Reply to "Comment on 'Boosted Kerr black holes in general relativity'"

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We discuss the comments made by Emanuel Gallo and Thomas Mädler on our papers [Gen. Relativ. Gravit. **49**, 77 (2017) and Phys. Rev. D **99**, 084054 (2019)] on boosted Kerr black holes in general relativity. We considered their criticisms carefully, as they added a significant contribution to the question of boosted Kerr black holes in general relativity examined in our papers. Here, we expand on further results of our papers that were not duly included in the published papers, as noticed by Gallo and Mädler. We now consider the complete asymptotic Lorentz transformations of Robinson-Trautman (RT) coordinates to Bondi-Sachs (BS) asymptotic coordinates of a radiative RT spacetime, which include the perturbation term of the boosted RT metric corresponding to the Lense-Thirring rotation originating from the Kerr metric. The transformation of the rotation parameter ω as the RT coordinates transform to BS coordinates is obtained, as well as the form of the boosted Kerr metric where we make use of the complete asymptotic Lorentz transformation.

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I. INTRODUCTION

Our aim in this Reply is to discuss the criticisms of our papers [1,2] made by Emanuel Gallo and Thomas Mädler in their Comment on Ref. [2] (see Ref. [14]). The authors correctly showed that the metrics presented in Refs. [1,2] contain an incomplete piece of a Lorentz transformation, as the additional transformations of the time and radial coordinates are still missing. Therefore, the chosen coordinates do not represent adapted coordinates with respect to an asymptotic inertial observer. Here, we now include the missing pieces of the Lorentz transformations that actually connect the Robinson-Trautman (RT) coordinates (u, r, θ, ϕ) used in radiative RT spacetimes to Bondi-Sachs (BS) coordinates (U, R, Θ, Φ) that satisfy the boundary conditions, as discussed in Sec. II of our Ref. [3]. These pieces of Lorentz transformations (missing in Refs. [1,2]) play an important role in approaching the issue of boosted Kerr black holes in general relativity.

In Secs. II and III we discuss RT radiative spacetimes, the relation between RT coordinates and the BS system that allows to extract physical quantities such as the Bondi mass aspect, the Bondi momentum aspect, and the total Bondi mass, and the Bondi momentum along the z direction. We also obtain a perturbative form of the boosted RT metric that includes a Lense-Thirring rotation term originating from the Kerr metric. We then obtain the transformation of the rotation parameter ω as the RT coordinates transform to BS coordinates, and obtain the form of the "boosted" Kerr metric where the Lorentz transformation is complete. We restrict this discussion to the axisymmetric metrics only [1], but it can be extended in a straightforward way to the nonaxisymmetric case [2,3].

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II. THE ROBINSON-TRAUTMAN RADIATIVE SPACETIMES, THE BONDI-SACHS CONDITIONS, AND THE SLOW-ROTATION LIMIT OF KERR METRICS

We start by discussing the case of nontwisting radiative spacetimes, namely, RT spacetimes, which are the only known radiative spacetimes. To our knowledge, twisting radiative spacetimes—which might have boosted Kerr black holes as remnants—have yet to be found.

Although RT spacetimes describe the exterior vacuum gravitational field of a bounded system radiating gravitational waves, the RT metrics do not satisfy the appropriate BS boundary conditions formulated for radiating systems. Therefore, we were led to implement transformations between RT and BS coordinates, which allow us to obtain some of the basic BS physical quantities, namely, the *news* functions and the BS energy-momentum fluxes of the emitted gravitational waves, as well as the BS energy-momentum conservation laws. This was explained in detail in our Ref. [3] (see also Ref. [4]). There we obtained the transformations of RT coordinates (u, r, θ, ϕ) of a general radiative RT spacetime to BS coordinates (U, R, Θ, Φ) of the BS metric for asymptotically flat isolated systems emitting gravitational waves and satisfying the appropriate boundary conditions. In this construction we see that the news functions of the nonaxisymmetric RT geometry, which constitute part of the data to be specified in the evolution of the system, are already completely specified once the metric function $K(u,\theta,\phi)$ of the RT geometry is given. On the other hand, $K(u,\theta,\phi)$ can be obtained, for all $u > u_0$, by integrating the RT equation from a given initial data $K(u_0,\theta,\phi)$ specified at the initial characteristic surface $u = u_0$. The news functions determine the dominant wave zone curvatures, and are fundamental in the construction of the net fluxes of energy and momentum of the emitted gravitational waves, when the initial data are evolved numerically.

Now, as can be seen in Sec. 2 of Ref. [3], the transformations of RT coordinates (u,r,θ,ϕ) to BS coordinates (U,R,Θ,Φ) (satisfying the appropriate BS asymptotic conditions) result, for a stationary axisymmetric configuration, in the BS mass aspect $m_B = m_0/K(\theta)^3$ with $\partial U/\partial u = K(\theta)$, where $K(\theta) = (\cosh\gamma + \cos\theta\sinh\gamma)$. Together with $\partial R/\partial r = K^{-1}(\theta)$ we have the complete Lorentz transformation. The mean value of the BS mass aspect over the sphere gives the total Bondi mass $M_B = m_0\cosh\gamma$. Therefore, the metric (1)–(2) of Ref. [3] in the stationary axisymmetric case,

$$ds^2 = \frac{r^2(d\theta^2 + \sin^2\theta d\phi^2)}{K^2(\theta)} - 2dudr - \left(1 - \frac{2m_0}{r}\right)du^2, \quad (1)$$

corresponds effectively to a boosted Schwarzschild black hole, as noted in the Comment below Eq. (30): "Note, we are not saying that the metric (29) could not be interpreted as a boosted black hole; what we are saying is that if these NU coordinates are used, they must be yet related to a Bondi system to extract physical quantities". The mentioned relation between RT coordinates and the BS system was actually implemented in Sec. 2 of Ref. [3] (see also Ref. [4]) and the physical quantities were extracted, namely, the Bondi mass aspect $m_B(\theta) = m_0/K^3(\theta)$, the Bondi momentum aspect $p_B^{(z)} = m_0\cos\theta/K^3(\theta)$, the total Bondi mass $M_B = m_0\cosh\gamma$, and the total Bondi linear momentum along z, $P_B^{(z)} = m_0\sinh\gamma$ [5,6].

Also, if we consider for instance the expansions of Eqs. (29) and (30) in Ref. [2], for large r and in the limit of small rotation parameter $\omega \ll m$ [namely, in the Lense-Thirring (L-T) approximation [7]], the same arguments used for the boosted Schwarzschild case [3] apply,

$$ds^2 = \frac{r^2(d\theta^2 + \sin^2\theta d\phi^2)}{K^2(\theta)} - 2dudr - \left(1 - \frac{2m_0}{r}\right)du^2 + \frac{4m_0\omega}{rK^2(\theta)}\sin^2\theta du d\phi + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (2)$$

including the perturbative L-T rotation term. This term results from the Kerr metric; see Eq. (23) in Ref. [1]. A careful analysis indicates that the rotation parameter transforms as

$$\omega \to \omega_B = \omega/K(\theta), \quad (3)$$

as RT coordinates transform to BS coordinates, cf. Sec. 2 of Ref. [3]. This transformation is considered a natural extension of the RT-BS transformation system when the Kerr rotation parameter is present, as for instance in Eq. (2).

Therefore, in BS coordinates (U,R,Θ,Φ) the metric (23) in Ref. [1] asymptotically assumes the form

$$ds^2 = (R^2 + \tilde{\Sigma}^2)(d\Theta^2 + \sin^2\Theta d\Phi^2) - 2(dU + \omega_B\sin^2\Theta d\Phi)\left(dR - \frac{\omega_B\sin^2\Theta}{K^2}d\Phi\right) - (dU + \omega_B\sin^2\Theta d\Phi)^2\left(\frac{1}{K^2} - \frac{2m_B R}{R^2 + \tilde{\Sigma}^2}\right) + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (4)$$

where the complete Lorentz transformation is applied. In the above

$$\tilde{\Sigma} = \omega_B\left(\frac{\sinh\gamma + \cosh\gamma\cos\Theta}{\cosh\gamma + \sinh\gamma\cos\Theta}\right),$$

and $\Theta = \theta$, $\Phi = \phi$. We note that $\tilde{\Sigma} = \Sigma/K$ [1]. A straightforward extension to the nonaxisymmetric case can be obtained by using the results of Ref. [3], cf. also the following sections.

III. BOOSTED BLACK HOLES AS REMNANTS OF NUMERICALLY EVOLVING INITIAL DATA

In past works we also examined radiative RT spacetimes by numerically evolving initial data corresponding to head-on and non-head-on collisions of two black holes, via the RT dynamics, from a given characteristic initial surface $u = u_0$ [8–11]. In the coordinate system (u,r,θ,ϕ) the RT metric is expressed as

$$ds^2 = \frac{r^2(d\theta^2 + \sin^2\theta d\phi^2)}{K^2(u,\theta,\phi)} - 2dudr - \left(\lambda(u,\theta,\phi) - \frac{2m_0}{r} - 2r\frac{K_{,u}}{K}\right)du^2,$$

where

$$\lambda = K^2 + \frac{K^2}{\sin\theta}\left(\sin\theta\frac{K_{,\theta}}{K}\right)_{,\theta} + \frac{K^2}{\sin^2\theta} + \left(\frac{K_{,\phi}}{K}\right)_{,\phi}.$$

Initial data are evolved for a sufficiently long time, from u_0 to the final retarded time u_f, when the net flux of energy and momentum carried by the gravitational waves ceases and the computation is stopped. The initial data correspond to a planar collision which we take in the plane (x,z) determined by the vector $\mathbf{n} = (n_1, 0, n_3)$, cf. Ref. [9].

The total energy $M(u)$ emitted and the total impulse imparted to the merged black hole due to the net momentum flux of the gravitational waves $\mathbf{P}_W(u)$ are evaluated from the integrated function $K(u,\theta,\phi)$ for $u \in [u_0, u_f]$, obtained from an accurate and numerically stable evolution of the initial data $K(u_0,\theta,\phi)$.

For all initial data the system settles down, at the final time u_f, to the form $K(u_f,\theta,\phi) \equiv K(\theta,\phi) = K_f(\cosh\gamma_f + \mathbf{n_f}\cdot\hat{\mathbf{x}}\sinh\gamma_f)$, with $\hat{\mathbf{x}} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$. Of course, the final parameters $(K_f, \gamma_f, \mathbf{n_f})$ [where $\mathbf{n_f} = (n_{1f}, 0, n_{3f})$] depend on the initial data. As can be seen in Refs. [8–11], the unit 3-vector $\mathbf{n_f}$ has the same direction as the final net total impulse imparted to the merged system by the momentum carried by the emitted gravitational waves. We note that $K(u_f,\theta,\phi)$ yields $\lambda = K_f^2$ (see, for instance, Ref. [12]).

Actually, the boost function $K(u_f,\theta,\phi) \equiv K(\theta,\phi)$ in the final metric is not a mere angle transformation, but rather results from the dynamical process by which the net momentum flux of the emitted gravitational waves leads to a Schwarzschild black hole boosted along the direction $\mathbf{n_f}$. We remark that γ_f turns out to be zero when the initial black holes have a mass ratio of 1 [13]. The Lorentz transformation $(u,r) \to (U,R)$ is completed with $K(\theta,\phi)$. The Bondi mass aspect is given as $m_B = m_0/K^3(\theta,\phi)$. After a straightforward integration, the final impulse is $P_f^x = n_{1f}P_f$, $P_f^z = n_{3f}P_f$, where $P_f = m_0 K_f^3 \sinh\gamma_f$.

Finally, even though twisting radiative spacetimes with appropriate BS boundary conditions are not currently known, from the above discussions we have that the metrics (23) in Ref. [1] and (27) in Ref. [2] can also be related to boosted Kerr black holes in light of the fact that the RT coordinates used can be related to a BS system, and that asymptotically the coordinate transformations $(u,r) \to (U,R)$ result in Eq. (4).

IV. THE HORIZONS AND ERGOSPHERE OF A BOOSTED KERR BLACK HOLE IN ASYMPTOTIC BS COORDINATES

The locations of the horizons were defined in Ref. [1] as the surfaces $g^{rr} = 0$, where g^{rr} corresponds to the metric in RT coordinates (namely, Eddington-Finkelstein-type coordinates) (u,r,θ,ϕ). However, these RT coordinates were not actually transformed via the complete Lorentz transformation [compare Eq. (23) of Ref. [1] and Eq. (4) of the present Reply].

The complete Lorentz transformation was given in Sec. 2 of Ref. [3], which now includes the transformation (3) and is used in Eq. (4) of this Reply [cf. also the text below Eq. (1)]. From Eq. (4) we obtain

$$g^{RR} = \frac{K^2R^2 - 2m_BK^4R + K^2\tilde{\Sigma}^2 + \omega_B^2\sin^2\Theta}{(R^2+\tilde{\Sigma}^2)K^4}, \quad (5)$$

which results in the expression

$$R^2 - 2m_BK^2R + \tilde{\Sigma}^2 + \omega_B^2\sin^2\Theta/K^2 = 0 \quad (6)$$

for the boosted horizons in BS coordinates. After some algebra we obtain the roots of Eq. (6),

$$R_H = m_BK^2 \pm \sqrt{m_B^2K^4 - \omega_B^2}. \quad (7)$$

In the limit $\omega = 0$ (the Schwarzschild limit in BS coordinates) we obtain $R_H = 2m_BK^2$ for the event horizon (in RT coordinates $r_H = 2m_0$). In the nonboosted limit $K = 1$ the horizon is given by $R_H = m_0^2 \pm \sqrt{m_0^2 - \omega^2}$.

The same procedure applies to the case of the ergosphere $g_{UU} = 0$, which corresponds to the limiting surface for static observers where the Killing vector $\partial/\partial U$ becomes null. We obtain

$$R^2 - 2m_BK^2R + \tilde{\Sigma}^2 = 0 \quad (8)$$

in BS coordinates, where the complete asymptotic Lorentz transformations are used.

The roots of Eq. (8) are

$$R_{\rm stat} = m_BK^2 + \sqrt{m_B^2K^4 - \tilde{\Sigma}^2}. \quad (9)$$

In Fig. 1 we plot the sections of the ergosphere $R_{\rm stat}$ (solid line) and the event horizon R_H (dashed line) by a plane containing the z axis. We should remark again that, contrary to the results obtained in Ref. [1], here we made use of the complete asymptotic Lorentz transformations so

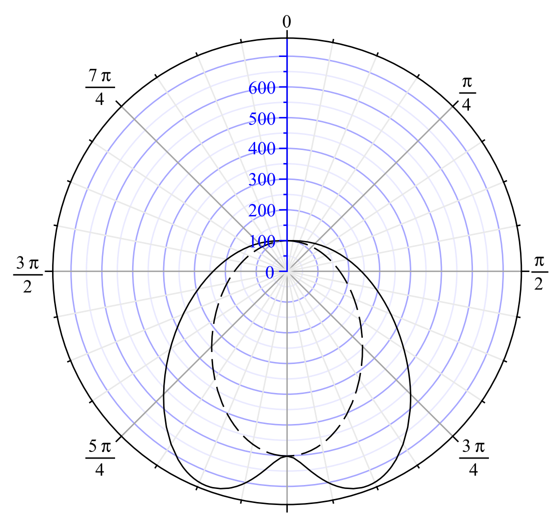

FIG. 1. Sections of the ergosphere $R_{\rm stat}$ (solid line) and the event horizon R_H (dashed line) by a plane containing the z axis. The parameters used are $\gamma = 0.9$, $m_0 = 200$, and $\omega = 195$.

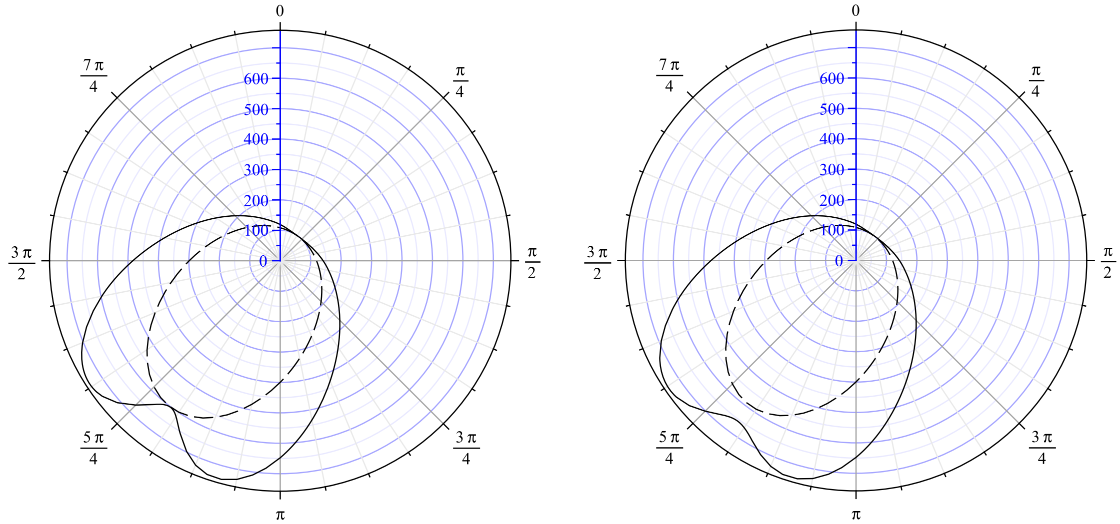

FIG. 2. (Left panel) Sections of the ergosphere $R_{\rm stat}(\Theta,\Phi)$ (solid line) and the event horizon $R_H(\Theta,\Phi)$ (dashed line) by a plane containing the z axis corresponding to $\Phi \simeq 0.185\pi$. This plane contains the two points that belong to both the ergosphere and the event horizon. (Right panel) Sections analogous to those in the left panel, corresponding to the plane of section at $\Phi = 0.25\pi$ containing the z axis. The dashed line is the section of the event horizon, and the solid line is the section of the ergosphere; in this plane, the event horizon and the ergosphere have only one common point. The parameters used are $\gamma = 0.9$, $m_0 = 200$, and $\omega = 195$, with $n_1 = 0.8$, $n_2 = 0.5$, and $n_3 = \sqrt{1 - n_1^2 - n_2^2} \simeq 0.33166$.

that not only the ergosphere $R_{\rm stat}(\Theta)$ but also the horizon $R_H(\Theta)$ are deformed by the boost, as seen in the figure.

An extension of the nonaxisymmetric case [2], with a view to completing the asymptotic Lorentz boost, yields for the event horizon and ergosphere, respectively,

$$R_H(\theta,\Phi) = m_B K^2(\theta,\Phi) + \sqrt{m_B^2 K^4(\theta,\Phi) - \omega_B^2} \quad (10)$$

and

$$R_{\rm stat}(\theta,\Phi) = m_B K^2(\theta,\Phi) + \sqrt{m_B^2 K^4(\theta,\Phi) - \tilde{\Sigma}^2}. \quad (11)$$

Here $K(\Theta,\Phi) = \cosh\gamma + \sinh\gamma\,\mathbf{n}\cdot\hat{\mathbf{x}}$, where $\mathbf{n} = (n_1, n_2, n_3)$ and $\hat{\mathbf{x}} = (\cos\Theta, \sin\Theta\cos\Phi, \sin\Theta\sin\Phi)$, with $n_1^2 + n_2^2 + n_3^2 = 1$.

In Fig. 2 we plot the sections of the ergosphere $R_{\rm stat}(\Theta,\Phi)$ (solid lines) and the event horizon $R_H(\Theta,\Phi)$ (dashed lines) by planes containing the z axis corresponding to $\Phi \simeq 0.185\pi$ (left panel) and $\Phi = 0.25\pi$ (right panel). We should remark again that, contrary to the results obtained in Refs. [1,2], here we made use of the complete asymptotic Lorentz transformations so that not only the ergosphere $R_{\rm stat}(\Theta)$ but also the horizon $R_H(\Theta)$ are deformed by the boost, as seen in the figure.

We do not comment on further results of Gallo and Mädler's comments on these questions for the family of RT geometries since they are complete, to our understanding, and we do not disagree with them.

V. ERRATA

An important remark: in Ref. [1] the coordinates used were RT coordinates (u, r, θ, ϕ), which are also known as outgoing Eddington-Finkelstein coordinates. In Ref. [2] these same RT coordinates were used but were inadvertently denoted as "Bondi-Sachs-type" coordinates. The terms "Robinson-Trautman (RT) coordinates (u, r, θ, ϕ)" and "Bondi-Sachs (BS) coordinates (U, R, Θ, Φ)" are have been maintained in the present Reply, and should also be replaced in Ref. [2]. Our main reason to maintain this notation is for the use of RT metrics (Secs. II and III of this Reply) and BS metrics (cf. also Sec. 2 of Ref. [3]).

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