Covariant Hamiltonian for gravity coupled to p-forms

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We review the covariant canonical formalism initiated by D'Adda, Nelson, and Regge in 1985, and extend it to include a definition of form-Poisson brackets (FPBs) for geometric theories coupled to p-forms. The form-Legendre transformation and the form-Hamilton equations are derived from a d-form Lagrangian with p-form dynamical fields ϕ. Momenta are defined as derivatives of the Lagrangian with respect to the "velocities" $d\phi$ and no preferred time direction is used. Action invariance under infinitesimal form-canonical transformations can be studied in this framework, and a generalized Noether theorem is derived, for both global and local symmetries. We apply the formalism to vielbein gravity in $d = 3$ and $d = 4$. In the $d = 3$ theory we can define form-Dirac brackets, and use an algorithmic procedure to construct the canonical generators for local Lorentz rotations and diffeomorphisms. In $d = 4$ the canonical analysis is carried out using FPBs, since the definition of form-Dirac brackets is problematic. Lorentz generators are constructed, while diffeomorphisms are generated by the Lie derivative. A "doubly covariant" Hamiltonian formalism is presented, allowing to maintain manifest Lorentz covariance at every stage of the Legendre transformation. The idea is to take curvatures as "velocities" in the definition of momenta.

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I. INTRODUCTION

Geometric theories like gravity or supergravity are conveniently formulated in the language of differential forms. Because the Lagrangian of a d-dimensional theory is written as a d-form, it is invariant by construction under diffeomorphisms (up to a total derivative). This framework is also well suited to the case of p-form fields coupled to (super)gravity, and a group-geometric approach has been developed since the late 1970s based on free differential algebras [1–8] (for a recent review see, e.g., Ref. [9]). In the 1980s a form-Hamiltonian formalism was proposed in a series of papers [10–14], where momenta π conjugate to basic p-form fields ϕ are defined as "derivatives" of the d-form Lagrangian with respect to the "velocities" $d\phi$, and the d-form Hamiltonian is defined as $H = (d\phi)\pi - L$. This form-Hamiltonian setting is *covariant*, since no preferred (time) direction is used to define momenta.

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Other covariant Hamiltonian formalisms have been proposed in the literature, and a very partial list of references on multimomentum and multisymplectic canonical frameworks is given in Refs. [15–28]. The essential ideas appeared in papers by De Donder and Weyl more than 70 years ago [15,16]. Some of these approaches are quite similar in spirit to the one we discuss here, but to our knowledge the first proposal of a d-form Hamiltonian, together with its application to gravity, can be found in Ref. [10].

In this paper we further develop the form-Hamiltonian approach of Refs. [10–14], and derive the Hamilton equations for all p-form degrees of freedom. The form-Legendre transformation is discussed in detail, keeping track of all necessary signs due to the presence of forms of various degrees. A definition of form-Poisson brackets (FPBs) is introduced, and generalizes the usual Poisson brackets to arbitrary p-forms. These FPBs satisfy generalized Jacobi identities, and (anti)symmetry and derivation properties, with signs depending on the form degrees. In this language we discuss infinitesimal canonical transformations and generators. A form-Noether theorem is derived, for both global and local invariances of the action.

We apply the formalism to $d = 4$ tetrad gravity, and complete the analysis of Refs. [10,11] by constructing the (Hamiltonian) Lorentz gauge generators, acting on the basic fields via Poisson brackets. Diffeomorphisms are discussed, and expressed in the Hamiltonian setting by means of the Lie derivative.

Vielbein gravity in $d=3$ is reformulated in the covariant Hamiltonian framework, and with the use of form-Dirac brackets we find the canonical generators for local Lorentz rotations and diffeomorphisms.

Finally, we discuss a "doubly covariant" Hamiltonian formalism for gravity (possibly coupled to p-forms), where the "velocities" $d\phi$ are replaced by their covariant version, i.e., the curvatures R. Momenta are then defined as the derivatives of L with respect to R, and all formulas (e.g., the Hamilton equations of motion) become automatically Lorentz covariant, with derivatives being replaced throughout by covariant derivatives.

II. VARIATIONAL PRINCIPLE FOR GEOMETRIC THEORIES WITH p-FORMS

We consider geometrical theories in d dimensions with a collection of dynamical fields ϕ_i that are p_i-forms. The action S is an integral on a manifold $\mathcal{M}^d$ of a d-form Lagrangian L that depends on ϕ_i and $d\phi_i$:

$$S=\int_{\mathcal{M}^d} L(\phi_i, d\phi_i). \tag{2.1}$$

The variational principle yields

$$\delta S=\int_{\mathcal{M}^d} \delta\phi_i \frac{\vec{\partial} L}{\partial\phi_i}+d(\delta\phi_i)\frac{\vec{\partial} L}{\partial(d\phi_i)}=0. \tag{2.2}$$

All products are exterior products between forms. The symbol $\frac{\vec{\partial} L}{\partial\phi_i}$ indicates the right derivative of L with respect to a p-form ϕ_i, defined by first bringing ϕ_i to the left in L (taking into account the sign changes due to the gradings) and then canceling it against the derivative. In other words, we use the graded Leibniz rule, considering $\frac{\partial}{\partial\phi_i}$ to have the same grading as ϕ_i. Integrating by parts,[1] and since the $\delta\phi_i$ variations are arbitrary, we find the Euler-Lagrange equations:

$$d\frac{\vec{\partial} L}{\partial(d\phi_i)}-(-)^{p_i}\frac{\vec{\partial} L}{\partial\phi_i}=0. \tag{2.3}$$

III. FORM HAMILTONIAN

Here we further develop a covariant Hamiltonian formalism well adapted to geometrical theories, initiated in Refs. [10–14]. We start by defining the $(d-p_i-1)$-form momenta

$$\pi^i\equiv\frac{\vec{\partial} L}{\partial(d\phi_i)}, \tag{3.1}$$

and a d-form Hamiltonian density (sum on i),

$$H\equiv d\phi_i\pi^i-L. \tag{3.2}$$

This Hamiltonian density does not depend on the "velocities" $d\phi_i$ since

$$\frac{\vec{\partial} H}{\partial(d\phi_i)}=\pi^i-\frac{\vec{\partial} L}{\partial(d\phi_i)}=0. \tag{3.3}$$

Thus, H depends on the ϕ_i and π^i,

$$H=H(\phi_i,\pi^i), \tag{3.4}$$

and the form analogues of the Hamilton equations read

$$d\phi_i=(-)^{(d+1)(p_i+1)}\frac{\vec{\partial} H}{\partial\pi^i},\qquad d\pi^i=(-)^{p_i+1}\frac{\vec{\partial} H}{\partial\phi_i}. \tag{3.5}$$

The first equation is equivalent to the momentum definition, and is obtained by taking the right derivative of H as given in Eq. (3.2) with respect to π^i,

$$\frac{\vec{\partial} H}{\partial\pi^i}=\frac{\vec{\partial} d\phi_j}{\partial\pi^i}\pi^j+(-)^{(d-p_i-1)(p_i+1)}d\phi_i-\frac{\vec{\partial} d\phi_j}{\partial\pi^i}\frac{\vec{\partial} L}{\partial(d\phi_j)}, \tag{3.6}$$

and then using Eq. (3.1), and $(d-p_i-1)(p_i+1)=(d+1)(p_i+1)(\mathrm{mod}\,2)$.

The second is equivalent to the Euler-Lagrange form equations, since

$$\frac{\vec{\partial} H}{\partial\phi_i}=\frac{\vec{\partial} d\phi_j}{\partial\phi_i}\pi^j-\frac{\vec{\partial} L}{\partial\phi_i}-\frac{\vec{\partial} d\phi_j}{\partial\phi_i}\frac{\vec{\partial} L}{\partial(d\phi_j)}=-\frac{\vec{\partial} L}{\partial\phi_i}, \tag{3.7}$$

because of the momenta definitions (3.1). Then, using Eq. (2.3) yields the form-Hamilton equation for $d\pi^i$.

IV. EXTERIOR DIFFERENTIAL AND FORM-POISSON BRACKET

The form-Hamilton equations allow to express the (on-shell) exterior differential of any p-form $F(\phi_i,\pi^i)$ as

$$dF=d\phi_i\frac{\vec{\partial} F}{\partial\phi_i}+d\pi^i\frac{\vec{\partial} F}{\partial\pi^i}=(-)^{(d+1)(p_i+1)}\frac{\vec{\partial} H}{\partial\pi^i}\frac{\vec{\partial} F}{\partial\phi_i}+(-)^{p_i+1}\frac{\vec{\partial} H}{\partial\phi_i}\frac{\vec{\partial} F}{\partial\pi^i}. \tag{4.1}$$

Using left derivatives, this expression simplifies to

[1] With a trivial boundary of $\mathcal{M}^d$, or appropriate boundary conditions.

$$dF = \frac{\overleftarrow{\partial} H}{\partial \pi^i}\frac{\overrightarrow{\partial} F}{\partial \phi_i} - (-)^{p_i d}\frac{\overleftarrow{\partial} H}{\partial \phi_i}\frac{\overrightarrow{\partial} F}{\partial \pi^i}. \tag{4.2}$$

Note: Left derivatives are defined as "acting on the left" and, for example, $\frac{\overleftarrow{\partial} H}{\partial \phi_i}$ really means $\frac{H\overleftarrow{\partial}}{\partial \phi_i}$. It is easy to verify[2] that the left and right derivatives of a f-form F with respect to a a-form A satisfy

$$\frac{\overleftarrow{\partial} F}{\partial A} = (-)^{a(f+1)}\frac{\overrightarrow{\partial} F}{\partial A}, \tag{4.3}$$

and this relation is used to prove Eq. (4.2).

The expression for the differential (4.2) suggests the definition of the *form-Poisson bracket*,

$$\{A, B\} \equiv \frac{\overleftarrow{\partial} B}{\partial \pi^i}\frac{\overrightarrow{\partial} A}{\partial \phi_i} - (-)^{p_i d}\frac{\overleftarrow{\partial} B}{\partial \phi_i}\frac{\overrightarrow{\partial} A}{\partial \pi^i}, \tag{4.4}$$

so that

$$dF = \{F, H\}. \tag{4.5}$$

Note 1: The form-Poisson bracket between the a-form A and the b-form B is a $(a+b-d+1)$-form, and canonically conjugated forms satisfy

$$\{\phi_i, \pi^j\} = \delta_i^j. \tag{4.6}$$

Note 2: A different definition of the form-Poisson bracket was given in Ref. [10], based on postulated properties of the FPB rather than on the Legendre transformation that leads to the evolution Eq. (4.5). In fact, the properties of the FPB in Ref. [10] differ from the ones given in the next section, which are *deduced* from the definition (4.4).

V. PROPERTIES OF THE FORM-POISSON BRACKET

Using the definition (4.4), the following relations can be shown to hold:

$$\{B, A\} = -(-)^{(a+d+1)(b+d+1)}\{A, B\}, \tag{5.1}$$

$$\{A, BC\} = B\{A, C\} + (-)^{c(a+d+1)}\{A, B\}C, \tag{5.2}$$

$$\{AB, C\} = \{A, C\}B + (-)^{a(c+d+1)}A\{B, C\}, \tag{5.3}$$

$$(-)^{(a+d+1)(c+d+1)}\{A, \{B, C\}\} + \text{cyclic} = 0, \tag{5.4}$$

$$(-)^{(a+d+1)(b+d+1)}\{\{B, C\}, A\} + \text{cyclic} = 0, \tag{5.5}$$

i.e., graded antisymmetry, the derivation property, and the form-Jacobi identities.

VI. INFINITESIMAL CANONICAL TRANSFORMATIONS

We can define the action of infinitesimal form-canonical transformations on any a-form A as follows:

$$\delta A = \varepsilon\{A, G\}, \tag{6.1}$$

where G is a $(d-1)$-form, the generator of the canonical transformation, and ε is an infinitesimal parameter depending only on the $\mathcal{M}^d$ coordinates. Then, $\{A, G\}$ is a a-form like A. We now prove that these transformations preserve the canonical FPB relations (4.6), thus deserving the name form-canonical transformations. As in the usual case, the proof involves the Jacobi identities applied to ϕ_i, π^j, and G:

$$\begin{aligned}&\{\{\phi_i, \pi^j\}, G\} + (-)^{p_i(p_i+d+1)}\{\{\pi^j, G\}, \phi_i\}\\ &\quad + \{\{G, \phi_i\}, \pi^j\} = 0.\end{aligned} \tag{6.2}$$

Using the graded antisymmetry of the FPB, this reduces to

$$\{\phi_i, \{\pi^j, G\}\} + \{\{\phi_i, G\}, \pi^j\} = \{\{\phi_i, \pi^j\}, G\} = 0, \tag{6.3}$$

since $\{\phi_i, \pi^j\} = \delta_i^j$ is a number. Then,

$$\begin{aligned}\{\phi'_i, \pi'^j\} &= \{\phi_i + \varepsilon\{\phi_i, G\}, \pi^j + \varepsilon\{\pi^j, G\}\}\\ &= \{\phi_i, \pi^j\} + \varepsilon\{\phi_i, \{\pi^j, G\}\} + \varepsilon\{\{\phi_i, G\}, \pi^j\}\\ &\quad + O(\varepsilon^2)\\ &= \{\phi_i, \pi^j\} + O(\varepsilon^2).\end{aligned} \tag{6.4}$$

Q.E.D.

VII. FORM-CANONICAL ALGEBRAS

The commutator of two infinitesimal canonical transformations generated by the $(d-1)$-forms G_1 and G_2 is again an infinitesimal canonical transformation, generated by the $(d-1)$-form $\{G_1, G_2\}$. This is due to the fact that

$$\{G_1, G_2\} = -\{G_2, G_1\} \tag{7.1}$$

for $(d-1)$-form entries, and the form-Jacobi identity

$$\{\{A, G_1\}, G_2\} - \{\{A, G_2\}, G_1\} = \{A, \{G_1, G_2\}\} \tag{7.2}$$

holds for any p-form A. Therefore, the form-canonical transformations close an algebra. This algebra is finite dimensional if all fundamental fields ("positions and momenta") are p-forms with $p \geq 1$, since there is only a finite number of $(d-1)$-form polynomials made out of

[2]Suppose that A is contained in F as $F = F_1 A F_2$. Then, $\frac{\overrightarrow{\partial} F}{\partial A} = (-)^{af_1}F_1F_2$ and $\frac{\overleftarrow{\partial} F}{\partial A} = (-)^{af_2}F_1F_2$, so that $\frac{\overleftarrow{\partial} F}{\partial A} = (-)^{a(f_1+f_2)}\frac{\overrightarrow{\partial} F}{\partial A} = (-)^{a(f-a)}\frac{\overrightarrow{\partial} F}{\partial A}$ and Eq. (4.3) follows.

the fundamental fields. On the other hand, if there are fundamental 0-forms, the algebra becomes infinite dimensional because there are infinitely many $(d-1)$-form polynomials.

Consider as an example a collection of 1-form fundamental fields ϕ_i $(i = 1, \ldots n)$ in $d = 4$. Their conjugate momenta are 2-form fields π^i. There are only two types of 3-form polynomials in these fields:

$$G_{ijk} = \phi_i \phi_j \phi_k, \qquad G_i^j = \phi_i \pi^j. \tag{7.3}$$

Their (finite) Poisson bracket algebra reads

$$\{G_{ijk}, G_{lmn}\} = 0, \qquad \{G_{ijk}, G_l^m\} = 3\delta^m_{[k} G_{ij]l},$$
$$\{G_i^j, G_k^l\} = \delta_i^l G_k^j - \delta_k^j G_i^l, \tag{7.4}$$

with $m = \binom{n}{3}$ generators G_{ijk} closing on a $U(1)^m$ subalgebra and n^2 generators G_i^j closing on a $U(n)$ subalgebra. The whole algebra is then a semidirect sum of $U(n)$ with $U(1)^m$.

VIII. ACTION INVARIANCE AND NOETHER'S THEOREM

A. Global invariances

Consider the action

$$S = \int_{\mathcal{M}^d} d\phi_i \pi^i - H. \tag{8.1}$$

Its variation under an infinitesimal form-canonical transformation generated by a $(d-1)$-form G is

$$\begin{aligned}
\delta S &= \int_{\mathcal{M}^d} d(\{\phi_i, G\})\pi^i + d\phi_i \{\pi^i, G\} - \{H, G\} \\
&= \int_{\mathcal{M}^d} d(\{\phi_i, G\}\pi^i) + (-)^{p_i+1}\{\phi_i, G\} d\pi^i \\
&\quad + d\phi_i \{\pi^i, G\} - \{H, G\} \\
&= \int_{\mathcal{M}^d} d(\{\phi_i, G\}\pi^i) + (-)^{p_i+1} \frac{\overleftarrow{\partial} G}{\partial \pi^i} d\pi^i - (-)^{p_i d} d\phi_i \frac{\overleftarrow{\partial} G}{\partial \phi_i} \\
&\quad - \{H, G\} \\
&= \int_{\mathcal{M}^d} d(\{\phi_i, G\}\pi^i) + (-)^{p_i+1}(-)^{p_i} d\pi^i \frac{\vec{\partial} G}{\partial \pi^i} \\
&\quad - (-)^{p_i d}(-)^{p_i d} d\phi_i \frac{\vec{\partial} G}{\partial \phi_i} - \{H, G\} \\
&= \int_{\mathcal{M}^d} d(\{\phi_i, G\}\pi^i) - d\pi^i \frac{\vec{\partial} G}{\partial \pi^i} - d\phi_i \frac{\vec{\partial} G}{\partial \phi_i} - \{H, G\} \\
&= \int_{\mathcal{M}^d} d(\{\phi_i, G\}\pi^i) - dG - \{H, G\} \\
&= \int_{\partial \mathcal{M}^d} (\{\phi_i, G\}\pi^i - G) - \int_{\mathcal{M}^d} \{H, G\}.
\end{aligned} \tag{8.2}$$

Thus, the action is invariant (up to a boundary term) under the infinitesimal canonical-form transformation generated by G if and only if

$$\{H, G\} = 0 \tag{8.3}$$

up to a total derivative. This result reproduces Noether's theorem in form language.

Note: Here G is a polynomial in the ϕ_i and π^i. In this case,

$$dG = d\pi^i \frac{\vec{\partial} G}{\partial \pi^i} + d\phi_i \frac{\vec{\partial} G}{\partial \phi_i} \tag{8.4}$$

has been used in the sixth line of Eq. (8.2). Generators containing spacetime functions $f(x)$ ("external fields") are considered in the next paragraph. On shell, we have

$$dG = \{G, H\}. \tag{8.5}$$

Thus, if G generates an invariance of the action, on shell its exterior derivative vanishes. Consider then the d-dimensional integral

$$\int dG \tag{8.6}$$

between two $(d-1)$-dimensional spacelike slices $\mathcal{S}_{t_1}$ and $\mathcal{S}_{t_2}$ of the $\mathcal{M}^d$ manifold corresponding to the times t_1 and t_2. By Stokes' theorem this integral is equal to the difference between the integrals of G on the $\mathcal{S}_{t_2}$ and $\mathcal{S}_{t_1}$ slices, and since $dG = 0$, this difference vanishes, implying that the 0-form quantity

$$\mathcal{G}(t) = \int_{\mathcal{S}_t} G \tag{8.7}$$

is conserved in time on the shell of the equations of motion.[3]

B. Gauge invariances generated by $\varepsilon(x)G$

Here we consider generators of the type $\varepsilon(x)G$, generating x-dependent infinitesimal form-canonical transformations:

$$\delta\phi_i = \varepsilon(x)\{\phi_i, G\}, \qquad \delta\pi^i = \varepsilon(x)\{\pi^i, G\}. \tag{8.8}$$

The variation of the action is computed along the same lines as in the preceding subsection, with an additional term due to the infinitesimal parameter ε being nonconstant:

[3] If $\{G, H\} = dW$, then $d(G - W) = 0$ on shell and $\int_{\mathcal{S}_t} G - W$ is conserved in time.

$$\delta S = \int_{\partial\mathcal{M}^d} \varepsilon(\{\phi_i, G\}\pi^i - G) + \int_{\mathcal{M}^d} (d\varepsilon G - \varepsilon\{H, G\}). \tag{8.9}$$

Thus, $\varepsilon(x)G$ is a gauge generator, leaving the action invariant (up to boundary terms) if and only if

$$G = 0, \qquad \{H, G\} = 0 \tag{8.10}$$

since $\varepsilon(x)$ is an arbitrary function. Thus, G and $\{H, G\}$ must be *constraints*.

If there is a collection of $(d-1)$-forms G_A generating local invariances of the action,[4] the commutator of two transformations generated by G_1 and G_2 must leave the action invariant. This commutator is generated by $\{G_1, G_2\}$ because of the Jacobi identities. Therefore, $\{G_A, G_B\}$ is a gauge generator. The gauge algebra can involve structure constants

$$\{G_A, G_B\} = C^C_{AB} G_C, \tag{8.11}$$

as in ordinary finite Lie algebras, or structure functions, as is the case for diffeomorphisms in gravity theories.

Finally, the infinitesimal transformations generated by $\varepsilon(x)G$ must preserve the constraints, and therefore

$$\{\text{constraints}, G\} \approx 0, \tag{8.12}$$

where $\approx$ means a weak equality, i.e., it holds on the constraint surface.

C. Gauge invariances generated by $\varepsilon(x)G + (d\varepsilon)F$

In gauge and gravity theories the infinitesimal symmetry transformations on the fields also contain derivatives of the x-dependent parameter. Therefore, we need to consider generators of the form $\varepsilon(x)G + (d\varepsilon)F$, where F is a $(d-2)$-form, and investigate how they transform the action. The answer is

$$\begin{aligned}\delta S = &\int_{\partial\mathcal{M}^d} \varepsilon(\{\phi_i, G\}\pi^i - G) + d\varepsilon(\{\phi_i, F\}\pi^i - F) \\ &+ \int_{\mathcal{M}^d} [d\varepsilon(G - \{H, F\}) - \varepsilon\{H, G\}].\end{aligned} \tag{8.13}$$

Thus, $\varepsilon(x)G + (d\varepsilon)F$ is a gauge generator leaving the action invariant if and only if

$$G - \{H, F\} = 0, \qquad \{H, G\} = 0. \tag{8.14}$$

Moreover, the infinitesimal transformation generated by $\varepsilon(x)G + (d\varepsilon)F$ must preserve the constraints, implying

$$\{\text{constraints}, G\} \approx 0, \qquad \{\text{constraints}, F\} \approx 0. \tag{8.15}$$

The conditions (8.14) and (8.15) generalize the conditions for gauge generators found in Ref. [29] to the case of geometric theories with fundamental p-form fields, and provide the basis for a constructive algorithm yielding all of the gauge generators. We illustrate the procedure in the next sections.

Note 1: F and G must be first-class quantities, i.e., have weakly vanishing FPBs with all the constraints, but they do not necessarily have to be constraints.

Note 2: This section reproduces the results of Ref. [29], in the present context of geometric theories with fundamental p-forms.

Note 3: In the form setting the time derivatives of the usual canonical formalism become exterior derivatives, and due to $d^2 = 0$ gauge generators cannot contain second or higher derivatives of ε. Thus, geometric theories do not give rise to tertiary constraints, since these would multiply second derivatives of the gauge parameter in the gauge generator chains [29].

IX. GRAVITY IN $d = 4$

A. Form Hamiltonian and constraints

The fields ϕ_i in this case are 1-forms: the vierbein V^a and the spin connection ω^{ab}. Torsion and Lorentz curvature are defined as usual,

$$R^a = dV^a - \omega^a{}_b V^b, \qquad R^{ab} = d\omega^{ab} - \omega^a{}_e \omega^{eb}, \tag{9.1}$$

and the Einstein-Hilbert 4-form Lagrangian is

$$\begin{aligned}L(\phi, d\phi) = R^{ab} V^c V^d \varepsilon_{abcd} = d\omega^{ab} V^c V^d \varepsilon_{abcd} \\ - \omega^a{}_e \omega^{eb} V^c V^d \varepsilon_{abcd}.\end{aligned} \tag{9.2}$$

The 2-form momenta conjugate to V^a and ω_{ab} are, respectively,[5]

$$\pi_a = \frac{\partial L}{\partial(dV^a)} = 0, \tag{9.3}$$

$$\pi_{ab} = \frac{\partial L}{\partial(d\omega^{ab})} = V^c V^d \varepsilon_{abcd}. \tag{9.4}$$

Both momenta definitions are *primary constraints*,

$$\Phi_a \equiv \pi_a = 0, \qquad \Phi_{ab} \equiv \pi_{ab} - V^c V^d \varepsilon_{abcd} = 0, \tag{9.5}$$

since they do not involve the "velocities" dV^a and $d\omega^{ab}$. The form Hamiltonian is

[4] Here and in the following, the invariance of the action will be understood up to surface terms.

[5] Unless stated otherwise, all partial derivatives act from the left in the following.

$$
\begin{aligned}
H &= dV^a \pi_a + d\omega^{ab}\pi_{ab} - d\omega^{ab} V^c V^d \varepsilon_{abcd} \\
&\quad + \omega^a{}_e \omega^{eb} V^c V^d \varepsilon_{abcd} \\
&= dV^a \Phi_a + d\omega^{ab}\Phi_{ab} + \omega^a{}_e \omega^{eb} V^c V^d \varepsilon_{abcd}. \qquad (9.6)
\end{aligned}
$$

The "velocities" dV^a and $d\omega^{ab}$ are undetermined at this stage. Indeed, the Hamilton equations of motion for dV^a and $d\omega^{ab}$ are just identities ($dV^a = dV^a$, $d\omega^{ab} = d\omega^{ab}$), whereas for the momenta they read

$$
d\pi_a = \frac{\partial H}{\partial V^a} = -2R^{bc} V^d \epsilon_{abcd}, \qquad (9.7)
$$

$$
d\pi_{ab} = \frac{\partial H}{\partial \omega^{ab}} = 2\omega^c_{[a} V^d V^e \epsilon_{b]cde}. \qquad (9.8)
$$

Requiring the "conservation" of Φ_a and Φ_{ab}, i.e., their *closure* in the present formalism, leads to the conditions

$$
d\Phi_a = \{\Phi_a, H\} = 0 \Rightarrow R^{bc} V^d \varepsilon_{abcd} = 0, \qquad (9.9)
$$

$$
d\Phi_{ab} = \{\Phi_{ab}, H\} = 0 \Rightarrow R^c V^d \varepsilon_{abcd} = 0. \qquad (9.10)
$$

To derive Eq. (9.10) we also made use of the identity

$$
F^e_{[a} \varepsilon_{bcd]e} = 0, \qquad (9.11)
$$

which holds for any antisymmetric F. The conditions (9.9) and (9.10) are, respectively, equivalent to the Einstein field equations and the zero-torsion condition $R^a = 0$, which enables to express the spin connection in terms of the vierbein. Note that we cannot call them secondary constraints, since they contain the "velocities" dV^a and $d\omega^{ab}$. In fact, they determine dV^a as

$$
dV^a = \omega^a{}_b V^b \qquad (9.12)
$$

and determine some (combinations of) components of $d\omega^{ab}$ by constraining R^{ab} via the Einstein equations.

Using the form bracket, we find the constraint algebra

$$
\{\Phi_a, \Phi_b\} = \{\Phi_{ab}, \Phi_{cd}\} = 0; \quad \{\Phi_a, \Phi_{bc}\} = -2\varepsilon_{abcd} V^d, \qquad (9.13)
$$

showing that the constraints are not all first class. This is consistent with the fact that some of the undetermined "velocities" get fixed by requiring conservation of the primary constraints. Classical works on constrained Hamiltonian systems can be found in Refs. [30–32], while the (usual) canonical treatment of vierbein first-order gravity in Dirac's formalism was first given in Ref. [33].

Note: The action variations (8.9) and (8.13) have been deduced assuming that H depends only on basic fields and momenta. This is not the case in constrained systems, where some of the velocities remain undetermined, and therefore appear in the Hamiltonian. However, they always appear multiplied by primary constraints, and the variation of these terms always vanishes weakly.

B. Gauge generators

1. Lorentz gauge transformations

We start from the first-class 2-forms π_{ab}, which have vanishing FPBs with the constraints Φ_a and Φ_{ab}. They will play the role of the $(d-2)$-forms F of Sec. VIII C, with two antisymmetric indices, and thus $F_{ab} = \pi_{ab}$. To find the corresponding $(d-1)$-form G_{ab} that completes the gauge generators one uses the first condition in Eq. (8.14), yielding G_{ab} as the PB of H with F_{ab}, up to constraints. Since

$$
\{H, \pi_{ab}\} = 2\omega_{[a}{}^e V^c V^d \epsilon_{b]ecd}, \qquad (9.14)
$$

we find that

$$
G_{ab} = 2\omega_{[a}{}^e V^c V^d \epsilon_{b]ecd} + \alpha^c_{ab}\Phi_c + \beta^{cd}_{ab}\Phi_{cd}, \qquad (9.15)
$$

where α^c_{ab} and β^{cd}_{ab} are 1-form coefficients to be determined by the second condition in Eq. (8.14), i.e., the weak vanishing of the PB between H and G_{ab}. This yields

$$
\alpha^c_{ab} = \delta^c_{[a} V_{b]}, \beta^{cd}_{ab} = -2\omega^c_{[a}\delta^d_{b]}, \qquad (9.16)
$$

so that G_{ab} becomes

$$
G_{ab} = 2\omega^c{}_{[a}\pi_{b]c} - V_{[a}\pi_{b]}. \qquad (9.17)
$$

It is easy to check that this G_{ab} has weakly vanishing PBs with the constraints Φ_a and Φ_{ab} and is therefore a first-class 3-form. We have thus constructed the gauge generator

$$
\begin{aligned}
\mathbb{G} &= \varepsilon^{ab} G_{ab} + d\varepsilon^{ab} F_{ab} = \varepsilon^{ab}(2\omega^c{}_a \pi_{bc} - V_a \pi_b) + (d\varepsilon^{ab})\pi_{ab} \\
&= \mathcal{D}\varepsilon^{ab}\pi_{ab} - \varepsilon^{ab} V_a \pi_b. \qquad (9.18)
\end{aligned}
$$

It generates the Lorentz gauge rotations on all canonical variables. Indeed,

$$
\delta V^a = \{V^a, \mathbb{G}\} = \varepsilon^a{}_b V^b, \qquad \delta\omega^{ab} = \{\omega^{ab}, \mathbb{G}\} = \mathcal{D}\varepsilon^{ab}, \qquad (9.19)
$$

$$
\delta\pi_a = \{\pi_a, \mathbb{G}\} = \varepsilon_a{}^b \pi_b, \qquad \delta\pi_{ab} = \{\pi_{ab}, \mathbb{G}\} = \varepsilon^c{}_{[a}\pi_{b]c}, \qquad (9.20)
$$

and it satisfies all of the conditions required to be a symmetry generator of the action.

X. LIE DERIVATIVE AND DIFFEOMORPHISMS

Infinitesimal diffeomorphisms on p-forms A are expressed by means of the Lie derivative ℓ_ε:

$$\delta A = \ell_\varepsilon A \equiv (\iota_\varepsilon d + d\iota_\varepsilon)A, \tag{10.1}$$

where ι_ε is the contraction along the tangent vector $\varepsilon(x) = \varepsilon^\mu(x)\partial_\mu$. Geometric theories are by construction invariant under diffeomorphisms, since the action is an integral of a d-form on a d-dimensional manifold.

The variations under infinitesimal diffs of the basic fields of $d = 4$ first-order tetrad gravity are

$$\delta V^a = \iota_\varepsilon dV^a + d(\iota_\varepsilon V^a) = \mathcal{D}\varepsilon^a + 2R^a{}_{bc}\varepsilon^b V^c + (\varepsilon^\mu \omega_\mu^{ab})V_b, \tag{10.2}$$

$$\delta\omega^{ab} = \iota_\varepsilon d\omega^{ab} + d(\iota_\varepsilon \omega^{ab}) = 2R^{ab}{}_{cd}\varepsilon^c V^d + 2(\varepsilon^\mu \omega_\mu^{c[a})\omega^{b]}{}_c, \tag{10.3}$$

where $\varepsilon^a \equiv \varepsilon^\mu V_\mu^a$, $\mathcal{D}$ is the Lorentz covariant derivative $\mathcal{D}\varepsilon^a \equiv d\varepsilon^a - \omega^a{}_b\varepsilon^b$, and R^a_{bc} are the flat components of the torsion 2-form R^a, and thus $R^a = R^a_{bc}V^bV^c$ and similarly for the Lorentz curvature R^{ab}.

The infinitesimal diffs on the momenta 2-forms are given by

$$\delta\pi_a = \iota_\varepsilon d\pi_a + d(\iota_\varepsilon\pi_a) = \iota_\varepsilon(\mathcal{D}\pi_a) + \mathcal{D}(\iota_\varepsilon\pi_a) + (\varepsilon^\mu\omega_a{}^b{}_\mu)\pi_b, \tag{10.4}$$

$$\delta\pi_{ab} = \iota_\varepsilon d\pi_{ab} + d(\iota_\varepsilon\pi_{ab}) = \iota_\varepsilon(\mathcal{D}\pi_{ab}) + \mathcal{D}(\iota_\varepsilon\pi_{ab}) + 2(\varepsilon^\mu\omega^c{}_{[a\mu})\pi_{b]c}. \tag{10.5}$$

We see that in all of these variations the last term is really a Lorentz rotation with parameter $\eta^{ab} = \varepsilon^\mu\omega_\mu^{ab}$. As the action is invariant under Lorentz transformations, the variations

$$\delta V^a = \mathcal{D}\varepsilon^a + 2R^a{}_{bc}\varepsilon^b V^c, \tag{10.6}$$

$$\delta\omega^{ab} = 2R^{ab}{}_{cd}\varepsilon^c V^d, \tag{10.7}$$

$$\delta\pi_a = \iota_\varepsilon(\mathcal{D}\pi_a) + \mathcal{D}(\iota_\varepsilon\pi_a), \tag{10.8}$$

$$\delta\pi_{ab} = \iota_\varepsilon(\mathcal{D}\pi_{ab}) + \mathcal{D}(\iota_\varepsilon\pi_{ab}) \tag{10.9}$$

generate the symmetries of the action by themselves. In fact, Eqs. (10.6) and (10.7) are the diff transformations deduced from the group manifold approach to first-order tetrad gravity; see, e.g., Refs. [5,9].

We may wonder whether the infinitesimal diffs could be expressed as canonical transformations via the FPB. In the present form-canonical scheme this seems impossible. The reason is that the would-be generator of the diffs, of the type

$$\mathbb{G} = \varepsilon(x)G + (d\varepsilon)F, \tag{10.10}$$

should be such that the 2-form F is a first-class quantity. However, there is only one such quantity, namely, π_{ab}, which we have already used in the construction of the Lorentz canonical generators. Indeed, π_a does not have weakly vanishing FPBs with the constraints Φ_{ab}. We can write down a canonical generator that reproduces the correct infinitesimal diffs on V^a and ω^{ab},

$$\mathbb{G} = \varepsilon^a(2R^b{}_{ac}V^c\pi_b + 2R^{bc}{}_{ad}V^d\pi_{bc}) + (\mathcal{D}\varepsilon^a)\pi_a, \tag{10.11}$$

but this $\mathbb{G}$ does not generate the correct diffs on the momenta π_a and π_{ab} and does not satisfy all of the conditions of Sec. VIII for a gauge generator.

XI. GRAVITY IN $d = 3$

A. Form Hamiltonian and constraints

The fields ϕ_i are the $d = 3$ vierbein V^a and the spin connection ω^{ab}. The torsion R^a and Lorentz curvature R^{ab} are defined as in Eq. (9.1), and the Einstein-Hilbert 3-form Lagrangian is

$$L(\phi, d\phi) = R^{ab}V^c\varepsilon_{abc} = d\omega^{ab}V^c\varepsilon_{abc} - \omega^a{}_e\omega^{eb}V^c\varepsilon_{abc}. \tag{11.1}$$

The 1-form momenta conjugate to V^a and ω_{ab} are, respectively,

$$\pi_a = \frac{\partial L}{\partial(dV^a)} = 0, \tag{11.2}$$

$$\pi_{ab} = \frac{\partial L}{\partial(d\omega^{ab})} = V^c\varepsilon_{abc}. \tag{11.3}$$

Both momenta definitions are *primary constraints*,

$$\Phi_a \equiv \pi_a = 0, \qquad \Phi_{ab} \equiv \pi_{ab} - V^c\varepsilon_{abc} = 0 \tag{11.4}$$

since they do not involve the "velocities" dV^a and $d\omega^{ab}$. The 3-form Hamiltonian is

$$H = dV^a\pi_a + d\omega^{ab}\pi_{ab} - d\omega^{ab}V^c\varepsilon_{abc} + \omega^a{}_e\omega^{eb}V^c\varepsilon_{abc} = \tag{11.5}$$

$$= dV^a\Phi_a + d\omega^{ab}\Phi_{ab} + \omega^a{}_e\omega^{eb}V^c\varepsilon_{abc}. \tag{11.6}$$

The Hamilton equations of motion for dV^a and $d\omega^{ab}$ are identities, while for the momenta they read

$$d\pi_a = \frac{\partial H}{\partial V^a} = -R^{bc}\epsilon_{abc}, \tag{11.7}$$

$$d\pi_{ab} = \frac{\partial H}{\partial\omega^{ab}} = 2\omega^c{}_{[a}V^d\epsilon_{b]cd}. \tag{11.8}$$

Requiring the "conservation" of Φ_a and Φ_{ab} leads to the conditions

$$d\Phi_a = \{\Phi_a, H\} = 0 \Rightarrow R^{bc}\varepsilon_{abc} = 0, \tag{11.9}$$

$$d\Phi_{ab} = \{\Phi_{ab}, H\} = 0 \Rightarrow R^{c}\varepsilon_{abc} = 0, \tag{11.10}$$

implying the vanishing of both curvatures: $R^a = 0$, $R^{ab} = 0$. These are the equations of motion of $d = 3$ first-order vielbein gravity. These equations completely determine the "velocities" dV^a and $d\omega^{ab}$:

$$dV^a = \omega^a{}_b V^b, \qquad d\omega^a{}_b = \omega^a{}_c \omega^{cb}. \tag{11.11}$$

Using the form bracket, we find the constraint algebra

$$\{\Phi_a, \Phi_b\} = \{\Phi_{ab}, \Phi_{cd}\} = 0, \qquad \{\Phi_a, \Phi_{bc}\} = -\varepsilon_{abc}, \tag{11.12}$$

and all other FPBs vanish. Thus, constraints are second class, which is consistent with the fact that all of the "velocities" are fixed by requiring conservation of the primary constraints. The three constraints Φ_{ab} ($ab = 12$, 13, 23) are equivalent to the three linear combinations $\Xi^a = \frac{1}{2}\epsilon^{abc}\Phi_{bc}$, and we find

$$\{\Phi_a, \Xi^b\} = \delta^b_a. \tag{11.13}$$

We will use the Ξ^a in the definition of Dirac brackets of next section. Note that form-Poisson brackets between 1-forms are symmetric in $d = 3$, and in all odd dimensions; see Eq. (5.1). Also, the FPBs between constraints yield numbers in $d = 3$ gravity, and this allows a definition of form-Dirac brackets (see next section). A similar definition is not available in $d = 4$, since the FPBs between constraints yield 1-forms, and the corresponding FPB matrix has no obvious inverse.

B. Form-Dirac brackets

We define form-Dirac brackets as follows:

$$\{f, g\}^* \equiv \{f, g\} - \{f, \Phi_a\}\{\Xi^a, g\} - \{f, \Xi^a\}\{\Phi_a, g\}. \tag{11.14}$$

These brackets vanish strongly if any entry is a constraint (Φ_a or Ξ^a). With the help of the general formulas (5.1)–(5.5), with $d = 3$ it is straightforward to show that the Dirac brackets inherit the same properties as the Poisson brackets, i.e.,

$$\{B, A\}^* = -(-)^{ab}\{A, B\}^*, \tag{11.15}$$

$$\{A, BC\}^* = B\{A, C\}^* + (-)^{ca}\{A, B\}^* C, \tag{11.16}$$

$$\{AB, C\}^* = \{A, C\}^* B + (-)^{ac} A\{B, C\}^*, \tag{11.17}$$

$$(-)^{ac}\{A, \{B, C\}^*\}^* + \text{cyclic} = 0, \tag{11.18}$$

$$(-)^{ab}\{\{B, C\}^*, A\}^* + \text{cyclic} = 0. \tag{11.19}$$

Using Dirac brackets, the second-class constraints (i.e., all of the constraints of the $d = 3$ theory) disappear, and we can use the 3-form Hamiltonian

$$H = \omega^a{}_e \omega^{eb} V^c \varepsilon_{abc}. \tag{11.20}$$

The Dirac brackets between the basic fields and their momenta are given by

$$\{V^a, V^b\}^* = 0, \qquad \{\omega^{ab}, \omega^{cd}\}^* = 0,$$
$$\{V^a, \omega^{bc}\}^* = -\frac{1}{2}\epsilon^{abc}, \tag{11.21}$$

$$\{\text{any}, \pi_a\}^* = 0, \qquad \{V^a, \pi_{bc}\}^* = 0,$$
$$\{\omega^{ab}, \pi_{cd}\}^* = \delta^{ab}_{cd}, \qquad \{\pi_{ab}, \pi_{cd}\}^* = 0. \tag{11.22}$$

Thus, V^a and $\Omega_b \equiv \epsilon_{bcd}\omega^{cd}$ become canonically conjugate variables:

$$\{V^a, \Omega_b\}^* = \delta^a_b. \tag{11.23}$$

The Hamilton equations expressed via the Dirac bracket become

$$dV^a = \{V^a, H\}^* = \{V^a, \omega^d{}_e \, \omega^{eb} V^c \varepsilon_{bcd}\}^*$$
$$= \omega^a{}_b V^b \Rightarrow R^a = 0, \tag{11.24}$$

$$d\omega^{ab} = \{\omega^{ab}, H\}^* = \{\omega^{ab}, \omega^d{}_e \, \omega^{ef} V^c \varepsilon_{fcd}\}^*$$
$$= \omega_e{}^{[a}\omega^{b]e} \Rightarrow R^{ab} = 0, \tag{11.25}$$

i.e., the field equations of $d = 3$ first-order vielbein gravity. For the "evolution" of the momenta we find

$$d\pi_a = \{\pi_a, H\}^* = 0, \tag{11.26}$$

$$d\pi_{ab} = \{\pi_{ab}, H\}^* = 2\omega^c{}_{[a} V^d \epsilon_{b]cd}$$
$$= \epsilon_{abc}\omega^c{}_d V^d \Rightarrow d\Phi_{ab} = 0, \tag{11.27}$$

where in the last line we used the identity

$$\omega_{[a}{}^d \epsilon_{bc]d} = 0. \tag{11.28}$$

The momenta evolutions reexpress the fact that the constraints are conserved, or equivalently, that the exterior derivative of the momenta is in agreement with their expression given by the second-class constraints.

C. Gauge generators

Now we apply our procedure to find the gauge generators. Here, besides the Lorentz generators, we will also find the canonical generators for diffeomorphisms.

1. Lorentz gauge transformations

We start from the first-class 1-forms π_{ab}. They are first class in the sense that they have vanishing Dirac brackets with all of the constraints. Actually, as the constraints are all second class, they have been effectively eliminated from the theory by the use of Dirac brackets. We take these 1-forms π_{ab} as the $(d-2)$-forms F in Eq. (8.14), and find the $(d-1)$-forms G that complete the gauge generator:

$$G_{ab} = \{H, F_{ab}\}^* = \{H, \pi_{ab}\}^* = 2\omega^c{}_{[a} V^d \epsilon_{b]cd}, \qquad (11.29)$$

Next, we have to check that $\{H, G_{ab}\} = 0$. Notice that here it is useless to add any combination of constraints to G_{ab}, since second-class constraints have no effect in a generator when using Dirac brackets. So $\{H, G_{ab}\}^* = 0$ must hold, with the G_{ab} as given in Eq. (11.29), and indeed this is the case: the bracket yields terms $\omega\omega V$ that sum to zero when using the $\{V, \omega\}^*$ bracket and the properties (11.16)–(11.17). Thus,

$$\mathbb{G} = d\epsilon^{ab} F_{ab} + \epsilon^{ab} G_{ab} = d\epsilon^{ab}\pi_{ab} + 2\epsilon^{ab}\omega^c{}_{[a} V^d \epsilon_{b]cd} \qquad (11.30)$$

generates gauge transformations via the Dirac bracket. Using the (second-class) constraint $\pi_{ab} = \epsilon_{abc} V^c$ in the second term of the generator yields

$$\mathbb{G} = d\epsilon^{ab}\pi_{ab} + 2\epsilon^{ab}\omega^c{}_{[a}\pi_{b]c} = (\mathcal{D}\epsilon^{ab})\pi_{ab}. \qquad (11.31)$$

It generates local Lorentz transformations with parameter $\epsilon_{ab}(x)$, since

$$\delta V^a = \{V^a, \mathbb{G}\}^* = 2\{\omega^{[b}{}_d, V^a\}^* \epsilon^{c]d}\pi_{bc} = \epsilon^a{}_b V^b, \qquad (11.32)$$

$$\delta\omega^{ab} = \{\omega^{ab}, \mathbb{G}\}^* = \mathcal{D}\epsilon^{ab}, \qquad (11.33)$$

$$\delta\pi_a = \{\pi_a, \mathbb{G}\}^* = 0, \qquad (11.34)$$

$$\delta\pi_{ab} = \{\pi_{ab}, \mathbb{G}\}^* = \{\epsilon_{abc} V^c, \mathbb{G}\}^* = \epsilon^c{}_{[a}\pi_{b]c}. \qquad (11.35)$$

Note that $\delta\pi_a = 0$ since $\mathbb{G}$ has no effect on second-class constraints.

2. Diffeomorphisms

The procedure of the preceding paragraph can be started with any 1-form; indeed, here any 1-form has vanishing Dirac brackets with the constraints. We choose F_a to be $\epsilon_{abc}\omega^{bc}$, since this 1-form is conjugate to V^a and therefore a good candidate to multiply the $d\varepsilon^a$ term in the generator of the diffeomorphisms. Then, G_a is found in the usual way:

$$G_a = \{H, F_a\}^* = \epsilon_{abc}\omega^b{}_d\omega^{dc}. \qquad (11.36)$$

Now we have to check that the second condition in Eq. (8.14) is satisfied, i.e., that

$$\{H, G_a\}^* = \{H, \epsilon_{abc}\omega^b{}_d\omega^{dc}\}^* = \epsilon_{abc}\omega^b{}_d\omega^d{}_e\omega^{ec} = 0. \qquad (11.37)$$

This is indeed so, as we can verify by specializing indices (for example, choose $a = 1$ and explicitly perform the sum on the other indices; the result vanishes because in each term $\omega\omega\omega$ two ω's always have the same indices). Therefore,

$$\mathbb{G} = d\varepsilon^a F_a + \varepsilon^a G_a = (d\varepsilon^a)\epsilon_{abc}\omega^{bc} + \varepsilon^a\epsilon_{abc}\omega^b{}_d\omega^{dc} = (\mathcal{D}\varepsilon^a)\varepsilon_{abc}\omega^{bc} \qquad (11.38)$$

generates a symmetry. Its action on the basic fields is given by

$$\delta V^a = \{V^a, \mathbb{G}\}^* = \mathcal{D}\varepsilon^a, \qquad (11.39)$$

$$\delta\omega^{ab} = \{\omega^{ab}, \mathbb{G}\}^* = 0, \qquad (11.40)$$

$$\delta\pi_a = \{\pi_a, \mathbb{G}\}^* = 0, \qquad (11.41)$$

$$\delta\pi_{ab} = \{\pi_{ab}, \mathbb{G}\}^* = \{\epsilon_{abc} V^c, \mathbb{G}\}^* = \epsilon_{abc}\mathcal{D}\varepsilon^c. \qquad (11.42)$$

This infinitesimal transformation has to be compared with the infinitesimal diffeomorphisms discussed in Sec. X. In the second-order formalism, i.e., when $R^a = 0$ holds, the above transformations of V^a and ω^{ab} are indeed diffeomorphisms, since the R^a term of Eq. (10.6) vanishes, and the variation of the spin connection can be taken equal to zero since it multiplies its own field equation when varying the action (this is the essence of the so-called *1.5 order formalism* used to prove the invariance of the $d = 4$ supergravity action under local supersymmetry variations [34]). Since the ω^{ab} field equation is equivalent to $R^a = 0$, any variation of ω^{ab} has no effect on the action when using $R^a = 0$. Thus, we can consider Eq. (11.38) to be the diffeomorphism generator of $d = 3$ gravity in the second-order formalism.

Note: The invariance of the action under the transformations (11.39)–(11.42) can be checked directly using integration by parts and the Bianchi identity $\mathcal{D}R^{ab} = 0$.

XII. A "DOUBLY COVARIANT" HAMILTONIAN FOR GRAVITY

Exploiting Lorentz symmetry, we can reformulate the form-canonical scheme for gravity in an even more covariant way. We call this scheme "doubly covariant," in the sense that not only is there no preferred time direction in the definition of the form-momenta, but all tensors appearing in the Hamiltonian and the equations of motion are Lorentz-covariant tensors.

To achieve this, it is sufficient to take as "velocities" not the exterior derivatives of V^a and ω^{ab}, but rather their Lorentz-covariant versions, i.e., the curvatures R^a and R^{ab}. The momenta are then defined as

$$\pi_a = \frac{\partial L}{\partial R^a} = 0, \tag{12.1}$$

$$\pi_{ab} = \frac{\partial L}{\partial R^{ab}} = V^c V^d \varepsilon_{abcd}. \tag{12.2}$$

Both momenta definitions coincide with those of Sec. IX and yield the same primary constraints,

$$\Phi_a \equiv \pi_a = 0, \qquad \Phi_{ab} \equiv \pi_{ab} - V^c V^d \varepsilon_{abcd} = 0, \tag{12.3}$$

since they do not involve the "velocities" R^a and R^{ab}. The doubly covariant form Hamiltonian is

$$H = R^a \pi_a + R^{ab} \pi_{ab} - R^{ab} V^c V^d \varepsilon_{abcd} = R^a \pi_a + R^{ab} \Phi_{ab}, \tag{12.4}$$

and it is a sum of primary constraints. It differs from the Hamiltonian of Sec. IX, which was not a sum of primary constraints. The Hamilton equations of motion are

$$R^a = \{V^a, H\} = R^a, \tag{12.5}$$

$$R^{ab} = \{\omega^{ab}, H\} = R^{ab}, \tag{12.6}$$

$$\mathcal{D}\pi_a = \{\pi_a, H\} = -2R^{bc} V^d \epsilon_{abcd}, \tag{12.7}$$

$$\mathcal{D}\pi_{ab} = \{\pi_{ab}, H\} = 0. \tag{12.8}$$

The FPBs here are defined so as to leave the "velocities" R^a and R^{ab} untouched.

Requiring the "covariant conservation" of Φ_a and Φ_{ab} leads to the conditions

$$\mathcal{D}\Phi_a = \{\Phi_a, H\} = 0 \Rightarrow R^{ab} V^d \varepsilon_{abcd} = 0, \tag{12.9}$$

$$\mathcal{D}\Phi_{ab} = \{\Phi_{ab}, H\} = 0 \Rightarrow R^c V^d \varepsilon_{abcd} = 0. \tag{12.10}$$

Note that to derive Eq. (12.10) we did not need the identity (9.11).

The conditions (12.9)–(12.10) are the same as those derived in Sec. IX, and likewise the constraint algebra is the same.

The doubly covariant formalism can be applied to geometric theories with a Lagrangian d-form $L = L(\phi, R)$ invariant under local gauge tangent-space symmetries, and where the variation of the "velocities" (i.e., curvatures) R is given by $\delta R = \mathcal{D}(\delta\phi)$, where $\mathcal{D}$ is the (Lorentz) covariant derivative. Indeed, consider the variational principle applied to the action

$$S = \int_{\mathcal{M}^d} L(\phi_i, R_i), \tag{12.11}$$

yielding

$$\delta S = \int_{\mathcal{M}^d} \delta\phi_i \frac{\vec{\partial} L}{\partial \phi_i} + \mathcal{D}(\delta\phi_i) \frac{\vec{\partial} L}{\partial R_i} = 0 \tag{12.12}$$

and leading to the Euler-Lagrange equations

$$\mathcal{D} \frac{\vec{\partial} L}{\partial R_i} - (-)^{p_i} \frac{\vec{\partial} L}{\partial \phi_i} = 0. \tag{12.13}$$

Defining the momenta

$$\pi^i \equiv \frac{\vec{\partial} L}{\partial R_i}, \tag{12.14}$$

the d-form Hamiltonian density

$$H \equiv R_i \pi^i - L \tag{12.15}$$

does not depend on the "velocities" R_i since

$$\frac{\vec{\partial} H}{\partial R_i} = \pi^i - \frac{\vec{\partial} L}{\partial R_i} = 0. \tag{12.16}$$

Thus, H depends on the ϕ_i and π^i,

$$H = H(\phi_i, \pi^i), \tag{12.17}$$

and the form analogues of the Hamilton equations read

$$R_i = (-1)^{(d+1)(p_i+1)} \frac{\vec{\partial} H}{\partial \pi^i}, \qquad \mathcal{D}\pi^i = (-)^{p_i+1} \frac{\vec{\partial} H}{\partial \phi_i}. \tag{12.18}$$

These equations are derived using the same reasoning as for Eq. (3.5).

XIII. CONCLUSIONS

We have extended the covariant Hamiltonian approach of Refs. [10–14] with a form-Legendre transformation that leads to a consistent definition of form-Poisson brackets. In the $d = 3$ vielbein gravity case, form-Dirac brackets can be defined. The algorithmic procedure of Ref. [29] can be generalized in this formalism, and was applied to find gauge generators for gravity in $d = 3$ and $d = 4$. Finally, a "doubly covariant" Hamiltonian was used in $d = 4$ gravity.

The formalism proposed here can be applied as it stands to supergravity theories, where p-forms abound. It could be worthwhile to use it for superspace Lagrangians with integral forms; see, e.g., Refs. [35,36]. Also, it appears to be particularly suited to noncommutative generalizations of gravity along the lines of Refs. [37,38], where the twist is defined in form language.

Finally, the form-Hamiltonian setting could be used in the quantization of geometric theories, replacing form-Poisson (or Dirac) brackets with commutators between operator-valued p-forms. The built-in Lorentz covariance would lead to a covariant quantization procedure, thus improving on conventional Hamiltonian methods where a time direction is singled out. This investigation is left to future work.

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