistency. We recall the usual expressions for the tilt of the spectrum and the tensor-to-scalar ratio,

$$n_s = 1 - 2\epsilon + \eta,$$
$$r = 16\epsilon, \tag{8.12}$$

where

$$\epsilon = \frac{1}{2}M_P^2 \left(\frac{V'}{V}\right)^2,$$
$$\eta = M_P^2 \frac{\epsilon'}{\epsilon}\frac{V'}{V}, \tag{8.13}$$

where the prime now denotes a derivative with respect to the inflaton field. These definitions for the slow-roll parameters agree with the Hubble slow-roll parameters $\epsilon_H = -\dot{H}/H^2$ and $\eta_H = \dot{\epsilon}_H/(H\epsilon_H)$ in the limit of slow roll.

As noted in the previous section, the plateau shape of the potential, which is generic to attractor models, will give rise to results that should match observations quite well. Calculating the slow-roll parameters, we find

$$\epsilon = \frac{4}{3\alpha} \operatorname{csch}^2\left(\sqrt{\frac{2}{3\alpha}} \phi/M_P\right), \qquad (8.14)$$

$$\eta = \frac{2}{3\alpha}\left[1 + \coth^2\left(\frac{\phi/M_P}{\sqrt{6\alpha}}\right)\right] \operatorname{sech}^2\left(\frac{\phi/M_P}{\sqrt{6\alpha}}\right). \qquad (8.15)$$

The value of ϕ at the end of inflation is found as usual by setting $\epsilon = 1$, returning

$$\phi_{\text{end}} = \sqrt{\frac{3\alpha}{2}} M_P \sinh^{-1}\left(\frac{2}{\sqrt{3\alpha}}\right). \qquad (8.16)$$

With this result, the number of e-folds is related to the field value ϕ as

$$\phi = \sqrt{\frac{3\alpha}{2}} M_P \cosh^{-1}\left(\sqrt{1 + \frac{4}{3\alpha} + \frac{4N}{3\alpha}}\right). \qquad (8.17)$$

We may therefore easily calculate the value of the spectral index and the tensor-to-scalar ratio:

$$n_s = \frac{1}{6\alpha}\left[\left(\sqrt{\frac{12}{\alpha} + 9} - 3\right)\alpha + 4N - 8\right]$$
$$\times \operatorname{csch}^2\left[\frac{1}{2}\cosh^{-1}\left(\sqrt{\frac{4}{3\alpha} + 1 + \frac{4N}{3\alpha}}\right)\right], \qquad (8.18)$$

$$r = \frac{48\alpha}{4N^2 + 2\alpha N\sqrt{\frac{12}{\alpha} + 9} + 3\alpha}. \qquad (8.19)$$

In particular, the current constraint on the tensor-to-scalar ratio $r < 0.06$ sets the allowed range of α as

$$0.06 \geq \frac{48\alpha}{4N^2 + 2\alpha N\sqrt{\frac{12}{\alpha} + 9} + 3\alpha}, \qquad (8.20)$$

returning $\alpha < 33.5$. Moreover, for the scalar spectrum, we find

$$\mathcal{P}_\zeta = \frac{3\alpha^2 m^2}{4\pi^2 M_P^2} \sinh^4\left[\frac{1}{2}\cosh^{-1}\left(\sqrt{\frac{4}{3\alpha} + 1 + \frac{4N}{3\alpha}}\right)\right]. \qquad (8.21)$$

The value of N depends on the details of reheating, but we expect it to be very close to $N = 60$ for the pivot scale. Using the COBE normalization $\mathcal{P}_\zeta = (2.099 \pm 0.029) \times 10^{-9}$, we may find a relation between α and m. For small values of α, we find that the value of m is mostly insensitive to the model, as we would expect from an attractor; we find $m \approx 7.25 \times 10^{-6} M_P$. As a result, in order to avoid the curvature effects from being dominated by the potential, we set $m^2 \lesssim 6\alpha M_P^2$, which sets $\alpha > 8.78 \times 10^{-12}$. This gives us the window for α as

$$10^{-11} \lesssim \alpha \lesssim 34. \qquad (8.22)$$

As a result, the predicted values for n_S and r are

$$0.9667 \lesssim n_s \lesssim 0.9669,$$
$$10^{-14} < r \lesssim 0.06. \qquad (8.23)$$

For large values of α, roughly $10 \lesssim \alpha \lesssim 33$, the predictions are in slight tension with observations, as they depend on the particulars of the model. However, for small α, the attractor nature of the theory becomes manifest, and the predictions of the model are well within the Planck contour.

While such a model is concordant with observations following a bounce, we must finally ask whether a model with these parameters is consistent with the observed curvature in the Universe today. As noted in the previous section, at the bounce, we have $a_b = M_P^2\sqrt{-\Omega_k/\Lambda}$. This can be used to find Ω_k in the convention that $a = 1$ at the bounce, which roughly corresponds to a difference in convention of roughly $e^{-2N_{\text{tot}}}$ (thanks to the small evolution of the curvature after inflation subsides). However, if $a = 1$ is taken to hold today and $a = a_b$ at the bounce, we may find that the value of the curvature today is

$$\Omega_k = \frac{\Lambda e^{-2N_{\text{tot}}}}{M_P^4}, \qquad (8.24)$$

where we used the relation $\Lambda = 3\alpha m^2 M_P^2$. For the range $1.58 \times 10^{-21} \lesssim \Lambda/M_P^4 \lesssim 1.57 \times 10^{-9}$ (which is realized for the allowed values of α), we find that the spatial curvature today is indeed set to be very small but nonzero nonetheless for these models. This tells us that a bouncing cosmology followed by a period of inflation is particularly robust; it can be realized without requiring extremely large (negative) curvature close to the bouncing point, and once a double bounce is realized, the curvature will be driven down to small levels as usual, in agreement with observations.

IX. CONCLUSIONS

We have proposed a bouncing scenario in the presence of a plateau potential stemming from a conformal action featuring a SO(1, 1) symmetry, allowing a single conformally coupled field to provide both the inflationary dynamics and the dominant matter content at the bounce. We demonstrated that in a contracting universe a bounce can be realized quite generically in the α-attractor paradigm when potential domination is satisfied and the field is sufficiently high up on the plateau. Performing a phase space analysis, we have shown that the measure on the space of solutions is dominated by bouncing solutions when equipped with an appropriate cutoff due to large field values being physically indistinguishable. By performing both analytic and numeric evolutions, we demonstrate that across the bounce low wave number modes of perturbations

are amplified. However, if the primordial perturbations at observable are normalized with respect to a time-independent maximally symmetric vacuum at early times, we find that there will be a percent-level suppression at low multipoles, which agrees with CMB observations. This is a distinct feature from the suppression that occurs due to finite-size effects. At large multipoles, the de Sitter evolution is reached, and we find that the predictions of the double bounce model agree with those of standard attractor inflation for the Planck pivot scale.

We should briefly comment on an implicit assumption which we made to simplify our treatment. We have considered a roughly homogeneous Universe before the bounce, despite the fact that a contracting universe would amplify inhomogeneities, which could threaten the bounce. However, this danger of excessive inhomogeneities can be alleviated in a number of ways. One possibility is considering an early stage of inflation before the bounce, which can take the system from the Planck scale down to the T-model plateau. This can be introduced by another field, such that the plateau lies actually at the bottom of the potential valley, the walls of which rise up to the Planck density. Examples of this kind of structure in the context of α attractors can be seen in Refs. [13,22], in which the early inflation is power law and lasts for only a short duration. Similarly, but without the need for an additional field, a slight modification of the inflaton potential can achieve a limited early inflation stage, by lifting the valley toward the Planck scale (see, for example, Fig. 5 of Ref. [43]).[6] One other way, which would not invoke early inflation, is to consider that the universe topology is not trivial, such that homogenization is achieved through "chaotic mixing" [44–46] due to the fact that the entire compact universe comes in causal contact soon after its quantum nucleation. Needless to say, slow-roll inflation after the bounce may easily enlarge the boundaries of the compact universe to super-horizon scales today so that any trace of nontrivial topology is lost. Finally, one could also perceive the bounce acting as a filter on excessive inhomogeneity. If we start with a lot of universes contracting, then those that are not homogeneous enough would not experience a bounce. Therefore only sufficiently homogeneous universes would make it through to inflation and expand like our Universe. How likely or unlikely the above scenarios are touches upon the infamous measure problem of inflationary cosmology [27–29,47], which goes beyond the scope of our paper.

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[6]In terms of α attractors, we may consider a change in the kinetic term from $1/[1 - \phi^2/(6\alpha)]$ to $1/[1 - \phi^2/(6\alpha) \exp(-\lambda\phi)]$, essentially "smearing" out the poles. Solving the equations of motion, it is possible to show that, due to the canonicalisation, any kinetic term which is roughly this shape produces an inflection point–type plateau irrespective of the potential. By tuning α and λ, it is possible to arrive at a steep "drop," corresponding to a long but finite plateau, which then rises up and makes contact with the Planck scale.

A possible extension of our analysis could include multiple fields, acting on separate scales for the bounce and the subsequent de Sitter expansion of the universe. A simple example would be the potential $V = \Lambda_1 \tanh^{2n}(\frac{\theta}{\sqrt{6\alpha}}) + \Lambda_2 \tanh^{2m}(\frac{\phi}{\sqrt{6\beta}})$, where the two scales could allow separate scales at which the bounce and inflation could occur. This effect would be particularly pronounced if $m \gg n$ such the potential is very steep and narrow in the direction of highest energy.

Note that if the bounce is marginal (i.e., we bounce at $w = -1/3$ and nearby $w > -1/3$), we expect there to be singularities just beyond the horizon. Specifically, if we find ourselves in a patch which has $w < -1/3$ while $w > -1/3$ outside it, everything beyond the horizon becomes singular. If we find ourselves in the only patch that has this feature, the universe will have closed up on a point.

ACKNOWLEDGMENTS

APPENDIX: THE HYPERBOLIC TANGENT OSCILLATOR

Throughout this paper, we have considered a scalar field of which the potential was $\tanh^2(x)/2$ (here, we include the factor of $1/2$ to facilitate an easier comparison with the usual $x^2/2$ potential of a harmonic oscillator). Let us now establish some of the relevant properties of such a field. The Hamiltonian for the field in the absence of gravitational coupling (and hence Hubble friction) is simply

$$\mathcal{H} = \frac{1}{2}P_\phi^2 + \frac{1}{2}\tanh^2(\phi) = E. \tag{A1}$$

Here, we have fixed the size of the potential to unity without loss of generality. The equation of motion for our system is

$$\ddot{\phi} = -\tanh(\phi)\text{sech}^2(\phi). \tag{A2}$$

The first thing to notice is that the qualitative behavior of the system is now dependent upon the total energy E. In the case in which $E \gg 1$, the system appears as that of a free field as the potential has little effect on the motion. When $E \ll 1$, the system behaves like a harmonic oscillator. For a harmonic oscillator, the kinetic energy and the potential follow $\cos^2(t)$ and $\sin^2(t)$, respectively. Hence,
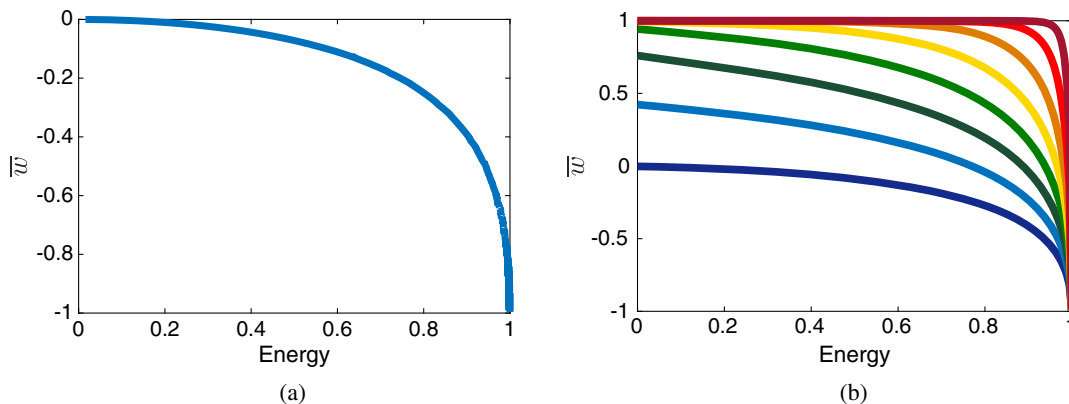
FIG. 7.   The average value of w across a cycle against the energy of the oscillator for increasing values of n in the potential. Shown on the left (a) is the behavior of w in the case of a quadratic tanh potential. Shown on the right (b) is the same for a range of even monomials. To show the overall behavior, the lines are each 2^m for m between 1 and 7, with the colors chosen such that the blue line corresponds to $m = 2$ and m increases as we move toward the red end of the spectrum. Here, we see that as n increases the average value of $w - \bar{w}$ increases, asymptoting to $\bar{w} = 1$ (corresponding to the square well). When $n = 2$, the results coincide with the above behavior, and the maximum value of $\bar{w}$ is 0. For higher n, this value increases, as does the range of energies over which $\bar{w}$ is close to 1.

$$w_{\text{SHO}} = \frac{KE - PE}{KE + PE} = \frac{\cos^2(t) - \sin^2(t)}{\cos^2(t) + \sin^2(t)} = \cos(2t). \quad \text{(A3)}$$

As a result, across a complete cycle, the average of w is zero. However, when we consider the behavior of the tanh potential, we come across a subtlety. When $E > 1$, the field no longer oscillates since the system is unbound. Hence, the equivalent behavior is captured by the long-term average of w, which tends to $w_u = \frac{E-2}{E}$. When $E < 1$, the average is difficult to find analytically. In the limit where $E \ll 1$, the harmonic oscillator becomes a good approximation to the dynamics of the system, and hence the average across a cycle is zero. For $0 < E < 1$, we can find the average behavior from numerical averaging shown in Fig. 7.

We can observe from Fig. 7 that when $E > \sqrt{3}/2$ we find that the average of $w < -1/3$ However, we note that this approximation would only hold in a limit in which we could neglect the Hubble friction, which would not be valid in this regime; in other words, $\ddot{a} < 0$ whenever the field is oscillating sufficiently quickly that we can use the average field approximation. Hence, a bounce that occurs during the oscillating phase of the field will not yield a large universe; a recollapse will occur before an oscillation is complete. We are therefore motivated to consider bounces that occur when the field is strongly damped.

Let us now turn to the case in which we consider higher powers of $\tanh(\phi)$ to form our potential. In this situation, we note that taking the limit as $n \to \infty$ yields no potential at all, since the $\tanh(\phi) < 1$ for all finite values of ϕ.

However, by changing the width of the potential, we can produce a more reasonable function to use; $V_n = \tanh^n(k\phi)$ for even integers n. In particular, consider fixing the potential at a given point irrespective of the value of n— as an example, let us fix the value of V_n at $\phi = \phi_o$:

$$V_n = \frac{1}{2}\tanh^n(k\phi) = \frac{1}{2}\tanh^2(\phi_o). \quad \text{(A4)}$$

From this prescription, we find that k is given:

$$k = \frac{\tanh^{-1}\left[\tanh^{2/n}(\phi_o)\right]}{\phi_o}. \quad \text{(A5)}$$

It is apparent that in the infinite limit ϕ_o will be the width of our square well, as the potential can only pass through the fixed value at the wall.

The equation of motion for our inflaton is now

$$\ddot{\phi} = -\frac{kn}{2}\text{sech}^2(k\phi)\tanh^{n-1}(k\phi). \quad \text{(A6)}$$

Figure 7 was generated by numerically solving this system for a range of energies of the oscillator. We see that as the power n increases the behavior of the system approaches more and more closely that of a square well. In particular, we note that the average behavior of the field across an oscillation is not well approximated by dust but becomes better and better approximated by that of a stiff fluid ($p = \rho$).

[1] A. A. Starobinsky, A new type of isotropic cosmological models without singularity, Phys. Lett. **91B,** 99 (1980); Adv. Ser. Astrophys. Cosmol. **3,** 130 (1987).

[2] K. Sato, Cosmological baryon number domain structure and the first order phase transition of a vacuum, Phys. Lett. **99B,** 66 (1981); Adv. Ser. Astrophys. Cosmol. **3,** 134 (1987).

[3] D. Kazanas, Dynamics of the universe and spontaneous symmetry breaking, Astrophys. J. **241,** L59 (1980).

[4] A. H. Guth, The inflationary universe: A possible solution to the horizon and flatness problems, Phys. Rev. D **23,** 347 (1981); Adv. Ser. Astrophys. Cosmol. **3,** 139 (1987).

[5] D. Baumann, Primordial Cosmology, Proc. Sci TASI2017 (**2017**) 009.

[6] Y. Akrami *et al.* (Planck Collaboration), Planck 2018 results. X. Constraints on inflation, arXiv:1807.06211.

[7] N. Aghanim *et al.* (Planck Collaboration), Planck 2018 results. VI. Cosmological parameters, arXiv:1807.06209.

[8] R. Kallosh, A. Linde, and D. Roest, Superconformal inflationary α-attractors, J. High Energy Phys. 11 (2013) 198.

[9] M. Galante, R. Kallosh, A. Linde, and D. Roest, Unity of Cosmological Inflation Attractors, Phys. Rev. Lett. **114,** 141302 (2015).

[10] R. Kallosh and A. Linde, Planck, LHC, and α-attractors, Phys. Rev. D **91,** 083528 (2015).

[11] A. Ijjas, P. J. Steinhardt, and A. Loeb, Inflationary schism, Phys. Lett. B **736,** 142 (2014).

[12] R. Brandenberger, Initial conditions for inflation—A short review, Int. J. Mod. Phys. D **26,** 1740002 (2017).

[13] J. J. M. Carrasco, R. Kallosh, and A. Linde, Cosmological attractors and initial conditions for inflation, Phys. Rev. D **92,** 063519 (2015).

[14] P. J. E. Peebles and A. Vilenkin, Quintessential inflation, Phys. Rev. D **59,** 063505 (1999).

[15] J. J. M. Carrasco, R. Kallosh, and A. Linde, α-attractors: Planck, LHC and dark energy, J. High Energy Phys. 10 (2015) 147.

[16] K. Dimopoulos and C. Owen, Quintessential Inflation with α-attractors, J. Cosmol. Astropart. Phys. 06 (2017) 027; K. Dimopoulos, L. Donaldson Wood, and C. Owen, Instant preheating in quintessential inflation with α-attractors, Phys. Rev. D **97,** 063525 (2018).

[17] H. Matsui, F. Takahashi, and T. Terada, Non-singular bouncing cosmology with positive spatial curvature and flat scalar potential, Phys. Lett. B **795,** 152 (2019).

[18] C. Brans and R. H. Dicke, Mach's principle and a relativistic theory of gravitation, Phys. Rev. **124,** 925 (1961).

[19] P. G. Ferreira, C. T. Hill, and G. G. Ross, No fifth force in a scale invariant universe, Phys. Rev. D **95,** 064038 (2017).

[20] Y. F. Cai, J. O. Gong, and S. Pi, Inflation beyond T-models and primordial B-modes, Phys. Lett. B **738,** 20 (2014).

[21] S. Karamitsos, Beyond the poles in attractor models of inflation, J. Cosmol. Astropart. Phys. 09 (2019) 022.

[22] K. Dimopoulos and M. Artymowski, Initial conditions for inflation, Astropart. Phys. **94,** 11 (2017).

[23] A. Alho and C. Uggla, Inflationary α-attractor cosmology: A global dynamical systems perspective, Phys. Rev. D **95,** 083517 (2017).

[24] G. F. R. Ellis, E. Platts, D. Sloan, and A. Weltman, Current observations with a decaying cosmological constant allow for chaotic cyclic cosmology, J. Cosmol. Astropart. Phys. 04 (2016) 026.

[25] M. G. Brown, K. Freese, and W. H. Kinney, The phantom bounce: A new oscillating cosmology, J. Cosmol. Astropart. Phys. 03 (2008) 002.

[26] J. D. Barrow and A. A. H. Graham, General dynamics of varying-alpha universes, Phys. Rev. D **88,** 103513 (2013).

[27] D. Sloan and J. Silk, Volume weighting the measure of the universe from classical slow-roll expansion, Phys. Rev. D **93,** 104030 (2016).

[28] D. Sloan, $w = -1$ as an attractor, Open J. Astrophys. (2016).

[29] A. Corichi and D. Sloan, Inflationary attractors and their measures, Classical Quantum Gravity **31,** 062001 (2014).

[30] D. Sloan, Minimal coupling and attractors, Classical Quantum Gravity **31,** 245015 (2014).

[31] W. H. Kinney, Horizon crossing and inflation with large eta, Phys. Rev. D **72,** 023515 (2005); C. Germani and T. Prokopec, On primordial black holes from an inflection point, Phys. Dark Universe **18,** 6 (2017); K. Dimopoulos, Ultra slow-roll inflation demystified, Phys. Lett. B **775,** 262 (2017).

[32] M. J. P. Morse and W. H. Kinney, Large-η constant-roll inflation is never an attractor, Phys. Rev. D **97,** 123519 (2018).

[33] W. C. Lin, M. J. P. Morse, and W. H. Kinney, Dynamical analysis of attractor behavior in constant roll inflation, J. Cosmol. Astropart. Phys. 09 (2019) 063.

[34] A. D. Linde, Can we have inflation with $\Omega > 1?*$, J. Cosmol. Astropart. Phys. 05 (2003) 002.

[35] K. Dimopoulos, Steep eternal inflation and the swampland, Phys. Rev. D **98,** 123516 (2018).

[36] B. Bonga, B. Gupt, and N. Yokomizo, Inflation in the closed FLRW model and the CMB, J. Cosmol. Astropart. Phys. 10 (2016) 031.

[37] B. Bonga, B. Gupt, and N. Yokomizo, Tensor perturbations during inflation in a spatially closed Universe, J. Cosmol. Astropart. Phys. 05 (2017) 021.

[38] I. Agullo, A. Ashtekar, and W. Nelson, Extension of the quantum theory of cosmological perturbations to the Planck era, Phys. Rev. D **87,** 043507 (2013).

[39] I. Agullo, A. Ashtekar, and W. Nelson, A Quantum Gravity Extension of the Inflationary Scenario, Phys. Rev. Lett. **109,** 251301 (2012).

[40] J. Quintin, Y. F. Cai, and R. H. Brandenberger, Matter creation in a nonsingular bouncing cosmology, Phys. Rev. D **90,** 063507 (2014).

[41] J. Quintin and R. H. Brandenberger, Black hole formation in a contracting universe, J. Cosmol. Astropart. Phys. 11 (2016) 029.

[42] L. E. Allen and D. Wands, Cosmological perturbations through a simple bounce, Phys. Rev. D **70,** 063515 (2004).

[43] A. Linde, On the problem of initial conditions for inflation, Found. Phys. **48**, 1246 (2018).

[44] G. F. R. Ellis, Topology and cosmology, Gen. Relativ. Gravit. **2**, 7 (1971).

[45] N. J. Cornish, D. N. Spergel, and G. D. Starkman, Does Chaotic Mixing Facilitate Omega < 1 Inflation?, Phys. Rev. Lett. **77**, 215 (1996).

[46] A. D. Linde, Creation of a compact topologically nontrivial inflationary universe, J. Cosmol. Astropart. Phys. 10 (2004) 004.

[47] G. W. Gibbons and N. Turok, The measure problem in cosmology, Phys. Rev. D **77**, 063516 (2008).