Symmetries of deformed supersymmetric mechanics on Kähler manifolds

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Based on the systematic Hamiltonian and superfield approaches, we construct the deformed $\mathcal{N}=4$, 8 supersymmetric mechanics on Kähler manifolds interacting with a constant magnetic field and study their symmetries. First, we construct the deformed $\mathcal{N}=4$, 8 supersymmetric Landau problem via the minimal coupling of standard (undeformed) $\mathcal{N}=4$, 8 supersymmetric free particle systems on a Kähler manifold with a constant magnetic field. We show that the initial "flat" supersymmetries are necessarily deformed to $SU(2|1)$ and $SU(4|1)$ supersymmetries, with the magnetic field playing the role of a deformation parameter, and that the resulting systems inherit all the kinematical symmetries of the initial ones. Then we construct $SU(2|1)$ supersymmetric Kähler oscillators and find that they include, in particular cases, the harmonic oscillator models on complex Euclidian and complex projective spaces, as well as super-integrable deformations thereof, viz. $\mathbb{C}^N$-Smorodinsky-Winternitz and $\mathbb{CP}^N$-Rosochatius systems. We show that the supersymmetric extensions proposed inherit all the kinematical symmetries of the initial bosonic models. They also inherit, at least in the case of the $\mathbb{C}^N$ systems, hidden (nonkinematical) symmetries. The superfield formulation of these supersymmetric systems is presented, based on the worldline $SU(2|1)$ and $SU(4|1)$ superspace formalisms.

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I. INTRODUCTION

The models of supersymmetric mechanics were initially introduced as toy models for supersymmetric field theories. However, it was quickly realized that such models are of a big interest in their own right. An important feature of the supersymmetric mechanics models is that the main new ingredient they bring in, the fermionic variables, after quantization become the operators representing the spin of particle. As the result, the fermionic parts of the relevant Hamiltonians play the role of generalized Pauli terms describing an interaction of a spin with external fields, in particular, with the magnetic field. From this viewpoint, the study of supersymmetric extensions of the mechanical systems interacting with the magnetic field is of obvious importance. However, such systems seem not to have attracted enough attention, despite an enormous number of publications on supersymmetric mechanics.

This is rather surprising, having in mind that the first practical application of ($\mathcal{N}=2$) supersymmetric mechanics technique was the explanation of the "accidental" double degeneracy of the spectrum of the (planar) Landau problem (see, e.g., [1]). The Landau problem is the problem of the planar motion of a nonrelativistic electron (charged $\frac{1}{2}$-spin particle) in a constant magnetic field. For a long time, it has been one of the central issues treated in the textbooks on quantum mechanics [2]. However, nowadays, saying "Landau problem", people sometimes ignore the spin of the original system.

The compact (spherical) analog of the planar Landau problem is associated with a particle moving on the two-sphere in the presence of a constant magnetic field generated by a Dirac monopole placed in the center of the sphere. The spherical Landau problem enjoys a $SO(3)$ invariance, which is also characteristic of the "free" particle on the two sphere. The higher-dimensional generalization of this problem, a particle on $\mathbb{CP}^N$ interacting with a constant magnetic field, inherits the $SU(N+1)$ invariance of the relevant free system. Quantum mechanically, the inclusion of a constant magnetic field supplies the system with a degenerate ground state. This is due to the preservation of the symmetries of a free particle. Thanks to this degeneracy, the quantum-mechanical Landau

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problem constitutes the basis of the theory of the quantum Hall effect [3], equally as if its higher-dimensional generalizations to complex projective spaces [4].

It is more or less obvious that the inclusion of constant fields preserves the initial symmetries of the free particle moving on the generic Kähler manifold as well. So the (spinless) Landau problem can be defined for any Kähler manifold. In order to restore the initial meaning of the Landau problem in the context of these systems, one should try to construct supersymmetric extensions of the (spinless) Landau problem on a Kähler manifold, such that they preserve the initial kinematical symmetries. However, in the existing literature devoted to supersymmetric extensions of the (generalized) Landau problem, the discussion of the symmetry properties of the supersymmetric systems constructed is as a rule left aside (see, e.g., [5,6]).

While for $\mathcal{N} = 2$, the construction of such supersymmetric extensions is a rather trivial task, it is not the case for the $\mathcal{N} \geq 4$ supersymmetric extensions. Generically, one may pose the question:

How should systems in Kähler manifolds interact with constant magnetic fields (in particular, the Landau problem) be supersymmetrized, so that their initial symmetries be preserved?

We guess that the general answer is as follows. Instead of considering $\mathcal{N}$, $d = 1$ Poincaré supersymmetric extensions of given bosonic systems, one should deal with superextensions based on the proper deformations of a standard $d = 1$ Poincaré supersymmetry.

An attempt towards proving this conjecture was performed years ago in [7]. It was observed there that the oscillator and the Landau problem on a complex projective space admit the deformed $\mathcal{N} = 4$ supersymmetric extension (later on called the "weak $\mathcal{N} = 4$ supersymmetric extension" [8]), which preserves the initial kinematical symmetries of those systems. Departing from this model, the class of systems with nonzero potentials called the "Kähler oscillator" was introduced [7,9]. These systems admit similar deformed supersymmetric extensions respecting the inclusion of a constant magnetic field. The relevant bosonic Hamiltonian reads

$$H_{osc} = g^{\bar{a}b}(\bar{\pi}_a \pi_b + |\omega|^2 \partial_{\bar{a}} K \partial_b K), \tag{1}$$

where $K(z, \bar{z})$ is the Kähler potential.

A few years later, the one-dimensional version of that Kähler superoscillator model was rederived within a $d = 1$ superfield formalism. It was based on $SU(2|1)$ superalgebra that was treated as a deformation of $\mathcal{N} = 4$, $d = 1$ Poincaré superalgebra [10,11]. Thereby, the "weak $\mathcal{N} = 4$ supersymmetry" was identified with $su(2|1)$ superalgebra (this fact was also independently noticed in the paper [12] treating the supersymmetric quantum Landau problem on $\mathbb{CP}^1$). Using similar techniques, the deformed $\mathcal{N} = 8$ one-dimensional Landau problem associated with $su(4|1)$ superalgebra was also defined [13]. This study was to a large extent inspired by the activity of building field-theoretical models with the "rigid supersymmetry on curved superspaces" initiated in [14].

Having in mind the "practical importance" of supersymmetrization respecting symmetries of the initial bosonic system and the field-theoretical importance of the "curved superspace approach", we develop here the systematic approach to the deformed supersymmetrization of various systems. These systems "live" on Kähler manifolds and interact with a constant magnetic field by the use of a *supersymmetric analog of a minimal coupling*. In the superfield formulation, such a coupling naturally comes out under some minimal choice of the related superfield Lagrangians.

Resorting first to the Hamiltonian formalism, we construct in this way the $SU(2|1)$ supersymmetric extensions of the Kähler oscillator (and of the Landau problem) on the generic Kähler space. Furthermore we also discuss the $SU(4|1)$ supersymmetric Landau problem on the special Kähler manifolds of the rigid type (that is the Kähler manifold equipped with the holomorphic symmetric tensor of the third rank obeying some compatibility condition [15]). We show that this approach perfectly matches with the requirement that the supersymmetric Landau problem exhibits all the kinematical symmetries of the original system and involves the appropriate spin interaction. It is demonstrated that both the $SU(2|1)$ and $SU(4|1)$ supersymmetric Landau problems inherit all the kinematical symmetries of the initial systems. Requiring the Hamiltonian in the $SU(2|1)$ case to commute with all the supercharges amounts to adding the appropriate Zeeman term to it. In the superspace language, this means that we should start from the properly central-charge extended superalgebra, with the Hamiltonian being identified with the relevant central charge. Analogously, the general $SU(2|1)$ Kähler superoscillator systems as superextensions of those with the Hamiltonian (1) can be constructed and then reproduced from the superfield approach.

Exemplifying the general analysis, we set up and study $SU(2|1)$ supersymmetric extensions of the following particular superintegrable Kähler oscillator models:

(i) $\mathbb{C}^N$ oscillator (the sum of N two-dimensional isotropic oscillators);
(ii) $\mathbb{C}^N$-Smorodinsky-Winternitz system (the sum of N copies of two-dimensional isotropic oscillators deformed by ring-shaped potentials) [16];
(iii) $\mathbb{CP}^N$ oscillator [7,17], i.e., the $\mathbb{CP}^N$- counterpart of the $\mathbb{C}^N$ oscillator;
(iv) $\mathbb{CP}^N$-Rosochatius system [18], i.e., the $\mathbb{CP}^N$-counterpart of the $\mathbb{C}^N$-Smorodinsky-Winternitz system.

We show that these models also inherit all the kinematical symmetries of the initial systems. In addition, we find the explicit expressions for the superanalogs of the hidden symmetry generators of the $\mathbb{C}^N$-oscillator and $\mathbb{C}^N$-Smorodinsky-Winternitz system (i.e., of the Fradkin and

Uhlenbeck tensors). Unfortunately, we were not yet able to find the superanalogs of such hidden symmetry generators for the $\mathbb{CP}^N$-oscillator and of the $\mathbb{CP}^N$-Rosochatius system, though they hopefully exist.

The paper is organized as follows:

In Sec. II, we describe the phase superspace as a proper setting for the supersymmetrization of systems on Kähler manifolds in an interaction with a constant magnetic field. The Legendre transformation relating the Hamiltonian and Lagrangian formulations of those systems is given. In Sec. III, we present the Hamiltonian formulations of $SU(2|1)$ and $SU(4|1)$ supersymmetric Landau problems. The general Hamiltonian formulation of the $SU(2|1)$ Kähler superoscillator is described in Sec. IV. As an example, we show that this class of Hamiltonians incorporates the supersymmetric version of a two-dimensional anisotropic oscillator. In Sec. V, the previously considered systems are recovered within the manifestly $SU(2|1)$ and $SU(4|1)$ covariant off shell superfield approaches. Section VI is devoted to a more detailed discussion of the $SU(2|1)$ supersymmetric extensions of the oscillatorlike systems on $\mathbb{C}^N$ and $\mathbb{CP}^N$ that are listed above and to the study of their symmetries.

II. PHASE SUPERSPACE, KINEMATICAL SYMMETRIES, AND LAGRANGIANS

The Kähler manifold M is the Hermitian manifold with the Hermitian metrics, $ds^2 = g_{a\bar b}dz^a d\bar z^b$, which also defines the symplectic structure,

$$\omega_M = i g_{a\bar b} dz^a \wedge d\bar z^b, \qquad d\omega_M = 0 \Rightarrow g_{a\bar b} = \partial_a \partial_{\bar b} K,$$
$$\partial_a = \frac{\partial}{\partial z^a}, \qquad \partial_{\bar b} = \frac{\partial}{\partial \bar z^b}, \tag{2}$$

where the real function $K(z,\bar z)$, Kähler potential, is defined up to the holomorphic and antiholomorphic functions, $K(z,\bar z) \to K(z,\bar z) + U(z) + \bar U(\bar z)$.

The Kähler manifold can be equipped with the Poisson brackets associated with the above symplectic structure,

$$\{f,g\}_M = i g^{\bar a b}\left(\frac{\partial f}{\partial \bar z^a}\frac{\partial g}{\partial z^b} - \frac{\partial g}{\partial \bar z^a}\frac{\partial f}{\partial z^b}\right), \qquad g^{\bar a b} g_{\bar b c} = \delta^a_c. \tag{3}$$

Therefore, the isometries of the Kähler structure should preserve both complex and symplectic structures; i.e., they are generated by the holomorphic Hamiltonian vector fields,

$$\mathbf{V}_\mu = \{\mathsf{h}_\mu, \}_M = V^a_\mu(z)\frac{\partial}{\partial z^a} + V^{\bar a}_\mu(\bar z)\frac{\partial}{\partial \bar z^a},$$
$$V^a_\mu = i g^{\bar b a}\partial_{\bar b}\mathsf{h}_\mu(z,\bar z), \qquad V^{\bar a}_\mu = \overline{V^a_\mu}, \tag{4}$$

where the real function $\mathsf{h}_\mu(z,\bar z)$ is a momentum map sometimes called the Killing potential. The holomorphicity of the vector field yields the following equation to the Killing potential:

$$\frac{\partial^2 \mathsf{h}_\mu}{\partial z^a \partial z^b} - \Gamma^c_{ab}\frac{\partial \mathsf{h}_\mu}{\partial z^c} = 0, \tag{5}$$

with $\Gamma^c_{ab} = g^{c\bar d}\partial_a g_{b\bar d}$.[1] The same result can be obtained by the direct solving of the Killing equations,

$$\text{(a)}\ V_{\mu a;b} + V_{\mu b;a} = 0,$$
$$\text{(b)}\ V_{\mu a;\bar b} + V_{\mu \bar b;a} = 0, \quad \text{with} \quad V_{\mu a} = g_{a\bar b}V^{\bar b}_\mu. \tag{6}$$

The action of the vector field $\mathbf{V}_\mu$ on an arbitrary function $f(z,\bar z)$ can be expressed through the Poisson bracket with the Killing potential,

$$\mathbf{V}_\mu f = \{\mathsf{h}_\mu, f\}_M.$$

Thus, the requirement that the vector fields $\mathbf{V}_\mu$ form Lie algebra amounts to the same Lie algebra relations for the Killing potentials,

$$[\mathbf{V}_\mu, \mathbf{V}_\nu] = C^\lambda_{\mu\nu}\mathbf{V}_\lambda, \Leftrightarrow \{\mathsf{h}_\mu, \mathsf{h}_\nu\}_M = C^\lambda_{\mu\nu}\mathsf{h}_\lambda + \text{const.}, \tag{7}$$

where the constant term either corresponds to a cocycle in that Lie algebra or can be absorbed by the appropriate constant shift of Killing potentials.

Let us consider the electrically charged particle moving on a Kähler manifold and interacting with the constant magnetic field of strength B, i.e., the $U(1)$-Landau problem on Kähler manifold. For this aim, we equip the cotangent bundle of the Kähler manifold with the following symplectic structure and Hamiltonian:

$$\omega_B = d\pi_a \wedge dz^a + d\bar\pi_a \wedge d\bar z^a - iBg_{a\bar b}dz^a \wedge d\bar z^b,$$
$$H_0 = g^{\bar a b}\bar\pi_a \pi_b. \tag{8}$$

The corresponding Poisson brackets are given by

$$\{\pi_a, z^b\} = \delta^b_a, \qquad \{\pi_a, \bar\pi_b\} = iBg_{a\bar b}. \tag{9}$$

The isometries of a Kähler structure discussed earlier define the Noether constants of motion,

$$J_\mu = V^a_\mu \pi_a + \bar V^{\bar a}_\mu \bar\pi_{\bar a} - B\mathsf{h}_\mu(z\bar z),$$
$$V^a_\mu = i g^{\bar b a}\partial_{\bar b}\mathsf{h}_\mu(z,\bar z)\colon \left\{\begin{array}{c} \{H_0, J_\mu\}_B = 0 \\ \{J_\mu, J_\nu\}_B = C^\lambda_{\mu\nu}J_\lambda \end{array}\right\}, \tag{10}$$

where the brackets $\{\cdot,\cdot\}_B$ are calculated according to (9). Notice that the vector fields generated by J_μ are independent of B,

[1] The only nonvanishing components of the Christoffel symbol in the Kähler geometry are Γ^c_{ab} and $\Gamma^{\bar c}_{\bar a \bar b} = g^{d\bar c}\partial_{\bar a} g_{d\bar b}$.

$$\tilde{\mathbf{V}}_\mu = \{J_\mu,\ \}_B = V^a_\mu(z)\frac{\partial}{\partial z^a} - V^a_{\mu,b}\pi_a\frac{\partial}{\partial \pi_b} + \text{c.c.} \tag{11}$$

Hence, coupling to a constant magnetic field preserves the whole symmetry algebra of a free particle moving on a Kähler manifold. This implies that the Landau problem can be properly defined on any Kähler manifold.

To construct fermionic extensions of the systems on Kähler manifolds interacting with constant magnetic field, we define the $(2N|MN)_C$-dimensional phase superspace equipped with the symplectic structure,

$$\begin{aligned}\Omega &= d\pi_a \wedge dz^a + d\bar\pi_a \wedge d\bar z^a \\ &\quad - i(Bg_{a\bar b} - R_{a\bar b c\bar d}\eta^{c\alpha}\bar\eta^d_\alpha)dz^a \wedge d\bar z^b \\ &\quad + i g_{a\bar b} D\eta^{a\alpha} \wedge D\bar\eta^b_\alpha,\end{aligned} \tag{12}$$

where $\alpha = 1, \ldots M$ are spinorial indices, $D\eta^{a\alpha} = d\eta^{a\alpha} + \Gamma^a_{bc}\eta^{b\alpha}dz^c$, and Γ^a_{bc}, $R_{a\bar b c\bar d} = g_{e\bar b}(\Gamma^e_{ac})_{,\bar d}$ are, respectively, the components of the connection and curvature of the Kähler structure.

The Poisson brackets corresponding to the symplectic structure (12) amount to the relations,

$$\begin{aligned}&\{\pi_a, z^b\} = \delta^b_a, \qquad \{\pi_a, \eta^{b\alpha}\} = -\Gamma^b_{ac}\eta^{c\alpha}, \\ &\{\pi_a, \bar\pi_b\} = i(Bg_{a\bar b} - R_{a\bar b c\bar d}\eta^{c\alpha}\bar\eta^d_\alpha), \qquad \{\eta^{a\alpha}, \bar\eta^b_\beta\} = i g^{a\bar b}\delta^\alpha_\beta,\end{aligned} \tag{13}$$

and their complex conjugates. They induce the following generic Poisson bracket for the functions on the phase superspace:

$$\begin{aligned}\{f,g\} &= \frac{\partial f}{\partial \pi_a} \wedge \nabla_a g + \frac{\partial f}{\partial \bar\pi_a} \wedge \bar\nabla_a g \\ &\quad + i(Bg_{a\bar b} - R_{a\bar b c\bar d}\eta^{c\alpha}\bar\eta^d_\alpha)\frac{\partial f}{\partial \pi_a} \wedge \frac{\partial g}{\partial \bar\pi_b} \\ &\quad + i g^{\bar a b}\left(\frac{\partial^l f}{\partial \eta^{a\alpha}} \wedge \frac{\partial^r g}{\partial \bar\eta^b_\alpha}\right),\end{aligned} \tag{14}$$

where $A \wedge B = AB - (-1)^{p(A)p(B)}BA$ and

$$\nabla_a \equiv \frac{\partial}{\partial z^a} - \Gamma^c_{ab}\eta^{b\alpha}\frac{\partial}{\partial \eta^{c\alpha}}. \tag{15}$$

The extended symplectic structure (12) and Poisson brackets (14) are manifestly covariant with respect to the transformation,

$$\tilde z^a = \tilde z^a(z), \qquad \tilde\pi_a = \frac{\partial z^b}{\partial \tilde z^a}\pi_b, \qquad \tilde\eta^{a\alpha} = \frac{\partial \tilde z^a}{\partial z^b}\eta^{b\alpha}. \tag{16}$$

Hence, we can lift the isometries (11) to the whole phase superspace and define the respective super-Hamiltonian vector fields as

$$\begin{aligned}\mathbf{V}_\mu &= \{\mathcal{J}_\mu,\}\\ &= V^a_\mu(z)\frac{\partial}{\partial z^a} - V^a_{\mu,b}\pi_a\frac{\partial}{\partial \pi_b} + V^a_{\mu,b}\eta^{b\alpha}\frac{\partial}{\partial \eta^{a\alpha}} + \text{c.c.},\end{aligned} \tag{17}$$

where

$$\mathcal{J}_\mu = J_\mu + \frac{\partial^2 \mathsf{h}_\mu}{\partial z^c \partial \bar z^d}\eta^{c\alpha}\bar\eta^d_\alpha, \tag{18}$$

with J_μ defined by (10).

Note that the symplectic structure (12) can be represented as a locally exact one form,

$$\begin{aligned}&\Omega = d\mathcal{A} \\ &\mathcal{A} = \pi_a dz^a + \bar\pi_a d\bar z^a + i\frac{B}{2}(\partial_a K dz^a - \partial_{\bar a} K d\bar z^a) \\ &\qquad + \frac{i}{2} g_{a\bar b}(\eta^{a\alpha} D\bar\eta^b_\alpha + \bar\eta^b_\alpha D\eta^{a\alpha}).\end{aligned} \tag{19}$$

Then, by the Hamiltonian,

$$\mathcal{H} = g^{\bar a b}\bar\pi_a\pi_b + \mathcal{U}(z,\bar z,\eta,\bar\eta), \tag{20}$$

where the potential term $\mathcal{U}(z,\bar z,\eta,\bar\eta)$ will be defined later for each specific system, we can immediately write down the first order-Lagrangian with the action,

$$\mathcal{S} = \int \mathcal{A} - \mathcal{H}dt. \tag{21}$$

Eliminating cyclic variables $\pi_a, \bar\pi_a$, we arrive at the second-order Lagrangian,

$$\begin{aligned}\mathcal{L} &= g_{a\bar b}\dot z^a \dot{\bar z}^b + i\frac{B}{2}(\partial_a K \dot z^a - \partial_{\bar a} K \dot{\bar z}^a) \\ &\quad + \frac{i}{2} g_{a\bar b}(\eta^{a\alpha} D_t\bar\eta^b_\alpha + \bar\eta^b_\alpha D_t\eta^{a\alpha}) - \mathcal{U}(z,\bar z,\eta,\bar\eta) \\ &\quad \text{with} \quad D_t\eta^a_\alpha = \dot\eta^a_\alpha + \Gamma^a_{bc}\eta^b_\alpha \dot z^c.\end{aligned} \tag{22}$$

Now we can rederive (and so check) all the previous formulas by applying the standard Legendre transformation just to this Lagrangian. We define the canonical bosonic momenta,

$$\begin{aligned}P_a &:= \frac{\partial \mathcal{L}}{\partial \dot z^a} = g_{a\bar b}\dot{\bar z}^b + i\frac{B}{2}\partial_a K - \frac{i}{2}\partial_c g_{a\bar b}(\eta^{c\alpha}\bar\eta^{\bar b}_\alpha), \\ P_{\bar a} &:= \frac{\partial \mathcal{L}}{\partial \dot{\bar z}^a} = \dot z^b g_{b\bar a} - i\frac{B}{2}\partial_{\bar a} K + \frac{i}{2}\partial_{\bar c} g_{b\bar a}(\eta^{c\alpha}\bar\eta^b_\alpha),\end{aligned} \tag{23}$$

and the canonical fermionic ones,

$$P_{a\alpha} := \frac{\partial^R \mathcal{L}}{\partial \dot\eta^{a\alpha}} = \frac{i}{2} g_{a\bar b}\bar\eta^b_\alpha, \qquad P^\alpha_{\bar a} := \frac{\partial^R \mathcal{L}}{\partial \dot{\bar\eta}^a_\alpha} = \frac{i}{2} g_{\bar a b}\eta^{b\alpha}. \tag{24}$$

The above expressions indicate the appearance of second-class constraints,

$$\phi_{a\alpha}=P_{a\alpha}-\frac{i}{2}g_{a\bar{b}}\bar{\eta}^{b}_{\alpha}\simeq 0,\qquad \phi^{\alpha}_{\bar{a}}=P^{\alpha}_{\bar{a}}-\frac{i}{2}g_{\bar{a}b}\eta^{b\alpha}\simeq 0. \tag{25}$$

Thus, for the Hamiltonian formulation, we need to eliminate these constraints in accordance with the Dirac's method. The standard procedure yields the following nonvanishing Dirac brackets (and their complex conjugates):

$$\begin{aligned}
&\{P_a, z^b\} = \delta^b_a, \qquad \{P_a, \eta^{b\alpha}\} = -\frac{1}{2}\Gamma^b_{ac}\eta^{c\alpha},\\
&\{P_a, \bar{\eta}^b_\alpha\} = -\frac{1}{2}\partial_a g_{c\bar{d}} g^{c\bar{b}}\bar{\eta}^d_\alpha, \qquad \{\eta^{a\beta}, \bar{\eta}^b_\alpha\} = i g^{a\bar{b}}\delta^\beta_\alpha,\\
&\{P_a, P_{\bar{b}}\} = -\frac{i}{4}[\partial_a g_{c\bar{d}}\partial_{\bar{b}} g_{f\bar{e}} - (a\leftrightarrow\bar{b})]g^{c\bar{e}}(\eta^{f\alpha}\bar{\eta}^d_\alpha),\\
&\{P_a, P_b\} = -\frac{i}{4}[\partial_a g_{c\bar{d}}\partial_b g_{f\bar{e}} - (a\leftrightarrow b)]g^{c\bar{e}}(\eta^{f\alpha}\bar{\eta}^d_\alpha).
\end{aligned} \tag{26}$$

Introducing the noncanonical bosonic momenta, $\pi_a = g_{a\bar{b}}\dot{\bar{z}}^b$, $\bar{\pi}_a = \dot{z}^b g_{b\bar{a}}$, and taking into account the relations between the momenta P_a, $P_{\bar{b}}$, π_a, $\pi_{\bar{b}}$ in (23), it is straightforward to recover the brackets involving π_a, $\bar{\pi}_a$ and defined earlier in Eqs. (13). In particular, it is easy to show that $\{\pi_a, \pi_b\} = \{\bar{\pi}_a, \bar{\pi}_b\} = 0$. It is also straightforward, applying the Noether procedure directly to (22) and assuming that the potential term $\mathcal{U}$ is invariant, to reproduce the conserved isometry current $\mathcal{J}_\mu$ defined in (18). With all these ingredients at hand, we are prepared to turn to supersymmetrizing the Landau problem on Kähler manifold.

III. SUPERSYMMETRIC LANDAU PROBLEM

To define the (deformed) $\mathcal{N}=2M$ supersymmetric extension of the Landau problem (i.e., of the free particle interacting with a constant magnetic field), we make use of the strategy similar to symplectic coupling in the pure bosonic case. The starting point is some supersymmetric Hamiltonian system supplied by supercharges Q^α and $\bar{Q}_\alpha$, which close on a Hamiltonian $\mathcal{H}_0$,

$$\begin{aligned}
&\{Q^\alpha, Q^\beta\}_0 = \{\bar{Q}_\alpha, \bar{Q}_\beta\}_0 = 0, \qquad \{Q^\alpha, \bar{Q}_\beta\}_0 = i\delta^\alpha_\beta\mathcal{H}_0,\\
&\{Q^\alpha, \mathcal{H}_0\}_0 = \{\bar{Q}_\alpha, \mathcal{H}_0\}_0 = 0,
\end{aligned} \tag{27}$$

where the Poisson brackets are defined by (13) with a zero magnetic field, $B=0$.

To introduce an interaction with an external magnetic field, we deform the supersymplectic structure, still preserving the form of the supercharges, $(\Omega_{B=0}, Q^\alpha, \bar{Q}_\alpha) \to (\Omega_B, Q^\alpha, \bar{Q}_\alpha)$. Now, the graded Poisson bracket $\{\cdot,\cdot\}$ is defined through the symplectic form Ω_B defined in (12), and one has to check whether the supersymmetry algebra (27) remains unaltered.

If this is the case, then the Hamiltonian can be defined as $\mathcal{H}_0 \coloneqq \frac{i}{M}\{Q^\alpha, \bar{Q}_\alpha\}$. Otherwise, we end up with some deformed superalgebra, which is different from the standard $d=1$, $\mathcal{N}=2M$ super Poincaré' algebra (27), and there we have to select the generator admitting an interpretation as the appropriate Hamiltonian, i.e.,

$$\{Q^\alpha, Q^\beta\} = 0 + iB\ldots, \qquad \{Q^\alpha, \bar{Q}_\beta\} = i\delta^\alpha_\beta\mathcal{H}_0 + iB\ldots \tag{28}$$

Here, dots stand for some possible extra generators, which should be further commuted with supercharges and among themselves in order to obtain a closed superalgebra.

Below we will show that this program works perfectly well for the cases of (deformed) $\mathcal{N}=4, 8$ supersymmetric Landau problems.

A. The $SU(2|1)$ (deformed $\mathcal{N}=4$) supersymmetric Landau problem

In order to set up the $\mathcal{N}=4$ Landau problem, we choose the standard "chiral" supercharges Q^α, $\bar{Q}_\alpha$ $(\alpha=1,2)$ with the same ansatz for them as in the absence of a magnetic field and introduce the charges generating the $SU(2)$ R symmetry,

$$\begin{aligned}
&Q^\alpha = \pi_a\eta^{a\alpha}, \qquad \bar{Q}_\alpha = \bar{\pi}_a\bar{\eta}^a_\alpha,\\
&\mathcal{R}^\alpha_\beta = g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\beta - \frac{1}{2}\delta^\alpha_\beta g_{a\bar{b}}\eta^{a\gamma}\bar{\eta}^b_\gamma.
\end{aligned} \tag{29}$$

The closure of their Poisson brackets yields the superalgebra,

$$\begin{aligned}
&\{Q^\alpha, Q^\beta\} = 0, \qquad \{\mathcal{R}^\alpha_\beta, \mathcal{R}^\gamma_\delta\} = -i\delta^\gamma_\beta\mathcal{R}^\alpha_\delta + i\delta^\alpha_\delta\mathcal{R}^\gamma_\beta,\\
&\{Q^\alpha, \mathcal{R}^\beta_\gamma\} = i\delta^\alpha_\gamma Q^\beta - \frac{i}{2}\delta^\beta_\gamma Q^\alpha, \qquad \{Q^\alpha, \bar{Q}_\beta\} = i\delta^\alpha_\beta\mathcal{H}_0 + iB\mathcal{R}^\alpha_\beta,\\
&\{Q^\alpha, \mathcal{H}_0\} = i\frac{B}{2}Q^\alpha, \qquad \{\mathcal{R}^\alpha_\beta, \mathcal{H}_0\} = 0,
\end{aligned} \tag{30}$$

where

$$\mathcal{H}_0 = g^{\bar{a}b}\bar{\pi}_a\pi_b - \frac{1}{2}R_{a\bar{b}c\bar{d}}\eta^{a\alpha}\bar{\eta}^b_\alpha\eta^{c\beta}\bar{\eta}^d_\beta + \frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha. \tag{31}$$

Extending the set (29) by the generator (31), we arrive at the $su(2|1)$ superalgebra (or "weak $\mathcal{N}=4$ superalgebra" in the terminology of [8]). We observe, however, that the supercharges do not commute with the Hamiltonian. This drawback can be remedied via the appropriate modification of the Hamiltonian,

$$\begin{aligned}
\tilde{\mathcal{H}}_0 &= \mathcal{H}_0 - \frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha\\
&= g^{a\bar{b}}\pi_a\bar{\pi}_b - \frac{1}{2}R_{a\bar{b}c\bar{d}}\eta^{a\alpha}\bar{\eta}^b_\alpha\eta^{c\beta}\bar{\eta}^d_\beta + Bg_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha:\\
&\quad\{Q^\alpha, \tilde{\mathcal{H}}_0\} = 0.
\end{aligned} \tag{32}$$

The last term in the Hamiltonians (31), (32) is obviously a Zeeman term describing the interaction of the spin with an external magnetic field. From the mathematical point of view, the shift in (32) is the new R-symmetry $U(1)$ generator $\mathcal{R} =: \frac{1}{2} g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha$. It extends $SU(2)$ R symmetry generated by $\mathcal{R}^\alpha_\beta$ to $U(2)$ R symmetry. Since $\tilde{H}_0$ commutes with all other generators of the extended superalgebra, it can be interpreted as the central charge generator promoting the standard $su(2|1)$ superalgebra to its central extension $\widehat{su}(2|1)$ [11].

All the generators of $su(2|1)$ superalgebra (and of its central extension) are manifestly invariant under the action of the isometry current (18),

$$\{Q^\alpha, \mathcal{J}_\mu\} = \{\bar{Q}_\alpha, \mathcal{J}_\mu\} = \{\mathcal{R}^\alpha_\beta, \mathcal{J}_\mu\} = \{\mathcal{H}_0, \mathcal{J}_\mu\} = 0. \quad (33)$$

This means that the supersymmetric system constructed inherits all the kinematical symmetries of the initial system. In particular, in the case of the $\mathbb{CP}^N$-Landau problem, the extended system respects $SU(N+1)$ symmetry.

Thus, we have accomplished the well-defined "weak $\mathcal{N}=4$ supersymmetrization" of the Landau problem on a generic Kähler manifold and found that its supersymmetry algebra is $\widehat{su}(2|1)$.

Finally, it is straightforward to write down the Lagrangian corresponding to (31),

$$\begin{aligned}\mathcal{L}_0 = {}& g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^b + i\frac{B}{2}(\partial_a K\dot{z}^a - \partial_{\bar{a}}K\dot{\bar{z}}^a) \\ &+ \frac{i}{2}g_{a\bar{b}}(\eta^{a\alpha}D_t\bar{\eta}^b_\alpha + \bar{\eta}^b_\alpha D_t\eta^{a\alpha}) \\ &+ \frac{1}{2}R_{a\bar{b}c\bar{d}}\eta^{a\alpha}\bar{\eta}^b_\alpha\eta^{c\beta}\bar{\eta}^d_\beta - \frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha. \end{aligned} \quad (34)$$

The Lagrangian corresponding to the shifted Hamiltonian (32) is obviously $\tilde{\mathcal{L}}_0 = \mathcal{L}_0 - \frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha$. These Lagrangians provide a higher-dimensional generalization of those constructed in [19], [10], using the $SU(2|1)$ superfield techniques. The superfield derivation of (34) will be given in Sec. VI. The relevant $SU(2|1)$ off shell multiplet content is N chiral multiplets (**2**, **4**, **2**). Note that the Lagrangian and Hamiltonian $\mathcal{L}_0$ and $\mathcal{H}_0$ coincide with the previously derived general expressions (22) and (20) for $\alpha = 1, 2$ and the choice $\mathcal{U} = \frac{1}{2}R_{a\bar{b}c\bar{d}}\eta^{a\alpha}\bar{\eta}^b_\alpha\eta^{c\beta}\bar{\eta}^d_\beta - Bg_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha$.

B. $SU(4|1)$ (deformed $\mathcal{N}=8$) supersymmetric Landau problem

In the previous subsection, we considered the coupling of $\mathcal{N}=4$ supersymmetric particle on Kähler manifold to a constant magnetic field and showed that the resulting system yields the deformed $SU(2|1)$ supersymmetric Landau problem. We have shown that the latter inherits the whole isometry group of the original system. Now we perform a similar construction for $\mathcal{N}=8$ supersymmetric mechanics on the *special Kähler manifolds of the rigid type* [20].

The special Kähler manifold of the rigid type is the Kähler manifold equipped with the symmetric tensor $f_{abc}dz^a dz^b dz^c$ and its complex conjugate which obey the following compatibility conditions:

$$\frac{\partial}{\partial\bar{z}^d}f_{abc} = 0, \quad f_{abc;d} = f_{abd;c}, \quad R_{a\bar{b}c\bar{d}} = -\bar{f}_{\bar{b}\bar{d}\bar{n}}g^{\bar{n}m}f_{mac}, \quad (35)$$

where $f_{abc;d} = f_{abc,d} - \Gamma^e_{da}f_{ebc} - \Gamma^e_{db}f_{aec} - \Gamma^e_{dc}f_{abe}$ is the covariant derivative of the third-rank covariant tensor. The special Kähler manifolds of the rigid type are widely known because of their close relevance to T duality that relates the UV and IR limits of the $\mathcal{N}=2$, $d=4$ super Yang-Mills theory [21].

To construct the relevant supersymmetric Landau problem, we choose the symplectic structure (12) and Poisson brackets (14) with the $su(4)$ spinor indices $\alpha, \beta = 1, \ldots, 4$. To avoid a possible confusion, we relabel them by the capital latin letters I, J, K, L. With this notation, the "flat" $\mathcal{N}=8$ supersymmetry algebra reads

$$\{Q^I, Q^J\} = \{\bar{Q}_I, \bar{Q}_J\} = 0, \quad \{Q^I, \bar{Q}_J\} = i\delta^I_J\mathcal{H}_{\mathrm{SUSY}}. \quad (36)$$

Following [20], we define the supercharges as

$$Q^I = \pi_a\eta^{aI} + \frac{i}{3}\bar{f}_{abc}\bar{T}^{abcI}, \qquad \bar{Q}_I = \bar{\pi}_a\bar{\eta}^a_I + \frac{i}{3}f_{abc}T^{abc}_I,$$
$$T^{abc}_I \equiv \frac{1}{2}\varepsilon_{IJKL}\eta^{aJ}\eta^{bK}\eta^{cL}, \quad (37)$$

where the symmetric tensor f_{abc} obeys the relations (35).[2] Also, we introduce the following deformation of the Poisson brackets used in [20]:

$$\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \eta^{bI}\} = -\Gamma^b_{ac}\eta^{cI},$$
$$\{\pi_a, \bar{\pi}_b\} = i(Bg_{a\bar{b}} - R_{a\bar{b}c\bar{d}}\eta^{cI}\bar{\eta}^d_I), \quad \{\eta^{aI}, \bar{\eta}^b_J\} = ig^{a\bar{b}}\delta^I_J. \quad (38)$$

Then we can construct R-symmetry charges forming $su(4)$ algebra by the same relations as in the undeformed case,

[2]Here, we introduced the antisymmetric symbol ε^{IJKL} satisfying the following identities:

$$\varepsilon^{1234} = \varepsilon_{1234} = 1, \qquad \varepsilon^{IJKL}\varepsilon_{IJKL} = 24,$$
$$\varepsilon^{IJKL}\varepsilon_{IJKM} = 6\delta^L_M, \qquad \varepsilon^{IJKL}\varepsilon_{IJMN} = 2(\delta^K_M\delta^L_N - \delta^K_N\delta^L_M),$$
$$\varepsilon^{IJKL}\varepsilon_{IMNP} = \delta^J_M\delta^K_N\delta^L_P - \delta^J_M\delta^K_P\delta^L_N + \delta^J_N\delta^K_P\delta^L_M - \delta^J_N\delta^K_M\delta^L_P + \delta^J_P\delta^K_M\delta^L_N - \delta^J_P\delta^K_N\delta^L_M.$$

The highest-degree monomial of the Grassmann variables can be represented as $\psi^I\psi^J\psi^K\psi^L = \frac{1}{24}\varepsilon^{IJKL}(\varepsilon_{MNPR}\psi^M\psi^N\psi^P\psi^R)$.

$$R^I_J = \eta^{aI} g_{a\bar b}\bar\eta^b_J - \frac{\delta^I_J}{4}\eta^{aK} g_{a\bar b}\bar\eta^b_K,$$
$$\{R^I_J, R^K_L\} = i(\delta^K_J R^I_L - \delta^I_L R^K_J). \qquad (39)$$

Calculating the modified Poisson brackets between the supercharges and R charges, we arrive at the generators $\mathcal{H}_{\rm SUSY}$, Q^I, R^I_J which form the superalgebra $su(4|1)$,

$$\{Q^I, Q^J\} = \{\bar Q_I, \bar Q_J\} = 0, \quad \{Q^I, \bar Q_J\} = i\delta^I_J\mathcal{H}_0 + iBR^I_J,$$
$$\{R^I_J, Q^K\} = i\delta^K_J Q^I - \frac{i}{4}\delta^I_J Q^K, \quad \{\mathcal{H}_0, Q^K\} = -\frac{3iB}{4}Q^K. \quad (40)$$

Here,

$$\mathcal{H}_0 = g^{\bar a b}\bar\pi_a\pi_b + R_{a\bar b c\bar d}\Lambda_0^{ac\bar b\bar d} + \frac{B}{4}\eta^{aK} g_{a\bar b}\bar\eta^b_K$$
$$-\frac{1}{3}f_{abc;d}\Lambda^{abcd} - \frac{1}{3}\bar f_{abc;d}\bar\Lambda^{abcd}, \qquad (41)$$

where, as before, $f_{abc;d}$ is the covariant derivative of the third-rank covariant symmetric tensor, and

$$\Lambda^{abcd} := -\frac{1}{8}\varepsilon_{IJKL}\eta^{aI}\eta^{bJ}\eta^{cK}\eta^{dL},$$
$$\Lambda_0^{ac\bar b\bar d} := \frac{1}{2}\eta^{aI}\eta^{cJ}\bar\eta^b_I\bar\eta^d_J. \qquad (42)$$

We observe that the inclusion of a constant magnetic field B deforms $\mathcal{N} = 8$, $d = 1$ Poincaré superalgebra to the $su(4|1)$ superalgebra.

Let us require that the isometry of the Kähler structure given by the vector field $\mathbf{V}_\mu$ preserves as well the third-order tensor $f_{abc}dz^a dz^b dz^c$; i.e., that the Lie derivative of the latter along this field equals to zero,

$$\mathcal{L}_{\mathbf{V}_\mu} f_{abc}dz^a dz^b dz^c = 0 \Leftrightarrow 3V^d_{\mu,(b}f_{ac)d} + V^d_\mu f_{abc,d} = 0. \quad (43)$$

Using these additional relations, one can check that the isometry generator (18) commutes with all the elements of $SU(4|1)$ superalgebra,

$$\{\mathcal{J}_\mu, Q_I\} = \{\mathcal{J}_\mu, \bar Q_I\} = \{\mathcal{J}_\mu, R^I_J\} = \{\mathcal{J}_\mu, \mathcal{H}_{\rm Lan}\} = 0. \quad (44)$$

Thus, we managed to define the consistent $SU(4|1)$ Landau problem on special Kähler manifolds of the rigid type.

In contrast to the $SU(2|1)$ Landau problem, we cannot bring the Hamiltonian to the form in which it commutes with the supercharges, except for the trivial case $f_{abc} = 0$.

Finally, taking into account the correspondence (22), we can write the expression for the relevant Lagrangian,

$$\mathcal{L}_0 = g_{a\bar b}\dot z^a\dot{\bar z}^b + i\frac{B}{2}(\partial_a K\dot z^a - \partial_{\bar a}K\dot{\bar z}^a)$$
$$+\frac{i}{2}g_{a\bar b}(\eta^{aI}D_t\bar\eta^b_I + \bar\eta^b_I D_t\eta^{aI}) - \frac{B}{4}\eta^{aK} g_{a\bar b}\bar\eta^b_K$$
$$+\frac{1}{3}(f_{abc;d}\Lambda^{abcd} + \bar f_{\bar a\bar b\bar c;\bar d}\bar\Lambda^{\bar a\bar b\bar c\bar d})$$
$$+ f_{abc}g^{c\bar c'}\bar f_{\bar c'\bar d\bar e}\Lambda_0^{ab\bar d\bar e}. \qquad (45)$$

The rederivation of this Lagrangian from the appropriate off shell $SU(4|1)$ superfield formalism is given in Sec. V, where the conditions (35) are resolved, in the *special coordinate frame*, through the single holomorphic function $\mathcal{F}(z)$ known as Seiberg-Witten prepotential,

$$g_{a\bar b} = \frac{\partial^2\mathcal{F}(z)}{\partial z^a\partial z^b} + \text{c.c.},$$
$$\Gamma_{ab\bar c} = \frac{\partial^3\mathcal{F}}{\partial z^a\partial z^b\partial z^c} \qquad f_{abc} = e^{i\nu}\frac{\partial^3\mathcal{F}(z)}{\partial z^a\partial z^b\partial z^c}. \qquad (46)$$

Clearly, the function $\mathcal{F}(z)$ is defined up to the redefinition,

$$\mathcal{F}(z) \to \mathcal{F}(z) + ic_{ab}z^a z^b + c_a z^a + c, \qquad (47)$$

where c_a, c are the arbitrary complex constants, and c_{ab} are the real ones, $\bar c_{ab} = c_{ab}$.

The corresponding Kähler potential is given by the expression,

$$K(z,\bar z) = \bar z^a\frac{\partial\mathcal{F}(z)}{\partial z^a} + z^a\frac{\partial\bar{\mathcal{F}}(\bar z)}{\partial\bar z^a}. \qquad (48)$$

In these coordinates, the T-duality transformation is realized as follows [21]:

$$(z^a, \mathcal{F}(z)) \to \left(u_a = \frac{\partial\mathcal{F}}{\partial z^a}, \tilde{\mathcal{F}}(u)\right),$$
$$\text{where } \frac{\partial^2\tilde{\mathcal{F}}(u)}{\partial u_a\partial u_c}\frac{\partial\mathcal{F}}{\partial z^c\partial z^b} = -\delta^a_b,$$
$$\tilde{\mathcal{F}}(u) = (u_a z^a - \mathcal{F}(z))|_{u_a = \partial_a\mathcal{F}(z)}. \qquad (49)$$

IV. $SU(2|1)$ KÄHLER SUPEROSCILLATOR

The Kähler oscillator is defined by the symplectic structure (8) and the Hamiltonian [9],

$$H_{osc} = g^{\bar a b}(\bar\pi_a\pi_b + |\omega|^2\partial_{\bar a}K\partial_b K), \qquad (50)$$

where $K(z,\bar z)$ is the Kähler potential.

This system is distinguished in that it is “friendly” to supersymmetrization: the addition of the potential (50) amounts to minor changes in the procedure of the $SU(2|1)$ supersymmetrization of the Landau problem described in the previous section. Namely, we can preserve the form

(29) of the $SU(2)$ R charges and adopt the following slightly modified expressions for the supercharges:[3]

$$\Theta^\alpha = \pi_a \eta^{a\alpha} + i\bar\omega \bar\partial_a K \varepsilon^{\alpha\beta} \bar\eta^a_\beta,$$
$$\bar\Theta_\alpha = \bar\pi_a \bar\eta^a_\alpha + i\omega \partial_a K \varepsilon_{\alpha\beta} \eta^{a\beta}. \quad (51)$$

Calculating their Poisson brackets, we obtain

$$\{\Theta^\alpha, \bar\Theta_\beta\} = i\delta^\alpha_\beta \mathcal{H}_{osc} + iB\mathcal{R}^\alpha_\beta, \qquad \{\Theta^\alpha, \Theta^\beta\} = 2i\bar\omega \mathcal{R}^{\alpha\beta},$$
$$\{\Theta^\alpha, \mathcal{R}^\beta_\gamma\} = -i\delta^\alpha_\gamma \Theta^\beta + \frac{i}{2}\delta^\beta_\gamma \Theta^\alpha, \quad (52)$$

where the Hamiltonian is now given by the expression,

$$\mathcal{H}_{osc} = g^{\bar a b}(\bar\pi_a \pi_b + |\omega|^2 \partial_{\bar a} K \partial_b K) - \frac{1}{2} R_{a\bar b c\bar d} \eta^{a\alpha} \bar\eta^b_\alpha \eta^{c\beta} \bar\eta^d_\beta$$
$$- \frac{1}{2}\omega K_{a;b} \eta^{a\alpha} \eta^b_\alpha - \frac{1}{2}\bar\omega K_{\bar a;\bar b} \bar\eta^a_\alpha \bar\eta^{b\alpha} + \frac{B}{2} g_{a\bar b} \eta^{a\alpha} \bar\eta^b_\alpha. \quad (53)$$

To close the superalgebra, we have to complete (52) by the $SU(2)$ algebra relations between the R charges as is given in (29) and by the full set of the Poisson brackets involving the supercharges $\bar\Theta^\beta$.

In order to bring this superalgebra into the conventional form, it is convenient to rotate the supercharges as

$$Q^\alpha = e^{i\nu/2} \cos\lambda \Theta^\alpha + e^{-i\nu/2} \sin\lambda \varepsilon^{\alpha\gamma} \bar\Theta_\gamma,$$
$$\bar Q_\alpha = e^{-i\nu/2} \cos\lambda \bar\Theta_\alpha - e^{i\nu/2} \sin\lambda \varepsilon_{\alpha\gamma} \Theta^\gamma, \quad (54)$$

where

$$\cos 2\lambda = \frac{B}{\sqrt{4|\omega|^2 + B^2}},$$
$$\sin 2\lambda = -\frac{2|\omega|}{\sqrt{4|\omega|^2 + B^2}}, \qquad \omega = |\omega| e^{i\nu}. \quad (55)$$

In terms of these newly defined quantities, the symmetry algebra is rewritten as

$$\{Q^\alpha, \bar Q_\beta\} = i\delta^\alpha_\beta \mathcal{H}_{osc} + \sqrt{4|\omega|^2 + B^2} \mathcal{R}^\alpha_\beta,$$
$$\{Q^\alpha, \mathcal{H}_{osc}\} = \frac{i}{2}\sqrt{4|\omega|^2 + B^2} Q^\alpha,$$
$$\{Q^\alpha, Q^\beta\} = \{\bar Q_\alpha, \bar Q_\beta\} = 0, \quad (56)$$

$$\{Q^\alpha, \mathcal{R}^\beta_\gamma\} = -i\delta^\alpha_\gamma Q^\beta + \frac{i}{2}\delta^\beta_\gamma Q^\alpha$$
$$\{\mathcal{R}^\alpha_\beta, \mathcal{R}^\gamma_\delta\} = i\delta^\gamma_\beta \mathcal{R}^\alpha_\delta - i\delta^\alpha_\delta \mathcal{R}^\gamma_\beta \quad \{\mathcal{R}^\alpha_\beta, \mathcal{H}_{osc}\} = 0. \quad (57)$$

Comparing these relations with those of the supersymmetric $\mathcal{N} = 4$ Landau problem (30), we can identify them as defining $SU(2|1)$ superalgebra with the deformation parameter $m = \sqrt{4|\omega|^2 + B^2}$.

The Lagrangian of $SU(2|1)$ supersymmetric Kähler oscillator is given by the general expression (22), with

$$\mathcal{U} = |\omega|^2 g^{a\bar b} \partial_a K \partial_{\bar b} K - \frac{1}{2} R_{a\bar b c\bar d} \eta^{a\alpha} \bar\eta^b_\alpha \eta^{c\beta} \bar\eta^d_\beta - \frac{\omega}{2} K_{a;b} \eta^{a\alpha} \eta^b_\alpha$$
$$- \frac{\bar\omega}{2} K_{\bar a;\bar b} \bar\eta^a_\alpha \bar\eta^{b\alpha} + \frac{B}{2} g_{a\bar b} \eta^{a\alpha} \bar\eta^b_\alpha. \quad (58)$$

The supersymmetrization procedure described above is capable of producing a family of nonequivalent Hamiltonians parametrized by an arbitrary holomorphic function. Namely, replacing the initial Kähler potential K by the gauge-equivalent one,

$$K(z, \bar z) \to K(z, \bar z) + \frac{1}{\omega} U(z) + \frac{1}{\bar\omega} \bar U(\bar z), \quad (59)$$

we obtain the class of Hamiltonians parametrized by an arbitrary holomorphic function $U(z)$,

$$\mathcal{H}_{osc} \to \mathcal{H}_{osc} = g^{\bar a b}(\bar\pi_a \pi_b + \partial_{\bar a} \bar U \partial_b U) - \frac{1}{2} R_{a\bar b c\bar d} \eta^{a\alpha} \bar\eta^b_\alpha \eta^{c\beta} \bar\eta^d_\beta$$
$$+ \frac{1}{2} U_{a;b} \eta^{a\alpha} \eta^b_\alpha + \frac{1}{2} \bar U_{\bar a;\bar b} \bar\eta^a_\alpha \bar\eta^{b\alpha} + \frac{B}{2} g_{a\bar b} \eta^{a\alpha} \bar\eta^b_\alpha$$
$$+ |\omega|^2 g^{\bar a b} \partial_{\bar a} K \partial_b K$$
$$+ |\omega| g^{\bar a b} (\partial_{\bar a} K \partial_b U + \partial_{\bar a} \bar U \partial_b K)$$
$$- \frac{\omega}{2} K_{a;b} \eta^{a\alpha} \eta^b_\alpha - \frac{\bar\omega}{2} K_{\bar a;\bar b} \bar\eta^a_\alpha \bar\eta^{b\alpha}. \quad (60)$$

In the limit $\omega = 0$, we arrive at the well-known Hamiltonian which admits, in the absence of magnetic field, the "flat" $\mathcal{N} = 4$ supersymmetry (see, e.g., [22]). It is given by the first line in the above expression with $B = 0$.

A. Two-dimensional anisotropic oscillator

The supersymmetrization procedure outlined above makes it possible to extend the class of the known systems admitting such a supersymmetrization. Here, we illustrate this on the case of a two-dimensional harmonic oscillator, which is the simplest system possessing the conventional $\mathcal{N} = 4$, $d = 1$ "Poincaré" supersymmetric extension. Take the one-dimensional complex space $(\mathbb{C}, ds^2 = dz d\bar z)$ and consider in it the Kähler oscillator defined by the potential,

$$K(z, \bar z) = z\bar z + \frac{igz^2}{2\omega} - \frac{i\bar g \bar z^2}{2\bar\omega}. \quad (61)$$

[3] We use here the following rules for complex conjugation and raising and lowering of $SU(2)$ spinor indices:

$$\overline{\varepsilon_{\alpha\beta}} = -\varepsilon^{\alpha\beta}, \qquad \varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha},$$
$$\varepsilon_{12} = \varepsilon^{21} = 1, \qquad \varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta} = \delta^\alpha_\delta \delta^\beta_\gamma - \delta^\alpha_\gamma \delta^\beta_\delta.$$

It gives rise to the following Kähler-oscillator system:

$$H = \pi\bar{\pi} + (\omega\bar{\omega} + g\bar{g})z\bar{z} + i\bar{\omega}gz^2 - i\omega\bar{g}\bar{z}^2,$$
$$\{\pi, z\} = \{\bar{\pi}, z\} = 1, \qquad \{\pi, \bar{\pi}\} = iB. \tag{62}$$

Diagonalizing this potential, we arrive at the two-dimensional anisotropic oscillator system with frequencies,

$$\omega^{\pm} = ||\omega| \pm |g||. \tag{63}$$

For the choice $\omega = 0$, it yields the two-dimensional isotropic oscillator with the frequency $|g|$, which admits, in the absence of a magnetic field, the standard $\mathcal{N} = 4$, $d = 1$ supersymmetrization. In the presence of a magnetic field, this supersymmetry is deformed to $SU(2|1)$. In the opposite limit, at $g = 0$, we once again obtain some $SU(2|1)$ supersymmetric extension of a two-dimensional isotropic oscillator, but different from the first option. In the generic case of $g \neq 0$, $\omega \neq 0$, the procedure proposed allows us to construct a $SU(2|1)$ superextension of the two-dimensional **anisotropic** oscillator interacting with a constant magnetic field perpendicular to the plane. Enlarging the above set of Poisson brackets by the relation $\{\eta^\alpha, \bar{\eta}_\beta\} = i\delta^\alpha_\beta$, we can write down the Hamiltonian of the supersymmetric extension of this system as

$$\mathcal{H}_{\rm anosc} = \pi\bar{\pi} + (\omega\bar{\omega} + g\bar{g})z\bar{z} + i\bar{\omega}gz^2 - i\omega\bar{g}\bar{z}^2 - \frac{ig}{2}\eta^\alpha\eta_\alpha + \frac{i\bar{g}}{2}\bar{\eta}_\alpha\bar{\eta}^\alpha + \frac{B}{2}\eta^\alpha\bar{\eta}_\alpha. \tag{64}$$

The relevant supercharges and R charges have the following simple form:

$$\Theta^\alpha = \pi\eta^\alpha + (i\bar{\omega}z + \bar{g}\,\bar{z})\varepsilon^{\alpha\beta}\bar{\eta}_\beta$$
$$\mathcal{R}^\alpha_\beta = \eta^\alpha\bar{\eta}_\beta - \frac{1}{2}\delta^\alpha_\beta\eta^\gamma\bar{\eta}_\gamma. \tag{65}$$

It is straightforward to extend this model to N-dimensional complex Euclidian space $\mathbb{C}^N$ (see Sec. VI).

V. SUPERFIELD FORMULATION

The one-particle [i.e., one-(complex)dimensional] versions of the Lagrangians presented above were derived from the $SU(2|1)$ and $SU(4|1)$ superfield approaches in [11] and [13]. The generalization of these models to the N-dimensional case is straightforward. We briefly describe it below.

A. $SU(2|1)$ case

As the first step, we reproduce the Lagrangian of the $SU(2|1)$ Kähler superoscillator corresponding to (53) and its particular case, the Lagrangian of $SU(2|1)$ supersymmetric Landau problem (34).

In [10] and [11], the coset method was used to define the world-line realizations of the supergroup $SU(2|1)$ on the $d = 1$ superspace $(t, \theta_\alpha, \bar{\theta}^\beta)$ identified with the coset of $SU(2|1)$ over its R-symmetry subgroup $SU(2)$. The basic objects of this realization are covariant spinor derivatives,

$$\mathcal{D}^\alpha = e^{-\frac{imt}{2}}\left[\left(1 + \frac{m}{2}\bar{\theta}^\beta\theta_\beta - \frac{3m^2}{16}(\bar{\theta}^\beta\theta_\beta)^2\right)\frac{\partial}{\partial\theta_\alpha} - \frac{m}{2}\bar{\theta}^\alpha\theta_\beta\frac{\partial}{\partial\theta_\beta} - \frac{i}{2}\bar{\theta}^\alpha\partial_t\right],$$
$$\bar{\mathcal{D}}_\alpha = e^{\frac{imt}{2}}\left[-\left(1 + \frac{m}{2}\bar{\theta}^\beta\theta_\beta - \frac{3m^2}{16}(\bar{\theta}^\beta\theta_\beta)^2\right)\frac{\partial}{\partial\bar{\theta}^\alpha} + \frac{m}{2}\bar{\theta}^\beta\theta_\alpha\frac{\partial}{\partial\bar{\theta}^\beta} + \frac{i}{2}\theta_\alpha\partial_t\right], \tag{66}$$

which, in the contraction limit $m = 0$, become standard covariant spinor derivatives of flat $\mathcal{N} = 4$, $d = 1$ supersymmetry. The chiral $SU(2|1)$ superfields $\Phi^a(t, \hat{\theta}, \bar{\hat{\theta}})$ satisfy the generalized $SU(2|1)$ covariant chirality constraints [11],

$$(\cos\lambda\bar{\mathcal{D}}_\alpha - \sin\lambda\mathcal{D}_\alpha)\Phi^a = 0. \tag{67}$$

In the appropriate superspace basis, the conditions (67) become "short" up to an overall factor,

$$(\cos\lambda\bar{\mathcal{D}}_\alpha - \sin\lambda\mathcal{D}_\alpha)\Phi^a = \left[1 + \frac{B}{4}\bar{\hat{\theta}}^\beta\hat{\theta}_\beta + \frac{\omega}{4}(\hat{\theta}_\beta\hat{\theta}^\beta + \bar{\hat{\theta}}^\beta\bar{\hat{\theta}}_\beta) - \frac{m^2}{32}(\bar{\hat{\theta}}^\beta\hat{\theta}_\beta)^2\right] \times \left[-\frac{\partial}{\partial\bar{\hat{\theta}}^\alpha} + \frac{i}{2}\hat{\theta}_\alpha\partial_t\right]\Phi^a, \tag{68}$$

and are solved by the expressions,

$$\Phi^a(t_{\rm L}, \hat{\theta}_\alpha) = z^a + \hat{\theta}_\alpha\eta^{a\alpha} + \frac{1}{2}\hat{\theta}_\alpha\hat{\theta}^\alpha A^a, \quad t_{\rm L} = t + \frac{i}{2}\bar{\hat{\theta}}^\alpha\hat{\theta}_\alpha. \tag{69}$$

The dependence on the new parameter λ is now hidden in the definition of the superspace coordinates $t_{\rm L}$ and $\hat{\theta}_\alpha$, which have the following $SU(2|1)$ transformation properties:

$$\delta\hat{\theta}_\alpha = \cos\lambda\left(\epsilon_\alpha e^{\frac{i}{2}mt_{\rm L}} + \frac{m}{2}\bar{\epsilon}^\beta\hat{\theta}_\beta\hat{\theta}_\alpha e^{-\frac{i}{2}mt_{\rm L}}\right) + \sin\lambda\left(\bar{\epsilon}_\alpha e^{-\frac{i}{2}mt_{\rm L}} + \frac{m}{2}\epsilon^\beta\hat{\theta}_\beta\hat{\theta}_\alpha e^{\frac{i}{2}mt_{\rm L}}\right), \tag{70}$$

$$\delta t_{\rm L} = i\cos\lambda\bar{\epsilon}^\beta\hat{\theta}_\beta e^{-\frac{i}{2}mt_{\rm L}} - i\sin\lambda\epsilon^\beta\hat{\theta}_\beta e^{\frac{i}{2}mt_{\rm L}}. \tag{71}$$

These coordinate transformations induce the off shell $SU(2|1)$ supersymmetry transformation of chiral superfields. On the component fields, they are realized as

$$\begin{aligned}\delta z^a &= -(\cos\lambda\epsilon_\alpha e^{\frac{i}{2}mt} + \sin\lambda\bar{\epsilon}_\alpha e^{-\frac{i}{2}mt})\eta^{a\alpha},\\ \delta\eta^{a\alpha} &= \bar{\epsilon}^\alpha(i\cos\lambda\dot{z}^a - \sin\lambda A^a)e^{-\frac{i}{2}mt}\\ &\quad - \epsilon^\alpha(i\sin\lambda\dot{z}^a + \cos\lambda A^a)e^{\frac{i}{2}mt},\\ \delta A^a &= -\cos\lambda\bar{\epsilon}_\alpha\left(i\dot{\eta}^{a\alpha} + \frac{m}{2}\eta^{a\alpha}\right)e^{-\frac{i}{2}mt}\\ &\quad + \sin\lambda\epsilon_\alpha\left(i\dot{\eta}^{a\alpha} - \frac{m}{2}\eta^{a\alpha}\right)e^{\frac{i}{2}mt},\end{aligned} \tag{72}$$

where ϵ_α are "infinitesimal" Grassmann parameters.

The corresponding off shell superfield Lagrangian is as follows (see [11] for the one-particle case):

$$\mathcal{L} = \int d^2\hat{\theta} d^2\bar{\hat{\theta}}\left[1 + \frac{B}{2}\bar{\hat{\theta}}^\alpha\hat{\theta}_\alpha + \frac{\omega}{2}(\hat{\theta}_\alpha\hat{\theta}^\alpha + \bar{\hat{\theta}}^\alpha\bar{\hat{\theta}}_\alpha)\right]K(\Phi^a,\bar{\Phi}^b), \tag{73}$$

where[4]

$$B = m\cos 2\lambda, \qquad \omega = -\frac{m}{2}\sin 2\lambda. \tag{74}$$

It is straightforward to check that the transformation of the factor within the square brackets in (73) precisely cancels the transformation of the integration measure $dt_L d^2\hat{\theta} d^2\bar{\hat{\theta}}$. Integrating in (73) over $\hat{\theta}$, $\bar{\hat{\theta}}$ and eliminating the auxiliary fields A^a, we recover the on shell Lagrangian (22) with the expression (58) for $\mathcal{U}$. In the particular case $\lambda = 0$ ($\omega = 0$), we arrive at the Lagrangian (34) of the Landau problem. Holomorphic terms (59) can be naturally inserted in (73) with $\omega \neq 0$ through the shift,

$$K(\Phi^a,\bar{\Phi}^b) \to K(\Phi^a,\bar{\Phi}^b) + \frac{1}{\omega}U(\Phi^a) + \frac{1}{\omega}\bar{U}(\bar{\Phi}^b), \tag{75}$$

which amounts to the introduction of the additional superpotential terms which, in components, induce the modified potential $\mathcal{U}$, as in (60).

It is instructive to see how the phenomenon of preserving the isometries under the deformation manifests itself in the superfield language. For this purpose, we need to know how the Kähler potential itself transforms under the isometry of the Kähler structure given by (4). To this end, we rewrite the equation (b) in (6) in the equivalent form as

$$\partial_c\partial_{\bar{d}}\{[V^a_\mu(z)\partial_a + V^{\bar{a}}_\mu(\bar{z})\partial_{\bar{a}}]K(z,\bar{z})\} = 0, \tag{76}$$

whence

$$[V^a_\mu(z)\partial_a + V^{\bar{a}}_\mu(\bar{z})\partial_{\bar{a}}]K(z,\bar{z}) = \varphi_\mu(z) + \bar{\varphi}_\mu(\bar{z}). \tag{77}$$

The holomorphic function $\varphi_\mu(z)$, in each specific case, can be defined up to a constant by differentiating (77) with respect to z^b.

The isometry transformations of the Kähler manifold in the superfield coordinates are obtained just by the changes $z^a \to \Phi^a$, $\bar{z}^a \to \bar{\Phi}^a$ in the relevant holomorphic Hamiltonian vector fields. Recalling the transformation (77) of $K(z,\bar{z})$ under isometry, we see that the superfield Lagrangian in (73) is transformed as

$$\delta^* K = b^\mu\varphi(\Phi^a)_\mu + \bar{b}^\mu\bar{\varphi}(\bar{\Phi}^a)_\mu, \tag{78}$$

where b_μ, $\bar{b}_\mu$ are constant isometry parameters. Taking the bar-spinor derivatives from the integration measure and making use of the chirality of Φ^a, it is easy to see that the holomorphic term in (78) does not contribute at $\omega = \lambda = 0$, $B = m$. The vanishing of the contribution from the conjugated antiholomorphic term in (78) can be proved after passing to the right-chiral basis in the $SU(2|1)$ superspace. This is the superfield proof of the property that the $SU(2|1)$ super Landau model inherits all the isometries of the undeformed case $\omega = \lambda = m = 0$. The isometries are not generically inherited by the Kähler superoscillator, when $\omega \neq 0$.

It should be pointed out that the input parameters of the above superfield formalism are just the contraction mass-dimension parameter m coming from the (anti)commutation relations of the $su(2|1)$ algebra and the angle λ coming from the chirality constraint (67). The physical meaning of these parameters as the strength of the external magnetic field and the oscillator frequency is revealed at the level of the component Lagrangians and Hamiltonians.

B. $SU(4|1)$ case

Next, let us present the $SU(4|1)$ superfield formulation for the Lagrangian of the $\mathcal{N} = 8$ Landau problem (45), based on the superspace approach developed in [23]. This superfield Lagrangian is written in terms of chiral (**2, 8, 6**) superfields as follows (its one-particle case was constructed in [13]):

$$\begin{aligned}S &= \int dt\mathcal{L}\\ &= -\int dt_{\rm L} d^4\theta e^{-3imt_{\rm L}}\mathcal{F}(\Phi^a) - \int dt_{\rm R} d^4\bar{\theta} e^{3imt_{\rm R}}\bar{\mathcal{F}}(\bar{\Phi}^a),\\ m &= |B|.\end{aligned} \tag{79}$$

Here, $\mathcal{F}(z)$ is the Seiberg-Witten prepotential (46), while the θ expansion of the superfields Φ^a reads

[4]We limit our attention to the real frequencies $\omega = |\omega|$ in order to match the superfield approach elaborated in [11]. In fact, one can easily generalize this consideration to $\omega \in \mathbb{C}$.

$$\Phi^a(t_{\rm L},\theta_I) = z^a + \theta_K \eta^{aK} e^{3imt_{\rm L}/4} + \frac{1}{2}\theta_I\theta_J A^{aIJ} e^{3imt_{\rm L}/2} - \frac{1}{6}\varepsilon^{IJKL}\theta_I\theta_J\theta_K\left(i\dot{\bar\eta}^a_L - \frac{m}{4}\bar\eta^a_L\right)e^{9imt_{\rm L}/4} + \frac{1}{24}\varepsilon^{IJKL}\theta_I\theta_J\theta_K\theta_L(\ddot{\bar z}^a + im\dot{\bar z}^a)e^{3imt_{\rm L}}, \quad (80)$$

with the following conjugation rules: $\overline{(A^{aIJ})} = A^a_{IJ} = \frac{1}{2}\varepsilon_{IJKL}A^{aKL}$, $\overline{(\eta^{aI})} = \bar\eta^a_I$.

The coordinate set $\{t_{\rm L},\theta^I\}$ is closed under the $SU(4|1)$ transformations,

$$\delta\theta_I = \epsilon_I + m\bar\epsilon^K\theta_K\theta_I, \qquad \delta t_{\rm L} = i\bar\epsilon^K\theta_K. \quad (81)$$

The corresponding off shell supersymmetry transformations of the component fields read

$$\delta z^a = -\epsilon_K\eta^{aK}e^{3imt/4}, \qquad \delta\bar z^a = \bar\epsilon^K\bar\eta^a_K e^{-3imt/4},$$
$$\delta A^{aIJ} = 2\bar\epsilon^{[I}\left(i\dot\eta^{aJ]} + \frac{m}{4}\eta^{aJ]}\right)e^{-3imt/4} + \varepsilon^{IJKL}\epsilon_{[K}\left(i\dot{\bar\eta}^a_{L]} - \frac{m}{4}\bar\eta^a_{L]}\right)e^{3imt/4},$$
$$\delta\eta^{aI} = \bar\epsilon^I(i\dot z^a)e^{-3imt/4} - \epsilon_K A^{aIK}e^{3imt/4},$$
$$\delta\bar\eta^a_I = -\epsilon_I(i\dot{\bar z}^a)e^{3imt/4} - \bar\epsilon^K A^a_{IK}e^{-3imt/4}. \quad (82)$$

Integration in (80) over θ, $\bar\theta$ gives the off shell Lagrangian,

$$\mathcal{L}_{\rm off-shell} = g_{a\bar b}\left[\dot z^a\dot{\bar z}^b - \frac{1}{4}A^{aIJ}A^b_{IJ} + \frac{i}{2}(\eta^{aK}\dot{\bar\eta}^b_K - \dot\eta^{aK}\bar\eta^b_K) - \frac{m}{4}\eta^{aK}\bar\eta^b_K\right] - \frac{i}{2}(\dot z^c\partial_c g_{a\bar b} - \dot{\bar z}^c\partial_{\bar c}g_{a\bar b})\eta^{aK}\bar\eta^b_K + im(\dot z^a\partial_{\bar a}\bar{\mathcal{F}} - \dot{\bar z}^a\partial_a\mathcal{F}) + \frac{1}{2}A^b_{IJ}\eta^{aI}\eta^{cJ}\partial_c g_{a\bar b} - \frac{1}{2}A^{aIJ}\bar\eta^b_I\bar\eta^c_J\partial_{\bar c}g_{a\bar b} - \frac{1}{24}[\varepsilon_{IJKL}\eta^{aI}\eta^{bJ}\eta^{cK}\eta^{dL}\partial_c\partial_d g_{a\bar b} + \varepsilon^{IJKL}\bar\eta^a_I\bar\eta^b_J\bar\eta^c_K\bar\eta^d_L\partial_{\bar c}\partial_{\bar d}g_{a\bar b}], \quad (83)$$

where the metric $g_{a\bar b}$ is identified with the metric defined in (46). The subsequent elimination of the auxiliary fields A^{aIJ} yields just the on shell Lagrangian (45).

It is important that the superfield action (79) is invariant under the transformations corresponding to (47) (see [24]),

$$\mathcal{F}(\Phi^a) \to \mathcal{F}(\Phi^a) + ic_{ab}\Phi^a\Phi^b + c_a\Phi^a + c,$$
$$\bar{\mathcal{F}}(\bar\Phi^a) \to \bar{\mathcal{F}}(\bar\Phi^a) - ic_{ab}\bar\Phi^a\bar\Phi^b + \bar c_a\bar\Phi^a + \bar c, \quad (84)$$

where c, c_a are complex numbers, and c_{ab} are real ones.

These transformations are just the $\mathcal{N}=8$ superfield version of the general transformations of the holomorphic prepotential $\mathcal{F}(z)$ under an arbitrary *isometry of the special Kähler structure*, i.e., of the isometry of Kähler structure preserving holomorphic third-order tensor (43) (see the Appendix). Hence, the invariance of (79) under (84) explicitly demonstrates that the deformed $\mathcal{N}=8$ supersymmetric mechanics we are considering inherits the full set of isometries of the undeformed case.

The proof of this superfield invariance is not too easy. To this end, one needs to represent the invariant chiral measure $d^4\theta e^{-3imt_{\rm L}}$ in the action (79) in terms of covariant derivatives (up to total time derivatives) as[5]

$$d^4\theta e^{-3imt_{\rm L}} = \frac{1}{24}e^{-3imt_{\rm L}}\varepsilon_{IJKL}\partial^I\partial^J\partial^K\partial^L = \frac{1}{24}\varepsilon_{IJKL}\mathcal{D}^I\mathcal{D}^J\mathcal{D}^K\mathcal{D}^L. \quad (85)$$

Covariant derivatives anticommute as

$$\{\bar{\mathcal{D}}_I,\bar{\mathcal{D}}_J\}=0, \quad \{\mathcal{D}^I,\mathcal{D}^J\}=0,$$
$$\{\mathcal{D}^I,\bar{\mathcal{D}}_J\}=\delta^I_J\mathcal{H}_0+m\tilde R^I_J, \quad \tilde R^I_J\mathcal{D}^K = \frac{1}{4}\delta^I_J\mathcal{D}^K - \delta^K_J\mathcal{D}^I, \quad (86)$$

where $\tilde R^I_J$ are $SU(4)$ matrix generators acting on external indices of superfields and covariant derivatives. The chiral superfield Φ^a ($a=1,\dots N$) describing N multiplets (**2**, **8**, **6**) satisfies the constraints [24],

$$\mathcal{D}^I\bar\Phi^a = 0, \qquad \bar{\mathcal{D}}_K\Phi^a = 0,$$
$$\mathcal{D}^I\mathcal{D}^J\Phi^a = \frac{1}{2}\varepsilon^{IJKL}\bar{\mathcal{D}}_K\bar{\mathcal{D}}_L\bar\Phi^a. \quad (87)$$

Exploiting (85)–(87) for the action (79), one can show its invariance under the transformations (84). Another, more direct proof is to substitute the explicit expressions (80) for Φ^a and the conjugated ones for $\bar\Phi^a$ into (84) and to be convinced that the coefficients of the higher-order monomials in $\theta_I(\bar\theta^I)$ in the holomorphic(antiholomorphic) shifts (84) either are combined into the total t derivatives or just vanish. Note that the reality condition for the

[5]Though expressions for $SU(4|1)$ covariant derivatives were not calculated, the function $\mathcal{D}^I\mathcal{D}^J\mathcal{D}^K\mathcal{D}^L\mathcal{F}(\Phi^a)$ is $SU(4|1)$ invariant. Hence, it must give the same invariant action (79).

coefficient c_{ab} in (84) is essential for ensuring the properties just mentioned.

Derivation of the purely bosonic counterpart of the transformations (84) from the isometry condition (43) is discussed in the Appendix.

VI. EXAMPLES OF SUPERINTEGRABLE KÄHLER OSCILLATOR MODELS

In the previous sections, we dealt with two classes of models admitting deformed supersymmetry: the Landau problems, and the Kähler oscillators. In the case of the Landau problem, we found that the supersymmetric extensions preserve all (kinematical) symmetries of the initial systems. But we were not able to prove the similar general proposition for the Kähler oscillators. In this section, we present supersymmetric extensions of two particular types of the Kähler oscillator systems which possess kinematical symmetries and the hidden symmetries generated by the constants of a motion quadratic in momenta. These two types are encompassed by the following models:

(i) $\mathbb{C}^N$-oscillator (the sum of N two-dimensional isotropic oscillators) and $\mathbb{C}^N$-Smorodinsky-Winternitz system (the sum of N copies of two-dimensional isotropic oscillators deformed by ring-shaped potentials).

(ii) $\mathbb{CP}^N$-oscillator and $\mathbb{CP}^N$-Rosochatius system, which are superintegrable counterparts of $\mathbb{C}^N$-oscillator and $\mathbb{C}^N$-Smorodinsky-Winternitz systems on the complex projective spaces.

Our main goal will be to inspect whether $SU(2|1)$ supersymmetric extensions of these systems inherit their hidden symmetries.

A. Euclidean spaces

We start by considering the Kähler oscillators on the complex Euclidian space $(\mathbb{C}^N, ds^2 = \sum_{a=1}^N dz^a d\bar{z}^a)$. The relevant phase space is defined by the Poisson brackets,

$$\{\pi_a, z^b\} = \delta_a^b, \quad \{\bar{\pi}_a, \bar{z}^b\} = \delta_a^b, \quad \{\pi_a, \bar{\pi}_b\} = iB\delta_{a\bar{b}}. \tag{88}$$

The set of symmetries of this space is constituted by the $SU(N)$ generators,

$$J_{a\bar{b}} = i\pi_a z^b - i\bar{\pi}_b \bar{z}^a - Bz^b\bar{z}^a\colon \{J_{\bar{a}b}, J_{\bar{c}d}\} = i\delta_{\bar{a}d}J_{\bar{b}c} - i\delta_{\bar{c}b}J_{\bar{a}d}, \tag{89}$$

and the translation generators,

$$J_a = i\pi_a - B\bar{z}^a\colon \{J_a, J_b\} = \{J_a, \bar{J}_b\} = 0,$$
$$\{J_a, J_{b\bar{c}}\} = -iJ_b\delta_{a\bar{c}}. \tag{90}$$

For the construction of $SU(2|1)$ supersymmetric Kähler oscillator models on this space, we have to complete the Poisson brackets (88) by the following ones:

$$\{\eta^{a\alpha}, \bar{\eta}^b_\beta\} = i\delta^{a\bar{b}}\delta^\alpha_\beta, \tag{91}$$

with $\alpha, \beta = 1, 2$. Then we should perform the $SU(2|1)$ supersymmetrization procedure described above, for the appropriate choice of the initial bosonic Kähler oscillator model.

1. Harmonic oscillator

We define the $\mathbb{C}^N$-harmonic oscillator defined as a Kähler oscillator with $K(z, \bar{z}) = \sum_{a=1}^N z^a\bar{z}^a$ and $\omega = \bar{\omega}$,

$$H_{osc} = \sum_{a=1}^N (\pi_a\bar{\pi}_a + \omega^2 z^a\bar{z}^a). \tag{92}$$

This system possesses $SU(N)$ kinematical symmetry generated by the generators (89) and hidden symmetries defined by the so-called Fradkin tensor,

$$I_{a\bar{b}} = \pi_a\bar{\pi}_b + \omega^2\bar{z}^a z^b\colon \{I_{a\bar{b}}, I_{c\bar{d}}\} = i\delta_{a\bar{d}}J_{c\bar{b}} - i\delta_{c\bar{b}}J_{a\bar{d}},$$
$$\{I_{a\bar{b}}, J_{c\bar{d}}\} = i\omega\delta_{a\bar{d}}I_{c\bar{b}} - i\omega\delta_{c\bar{b}}I_{a\bar{d}}. \tag{93}$$

In the $SU(2|1)$ supersymmetric extension of this system, the Hamiltonian, dynamical supercharges, and R charges are determined by those of the two-dimensional isotropic oscillator,

$$\mathcal{H} = \sum_{a=1}^N \mathcal{H}_a, \qquad \Theta^\alpha = \sum_{a=1}^N \Theta^{a\alpha}, \qquad \mathcal{R}^\alpha_\beta = \sum_{a=1}^N \mathcal{R}^{a\alpha}_\beta, \tag{94}$$

with

$$\mathcal{H}_a = \pi_a\bar{\pi}_a + \omega^2 z^a\bar{z}^a + \frac{B}{2}\eta^{a\alpha}\bar{\eta}^a_\alpha,$$
$$\Theta^{a\alpha} = \pi_a\eta^{a\alpha} + i\omega z^a\varepsilon^{\alpha\beta}\bar{\eta}^a_\beta, \quad \mathcal{R}^{a\alpha}_\beta = \eta^{a\alpha}\bar{\eta}^a_\beta - \frac{1}{2}\delta^\alpha_\beta i\eta^{a\gamma}\bar{\eta}^a_\gamma. \tag{95}$$

All constants of motion of the bosonic Hamiltonian become those of the supersymmetrized one, since all these quantities are just sums of the bosonic and fermionic parts. Moreover, in the supersymmetric system, there appear additional symmetry generators acting on the fermionic variables only. Thus, the system with the Hamiltonian (95) inherits kinematical $SU(N)$ symmetries of the bosonic sector (89), hidden symmetries generated by the Fradkin tensor (93), and reveals an additional $U(N)$ symmetry realized in the fermionic sector,

$$\mathcal{R}_{a\bar{b}} = \sum_\alpha \eta^{b\alpha}\bar{\eta}^a_\alpha\colon \{\mathcal{R}_{a\bar{b}}, \mathcal{R}_{c\bar{d}}\} = i\delta_{a\bar{d}}\mathcal{R}_{c\bar{b}} - i\delta_{c\bar{b}}\mathcal{R}_{a\bar{d}}. \tag{96}$$

Now we turn to considering a less trivial example of the $SU(2|1)$ supersymmetric Kähler oscillator with hidden symmetries.

2. $\mathbb{C}^N$-Smorodinsky-Winternitz system

The $\mathbb{C}^N$-Smorodinsky-Winternitz system is defined by the Hamiltonian [16],

$$H_{SW} = \sum_{a=1}^{N} I_a, \qquad I_a = \pi_a \bar{\pi}_a + |\omega|^2 z^a \bar{z}^a + \frac{|g_a|^2}{z^a \bar{z}^a}. \tag{97}$$

It has N manifest $U(1)$ symmetries $z^a \to e^{i\kappa}$, with the generators $J_{a\bar{a}}$, and the hidden symmetries spanned by the above generators I_a, as well as by the following ones (the so-called Uhlenbeck tensor):

$$I_{ab} = J_{a\bar{b}} J_{b\bar{a}} - \frac{1}{2} J_{a\bar{a}} J_{b\bar{b}} + \frac{|g_a|^2 z^b \bar{z}^b}{z^a \bar{z}^a} + \frac{|g_b|^2 z^a \bar{z}^a}{z^b \bar{z}^b}, : \\ \{I_{ab}, H_{SW}\} = 0, \tag{98}$$

where $J_{a\bar{b}}$ are $u(N)$ generators defined in (89).

This system can be identified as a Kähler oscillator with the following Kähler potential:

$$K = z\bar{z} + \frac{g_a}{\omega} \log z^a + \frac{\bar{g}_a}{\bar{\omega}} \log \bar{z}^a, \\ \arg \omega = \arg \sum_{a=1}^{N} g_a + \pi/2. \tag{99}$$

Its $SU(2|1)$ supersymmetric extension is found to be associated with the Hamiltonian,

$$\mathcal{H}_{SW} = \sum_{a=1}^{N} \mathcal{I}_a, \\ \mathcal{I}_a = \pi_a \bar{\pi}_a + |\omega|^2 z^a \bar{z}^a + \frac{|g_a|^2}{z^a \bar{z}^a} + \frac{g_a}{2} \frac{\eta^{a\alpha} \eta^a_\alpha}{z^a z^a} \\ + \frac{\bar{g}_a}{2} \frac{\bar{\eta}^a_\alpha \bar{\eta}^{a\alpha}}{\bar{z}^a \bar{z}^a} + \frac{B}{2} \eta^{a\alpha} \bar{\eta}^a_\alpha, \tag{100}$$

and the supercharges,

$$\Theta^{a\alpha} = \pi_a \eta^{a\alpha} + i\omega \epsilon^{\alpha\beta} \bar{\eta}^a_\beta \left(z^a + \frac{g_a}{\omega z^a} \right). \tag{101}$$

Clearly, the generators $\mathcal{I}_a$ commute with each other, and so they are the constants of motion of the supersymmetric $\mathbb{C}^N$-Smorodinsky-Winternitz system. This supersymmetric system possesses N manifest $U(1)$ symmetries $z^a \to e^{i\kappa} z^a$, $\eta^a_\alpha \to e^{i\kappa} \eta^a_\alpha$, with the generators,

$$\mathcal{J}_{a\bar{a}} = J_{a\bar{a}} + \eta^{a\alpha} \bar{\eta}^a_\alpha : \{\mathcal{J}_{a\bar{a}}, \mathcal{J}_{b\bar{b}}\} = \{\mathcal{J}_{a\bar{a}}, \mathcal{I}_b\} = 0. \tag{102}$$

The extensions of the hidden symmetry generators I_a, I_{ab} are given, respectively, by the generators $\mathcal{I}_a$ defined in (100) and by the following ones:

$$\mathcal{I}_{ab} = I_{ab} + \frac{g_a}{2} \frac{z^b \bar{z}^b}{z^a z^a} \eta^{a\alpha} \eta^a_\alpha + \frac{\bar{g}_a}{2} \frac{z^b \bar{z}^b}{\bar{z}^a \bar{z}^a} \bar{\eta}^a_\alpha \bar{\eta}^{a\alpha} + \frac{g_b}{2} \frac{z^a \bar{z}^a}{z^b z^b} \eta^{b\alpha} \eta^b_\alpha \\ + \frac{\bar{g}_b}{2} \frac{z^a \bar{z}^a}{\bar{z}^b \bar{z}^b} \bar{\eta}^b_\alpha \bar{\eta}^{b\alpha} : \{\mathcal{I}_{ab}, \mathcal{H}_{SW}\} = 0. \tag{103}$$

Thus, the $SU(2|1)$ supersymmetric extension of the $\mathbb{C}^N$-Smorodinsky-Winternitz system inherits all its hidden symmetries.

The conclusion is that the "Kähler superoscillator approach" yields the well-defined superextensions of both the isotropic oscillator and the Smorodinsky-Winternitz system on $\mathbb{C}^N$.

B. Complex projective spaces

In this section, we will deal with superintegrable systems on complex projective spaces $\mathbb{CP}^N$, which are specified by the presence of a constant magnetic field and belong to the class of the Kähler oscillator models.

Consider the complex projective space equipped with $su(N+1)$-invariant Fubini-Study metrics,

$$g_{a\bar{b}} dz^a d\bar{z}^b, \quad \text{with} \quad g_{a\bar{b}} = \frac{\log(1+z\bar{z})}{\partial z^a \partial \bar{z}^b} = \frac{\delta_{a\bar{b}}}{1+z\bar{z}} - \frac{\bar{z}^a z^b}{(1+z\bar{z})^2}. \tag{104}$$

The inverse metrics, nonzero Christoffel symbols, and Riemann tensor are defined by the expressions,

$$g^{\bar{a}b} = (1+z\bar{z})(\delta^{\bar{a}b} + \bar{z}^a z^b), \\ \Gamma^a_{bc} = -\frac{\delta^a_b \bar{z}^c + \delta^a_c \bar{z}^b}{1+z\bar{z}}. \quad R_{a\bar{b}c\bar{d}} = g_{a\bar{b}} g_{c\bar{d}} + g_{c\bar{b}} g_{a\bar{d}}, \tag{105}$$

The Killing potentials of $su(N+1)$ isometry algebra are of the form,

$$h_{a\bar{b}} = \frac{z^b \bar{z}^a}{1+z\bar{z}}, \qquad h_a = \frac{\bar{z}^a}{1+z\bar{z}}. \tag{106}$$

Equipping the cotangent bundle of $\mathbb{CP}^N$ with the twisted symplectic structure (8) and the related Poisson brackets, we obtain the mechanics systems involving an interaction with a constant magnetic field.

The $su(N+1)$ isometry generators are given by the expressions of the form,

$$J_{a\bar{b}} = i(z^b \pi_a - \bar{\pi}_b \bar{z}^a) - B \frac{\bar{z}^a z^b}{1+z\bar{z}}, \\ J_a = i(\pi_a + \bar{z}^a (\bar{z}\,\bar{\pi})) - B \frac{\bar{z}^a}{1+z\bar{z}} : \\ \{J_{\bar{a}b}, J_{\bar{c}d}\} = i\delta_{\bar{a}d} J_{\bar{b}c} - i\delta_{\bar{c}b} J_{\bar{a}d}, \\ \{J_a, \bar{J}_b\} = iJ_{a\bar{b}}, \qquad \{J_a, J_{b\bar{c}}\} = \mp i J_b \delta_{a\bar{c}}. \tag{107}$$

Extending these generators to this phase superspace as in (18), we obtain

$$\mathcal{J}_{a\bar{b}} = J_{a\bar{b}} + \frac{\partial^2 h_{a\bar{b}}}{\partial z^c \partial \bar{z}^d}\eta^{c\alpha}\bar{\eta}^d_\alpha, \qquad \mathcal{J}_a = J_a + \frac{\partial^2 h_a}{\partial z^c \partial \bar{z}^d}\eta^{c\alpha}\bar{\eta}^d_\alpha. \tag{108}$$

With these expressions at hand, we can construct superintegrable models admitting weak $SU(2|1)$ supersymmetry.

1. $\mathbb{CP}^N$-oscillator

The oscillator on a complex projective space is defined by the Hamiltonian [7],[6]

$$H_{osc} = g^{\bar{a}b}\bar{\pi}_a\pi_b + |\omega|^2 z\bar{z}. \tag{109}$$

The constants of motion of this system are given by the $u(N)$ generators $J_{a\bar{b}}$ (107) and by the analog of "Fradkin tensor",

$$I_{a\bar{b}} = J_a\bar{J}_b + |\omega|^2\bar{z}^a z^b. \tag{110}$$

This system belongs to the class of "Kähler oscillators" (1) with $K = \log(1 + z\bar{z})$, and hence admits a $SU(2|1)$ supersymmetric extension. The relevant Hamiltonian and supercharges read

$$\mathcal{H}_{osc} = g^{\bar{a}b}\bar{\pi}_a\pi_b + |\omega|^2 z\bar{z} - \frac{1}{2}(g_{a\bar{b}}g_{c\bar{d}} + g_{c\bar{b}}g_{a\bar{d}})\eta^{a\alpha}\bar{\eta}^b_\alpha\eta^{c\beta}\bar{\eta}^d_\beta - \frac{\omega}{2}\frac{\bar{z}^a\bar{z}^b\eta^{a\alpha}\eta^b_\alpha}{(1+z\bar{z})^2} - \frac{\bar{\omega}}{2}\frac{z^a z^b\bar{\eta}^a_\alpha\bar{\eta}^{b\alpha}}{(1+z\bar{z})^2} + \frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha, \tag{111}$$

$$\Theta^\alpha = \pi_a\eta^{a\alpha} + i\bar{\omega}\frac{z^a}{1+z\bar{z}}\varepsilon^{\alpha\beta}\bar{\eta}^a_\beta,$$
$$\bar{\Theta}_\alpha = \bar{\pi}_a\bar{\eta}^a_\alpha + i\omega\frac{\bar{z}^a}{1+z\bar{z}}\varepsilon_{\alpha\beta}\eta^{a\beta}. \tag{112}$$

This system has the manifest $u(N)$ symmetry defined by the generators $\mathcal{J}_{a\bar{b}}$: $\{\mathcal{J}_{a\bar{b}}, \mathcal{H}_{osc}\} = 0$.

One could expect that the appropriate generalization of the Fradkin tensor should still have the form (110), with J_a replaced by $\mathcal{J}_a$, and that just this minimal modification yields constants of motion of the superoscillator. However, one can check that it is not the case. So, for the time being, it is an open question whether a supersymmetric counterpart of the Fradkin tensor exists.

2. $\mathbb{CP}^N$-Rosochatius system

The $\mathbb{CP}^N$-Rosochatius system is defined by the symplectic structure (8) and by the Hamiltonian [18],

$$H_{Ros} = (1 + z\bar{z})\left(\pi\bar{\pi} + (z\pi)(\bar{z}\,\bar{\pi}) + |\omega_0|^2 + \sum_{a=1}^N\frac{|\omega_a|^2}{z^a\bar{z}^a}\right) - \sum_{i=0}^N|\omega_i|^2. \tag{113}$$

This system possesses N manifest $U(1)$ symmetries with the generators $J_{a\bar{a}}$ defined in (107), as well as symmetries generated by the second-order constants of motion,

$$I_a = J_a\bar{J}_{\bar{a}} + \omega_0^2 z^a\bar{z}^a + \frac{\omega_a^2}{\bar{z}^a z^a},$$
$$I_{ab} = J_{a\bar{b}}J_{b\bar{a}} - \frac{1}{2}J_{a\bar{a}}J_{b\bar{b}} + \left(\omega_a^2\frac{z^b\bar{z}^b}{z^a\bar{z}^a} + \omega_b^2\frac{z^a\bar{z}^a}{z^b\bar{z}^b}\right). \tag{114}$$

The Hamiltonian (113) can be cast, up to a constant shift, in the form of the "Kähler oscillator" Hamiltonian [7,9],

$$H_{Ros} = g^{a\bar{b}}(\pi_a\bar{\pi}_b + |\omega|^2\partial_a K\partial_{\bar{a}}K) - E_0, \tag{115}$$

where

$$K = \log(1 + z\bar{z}) - \sum_{a=1}^N\left(\frac{\omega_a}{\omega}\log z^a + \frac{\bar{\omega}_a}{\bar{\omega}}\log\bar{z}^a\right),$$
$$\omega = \sum_{i=0}^N\omega_i, \qquad E_0 = \left|\sum_{i=0}^N\omega_i\right|^2 - \sum_{i=0}^N|\omega_i|^2. \tag{116}$$

Thus, this system admits a $SU(2|1)$ supersymmetric extension given by the following Hamiltonian and supercharges:

$$\mathcal{H}_{Ros} = H_{Ros} - \frac{1}{2}(g_{a\bar{b}}g_{c\bar{d}} + g_{c\bar{b}}g_{a\bar{d}})\eta^{a\alpha}\bar{\eta}^b_\alpha\eta^{c\beta}\bar{\eta}^d_\beta - \left(\frac{\omega\bar{z}^a\bar{z}^b}{1+z\bar{z}} - \frac{\omega_a\bar{z}^b}{z^a} - \frac{\omega_b\bar{z}^a}{z^b}\right)\frac{\eta^{a\alpha}\eta^b_\alpha}{2(1+z\bar{z})} - \left(\frac{\bar{\omega}z^a z^b}{1+z\bar{z}} - \frac{\bar{\omega}_a z^b}{\bar{z}^a} - \frac{\bar{\omega}_b z^a}{\bar{z}^b}\right)\frac{\bar{\eta}^a_\alpha\bar{\eta}^{b\alpha}}{2(1+z\bar{z})} + \frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha, \tag{117}$$

$$\Theta^\alpha = \pi_a\eta^{a\alpha} + i\left(\bar{\omega}\frac{z^a}{1+z\bar{z}} - \frac{\bar{\omega}_a}{\bar{z}^a}\right)\varepsilon^{\alpha\beta}\bar{\eta}^a_\beta. \tag{118}$$

They are easily checked to constitute the $su(2|1)$ superalgebra (52) ($\mathcal{H}_{Ros} \equiv \mathcal{H}_{osc}$).

It is interesting that, in contrast to the $\mathbb{C}^N$-Smorodinsky-Winternitz system, in the absence of a magnetic field and under the special choice of the parameters ω_i, this system

[6]Hereafter, we use the notation $z\bar{z} \equiv \sum_{c=1}^N z^c\bar{z}^c$, $(\pi z) = \sum_{c=1}^N \pi_c z^c$ etc.

admits flat $\mathcal{N}=4$, $d=1$ "Poincaré" supersymmetry [18]. The choice just mentioned is as follows:

$$B=0, \qquad |\omega| = \left|\sum_{i=0}^{N}\omega_i\right| = 0. \tag{119}$$

The second equation has the simple graphical illustration: it defines the planar polygon with the edges $|\omega_a|$ and, therefore, corresponds to the inequality $|\omega_0| \le \sum_{a=1}^{N}|\omega_a|$, where, without a loss of generality, we assume that $|\omega_0| \ge |\omega_1| \ge \cdots \ge |\omega_N|$. In this case, we arrive at the well-known $\mathcal{N}=4$ supersymmetric mechanics on Kähler manifold with the holomorphic prepotential $U(z) = \sum_{a=1}^{N}\omega_a \log z^a$ (see, e.g., [22]).

Finally, we note that all symmetries respected by the systems considered in this section are symmetries of the appropriate superfield Lagrangians (73) at $B \neq 0$, $\omega \neq 0$, with Φ^a, $\bar{\Phi}^b$ standing for z^a, $\bar{z}^b$.

VII. DISCUSSION AND OUTLOOK

In this paper, we presented the systematic combined Hamiltonian and superfield approach to the construction of the multiparticle models of deformed $\mathcal{N}=4$, 8 supersymmetric mechanics on Kähler manifolds in interaction with a constant magnetic field. The latter are introduced via a supersymmetric version of minimal coupling. We applied this approach to the various (super)integrable models and demonstrated that such superextensions preserve all kinematical symmetries of the initial bosonic systems (and some hidden symmetries in a few particular cases). One of the basic features of our approach is that diverse isometries are realized on the $SU(2|1)$ multiplets of the same sort, without introducing any extra multiplet. This is a crucial difference of our approach from the models of Refs. [25–27] in which similar isometries were realized within the standard $\mathcal{N}=4$ supersymmetric mechanics at a cost of introducing extra degrees of freedom (coming back to the spin variables introduced in [28]).[7]

The next obvious task is the study of the *quantum mechanical* properties (spectra, etc.) of the $SU(2|1)$ supersymmetric Landau problem on $\mathbb{CP}^N$, as well as of the $SU(2|1)$ supersymmetric oscillatorlike models on $\mathbb{C}^N$ and $\mathbb{CP}^N$.

Some other tasks are

(i) Coupling, to a constant magnetic field, of "flat" $\mathcal{N}=8$ supersymmetric mechanics with a nonzero potential on special Kähler manifolds as suggested in [30] and studying the new deformed $\mathcal{N}=8$ mechanics models obtained in this way;

(ii) The construction of the deformed supersymmetric extensions of the Landau problem on quaternionic manifolds and, in particular, on quaternionic projective spaces $\mathbb{HP}^N$, having in mind their relevance to the so-called high-dimensional Hall effect [31];

(iii) The construction of the $\mathbb{HP}^N$-Rosochatius system and studying the symmetry properties of it and of the $\mathbb{HP}^N$ oscillator's [32], as well as of their supersymmetric extensions.

(iv) Introducing the notion of a quaternionic oscillator, by analogy with the Kähler one, and the study of its possible deformed supersymmetric extensions.

We plan to address this circle of problems in the near future.

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APPENDIX: ISOMETRIES OF SPECIAL KÄHLER STRUCTURE IN THE LOCAL COORDINATES

In this appendix, we formulate the conditions (43) defining the isometries of the *special Kähler structure* in the local coordinate frame, in which the Kähler metric and the tensor $f_{abc}(z)$ take the form (48). The Eq. (43) expresses, in the special coordinate frame, via Seiberg-Witten prepotential $\mathcal{F}(z)$ as follows:

$$3\partial_{(a}V^d_\mu\partial_b\partial_{c)}\partial_d\mathcal{F} + V^d_\mu\partial_a\partial_b\partial_c\partial_d\mathcal{F} = 0, \tag{A1}$$

with V^a_μ, $\bar{V}^{\bar{a}}_\mu$ being the components of the holomorphic Hamiltonian vector field (4).

To extract the necessary corollaries of this equation, we first act by the derivative ∂_a on (76), where the Kähler potential is defined by (48). Step by step, it yields

[7]Applications of the spin variables in the models of $SU(2|1)$ mechanics were considered, e.g., in [29].

$$\begin{aligned}
&\partial_a\partial_b\partial_{\bar c}[(V^d_\mu\partial_d+V^{\bar d}_\mu\partial_{\bar d})(\bar z^e\partial_e\mathcal{F}+z^e\partial_{\bar e}\bar{\mathcal{F}})]=0\Rightarrow\\
&\partial_a\partial_b\partial_{\bar c}[V^d_\mu(\bar z^e\partial_d\partial_e\mathcal{F}+\partial_{\bar d}\bar{\mathcal{F}})+V^{\bar d}_\mu(\partial_d\mathcal{F}+z^e\partial_{\bar d}\partial_{\bar e}\bar{\mathcal{F}})]=0\Rightarrow\\
&\partial_a\partial_{\bar c}[\partial_b V^d_\mu\partial_{\bar d}\bar{\mathcal{F}}+\bar z^e\partial_b(V^d_\mu\partial_d\partial_e\mathcal{F})+V^{\bar d}_\mu(\partial_{\bar d}\partial_{\bar b}\bar{\mathcal{F}}+\partial_d\partial_b\mathcal{F})]=0\Rightarrow\\
&\partial_{\bar c}[V^{\bar d}_\mu\partial_a g_{b\bar d}+\partial_a\partial_b V^d_\mu\partial_{\bar d}\bar{\mathcal{F}}+\bar z^e\partial_a\partial_b(V^d_\mu\partial_d\partial_e\mathcal{F})]=0\Rightarrow\\
&\partial_{\bar c}V^{\bar d}_\mu\partial_a\partial_b\partial_d\mathcal{F}+\partial_a\partial_b V^d_\mu g_{d\bar c}+\partial_a V^d_\mu\partial_d\partial_c\partial_b\mathcal{F}+\partial_b V^d_\mu\partial_d\partial_c\partial_a\mathcal{F}+V^d_\mu\partial_d\partial_c\partial_a\partial_b\mathcal{F}=0\Rightarrow\\
&3\partial_{(a}V^d_\mu\partial_b\partial_{c)}\partial_d\mathcal{F}-\partial_a\partial_b\partial_d\mathcal{F}(\partial_c V^d_\mu-\partial_{\bar c}V^{\bar d}_\mu)+V^d_\mu\partial_a\partial_b\partial_c\partial_d\mathcal{F}+g_{d\bar c}\partial_a\partial_b V^d_\mu=0.
\end{aligned}\tag{A2}$$

Using the last condition, we can rewrite (A1) as

$$g_{d\bar c}\partial_a\partial_b V^d_\mu-\partial_a\partial_b\partial_d\mathcal{F}(\partial_c V^d_\mu-\partial_{\bar c}V^{\bar d}_\mu)=0. \tag{A3}$$

Next, taking $\partial_{\bar e}$ derivative of this relation, we obtain

$$\partial_{\bar e}\partial_{\bar d}\partial_{\bar c}\bar{\mathcal{F}}\partial_a\partial_b V^d_\mu=-\partial_a\partial_b\partial_d\mathcal{F}\partial_{\bar e}\partial_{\bar c}V^{\bar d}_\mu. \tag{A4}$$

The left- and right-hand sides of this relation are products of holomorphic and antiholomorphic functions. Obviously, the factors of the same holomorphicity should be equal, which yields

$$\partial_a\partial_b V^c_\mu=iC^{cd}_\mu\partial_a\partial_b\partial_d\mathcal{F},\qquad C^{cd}_\mu=\bar C^{dc}_\mu, \tag{A5}$$

where C^{cd}_μ are some complex constant parameters.

Taking also into account (A3), the solution of (A5) can be written as

$$\begin{aligned}
V^d_\mu&=iC^{de}_\mu\partial_e\mathcal{F}+\beta^d_{\mu a}z^a+\alpha^d_\mu,\\
V^{\bar d}_\mu&=-iC^{de}_\mu\partial_e\mathcal{F}+\beta^d_{\mu a}\bar z^a+\bar\alpha^d_\mu,
\end{aligned}\tag{A6}$$

where $\beta^d_{\mu a}$ and α^d_μ are, respectively, real and complex constant parameters. From (A3) and (A5), it follows that C^{cd}_μ is a symmetric real matrix, $C^{cd}_\mu=C^{dc}_\mu$.

The variation of $\mathcal{F}$ is then equal to

$$\delta_\mu\mathcal{F}\equiv V^d_\mu\partial_d\mathcal{F}=(iC^{de}_\mu\partial_e\mathcal{F}+\beta^d_{\mu a}z^a+\alpha^d_\mu)\partial_d\mathcal{F}. \tag{A7}$$

Inserting this solution in (A1) yields the condition,

$$\partial_a\partial_b\partial_c(\delta_\mu\mathcal{F})=0, \tag{A8}$$

having the obvious general solution,

$$\delta_\mu\mathcal{F}=c_\mu+c_{a\mu}z^a+c_{ab\mu}z^a z^b, \tag{A9}$$

where c_μ, $c_{a\mu}$ and $c_{ab\mu}$ are complex parameters.

Next we insert the solution (A6) in the Killing equation (6) (b), with the metric defined by (46), and derive the additional condition on $\delta_\mu\mathcal{F}$,

$$\partial_a\partial_b(\delta_\mu\mathcal{F})+\partial_{\bar a}\partial_{\bar b}(\delta_\mu\bar{\mathcal{F}})=0. \tag{A10}$$

This equation amounts to the reality condition $\overline{(c_{ab\mu})}=-c_{ab\mu}$.

The superfield transformations (84) have precisely the form of the general isometry $\delta_\mu\mathcal{F}$, with the complex coordinates z^a, $\bar z^a$ being replaced by the chiral $SU(4|1)$ superfields Φ^a and their antichiral counterparts.

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