From gauged linear sigma models to geometric representation of $\mathbb{WCP}(N,\tilde{N})$ in 2D

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In this paper two issues are addressed. First, we discuss renormalization properties of a class of gauged linear sigma models (GLSM), which reduce to $\mathbb{WCP}(N,\tilde{N})$ nonlinear sigma models (NLSM) in the low-energy limit. Sometimes they are referred to as the Hanany-Tong models. If supersymmetry is $\mathcal{N}=(2,2)$ the ultraviolet-divergent logarithm in GLSM appears, in the renormalization of the Fayet-Iliopoulos parameter, and is exhausted by a single tadpole graph. This is not the case in the daughter NLSMs. As a result, the one-loop renormalizations are different in GLSMs and their daughter NLSMs. We explain this difference and identify its source. In particular, we show why at $N=\tilde{N}$ there is no UV logarithm in the parent GLSM, while they do appear in the corresponding NLSM. In the second part of the paper we discuss the same problem for a class of $\mathcal{N}=(0,2)$ GLSMs considered previously. In this case renormalization is not limited to one loop; all orders exact β functions for GLSMs are known. We discuss logarithmically divergent loops at one- and two-loop levels.

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I. INTRODUCTION

In 1979 Witten suggested [1] an ultraviolet (UV) completion for $\mathbb{CP}(N-1)$, one of the most popular nonlinear sigma models (NLSM), with the aim of large-N solution of the latter. He considered both nonsupersymmetric and $\mathcal{N}=(2,2)$ versions. In the supersymmetric case the UV completion is in fact a two-dimensional scalar supersymmetric quantum electrodynamics with the Fayet-Iliopoulos (FI) term and judiciously chosen n fields. UV completions of this type are referred to as gauged linear sigma models (GLSM).

The target space of $\mathbb{CP}(N-1)$ and similar models (see below) is Kählerian[1] and of the *Einstein* type. Such models are renormalizable since all higher-order corrections are proportional to the target-space metric, and, therefore, are characterized by a single coupling constant.[2] Thus, geometry of the target space is fixed up to a single scale factor.

The renormalization group (RG) flow from GLSM to NLSM is smooth; no change in the β function occurs on the way.[3] Moreover, for $\mathbb{CP}(N-1)$ we know the large-N solution that explicitly matches the dynamical scale following from the β function. For, say, the Grassmann model $\mathcal{G}(L,M)$ (here $M+L=N$) the solution is not worked out in full. However, the β functions in both regimes—GLSM and NLSM—coincide [2,3].

In [4–6] a generalization of $\mathcal{N}=(2,2)$ GLSMs was suggested and discussed. These generalizations include a number of n fields, with sign-alternating charges. Of special importance is the case in which the number of positive charges N is equal to that of the negative charges $\tilde{N}$.[4] In such GLSMs the Fayet-Iliopoulos parameter is not renormalized [assuming $\mathcal{N}=(2,2)$]. When these GLSMs are rewritten at low energies in the form of NLSMs they give rise to the so-called weighted $\mathbb{WCP}(N,\tilde{N})$ models. The target spaces in these cases are non-Einsteinian non-compact manifolds. Hence, these models are not renormalizable in the conventional sense of this

[1]More exactly, $\mathbb{CP}(N-1)$ is a particular case of the Grassmann model that, in turn, belongs to the class of compact, homogeneous symmetric Kähler manifolds.

[2]In (2,2) supersymmetric models the first loop is the only one that contributes to the coupling constant renormalization. In (0,2) models fermions do not contribute in the first loop, manifesting themselves starting from two loops.

[3]Strictly speaking, whether this statement survives beyond one loop in models other than $\mathcal{N}=(2,2)$ models is not fully known. This feature is by no means generic.

[4]The general condition is $\sum_i q_i + \sum_{\tilde{i}} \tilde{q}_{\tilde{i}} = 0$.

word.[5] We discuss these $\mathcal{N}=(0,2)$ models as well. Unlike the $\mathcal{N}=(2,2)$ case in the (0,2) models the second loop does not vanish in generic cases, resulting in "new" structures. (In those special cases when it does, the third and higher loops do not vanish.) In this paper we address the issue of RG running in the parent-daughter pairs GLSM/NLSM for such target spaces. Discussion of some previous results in the Hanany-Tong model [6] that inspired the current work can be found in [9]. Recently, a number of GLSMs with sign-alternating charges were considered, and the β functions calculated for (0,2) supersymmetric versions [10].

Our conclusions are as follows. For the class of GLSMs that upon reduction produce NLSMs of the $\mathbb{WCP}(N,\tilde{N})$ type the RG evolution is more complicated and is not smooth. The Kähler potential (and, hence, the Lagrangian) of the resulting NLSMs consists of two parts. The first part has exactly the same structure as the second term in the bare Kähler potential $K^{(0)}$; see Eq. (13). Its RG evolution produces the same formula for the renormalized coupling constant r as for the FI constant in the parent GLSM. The first term in (13) is not renormalized at all. Moreover, a new structure emerges upon RG evolution [see the second line in Eq. (26)] that receives a logarithmic in μ coefficient in the RG flow, totally unrelated to that of $r(\mu)$. Thus, the RG flow for the $\mathbb{WCP}(N,\tilde{N})$ models is not described by a single running coupling constant. The number of emergent structures grows in higher loops in the nonsupersymmetric case [see (66)], so that these NLSMs are not renormalizable in the conventional sense of this word. Is the number of the emergent structures limited in $\mathcal{N}=(0,2)$ supersymmetry? The answer to this question can be found in Sec. VIII. The RG evolution in the $N=\tilde{N}$ models at $\mu\to 0$ will be discussed in a separate publication.

The paper is organized as follows. In Sec. II we briefly outline the GLSM formalism and renormalization of the FI constant under the RG evolution. Section III is devoted to reduction to NLSMs of the $\mathbb{WCP}(N,\tilde{N})$ type. We derive geometry of the target space: metric, Riemann, and Ricci tensors, scalar curvature, etc. In Sec. IV we consider RG evolution in the $\mathbb{WCP}(N,\tilde{N})$ models. Distinct structures responsible for different effects are isolated and a general result is formulated. Section VI presents the simplest example of $\mathbb{WCP}(1,1)$ for illustration. In Sec. VII we work out the $\mathcal{N}=(0,2)$ versions of the $\mathbb{WCP}(N,\tilde{N})$ models.

[5]In [7] the notion of a generalized renormalizability of any two-dimensional NLSM is presented in the form of a quantum deformation of its geometry described by the NLSM metric. By nonrenormalizability we mean a more traditional definition for which generally speaking an infinite number of counterterms is needed to eliminate all ultraviolet logarithms. A thorough discussion of geometrical properties to be used below can be found in [8].

II. GENERAL CONSTRUCTION

We start from presenting the bosonic part of our "master" model; its versions are studied below. First, we introduce two types (or flavors) of complex fields n_i and ρ_a, with the electric charges $+1$ and -1, respectively,

$$S=\int d^2x\left\{|\nabla_\mu n_i|^2+|\tilde{\nabla}_\mu\rho_a|^2+\frac{1}{4e^2}F_{\mu\nu}^2+\frac{1}{e^2}|\partial_\mu\sigma|^2+\frac{1}{2e^2}D^2+2|\sigma|^2(|n_i|^2+|\rho_a|^2)+iD(|n_i|^2-|\rho_a|^2-r)\right\}+\text{fermions}. \tag{1}$$

The index i runs from $i=1,2,\ldots,N$ while $a=1,2,\ldots,\tilde{N}$. The action above is written in Euclidean conventions. The parameter r in the last term of Eq. (1) is dimensionless. It represents the two-dimensional Fayet-Iliopoulos term.

The U(1) gauge field A_μ acts on n and ρ through appropriately defined covariant derivatives,[6]

$$\nabla_\mu=\partial_\mu-iA_\mu,\qquad \tilde{\nabla}_\mu=\partial_\mu+iA_\mu, \tag{2}$$

reflecting the sign difference between the charges. The electric coupling constant e^2 has dimension of mass squared. A key physical scale is defined through the product

$$m_V^2=e^2r. \tag{3}$$

If $e^2\to\infty$ all auxiliary fields (i.e., D and σ) can be integrated out, and we are in the NLSM regime. All terms except the kinetic terms of n and ρ disappear from the action, while the last term reduces to the constraint

$$\sum_{i=1}^{N}|n_i|^2-\sum_{a=1}^{\tilde{N}}|\rho_a|^2=r. \tag{4}$$

However, if the normalization point $\mu^2\gg m^2$, the appropriate regime is that of GLSM. The parameter r is the only one that is logarithmically renormalized at one loop in GLSM. The only trivially calculable contribution comes from the tadpole diagram of Fig. 1. Namely,[7]

$$r(\mu)=r_{\rm UV}-\frac{N-\tilde{N}}{2\pi}\log\frac{M_{\rm UV}}{\mu}. \tag{5}$$

This renormalization vanishes if $N=\tilde{N}$ due to cancellation of charge $+1$ and -1 fields. Now we proceed to the discussion of the NLSM regime.

[6]For a generic situation,

$$\nabla_\mu=\partial_\mu-iq_iA_\mu,\qquad \tilde{\nabla}_\mu=\partial_\mu+i\tilde{q}_aA_\mu,$$

which reduces to (2) for $q_i=1$ and $\tilde{q}_a=-1$.

[7]Equation (5) assumes that r is positive and $N\geq\tilde{N}$. If $\tilde{N}>N$ one should consider negative r.

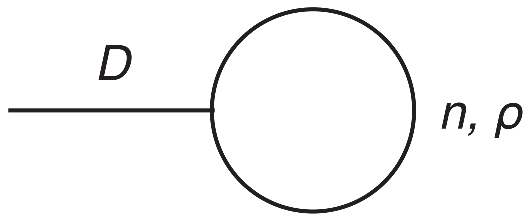

FIG. 1. Tadpole graph determining renormalization of r in the GLSM regime.

III. GEOMETRIC FORMULATION OF $\mathbb{WCP}(N,\tilde{N})$

To derive the geometric formulation we must take into account that the constraint (4) and the U(1) gauge invariance reduce the number of complex fields from $N+\tilde{N}$ in the set $\{n_i\}+\{\rho_a\}$ down to $N+\tilde{N}-1$. The choice of the coordinates on the target space manifold can be made through various patches. For the time being we choose one specific patch. Namely, the last ρ in the set $\{\rho_a\}$ (assuming it does not vanish on the selected patch) is denoted as

$$\rho_{\tilde{N}} = \varphi, \tag{6}$$

where φ is set *real*. Then the coordinates on the target manifold are

$$z_i = \varphi n_i, \qquad i = 1,2,\ldots,N,$$
$$w_a = \frac{\rho_a}{\varphi}, \qquad a = 1,2,\ldots,\tilde{N}-1. \tag{7}$$

Note that the variables introduced in (7) are not charged under U(1).

On the given patch φ can be expressed in terms of the above coordinates as

$$\varphi = \left[\frac{-r+H}{2(w_a\bar{w}_a+1)}\right]^{1/2}, \tag{8}$$

where

$$H \equiv \sqrt{r^2 + 4z^i\bar{z}_i(w_a\bar{w}_a+1)}. \tag{9}$$

The shorthand in Eq. (9) is used throughout the paper. Integrating out the gauge field we observe that

$$A_\mu = \frac{i}{2H}\left(\frac{1}{\varphi^2}z_i\overset{\leftrightarrow}{\partial}_\mu\bar{z}_i - \varphi^2 w_a\overset{\leftrightarrow}{\partial}_\mu\bar{w}_a\right). \tag{10}$$

Now we are ready to present the geometric data of the target space for $\mathbb{WCP}(N,\tilde{N})$ in the following form:

$$g_{i\bar{j}} = \frac{1}{\varphi^2}\left(\delta_{i\bar{j}} - \frac{1}{H\varphi^2}\bar{z}_i z_{\bar{j}}\right), \qquad g_{i\bar{a}} = \frac{1}{H}\bar{z}_i w_{\bar{a}}$$
$$g_{a\bar{b}} = \varphi^2\left(\delta_{a\bar{b}} - \frac{\varphi^2}{H}\bar{w}_a w_{\bar{b}}\right), \tag{11}$$

with the inverse

$$g^{i\bar{j}} = \varphi^2\left(\delta^{i\bar{j}} + \frac{1}{\varphi^4}\bar{z}^{\bar{j}}z^i\right), \qquad g^{i\bar{a}} = -\frac{1}{\varphi^2}z^i\bar{w}^{\bar{a}}$$
$$g^{a\bar{b}} = \frac{1}{\varphi^2}(\delta^{a\bar{b}} + w^a\bar{w}^{\bar{b}}). \tag{12}$$

This result can also be recovered by differentiating the corresponding Kähler potential,[8]

$$K^{(0)}(\phi_p,\bar{\phi}_{\bar{q}}) = H + 2r\log\varphi, \tag{13}$$

which was previously found in [9,11]. We consider the Kähler potential (13) and its one-loop correction in more detail in the next section. For the time being, let us move on to further discuss geometry of this target space. To this end, we first find the metric connections,

$$\Gamma^i_{jk} = \frac{-1}{H\varphi^2}(\bar{z}_j\delta^i_k + \bar{z}_k\delta^i_j) + \frac{2}{H^2\varphi^4}z^i\bar{z}_j\bar{z}_k,$$
$$\Gamma^i_{aj} = \frac{\varphi^2}{H}\bar{w}_a\delta^i_j - \frac{2}{H^2}z^i\bar{w}_a\bar{z}_j,$$
$$\Gamma^i_{ab} = \frac{2\varphi^4}{H^2}z^i\bar{w}_a\bar{w}_b,$$
$$\Gamma^a_{bc} = -\frac{\varphi^2}{H}(\bar{w}_b\delta^a_c + \bar{w}_c\delta^a_b),$$
$$\Gamma^a_{bi} = \frac{1}{H\varphi^2}\bar{z}_i\delta^a_b,$$
$$\Gamma^a_{ij} = 0, \tag{14}$$

where $1\le i,\, j\le N$ and $1\le a,\, b\le \tilde{N}-1$. Then the Ricci tensor takes the form

$$R_{i\bar{j}} = \frac{N-\tilde{N}}{r}g_{i\bar{j}} + \frac{(-r+H)[(\tilde{N}-N)H+r]}{rH^2\varphi^2}\delta_{i\bar{j}} - \frac{(-r+H)^2[(\tilde{N}-N)H+2r]}{rH^4\varphi^4}\bar{z}_i z_{\bar{j}},$$
$$R_{i\bar{a}} = \frac{N-\tilde{N}}{r}g_{i\bar{a}} + \frac{1}{rH^4}[(\tilde{N}-N)H^3 + (\tilde{N}-N)Hr^2 + 2r^3]\bar{z}_i w_{\bar{a}},$$
$$R_{a\bar{b}} = \frac{N-\tilde{N}}{r}g_{a\bar{b}} + \frac{\varphi^2(r+H)[(\tilde{N}-N)H+r]}{rH^2}\delta_{a\bar{b}} - \frac{\varphi^4(r+H)^2[(\tilde{N}-N)H+2r]}{rH^4}\bar{w}_a w_{\bar{b}}. \tag{15}$$

[8]Equation (13) coincides with (6.14) in Ref. [11] if we take into account that with our coordinate patch $4\pi/g^2$ in [11] should be replaced by $-r$.

According to (15), the target space is *not* an Einstein space. Furthermore, the scalar curvature is

$$R = \frac{2}{H^3}\{[(\tilde{N}-N)^2 + (\tilde{N}+N-2)]H^2 + 2(\tilde{N}-N)Hr + 2r^2\}. \quad (16)$$

Equation (16) implies that H is a function of scalar curvature and parameters N, $\tilde{N}$ and r, say $H = H(R, r, N, \tilde{N})$. Note that setting N equal to 0 and r negative, we should be able to recover all well-known results in $\mathbb{CP}(\tilde{N}-1)$ model. Indeed, all non-Einstein terms in Eq. (15), the last two lines, vanish because $r + H = 0$ [and so do the first five lines because we must put all terms with z_i to 0 in (15)]. Then the coefficients of the Ricci and scalar curvature also match, namely,

$$R_{a\bar{b}} \to -\frac{\tilde{N}}{r} g_{a\bar{b}} \quad \text{and} \quad R \to -\frac{2}{r}\tilde{N}(\tilde{N}-1), \quad (17)$$

with $r < 0$.

Next, we observe that the general theory of NLSMs implies at one loop (see, e.g., [7])

$$S_{\rm NLSM} = \int d^2x \bigg\{ g_{p\bar{q}}(\partial_\mu \phi^p \partial_\mu \bar{\phi}^{\bar{q}}) - \frac{1}{2\pi}\log\frac{M_{\rm UV}}{\mu} R_{p\bar{q}}(\partial_\mu \phi^p \partial_\mu \bar{\phi}^{\bar{q}}) \bigg\} + \cdots, \quad (18)$$

where $\{\phi^p\}$ is a generic coordinate of the target space. The question we address now is the relation between two results: Eq. (5) in GLSM and Eq. (15) in NLSM. Both expressions mentioned above are known in the literature [for (15) with a particular choice of N, $\tilde{N}$ see, e.g., [9]]. Clarification of their relationships is our starting goal.

IV. RENORMALIZATION IN GLSM VS NLSM

In this section, we study the renormalization of the $\mathbb{WCP}(\boldsymbol{N},\tilde{\boldsymbol{N}})$ model in the NLSM regime, and trace its origin from the parent GLSM. First of all, to discuss the renormalization structure, it is convenient to rephrase the previous results in terms of the Kähler potential.

For a Kähler manifold endowed with the Kähler potential $K(\phi_p, \bar{\phi}_{\bar{q}})$, the metric is determined by the relation

$$g_{p\bar{q}} = \frac{\partial}{\partial\phi^p}\frac{\partial}{\partial\bar{\phi}^{\bar{q}}} K(\phi_p, \bar{\phi}_{\bar{q}}), \quad (19)$$

while other components vanish. The corresponding Ricci tensor is given by

$$R_{p\bar{q}} = -\frac{\partial}{\partial\phi^p}\frac{\partial}{\partial\bar{\phi}^{\bar{q}}} \log\sqrt{g}, \quad (20)$$

where g represents the determinant of the metric tensor,

$$g = |\det\{g_{p\bar{q}}\}|. \quad (21)$$

If this manifold admits an Einstein-Kähler metric, the Ricci tenosr is proportional to its metric, in other words,

$$-\log\sqrt{g} = \alpha K(\phi_p, \bar{\phi}_{\bar{q}}) \quad (22)$$

for some constant α. Yet, our case does *not* belong to this class.

Back to our model, we can recover the result (11) by using (13) and (19). For convenience we represent it in a different form,

$$\begin{aligned}
g_{i\bar{j}} &= \frac{-r+H}{H\varphi^2}\delta_{i\bar{j}} - \frac{(-r+H)^2}{H^3\varphi^4}\bar{z}_i z_j + r\underbrace{\left(\frac{1}{H\varphi^2}\delta_{i\bar{j}} - \frac{-r+2H}{H^3\varphi^4}\bar{z}_i z_j\right)},\\
g_{i\bar{a}} &= \frac{H^2+r^2}{H^3}\bar{z}_i w_{\bar{a}} + \underbrace{\frac{-r^2}{H^3}}\bar{z}_i w_{\bar{a}},\\
g_{a\bar{b}} &= \frac{\varphi^2(r+H)}{H}\delta_{a\bar{b}} - \frac{\varphi^4(H+r)^2}{H^3} w_{\bar{b}}\bar{w}_a + r\underbrace{\left(-\frac{\varphi^2}{H}\delta_{a\bar{b}} + \frac{\varphi^4(2H+r)}{H^3} w_{\bar{b}}\bar{w}_a\right)}.
\end{aligned} \quad (23)$$

In the above formulas for the metric tensor [they are identical to (11)] we separate the contributions from H and $2r\log\varphi$, respectively, in the Kähler potential $K^{(0)}$, namely, the terms marked by the underbrace originate from $2r\log\varphi$ in Eq. (13).

From the expression (11), the metric determinant can be calculated in a straightforward manner, and we obtain

$$-\log\sqrt{g} = 2(N-\tilde{N})\log\varphi + \log H \quad (24)$$

for which the result coincides with the example in [9] with a particular pair of N, $\tilde{N}$. As a consistency check, we can apply (20) to (24) to see that it indeed reproduces (15). Also, it is instructive to explicitly indicate the Einstein part and non-Einstein part in the Ricci curvature. Namely,

$$R_{p\bar{q}} = \left(\frac{N-\tilde{N}}{r}\right) g_{p\bar{q}} + \frac{\partial}{\partial\phi^p}\frac{\partial}{\partial\bar{\phi}^{\bar{q}}}\left(\log H - \frac{N-\tilde{N}}{r}H\right). \quad (25)$$

At one-loop level, the Kähler potential acquires a correction following from (24); see also (25),

$$
\begin{aligned}
&K^{(0)}+K^{(1)} \\
&=K^{(0)}-\frac{1}{2\pi}\log\frac{M_{\rm UV}}{\mu}\left[\frac{N-\tilde{N}}{r}K^{(0)}+\left(\log H-\frac{N-\tilde{N}}{r}H\right)\right] \\
&=H+2\left(r-\frac{N-\tilde{N}}{2\pi}\log\frac{M_{\rm UV}}{\mu}\right)\log\varphi-\left(\frac{1}{2\pi}\log\frac{M_{\rm UV}}{\mu}\right)\log H.
\end{aligned} \tag{26}
$$

The correction of the coupling constant r and the $\log H$ term results from the first and the second terms in Eq. (24), respectively, while the corrections to the H term cancel. As a consistency check, considering the $\mathbb{CP}(\tilde{N}-1)$ case ($N=0$), we observe that H reduces to a constant and dropping all H terms we observe that the Kähler potential renormalizes multiplicatively. We recover the conventional $\mathbb{CP}(\tilde{N}-1)$ result.

We can immediately read off from Eq. (26) that the FI parameter is renormalized as[9]

$$
r(\mu)=r_{\rm UV}-\frac{N-\tilde{N}}{2\pi}\log\frac{M_{\rm UV}}{\mu}, \tag{27}
$$

in agreement with (5) obtained in the GLSM analysis.

As an essential example, we consider the model with the equal numbers of positive and negative charges. Then, as shown in (27), the FI parameter gets no correction and the corresponding β function vanishes. However, the Kähler potential is still modified by the one-loop contribution,

$$
K^{(0)}+K^{(1)}|_{N=\tilde{N}}=2r\log\varphi+H-\left(\frac{1}{2\pi}\log\frac{M_{\rm UV}}{\mu}\right)\log H. \tag{28}
$$

The emergent term proportional to $\log H$ does not vanish even if $N=\tilde{N}$, therefore making the theory nonrenormalizable [in the case of $\mathcal{N}=(0,2)$ supersymmetry, see Sec. VII].

Note that in the generic case $N\neq\tilde{N}$ but $N\sim\tilde{N}$ the renormalization of r scales as N while the coefficient of $\log H$ is $O(N^0)$. Then the latter can be ignored in the large-N limit.

V. WHERE DOES THE DISCREPANCY BETWEEN GLSM AND NLSM COME FROM?

The answer to the above question might seem paradoxical. Let us return to Sec. I in which it was stated that the only ultraviolet logarithm in GLSM comes from the Fayet-Iliopoulos term renormalization depicted in Fig. 1. This statement is correct. However, this does not mean that there are no other logarithms in this model (which is a two-dimensional reduction of supersymmetric quantum electrodynamics with matter fields possessing opposite charges). If we descend down in μ below m_V [see (3)], we discover logarithms of m_V/μ rather than $\log M_{\rm UV}/\mu$. The former in a sense might be called "infrared." They come from the Z factors of the matter fields in (1) and are determined by the graphs shown in Fig. 2.

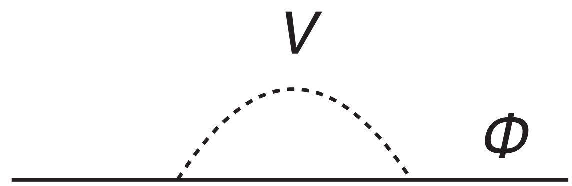

FIG. 2. Z factor of the matter fields in GLSM with $\mu\ll m_V$.

On dimensional grounds the one-loop contribution to the Z factor is proportional to

$$
e^2 m_V^{-2}\log\frac{m_V}{\mu}\sim\frac{1}{r}\log\frac{m_V}{\mu}. \tag{29}
$$

It is curious that the same type of infrared logarithms were found 45 years ago [12] in weak flavor-changing decays and are widely known now as penguins. They are typical of theories with multiple scales.

In passing from GLSMs to NLSMs we tend $m_V\to\infty$, thus identifying it with $M_{\rm UV}$. The distinction between two types of logarithms is lost.

We conclude that the Fayet-Iliopoulos parameter is related—in the NLSM formulation—to the cohomology class of the Kähler form of the target space generically defined as

$$
\omega=\frac{i}{2}g_{p\bar{q}}d\phi^p\wedge d\bar{\phi}^{\bar{q}}, \tag{30}
$$

where d is the de Rham operator.[10] Speaking in physical terms, the Kähler class can be viewed as a product of a complexified scale parameter r and an analog of the appropriately normalized topological (or θ) term. The latter takes integer values.

These remarks explain the structure of the first line in Eq. (26), as well as the emergence of extra logarithms. That is why in GLSM we recover Eq. (27), inherited from GLSM, in addition to an "extra" last term in the first line of Eq. (26).

VI. THE SIMPLEST EXAMPLE: $\mathbb{WCP}(1,1)$ MODEL

To further illustrate our analysis, let us have a closer look at the minimal example consisting of only two chiral fields,

[9] To factor out r in $K^{(0)}$ it is convenient to rescale z, namely, $z\to zr$. Then H becomes proportional to r and $K^{(0)}\to r\tilde{K}^{(0)}$ where $\tilde{K}^{(0)}$ is r independent.

[10] See [4,13] for more thorough discussions. We thank A. Gorsky, S. Ketov, A. Losev, and D. Tong for instructive correspondence on this issue.

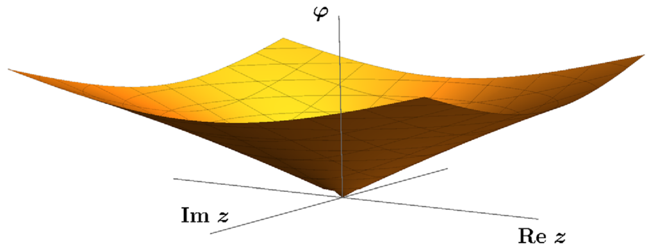

FIG. 3. Geometry of the chosen patch of the target space in the $\mathbb{WCP}(1,1)$ model.

one with the positive unit charge and the other with the negative unit charge (i.e., $N = \tilde{N} = 1$). Also, the appropriate number of Fermi superfields can be included, so we can consider either $\mathcal{N} = (2,2)$ or $\mathcal{N} = (0,2)$ theories. The bare Kähler potential in this problem is presented in [14], Sec. 52.

Note that in the given simplest case

$$H = \sqrt{r^2 + 4z\bar{z}}. \tag{31}$$

First of all, let us examine the structure of the vacuum manifold in the corresponding GLSM,

$$|n|^2 - |\rho|^2 = r. \tag{32}$$

This space is simply a four-dimensional hyperboloid. Gauging out a $U(1)$ phase we arrive at a two-dimensional target space in $\mathbb{WCP}(1,1)$ (two real dimensions).

Indeed, ρ in Eq. (6) can be chosen to be real and positive; then so is φ. Using the choice of coordinates in (6) and (7), we can reduce (32) to

$$\frac{|z|^2}{\varphi^2} - \varphi^2 = r, \tag{33}$$

illustrated in Fig. 3. From the graph, we see that the singularity at $\varphi = 0$ is 1 and the only one singular point on this patch. However, considering z and $\bar{z}$ as coordinates, we observe that

$$\varphi^2 = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + z\bar{z}} \tag{34}$$

becomes 0 at the origin, the point that must be punctured on the given patch. The constraints imposed on the fermion fields are of the type

$$\bar{n}\tau_+ - \bar{\rho}\xi_+ = 0, \tag{35}$$

which implies that the fermions live on the tangent bundle of the target manifold; see Sec. VII.

Following the same line of calculation as in Sec. III, we obtain the only nonvanishing element of the metric

$$g_{1\bar{1}} = \frac{1}{\sqrt{r^2 + 4z\bar{z}}}. \tag{36}$$

Its connections are

$$\Gamma^1_{11} = -\frac{2\bar{z}}{r^2 + 4z\bar{z}}, \quad \text{and} \quad \Gamma^{\bar{1}}_{\bar{1}\,\bar{1}} = -\frac{2z}{r^2 + 4z\bar{z}}. \tag{37}$$

In addition, it is not difficult for the curvature tensor,

$$R_{1\bar{1}1\bar{1}} = -\frac{2r^2}{(r^2 + 4z\bar{z})^{5/2}} \tag{38}$$

and the Ricci tensor,

$$R_{1\bar{1}} = \frac{2r^2}{(r^2 + 4z\bar{z})^2}. \tag{39}$$

From (36) and (39), we can explicitly see that the Ricci tensor is not proportional to the metric, and this is consistent with the fact that the target manifold is not of the Einstein type and our general analysis of Sec. IV. The scalar curvature is also computed; it reduces to

$$R = \frac{4r^2}{(r^2 + 4z\bar{z})^{3/2}}. \tag{40}$$

Now, it is time to talk about the quantum correction of this model. That is, the β function is computed as follows,

$$\beta(g_{1\bar{1}})_{\text{one-loop}} = \frac{r^2}{\pi(r^2 + 4z\bar{z})^2}. \tag{41}$$

For the $\mathcal{N} = (2,2)$ case, this is the end of the story. However, for a nonsupersymmetric model, it is not the case; i.e., it still receives the two-loop correction. Namely,

$$\beta(g_{1\bar{1}})_{\text{two-loop}} = \frac{r^2}{\pi(r^2 + 4z\bar{z})^2}\left[1 + \frac{r^2}{2\pi(r^2 + 4z\bar{z})^{3/2}}\right]; \tag{42}$$

see Sec. VIII for various $\mathcal{N} = (0,2)$ models. In this simplest example the Lagrangian (including one loop) can be written as follows:

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} = \partial_\mu z \partial^\mu \bar{z}\left[\frac{1}{H} - \left(\frac{r^2}{\pi}\log\frac{M_{\text{UV}}}{\mu}\right)\frac{1}{H^4}\right], \tag{43}$$

where H is given in (31). The first term on the right-hand side represents the bare Lagrangian that is not renormalized (remember that in the case at hand $N = \tilde{N} = 1$). The second term emerges at one loop—a different structure proportional to $\log\mu$ that is absent in the UV. It can be ignored at large $|z|$.

As an aside, if we take the $U(1)$ charges of two chiral superfields to be q and $-q$, the geometry of the target space does not change, but only the scale r from the FI term rescales as r/q. For example,

$$g_{1\bar{1}} = \frac{1}{\sqrt{(r/q)^2 + 4z\bar{z}}}. \tag{44}$$

VII. GENERALIZATION TO $\mathcal{N}=(0,2)$ $\mathbb{WCP}(N,\tilde{N})$

The $\mathcal{N}=(0,2)$ deformation discussed in this section was introduced in [10]. With the fermion fields taken into account, we can work out in this paper heterotic supersymmetric versions. As suggested in [15,16], we construct $\mathcal{N}=(0,2)$ GLSM with the gauge multiplet, Fermi multiplets, and two types of the boson chiral superfields with $U(1)$ charge $+1$ and -1.

Before proceeding to the invariant action, we recall the superfield representation for each multiplet. In this case, two chiral multiplets are

$$\begin{aligned}\mathcal{N}_i &= n_i+\sqrt{2}\theta^+\tau_{+,i}-i\theta^+\bar{\theta}^+\nabla_{++}n_i,\\ \varrho_a &= \rho_a+\sqrt{2}\theta^+\xi_{+,a}-i\theta^+\bar{\theta}^+\nabla_{++}\rho_a,\end{aligned}\tag{45}$$

the gauge multiplet is

$$\mathcal{U}_{--}=\sigma-2i\theta^+\bar{\lambda}_i-2i\bar{\theta}^+\lambda_i+2\theta^+\bar{\theta}^+D,\tag{46}$$

and, lastly, the Fermi multiplets are

$$\Gamma_{-,M}=\chi_{-M}-\sqrt{2}\theta^+G_M-i\theta^+\bar{\theta}^+\nabla_{++}\chi_{-M}.\tag{47}$$

Now, we are allowed to present the full expression of the $\mathcal{N}=(0,2)$ extension of the key model, which is nonminimal

$$\begin{aligned}S_{(0,2)}=\int d^2x\Big\{&|\nabla_\mu n_i|^2-i\bar{\tau}_{+,i}\nabla_{--}\tau^i_++|\tilde{\nabla}_\mu\rho_a|^2-i\bar{\xi}_{+,a}\tilde{\nabla}_{--}\xi^a_+\\ &+\frac{1}{4e^2}F^2_{\mu\nu}+\frac{1}{e^2}i\bar{\lambda}_-\nabla_{++}\lambda_--i\bar{\chi}_{-M}\nabla_{++}\chi^M_-\\ &+\frac{1}{2e^2}D^2+|G_M|^2+\sqrt{2}\bar{n}_i\lambda_-\tau^i_+-\sqrt{2}\bar{\tau}_{+,i}\bar{\lambda}_-n^i\\ &-\sqrt{2}\bar{\rho}_a\lambda_-\xi^a_++\sqrt{2}\bar{\xi}_{+,a}\bar{\lambda}_-\rho^a+iD(|n_i|^2-|\rho_a|^2-r)\Big\}.\end{aligned}\tag{48}$$

The covariant derivative for the χ_{-M} field is defined though its $U(1)$ charge, q_M, such that

$$\nabla_{++}\chi_{-M}=(\partial_{++}-iq_MA_{++})\chi_{-M}.\tag{49}$$

Note that the σ field is suppressed preserving (0,2) supereymmetry. Also, we notice that G_M is an auxiliary field.

In the NLSM regime, the gauge multiplet becomes auxiliary (all kinetic terms vanish in $e^2\to\infty$ limit), so the corresponding component fields (i.e., σ, λ_-, and D) can be integrated out to give the constraints. To be more precise, the D term again results in Eq. (4), and gauginos yield

$$\sum_{i=1}^{N}\bar{n}_i\tau^i_+-\sum_{a=1}^{\tilde{N}}\bar{\rho}_a\xi^a_+=0,\tag{50}$$

where the same condition applies to its Hermitian conjugate.

To obtain the geometric formulation of $\mathcal{N}=(0,2)$ $\mathbb{WCP}(N,\tilde{N})$, we follow the parallel treatment in Sec. III to eliminate $U(1)$ redundancy by setting

$$\varrho_{\tilde{N}}=\varphi+\sqrt{2}\theta^+\kappa_++\cdots,\tag{51}$$

in which φ is a *real* field and κ_+ is a *complex* Weyl fermion. Notice that $\varrho_{\tilde{N}}$ is assumed to be nowhere vanishing on the chosen patch. On the target manifold, bosonic coordinates are defined in the same way as Eq. (7) while the fermionic coordinates are

$$\begin{aligned}\zeta_{+,i}&=\kappa_+n_i+\tau_{+,i}\varphi &&\text{for } i=1,2,\ldots,N,\\ \eta_{+,a}&=\frac{1}{\varphi}\left(\xi_{+,a}-\frac{\rho_a}{\varphi}\kappa_+\right) &&\text{for } a=1,2,\ldots,\tilde{N}-1.\end{aligned}\tag{52}$$

This can be seen by taking the following parametrization for superfields:

$$\begin{aligned}Z_i&=z_i+\sqrt{2}\theta^+\zeta_{+,i}=:\mathcal{N}_i\varrho_{\tilde{N}},\\ W_a&=w_a+\sqrt{2}\theta^+\eta_{+,a}=:\varrho_a\varrho^{-1}_{\tilde{N}}.\end{aligned}\tag{53}$$

On this patch φ has the identical expression as Eq. (8) and κ_+ is written in terms of the above coordinates by

$$\kappa_+=\frac{\bar{z}_i\zeta^i_+-\varphi^4\bar{w}_a\eta^a_+}{H\varphi}.\tag{54}$$

Integrating out gauge fields we then find that

$$\begin{aligned}A_{--}&=\frac{i}{2H}\left(\frac{1}{\varphi^2}z^i\overset{\leftrightarrow}{\partial}_{--}\bar{z}_i-\varphi^2w^a\overset{\leftrightarrow}{\partial}_{--}\bar{w}_a\right)-\frac{1}{H}\sum_M q_M\bar{\chi}_{-M}\chi^M_-,\\ A_{++}&=\frac{i}{2H}\left(\frac{1}{\varphi^2}z^i\overset{\leftrightarrow}{\partial}_{++}\bar{z}_i-\varphi^2w^a\overset{\leftrightarrow}{\partial}_{++}\bar{w}_a\right)\\ &\quad+h_{i\bar{j}}\bar{\zeta}^{\bar{j}}_+\zeta^i_++h_{a\bar{b}}\bar{\eta}^{\bar{b}}_+\eta^a_++(h_{a\bar{i}}\bar{\zeta}^i_+\eta^a_++\text{H.c.}),\end{aligned}\tag{55}$$

where

$$\begin{aligned}h_{i\bar{j}}&=\frac{-1}{H\varphi^2}\left(\delta_{i\bar{j}}-\frac{2H-r}{H^2\varphi^2}\bar{z}_iz_{\bar{j}}\right),\\ h_{a\bar{b}}&=\frac{\varphi^2}{H}\left(\delta_{a\bar{b}}-\frac{\varphi^2}{H^2}(2H+r)\bar{w}_{\bar{a}}w_{\bar{b}}\right),\\ h_{a\bar{i}}&=\frac{r}{H^3}z_{\bar{i}}\bar{w}_a.\end{aligned}\tag{56}$$

As a remark, these coefficients can also be related to the connection associated with χ_M fields [see Eq. (61) with ∂_{++} replaced by the exterior derivative d] in the way

$$d\Omega=-ih_{p\bar{q}}d\phi^p\wedge d\bar{\phi}^{\bar{q}}.\tag{57}$$

Next, we present the final expression of the geometric formulation of $\mathcal{N}=(0,2)$ $\mathbb{WCP}(N,\tilde{N})$ by collecting the above ingredients.

$$S_{\rm NLSM}=\int d^2x\{g_{i\bar{j}}\partial_\mu\bar{z}^{\bar{j}}\partial^\mu z^i+g_{a\bar{b}}\partial_\mu\bar{w}^{\bar{b}}\partial^\mu w^a+g_{i\bar{a}}\partial_\mu\bar{w}^{\bar{a}}\partial^\mu z^i+g_{a\bar{i}}\partial_\mu\bar{z}^{\bar{i}}\partial^\mu w^a+ig_{i\bar{j}}\bar{\zeta}_+^{\bar{j}}\nabla^c_{--}\zeta^i_++ig_{a\bar{b}}\bar{\eta}^{\bar{b}}_+\nabla^c_{--}\eta^a_+ +ig_{i\bar{a}}\bar{\eta}^{\bar{a}}_+\nabla^c_{--}\zeta^i_++ig_{a\bar{i}}\bar{\zeta}^{\bar{i}}_+\nabla^c_{--}\eta^a_++i\bar{\chi}_{-M}\nabla^f_{++}\chi^M_- +[h_{i\bar{j}}\bar{\zeta}^{\bar{j}}_+\zeta^i_++h_{a\bar{b}}\bar{\eta}^{\bar{b}}_+\eta^a_++h_{i\bar{a}}\bar{\eta}^{\bar{a}}_+\zeta^i_++h_{a\bar{i}}\bar{\zeta}^{\bar{i}}_+\eta^a_+]\times\sum_M q_M\bar{\chi}_{-M}\chi^M_-\}. \tag{58}$$

The covariant derivative in (58) for chiral fields is defined as

$$\nabla^c_{--}\psi^p_+ := \partial_{--}\psi^p_+ + \Gamma^p_{qs}(\partial_{--}\phi^q)\psi^s_+, \tag{59}$$

where $\{\phi_p\}$ and $\{\psi_p\}$ are generic coordinates on the target space, say, $\{z_i, w_a\}$ and $\{\zeta_{+,i}, \eta_{+,a}\}$, respectively, and Γ^p_{qs} is defined in Eq. (14) while for Fermi multiplet, it is shown that

$$\nabla^f_{++}\chi^M_- := \partial_{++}\chi^M_- - iq_M\Omega_{++}\chi^M_-, \tag{60}$$

with

$$\Omega_{++}=\frac{i}{2H}\left(\frac{1}{\varphi^2}z^i\overleftrightarrow{\partial}_{++}\bar{z}_i-\varphi^2w^a\overleftrightarrow{\partial}_{++}\bar{w}_a\right). \tag{61}$$

Two remarks are in order here. First, we may wonder whether it is possible to enhance supersymmetry in (58) up to $\mathcal{N}=(2,2)$ under an appropriate choice of parameters. The answer is negative. This can be traced back to the original construction of the gauged formulation, Eq. (48). Evidently, the kinetic term of the left-handed fermions, τ_+ and ξ_+ (corresponding to ζ_+ and η_+), respectively, does not match that for the right-handed fermions χ_{-M}. In addition, interactions of these fermions are different. These two facts block the possibility of finding $\mathcal{N}=(2,2)$ models in the class of $\mathcal{N}=(0,2)$ models considered in this section. However, we see that once the anomaly-free condition is met, the two-loop term in the β function vanishes much in the same way as in $\mathcal{N}=(2,2)$; see Sec. IX for more details.

Second, in accordance with [10,17], we need to impose the constraints on the representation of the chiral and Fermi multiplets for the theories to be free of the gauge anomalies, which implies their internal quantum consistency. Namely,

$$\sum_{i,a}q_i^2+\tilde{q}_a^2=N+\tilde{N}=\sum_M q_M^2, \tag{62}$$

where the $U(1)$ charges on the left-hand side come from the (left-handed) fermions in the supermultiplets $\mathcal{N}_i$ and ϱ_a while those on the right-hand side are from the (right-handed) Fermi multiplets.

To wrap up this section, the geometry of the target manifold is identical to that obtained from the bosonic calculation, cf. Eqs. (11) and (14)–(16), at the classical level. Since the fermion fields play no role in one-loop renormalization, the FI parameter and the Kähler potential receive the same corrections at the first order as discussed in the previous section. Taking one step further, we show in the next two sections that the $\mathcal{N}=(0,2)$ case has no correction at the two-loop level.

VIII. MORE ON GEOMETRY OF $\mathbb{WCP}(N,\tilde{N})$

As a complement to the discussion of the $\mathbb{WCP}(N,\tilde{N})$ target manifold carried out above, here we present the Riemann curvature tensors needed for the second loop to be obtained in Sec. IX.

For a generic Kähler manifold, the Riemann curvature tensor can be written as

$$R_{\bar{p}qr}{}^s=-R_{q\bar{p}r}{}^s=\bar{\partial}_{\bar{p}}\Gamma^s_{qr}, \tag{63}$$

implying

$$\begin{aligned}
R_{\bar{i}jk}{}^l&=\frac{-1}{H\varphi^2}(\delta^l_k\delta_{j\bar{i}}+\delta^l_j\delta_{k\bar{i}})+\frac{-r+2H}{H^3\varphi^4}z_{\bar{i}}(\bar{z}_j\delta^l_k+\bar{z}_k\delta^l_j)+\frac{2}{H^2\varphi^4}z^l(\bar{z}_j\delta_{k\bar{i}}+\bar{z}_k\delta_{j\bar{i}})-\frac{4(-r+2H)}{H^4\varphi^6}z_{\bar{i}}\bar{z}_j\bar{z}_kz^l,\\
R_{\bar{i}aj}{}^k&=\frac{r}{H^3}z_{\bar{i}}\delta^k_j\bar{w}_a-\frac{2}{H^2}\delta_{j\bar{i}}z^k\bar{w}_a+\frac{4(-r+H)}{H^4\varphi^2}z_{\bar{i}}\bar{z}_jz^k\bar{w}_a,\qquad R_{\bar{i}ab}{}^j=\frac{4r\varphi^2}{H^4}z^jz_{\bar{i}}\bar{w}_a\bar{w}_b,\qquad R_{\bar{i}ab}{}^c=-\frac{r}{H^3}z_{\bar{i}}(\bar{w}_b\delta^c_a+\bar{w}_a\delta^c_b),\\
R_{\bar{i}bj}{}^a&=\frac{\delta^a_b}{H\varphi^2}\left[\delta_{j\bar{i}}-\frac{-r+2H}{H^2\varphi^2}\bar{z}_jz_{\bar{i}}\right],\qquad R_{\bar{i}jk}{}^a=0,\qquad R_{\bar{a}ij}{}^k=\frac{r}{H^3}(\bar{z}_j\delta^k_i+\bar{z}_i\delta^k_j)w_{\bar{a}}-\frac{4r}{H^4\varphi^2}z^k\bar{z}_j\bar{z}_iw_{\bar{a}},\\
R_{\bar{a}bi}{}^j&=\frac{\varphi^2}{H}\left(\delta^j_i-\frac{2}{H\varphi^2}z^j\bar{z}_i\right)\delta_{b\bar{a}}-\frac{\varphi^4(2H+r)}{H^3}\delta^j_iw_{\bar{a}}\bar{w}_b+\frac{4(H+r)\varphi^2}{H^4}\bar{z}_iz^jw_{\bar{a}}\bar{w}_b,\\
R_{\bar{a}bc}{}^i&=\frac{2\varphi^4}{H^2}z^i(\delta_{b\bar{a}}\bar{w}_c+\delta_{c\bar{a}}\bar{w}_b)-\frac{4\varphi^6(r+2H)}{H^4}z^iw_{\bar{a}}\bar{w}_b\bar{w}_c,\qquad R_{\bar{a}bc}{}^d=-\frac{\varphi^2}{H}(\delta_{b\bar{a}}\delta^d_c+\delta_{c\bar{a}}\delta^d_b)+\frac{\varphi^4(2H+r)}{H^3}(\bar{w}_b\delta^d_c+\bar{w}_c\delta^d_b)w_{\bar{a}},\\
R_{\bar{a}bi}{}^c&=-\frac{r}{H^3}\bar{z}_iw_{\bar{a}}\delta^c_b\qquad R_{\bar{a}ij}{}^b=0.
\end{aligned} \tag{64}$$

In what follows we also need a special quadratic combination of the Riemann tensors,

$$R^{(2)}_{p\bar{q}} = R^{rs}_{pt} R_{\bar{q}rs}{}^{t}. \tag{65}$$

This combination can be obtained by a tedious although straightforward calculation. Extensively employing (12) and (64) we derive

$$\begin{aligned}
R^{(2)}_{i\bar{j}} &= \frac{2}{H^4\varphi^2}[(N+\tilde{N}-1)H^2 + r^2]\delta_{i\bar{j}} \\
&\quad + \frac{1}{H^7\varphi^4}[-4(N+\tilde{N}-1)H^4 + 4(N+\tilde{N}-1)H^3 r \\
&\quad - 2(N+\tilde{N}+2)H^2r^2 + 8Hr^3 - 4r^4]\bar{z}_i z_{\bar{j}}, \\
R^{(2)}_{\imath\bar{a}} &= \frac{2r^2}{H^7}[(N+\tilde{N}-2)H^2 + 2r^2]\bar{z}_i w_{\bar{a}}, \\
R^{(2)}_{a\bar{b}} &= \frac{2\varphi^2}{H^4}[(N+\tilde{N}-1)H^2 + r^2]\delta_{a\bar{b}} \\
&\quad - \frac{\varphi^4}{H^7}[4(N+\tilde{N}-1)H^4 + 2(2N+2\tilde{N}-3)H^3 r \\
&\quad + 2(N+\tilde{N})H^2r^2 + 6Hr^3 + 4r^4]\bar{w}_a w_{\bar{b}}.
\end{aligned} \tag{66}$$

One can easily verify the above expressions in two simple limiting cases. First, we consider the $\mathbb{CP}$ models and, then, the simplest example $N = \tilde{N} = 1$ discussed in Sec. VI. To reduce the generic case to $\mathbb{CP}(\tilde{N}-1)$, we should again take $N = 0$ and r negative, arriving at

$$R^{(2)}_{a\bar{b}} \to \frac{2\tilde{N}}{r^2} g_{a\bar{b}}. \tag{67}$$

On the other hand, addressing the $\mathbb{WCP}(1,1)$ model, we set $N = \tilde{N} = 1$ and obtain

$$R^{(2)}_{1\bar{1}} \to \frac{4r^4}{H^7}, \tag{68}$$

cf. Eq. (42).

As is seen from Eq. (66), more and more structures emerge in higher order corrections. In comparison with the one-loop results in which only terms up to H^{-4} show up at two loops we find additional H^{-5} to H^{-7} terms. It is not possible to absorb them in $g_{p\bar{q}}$. This illustrates our statement of nonrenormalizability of the nonsupersymmetric Hanany-Tong model.

Similarly to the $\mathcal{N}=(2,2)$ case, the two-loop correction does not exist in the $\mathcal{N}=(0,2)$ sigma model since the imposition of (62) leads to a vanishing coefficient in front of the second order term in the beta function.

IX. SECOND LOOP

Let us explore the renormalization of $\mathbb{WCP}(N,\tilde{N})$ in higher loops. For a given bosonic two-dimensional nonlinear sigma model, the first two terms in the β function are known in the general form (see, e.g., [18]), namely,

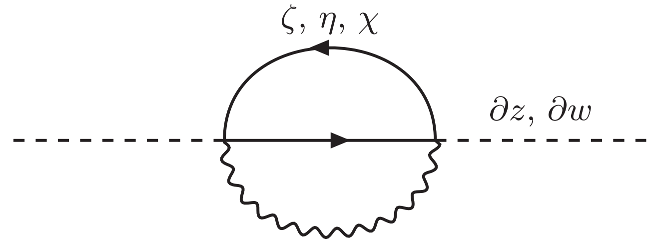

FIG. 4. The second-order fermion loop diagram contributing to the β function. The wavy line denotes the quantum part of the bosonic fields, z, and w.

$$\beta_{p\bar{q}} = \frac{1}{2\pi} R_{p\bar{q}} + \frac{1}{8\pi^2} R^{(2)}_{p\bar{q}} + \cdots, \tag{69}$$

where the first term is nothing but the Ricci tensor and the following term represents the second power of the Riemann tensors; see Eq. (66).

Note that in Eq. (69), the term proportional to the Ricci curvature stands for the one-loop correction while the second term composed of the square of the Riemann tensors relates to the two-loop calculation. The discussion of the first order renormalization is presented in Sec. IV. Now we briefly outline what happens in the second order.

In the $\mathcal{N}=(0,2)$ model we consider the two-loop fermionic contribution shown in Fig. 4. If we work in the vicinity of the origin on the given patch and keep only the lowest order terms, this is the only relevant diagram. Then, it is easy to see that

$$-\frac{1}{16\pi^2}\left(N + \tilde{N} + \sum_M q_M^2\right) = -\frac{1}{8\pi^2}(N+\tilde{N}). \tag{70}$$

In the above equality we employ the anomaly-free condition (62); see [10] for details. It is important to stress that we should take the coefficient in front of the fermion contribution to $R^{(2)}_{p\bar{q}}$ to be $-1/(16\pi^2)$ in the minimal $\mathcal{N}=(0,2)$ model [19]. The reason is that the fermion graph in Fig. 4 contributing at two loops acquires an extra factor $1/2$ in passing from the Driac to Weyl fermions.

Returning to the anomaly-free nonminimal $\mathcal{N}=(0,2)$ model we observe that the second order contributions from bosonic and fermionic fields cancel each other, and thus the second order coefficient vanishes.

Indeed, if we neglect for a short while the N and $\tilde{N}$ dependence in Eq. (70), the fermionic two-loop correction reduces, $-1/8\pi^2$. Together with the bosonic part in (69), we obtain the second order coefficient

$$-\frac{1}{8\pi^2} + \frac{1}{8\pi^2} = 0, \tag{71}$$

leading to the same result as in $\mathcal{N}=(2,2)$ model, in which only the first loop survives as a result of a nonrenormalization theorem. The two models above are expected to have different contributions starting from three loops.

The remaining question refers to the overall factor $N+\tilde{N}$ in Eq. (70). In the latter equation it was obtained by examining the vicinity of the origin of the given patch.

Now we have a closer look at the general form of $R^{(2)}_{p\bar{q}}$ near the origin starting from (66). It is important that in the vicinity of the origin of the origin $H\to r$ and therefore

$$\begin{aligned} R^{(2)}_{i\bar{j}} &\to (N+\tilde{N})\left[\frac{2}{H^2\varphi^2}\delta_{i\bar{j}} - \frac{1}{H^5\varphi^4}(2r^2)\bar{z}_i z_{\bar{j}}\right],\\ R^{(2)}_{i\bar{a}} &\to (N+\tilde{N})\left[\frac{2r^2}{H^5}\bar{z}_i w_{\bar{a}}\right],\\ R^{(2)}_{a\bar{b}} &\to (N+\tilde{N})\left[\frac{2\varphi^2}{H^2}\delta_{a\bar{b}} - \frac{\varphi^4}{H^7}(10r^4)\bar{w}_a w_{\bar{b}}\right]. \end{aligned} \tag{72}$$

Proportionality of $R^{(2)}_{p\bar{q}}$ to the overall $N+\tilde{N}$ factor near the coordinate origin is obvious in the above expressions.

Summarizing, from general covariance and the above calculation in the nonminimal anomaly-free $\mathcal{N}=(0,2)$ $\mathbb{WCP}(N,\tilde{N})$ models, we have

$$\beta(g_{p\bar{q}}) = \frac{1}{2\pi}R_{p\bar{q}} + \cdots, \tag{73}$$

where the ellipses stand for *three*-loop and higher order corrections.

In addition, we can compare with the results in [10] [see Eq. (4.10)]. Generally speaking, the β function in our notation has the form

$$\beta(g^2) = -\frac{g^2}{4\pi}\left(\sum_i q_i - \frac{1}{2}\sum_\alpha q_\alpha\gamma_\alpha + \frac{1}{2}\sum_M q_M\gamma_M\right), \tag{74}$$

where γ_α and γ_M are the anomalous dimensions of the chiral multiplets and the Fermi multiplets, respectively. Also, the coupling constant g is linked to the FI parameter through the relation

$$r = \frac{2}{g^2}. \tag{75}$$

Perturbatively, to obtain the *two*-loop β function, we only need γ at the *one*-loop level, and we know that

$$\gamma_\alpha|_{1-\text{loop}} = \gamma_M|_{1-\text{loop}} \equiv \gamma. \tag{76}$$

Thus, Eq. (74) is simplified as

$$\beta(g^2)_{\text{two-loop}} = -\frac{g^2\sum_i q_i}{4\pi} + \frac{\gamma g^2}{8\pi}\left(\sum_\alpha q_\alpha - \sum_M q_M\right). \tag{77}$$

Since this formula is universal, it is good enough to consider a simple example discussed in [10], in particular, the $\mathcal{N}=(0,2)$ $\mathbb{CP}(N-1)$ model. In this case, there are N chiral fields with positive unit charge and the same number of Fermi multiplets with the same charge as that of chiral fields. As a consequence, the second term in (77) vanishes and only the one-loop effect survives, namely,

$$\beta(g^2)_{\text{two-loop}} = -\frac{Ng^2}{4\pi}. \tag{78}$$

A similar argument can also be applied to the entire particular class of the $\mathcal{N}=(0,2)$ $\mathbb{WCP}(N,\tilde{N})$ models without internal anomalies that we consider in this paper. To proceed, let us first note that the anomaly-free condition (62) in this model again forces the left-handed fermions to "pair up" with the right-handed ones as is the case in $\mathcal{N}=(2,2)$ models. We can specify a particular choice for the set of q_Ms such that N of them have the $U(1)$ charge $+1$ and the rest of the $\tilde{N}$ fields have the $U(1)$ charge -1. Then, Eq. (77) further reduces to

$$\begin{aligned} \beta(g^2)_{\text{two-loop}} &= -\frac{(N-\tilde{N})g^4}{4\pi} + \frac{\gamma g^2}{8\pi}[N-\tilde{N}-(N-\tilde{N})]\\ &= -\frac{(N-\tilde{N})g^4}{4\pi}; \end{aligned} \tag{79}$$

i.e., the two-loop contribution vanishes in much the same way as we have seen in the $\mathbb{CP}(N-1)$ case.

X. CONCLUSIONS

In this paper, we studied the structures of a particular NLSM derived from a class of GLSM and its $\mathcal{N}=(0,2)$ family. The geometry of such NLSM is a weighted complex projective space, $\mathbb{WCP}(N,\tilde{N})$, where N and $\tilde{N}$ stand for the number of fields with the opposite $U(1)$ charges. This noncompact Kählerian manifold does not admit a Kähler-Einstein metric that leads to emergence of extra structures and two different types of logarithms. Renormalization of the Fayet-Iliopoulos in GLSM and that of the Kähler class in NLSM coincide. However, there are additional logarithms in NLSM.

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