Nonstationary energy in general relativity

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Using the time evolution equations of (cosmological) general relativity in the first order Fischer-Marsden form, we construct an integral that measures the amount of nonstationary energy on a given spacelike hypersurface in D dimensions. The integral vanishes for stationary spacetimes; and with a further assumption, reduces to Dain's invariant on the boundary of the hypersurface which is defined with the Einstein constraints and a fourth order equation defining approximate Killing symmetries.

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I. INTRODUCTION

Dain [1] constructed a geometric invariant that measures the *nonstationary* energy for an asymptotically flat hypersurface in $3+1$ dimensions for the case of time-symmetric initial data which, for vacuum, is an invariant that quantifies the total energy of the gravitational radiation. So this invariant is a component of the total Arnowitt-Deser-Misner (ADM) energy [2] assigned to an asymptotically flat hypersurface. That construction was extended to the time-nonsymmetric case recently in [3]. To give an example of how useful such a geometric invariant can be when constructing initial data for the gravitational field, let us recall the first observation of the merger of two black holes [4]. According to this observation, two initial black holes with masses (approximately) $36\mathrm{M}_{\odot}$ and $29\mathrm{M}_{\odot}$ merged to produce a single stationary black hole of mass $62\mathrm{M}_{\odot}$ plus gravitational radiation of total energy equivalent to $3\mathrm{M}_{\odot}$. Assuming this system to be isolated in an asymptotically flat spacetime, the total initial ADM energy of $65\mathrm{M}_{\odot}$ is certainly conserved. But this total ADM energy of the initial data needs a refinement as it clearly has a nonstationary part equal to $3\mathrm{M}_{\odot}$. The important question is to identify this nonstationary energy in the initial data.

Dain's construction and its extension to the nontime symmetric case by Kroon and Williams [3] are based on several earlier crucial works one of which is the Killing initial data (KID) concept of Moncrief [5] and Beig-Chruściel [6]; and a fourth order operator defined by Bartnik [7]. Of course all of the discussion is related to the Cauchy problem in general relativity and the related issue of constructing initial data for the time evolution equations. Here by using the time-evolution equations, in the form given by Fischer and Marsden [8], we construct a new representation of the nonstationary energy in generic D dimensional spacetimes with or without a cosmological constant.

The outline of the paper is as follows: in Sec. II we briefly summarize Dain's construction using the constraints and present a new approach using the evolution equations. In Sec. III we give the details of the relevant computations in D dimensions. The Appendix is devoted to the ADM decomposition.

II. DAIN'S INVARIANT IN BRIEF AND A NEW FORMULATION

Leaving the details of the construction to the next section, let us first briefly summarize the ingredients needed to define Dain's invariant on a spacelike hypersurface Σ of the spacetime $\mathcal{M} = \mathbb{R}\times\Sigma$. Then we shall discuss our new formulation via the evolution equations.

The initial data on the hypersurface is defined by the Riemannian metric γ_{ij} and the extrinsic curvature K_{ij} in local coordinates. Denoting D_i to be the covariant derivative compatible with γ_{ij} and assuming the usual ADM decomposition of the spacetime metric $g_{\mu\nu}$, the line element reads

$$ds^2 = (N_i N^i - N^2)dt^2 + 2N_i dt dx^i + \gamma_{ij}dx^i dx^j, \quad (1)$$

while the extrinsic curvature becomes[1]

[1]Our definition of the extrinsic curvature is as follows: given (X,Y) two vectors on the tangent space $T_p\Sigma$ and n be the unit normal to Σ, then $K(X,Y) := g(\nabla_X n, Y)$ with ∇ being the covariant derivative compatible with the spacetime metric g.

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$$K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - D_i N_j - D_j N_i), \tag{2}$$

with the lapse function $N = N(t, x^i)$ and the shift vector $N^i = N^i(t, x^i)$. The spatial indices can be raised and lowered with the $D-1$ dimensional spatial metric γ; over dot denotes the derivative with respect to t, and the Latin letters are used for the spatial dimensions, $i, j, k, \ldots = 1, 2, 3, \ldots D-1$, whereas the Greek letters are used to denote the spacetime dimensions, $\mu, \nu, \rho, \ldots = 0, 1, 2, 3, \ldots D-1$. All the relevant details of the ADM decomposition are given in the Appendix.

Under the above decomposition of spacetime, the D-dimensional Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{3}$$

yield the Hamiltonian and momentum constraints on the hypersurface Σ as

$$\begin{aligned} \Phi_0(\gamma, K) &\coloneqq -{}^{\Sigma}R - K^2 + K_{ij}^2 + 2\Lambda - 2\kappa T_{nn} = 0, \\ \Phi_i(\gamma, K) &\coloneqq -2 D_k K_i^k + 2 D_i K - 2\kappa T_{ni} = 0, \end{aligned} \tag{4}$$

where $K \coloneqq \gamma^{ij} K_{ij}$ and $K_{ij}^2 \coloneqq K^{ij} K_{ij}$. From now on we shall work in vacuum, hence $T_{\mu\nu} = 0$. Denoting $\Phi(\gamma, K)$ to be the constraint covector with components (Φ_0, Φ_i) and $D\Phi(\gamma, K)$ to be its linearization about a given solution (γ, K) to the constraints and $D\Phi^*(\gamma, K)$ to be the formal adjoint map, then following Bartnik [7], one defines another operator $\mathcal{P}$:

$$\mathcal{P} \coloneqq D\Phi(\gamma, K) \circ \begin{pmatrix} 1 & 0 \\ 0 & -D^m \end{pmatrix}. \tag{5}$$

The reason why we need this operator will be clear below. Using the formal adjoint $\mathcal{P}^*$ of Bartnik's operator, Dain [1] defines the following integral over the hypersurface

$$\mathcal{I}(N, N^i) \coloneqq \int_{\Sigma} dV \mathcal{P}^* \begin{pmatrix} N \\ N^k \end{pmatrix} \cdot \mathcal{P}^* \begin{pmatrix} N \\ N^k \end{pmatrix}, \tag{6}$$

where the multiplication is defined as

$$\begin{pmatrix} N \\ N^i \end{pmatrix} \cdot \begin{pmatrix} A \\ B_i \end{pmatrix} \coloneqq NA + N^i B_i. \tag{7}$$

The integral (6) is to be evaluated for specific vectors $\xi \coloneqq (N, N^i)$ that satisfy the fourth-order equation

$$\mathcal{P} \circ \mathcal{P}^*(\xi) = 0, \tag{8}$$

which Dain called the *approximate Killing initial data* (KID) equation. It is clear that if ξ satisfies the lower derivative equation $\mathcal{P}^*(\xi) = 0$, then it also satisfies (8). Moreover, these particular solutions, together with an assumption on their decay at infinity, also solve the KID equations which are simply $D\Phi^*(\gamma, K)(\xi) = 0$. In fact this point is crucial but well-established: Moncrief [5] proved that ξ is a spacetime Killing vector satisfying $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ if and only if it satisfies the KID equations. Namely one has

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \Leftrightarrow D\Phi^*(\gamma, K)(\xi) = 0, \tag{9}$$

with (N, N^i) being the projections off and onto the hypersurface of the Killing vector field ξ. The physical picture is clear: initial data on the hypersurface clearly encode the spacetime symmetries. There have been rigorous works on the KIDs in [6,9,10] which we shall employ in what follows.

Observe that for any Killing vector field $\mathcal{I}(N, N^i)$ vanishes identically. So by design, Dain's invariant identically vanishes for initial data with exact symmetries. Then Dain goes on to show that for asymptotically flat spaces, for the case of approximate translational KID's $\mathcal{I}(N, N^i)$ can measure the *nonstationary energy* contained in the hypersurface Σ. To simplify his calculations Dain considered the time symmetric initial data ($K_{ij} = 0$) in three spatial dimensions. There are two crucial points to note about Dain's construction: first, one can show that for any asymptotically flat three manifold, the approximate KID equation has nontrivial solutions which are not KIDs; second, using integration by parts, one can convert the volume integral (6) to a surface integral. We shall discuss these in the next section, but let us first give another formulation of this invariant.

A. Nonstationary energy via time-evolution equations

In Dain's construction, as is clear from the above summary, time evolution of the initial data has not played a role: in fact one only works with the constraints on the hypersurface. This fact somewhat obscures the interpretation of the proposed invariant as the nonstationary energy contained in the initial data. In what follows, we propose another formulation of this invariant with the help of the time evolution equations which makes the interpretation clearer. For this purpose let us consider the phase space variables to be the spatial metric γ_{ij} and the canonical momenta π^{ij}; the latter can be found from the Einstein-Hilbert Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{EH}} &= \frac{1}{\kappa} \sqrt{-g}(R - 2\Lambda) \\ &= \frac{1}{\kappa} \sqrt{\gamma} N({}^{\Sigma}R + K_{ij}^2 - K^2 + \Lambda) \\ &\quad + \text{boundary terms} \end{aligned} \tag{10}$$

which are

$$\pi^{ij} := \frac{\delta \mathcal{L}_{\mathrm{EH}}}{\delta \dot{\gamma}_{ij}} = \frac{1}{\kappa}\sqrt{\gamma}(K^{ij} - \gamma^{ij}K). \tag{11}$$

Using the canonical momenta, it pays to recast the densitized versions of the constraints (4) for $T_{\mu\nu} = 0$ and setting $\kappa = 1$ as

$$\begin{aligned} \Phi_0(\gamma,\pi) &:= \sqrt{\gamma}(-{}^{\Sigma}R + 2\Lambda) + G_{ijkl}\pi^{ij}\pi^{kl} = 0, \\ \Phi_i(\gamma,\pi) &:= -2\gamma_{ik}D_j\pi^{kj} = 0, \end{aligned} \tag{12}$$

where the *DeWitt metric* [11] G_{ijkl} in D dimensions reads

$$G_{ijkl} = \frac{1}{2\sqrt{\gamma}}\left(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \frac{2}{D-2}\gamma_{ij}\gamma_{kl}\right). \tag{13}$$

Ignoring the possible boundary terms, the ADM Hamiltonian density turns out to be a sum of the constraints as

$$\mathcal{H} = \int_{\Sigma} d^{D-1}x \langle \mathcal{N}, \Phi(\gamma,\pi)\rangle, \tag{14}$$

with $\mathcal{N}$ being the lapse-shift vector with components (N, N^i) which play the role of the Lagrange multipliers; and the angle-brackets denote the usual contraction. Given an $\mathcal{N}$, the remaining evolution equations can be written in a compact form (the Fischer-Marsden form [12]) as

$$\frac{d}{dt}\begin{pmatrix}\gamma \\ \pi\end{pmatrix} = J \circ D\Phi^*(\gamma,\pi)(\mathcal{N}), \tag{15}$$

where the J matrix reads

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{16}$$

The reason why the formal adjoint of the linearized constraint map $D\Phi^*(\gamma,\pi)$ appears in the time evolution can be seen as follows: the Hamiltonian form of the Einstein-Hilbert action

$$\mathcal{S}_{\mathrm{EH}}[\gamma,\pi] = \int dt \int d^{D-1}x (\pi^{ij}\dot{\gamma}_{ij} - \langle \mathcal{N}, \Phi(\gamma,\pi)\rangle), \tag{17}$$

when varied about a background (γ,π) gives

$$\begin{aligned} D\mathcal{S}_{\mathrm{EH}}[\gamma,\pi] = \int dt \int d^{D-1}x (&\delta\pi^{ij}\dot{\gamma}_{ij} + \pi^{ij}\delta\dot{\gamma}_{ij} \\ &- \langle \mathcal{N}, D\Phi(\gamma,\pi)\cdot(\delta\gamma,\delta\pi)\rangle). \end{aligned} \tag{18}$$

Here the linearized form of the constraint map can be computed to be

$$D\Phi\begin{pmatrix} h_{ij} \\ p^{ij}\end{pmatrix} = \begin{pmatrix} \sqrt{\gamma}({}^{\Sigma}R^{ij}h_{ij} - D^iD^jh_{ij} + \Delta h) \\ -hG_{ijkl}\pi^{ij}\pi^{kl} + 2G_{ijkl}p^{ij}\pi^{kl} + 2G_{njkl}h_{im}\gamma^{mn}\pi^{ij}\pi^{kl} \\ -2\gamma_{ik}D_jp^{kj} - \pi^{jk}(2D_kh_{ij} - D_ih_{jk}) \end{pmatrix}, \tag{19}$$

where $\delta\gamma_{ij} := h_{ij}$, $h := \gamma^{ij}h_{ij}$, $\delta\pi^{ij} := p^{ij}$ and $\Delta := D_kD^k$. We have used the vanishing of the constraints to simplify the expression. In (18) using integration by parts when necessary and dropping the boundary terms one arrives at the desired result

$$\begin{aligned} D\mathcal{S}_{\mathrm{EH}}[\gamma,\pi] = \int dt \int d^{D-1}x (&\delta\pi^{ij}\dot{\gamma}_{ij} - \dot{\pi}^{ij}\delta\gamma_{ij} \\ &- \langle (\delta\gamma,\delta\pi), D\Phi^*(\gamma,\pi)\cdot\mathcal{N}\rangle), \end{aligned} \tag{20}$$

where the adjoint constraint map appears in the process which reads

$$D\Phi^*\begin{pmatrix} N \\ N^i \end{pmatrix} = \begin{pmatrix} \sqrt{\gamma}({}^{\Sigma}R^{ij} - D^iD^j + \gamma^{ij}\Delta)N - N\gamma^{ij}G_{klmn}\pi^{kl}\pi^{mn} + 2NG_{klmn}\gamma^{ik}\pi^{jl}\pi^{mn} + 2\pi^{k(i}D_kN^{j)} - D_k(N^k\pi^{ij}) \\ 2NG_{ijkl}\pi^{kl} + 2D_{(i}N_{j)} \end{pmatrix}. \tag{21}$$

Setting the variation (20) to zero one obtains the evolution equations (15) or in more explicit form one has

$$\frac{d\gamma_{ij}}{dt} = 2NG_{ijkl}\pi^{kl} + 2D_{(i}N_{j)}, \tag{22}$$

and

$$\begin{aligned} \frac{d\pi^{ij}}{dt} = &\sqrt{\gamma}(-{}^{\Sigma}R^{ij} + D^iD^j - \gamma^{ij}\Delta)N + N\gamma^{ij}G_{klmn}\pi^{kl}\pi^{mn} \\ &- 2NG_{klmn}\gamma^{ik}\pi^{jl}\pi^{mn} - 2\pi^{k(i}D_kN^{j)} + D_k(N^k\pi^{ij}). \end{aligned} \tag{23}$$

Together with the constraints (12) these two tensor equations constitute a set of constrained dynamical system for a *given* lapse-shift vector an (N, N^i). The constraints have a dual role: they determine the viable initial data and also generate time evolution of the initial data once the lapse-shift vector is chosen. As noted above, if $D\Phi^*(\gamma,\pi)(\mathcal{N}) = 0$, namely $\mathcal{N} = \xi$ is a Killing vector field then the time evolution is trivial. In particular this would be the case for a stationary Killing vector.

Consider now an $\mathcal{N}$ which is *not* a Killing vector, which means $D\Phi^*(\gamma,\pi)(\mathcal{N}) \neq 0$; and in particular directly from the evolution equations we can find how much

$D\Phi^*(\gamma,\pi)(\mathcal{N})$ differs from zero (or how much a given $\mathcal{N}$ fails to be a Killing vector) as

$$D\Phi^*(\gamma,\pi)(\mathcal{N}) = J^{-1}\circ\frac{d}{dt}\begin{pmatrix}\gamma\\ \pi\end{pmatrix}. \tag{24}$$

To get a number from this matrix, first one should note that the units of γ and π are different by a factor of $1/L$ and so a naive approach of taking the "square" of this matrix does not work. At this stage to remedy this, one needs the (adjoint) operator of Bartnik that we have introduced above: so one has

$$\begin{aligned}\mathcal{P}^*(\mathcal{N}) &\coloneqq \begin{pmatrix}1 & 0\\ 0 & D_m\end{pmatrix}\circ D\Phi^*(\gamma,\pi)(\mathcal{N})\\ &= \begin{pmatrix}1 & 0\\ 0 & D_m\end{pmatrix}\circ J^{-1}\circ\frac{d}{dt}\begin{pmatrix}\gamma\\ \pi\end{pmatrix},\end{aligned} \tag{25}$$

which yields $\mathcal{P}^*(\mathcal{N}) = (-\dot{\pi}, D_m\dot{\gamma})$. Since π is a tensor density to get a number out of this vector, we further define

$$\tilde{\mathcal{P}}^*(\mathcal{N}) \coloneqq \begin{pmatrix}\gamma^{-1/2} & 0\\ 0 & 1\end{pmatrix}\circ\mathcal{P}^*(\mathcal{N}). \tag{26}$$

Then the integral of $\tilde{\mathcal{P}}^*(\mathcal{N})\cdot\tilde{\mathcal{P}}^*(\mathcal{N})$ over the hypersurface yields

$$\begin{aligned}\mathcal{I}(\mathcal{N}) &= \int_\Sigma dV\tilde{\mathcal{P}}^*(\mathcal{N})\cdot\tilde{\mathcal{P}}^*(\mathcal{N})\\ &= \int_\Sigma dV\left(|D_m\dot{\gamma}_{ij}|^2 + \frac{1}{\gamma}|\dot{\pi}^{ij}|^2\right),\end{aligned} \tag{27}$$

where $|D_m\dot{\gamma}_{ij}|^2 \coloneqq \gamma^{mn}\gamma^{ij}\gamma^{kl}D_m\dot{\gamma}_{ik}D_n\dot{\gamma}_{jl}$ and $|\dot{\pi}^{ij}|^2 \coloneqq \gamma_{ij}\gamma_{kl}\dot{\pi}^{ik}\dot{\pi}^{jl}$. This is another representation of Dain's invariant which explicitly involves the time derivatives of the canonical fields. We have also not assumed that the cosmological constant vanishes, hence our result is valid for generic spacetimes. Note that this expression is valid for any $\mathcal{N}$ which is not necessarily an approximate KID, hence given a solution to the constraint equations and a choice of the lapse-shift vector, one can compute this integral. But the volume integral becomes a surface integral when $\mathcal{N}$ is an approximate KID which is the case considered by Dain. Observe that by construction, $\mathcal{I}(\mathcal{N})$ is a non-negative number. To get the explicit expression as a volume integral in terms of the canonical fields and not their time derivatives, one should plug the two evolution equations (22) and (23) to (27). The resulting expression is

$$\begin{aligned}\mathcal{I}(\mathcal{N}) = \int_\Sigma dV\{&|D_mV^{ij}|^2 + {}^\Sigma R_{ij}^2N^2 + (D_iD_jN)^2\\ &- 2{}^\Sigma R^{ij}ND_iD_jN + 2{}^\Sigma RN\Delta N + (D-3)\Delta N\Delta N\\ &+ 2Q\Delta N + Q_{ij}^2 + 2{}^\Sigma R_{ij}NQ^{ij} - 2Q^{ij}D_iD_jN\\ &+ 4D_mD_{(i}N_{j)}D^mD^{(i}N^{j)} + 4D_mD_iN_jD^mV^{ij}\},\end{aligned} \tag{28}$$

where

$$V^{ij} \coloneqq \frac{2N}{\sqrt{\gamma}}\left(\pi^{ij} - \frac{1}{D-2}\pi\gamma^{ij}\right), \tag{29}$$

and

$$\begin{aligned}Q^{ij} \coloneqq{}& \frac{2N}{\gamma}\left(\pi^i_k\pi^{kj} - \frac{\pi\pi^{ij}}{D-2}\right) - \frac{N}{\gamma}\gamma^{ij}\left(\pi_{kl}^2 - \frac{\pi^2}{D-2}\right)\\ &- \frac{1}{\sqrt{\gamma}}D_k(N^k\pi^{ij}) + \frac{2}{\sqrt{\gamma}}\pi^{k(i}D_kN^{j)},\end{aligned} \tag{30}$$

and $Q \coloneqq \gamma_{ij}Q^{ij}$. Equation (28) is our main result: given a solution, that is an initial data, one an compute this integral which measures the deviation from stationarity. We can also write (28) in terms of γ_{ij} and the extrinsic curvature K_{ij}. For this purpose all one needs to do is to rewrite V^{ij} and Q^{ij} in terms of these variables. They are given as

$$V^{ij} = 2NK^{ij}, \tag{31}$$

and

$$\begin{aligned}Q^{ij} \coloneqq{}& 2N(K^i_kK^{kj} - KK^{ij}) - N\gamma^{ij}(K_{kl}^2 - K^2) - D_k(N^kK^{ij})\\ &+ \gamma^{ij}D_k(N^kK) + 2K^{k(i}D_kN^{j)} - 2KD^{(i}N^{j)}.\end{aligned} \tag{32}$$

Up to now we have not made a choice of gauge or coordinates. Let us now choose the Gaussian normal coordinates ($N=1$, $N^i=0$) on Σ for which the integral reads

$$\begin{aligned}\mathcal{I}(\mathcal{N}) = \int_\Sigma dV\Bigg\{&\frac{4}{\gamma}\left(|D_m\pi^{ij}|^2 - \frac{D-3}{(D-2)^2}|D_m\mathrm{Tr}(\pi)|^2\right)\\ &+ {}^\Sigma R_{ij}^2 + \frac{4}{\gamma}{}^\Sigma R_{ij}\pi^{ik}\pi^j_k - \frac{4}{(D-2)\gamma}{}^\Sigma R_{ij}\pi^{ij}\mathrm{Tr}(\pi)\\ &- \frac{4}{\gamma}\Lambda\left(\mathrm{Tr}(\pi^2) - \frac{1}{(D-2)}(\mathrm{Tr}(\pi))^2\right)\\ &+ \frac{D-7}{\gamma^2}\left(\mathrm{Tr}(\pi^2) - \frac{1}{D-2}(\mathrm{Tr}(\pi))^2\right)^2\\ &+ \frac{4}{\gamma^2}\left(\mathrm{Tr}(\pi^4) - \frac{2}{D-2}\mathrm{Tr}(\pi)\mathrm{Tr}(\pi^3)\right.\\ &\left.+ \frac{1}{(D-2)^2}(\mathrm{Tr}(\pi))^2\mathrm{Tr}(\pi^2)\right)\Bigg\},\end{aligned} \tag{33}$$

where $\mathrm{Tr}(\pi) := \gamma_{ij}\pi^{ij}$ and $\mathrm{Tr}(\pi^2) := \pi^{ij}\pi_{ij}$ and so on. In terms of the extrinsic curvature, in the Gaussian normal coordinates, one has

$$\mathcal{I}(\mathcal{N}) = \int_\Sigma dV\{4|D_m K_{ij}|^2 + {}^\Sigma R_{ij}^2 + 4{}^\Sigma R_{ij}(K^{ik}K_k^j - KK^{ij}) + 4\Lambda(K^2 - K_{ij}^2) + 4K_{ij}K^{jl}K_{lm}K^{mi} - 8KK_{ij}K^{jl}K_l^i - 2(D-9)K^2K_{ij}^2 + (D-7)((K_{ij}^2)^2 + K^4)\}. \tag{34}$$

For a physically meaningful solution whose ADM mass and angular momenta are finite for the asymptotically flat case, or in the case of $\Lambda \neq 0$ whose Abbott-Deser [13] charges are finite, this quantity is expected to be finite and represents the nonstationary part of the total energy by construction. Observe that while the ADM momentum ($P_i = \oint_{\partial\Sigma} K_{ij} dS^j$) and angular momenta ($J^{jk} = \oint_{\partial\Sigma}(x^j K^{km} - x^k K^{jm})dS_m$) are linear in the extrinsic curvature given as integrals over the boundary, $\mathcal{I}(\mathcal{N})$ has quadratic, cubic and quartic terms in the extrinsic curvature in the bulk integral$\oint$.

Before we lay out the details of the above discussion, let us note that our final formula (28) can be reduced in various ways depending on the physical problem or the numerical integration scheme: for example, one can choose the maximal slicing gauge for which $\mathrm{Tr}(\pi) = K = 0$. If the problem permits time-symmetric initial data $\pi^{ij} = K^{ij} = 0$, then in this restricted case, $V^{ij} = Q^{ij} = 0$, and the integral (28) reduces to

$$\mathcal{I}(\mathcal{N}) = \int_\Sigma dV({}^\Sigma R_{ij}^2 N^2 + (D_i D_j N)^2 - 2{}^\Sigma R^{ij} N D_i D_j N + 2{}^\Sigma R N \triangle N + 4 D_m D_i N_j D^m D^{(i} N^{j)} + (D-3)\triangle N \triangle N).$$

Let us go back to (27) which was the defining relation of the invariant and try to write it as a boundary integral over the boundary of the hypersurface Σ. Then one has

$$\mathcal{I}(\mathcal{N}) = \int_\Sigma dV \tilde{\mathcal{P}}^*(\mathcal{N}) \cdot \tilde{\mathcal{P}}^*(\mathcal{N}) = \int_\Sigma dV \mathcal{N} \cdot \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}}^*(\mathcal{N}) + \oint_{\partial\Sigma} dS n^k B_k, \tag{35}$$

which requires $\tilde{\mathcal{P}} \circ \tilde{\mathcal{P}}^*(\mathcal{N}) = 0$. This the approximate KID equation introduced by Dain [1] and B_k is the boundary term to be found below. Note that our bulk integral (28) is more general and does not assume the existence of approximate symmetries.

III. DETAILS OF THE CONSTRUCTION IN D DIMENSIONS

A. Boundary integral

The importance of the Einstein constraints (4) cannot be overstated: clearly the initial data is not arbitrary, one must solve these equations to feed the evolution equations; but, as importantly, the constraints also determine the evolution equations and they are related to the symmetries of the spacetime in a rather intricate way as we have seen above. One can consider the constraints (4) as the kernel of a map Φ

$$\Phi:\ \mathcal{M}_2 \times \mathcal{S}_2^* \to \mathcal{C}^* \times \mathcal{X}^*, \tag{36}$$

where $\mathcal{M}_2$ denotes the space of the Riemannian metrics and $\mathcal{S}_2^*$ denotes the space of symmetric rank-2 tensor densities, $\mathcal{C}^*$ denotes the space of scalar function densities and $\mathcal{X}^*$ the space of vector field densities on the hypersurface Σ. We can express the constraint map explicitly as

$$\Phi\begin{pmatrix}\gamma_{ij}\\ \pi^{ij}\end{pmatrix} = \begin{pmatrix}\sqrt{\gamma}(2\Lambda - {}^\Sigma R) + \gamma^{-1/2}(\pi_{ij}^2 - \frac{\pi^2}{D-2})\\ -2\gamma_{ki}D_j\pi^{kj}\end{pmatrix}, \tag{37}$$

whose linearization can be found to be

$$D\Phi\begin{pmatrix}h_{ij}\\ p^{ij}\end{pmatrix} = \begin{pmatrix}\sqrt{\gamma}({}^\Sigma R^{ij} - D^iD^j + \gamma^{ij}\triangle)h_{ij}\frac{1}{\sqrt{\gamma}}\left(\gamma^{ij}\left(\frac{\pi^2}{D-2} - \pi_{ij}^2\right) + 2(\pi^{ik}\pi_k^j - \frac{\pi^{ij}\pi}{D-2})\right)h_{ij} + \frac{2}{\sqrt{\gamma}}(\pi_{ij} - \frac{\pi\gamma_{ij}}{D-2})p^{ij}\\ (\pi^{ij}D_k - 2\delta_k^{(i}\pi^{j)l}D_l)h_{ij} - 2\gamma_{k(i}D_{j)}p^{ij}\end{pmatrix}. \tag{38}$$

We can define a 2×2 matrix as

$$D\Phi := \begin{pmatrix}\sqrt{\gamma}({}^\Sigma R^{ij} - D^iD^j + \gamma^{ij}\triangle) + \frac{1}{\sqrt{\gamma}}\left(\gamma^{ij}\left(\frac{\pi^2}{D-2} - \pi_{ij}^2\right) + 2(\pi^{ik}\pi_k^j - \frac{\pi^{ij}\pi}{D-2})\right) & \frac{2}{\sqrt{\gamma}}(\pi_{ij} - \frac{\pi\gamma_{ij}}{D-2})\\ \pi^{ij}D_k - 2\delta_k^{(i}\pi^{j)l}D_l & -2\gamma_{k(i}D_{j)}\end{pmatrix}, \tag{39}$$

such that

$$D\Phi\begin{pmatrix}h_{ij}\\ p^{ij}\end{pmatrix} = D\Phi\circ\begin{pmatrix}h_{ij}\\ p^{ij}\end{pmatrix}. \tag{40}$$

Defining [7]

$$\tilde{\mathcal{P}} := D\Phi\circ\begin{pmatrix}\gamma^{-1/2} & 0\\ 0 & -D^m\end{pmatrix}, \tag{41}$$

one finds

$$\tilde{\mathcal{P}} := \begin{pmatrix} {}^{\Sigma}R^{ij} - D^iD^j + \gamma^{ij}\triangle + \frac{1}{\gamma}\left(\gamma^{ij}\left(\frac{\pi^2}{D-2} - \pi^2_{ij}\right) + 2\left(\pi^{ik}\pi_k^j - \frac{\pi^{ij}\pi}{D-2}\right)\right) & \frac{2}{\sqrt{\gamma}}\left(\frac{\pi\gamma_{ij}}{D-2} - \pi_{ij}\right)D^m \\ \frac{1}{\sqrt{\gamma}}(\pi^{ij}D_k - 2\delta_k^{(i}\pi^{j)l}D_l) & 2\gamma_{k(i}D_{j)}D^m \end{pmatrix}, \tag{42}$$

which is a map as

$$\tilde{\mathcal{P}}\colon\ \mathcal{S}_2\times\mathcal{S}_{1,2}\to\mathcal{C}\times\mathcal{X}, \tag{43}$$

where $\mathcal{S}_2$ denotes the space of covariant rank-2 tensors, $\mathcal{S}_{1,2}$ denotes the space of covariant rank-3 tensors which are symmetric in last two indices, $\mathcal{C}$ denotes the space of scalar function and $\mathcal{X}$ the space of vector fields on the hypersurface Σ.

The formal adjoint of $\tilde{\mathcal{P}}$-operator was defined in (26) via the (21) and it is a map of the form

$$\tilde{\mathcal{P}}^*\colon\ \mathcal{C}\times\mathcal{X}\to\mathcal{S}_2\times\mathcal{S}_{1,2}. \tag{44}$$

Working out the details, one arrives at

$$\tilde{\mathcal{P}}^*\begin{pmatrix}N\\ N^k\end{pmatrix} = \begin{pmatrix}N{}^{\Sigma}R^{ij} - D^iD^jN + \gamma^{ij}\triangle N + Q^{ij}\\ D_m(2D_{(i}N_{j)} + V_{ij})\end{pmatrix}, \tag{45}$$

where V^{ij} and Q^{ij} were given (29), (30) respectively. We have used this expression in the previous section to find the bulk integral of the nonstationary energy. Now let us use this operator and its adjoint to find an expression on the boundary. For this purpose we need the following identity:

$$\int_\Sigma dV\begin{pmatrix}N\\ N^k\end{pmatrix}\cdot\tilde{\mathcal{P}}\begin{pmatrix}s_{ij}\\ s_{kij}\end{pmatrix} = \int_\Sigma dV\begin{pmatrix}s_{ij}\\ s_{kij}\end{pmatrix}\cdot\tilde{\mathcal{P}}^*\begin{pmatrix}N\\ N^k\end{pmatrix} + \oint_{\partial\Sigma} dSn^k\mathcal{B}_k, \tag{46}$$

with generic $s_{ij}\in\mathcal{S}_2$ and $s_{kij}\in\mathcal{S}_{1,2}$. After making use of (42) and (45), a slightly cumbersome computation yields the boundary term:

$$\begin{aligned}\mathcal{B}_k &= s_{kj}D^jN - ND^js_{kj} + ND_ks - sD_kN + 2N^iD^js_{jki}\\ &\quad - 2s_{kij}D^iN^j + \frac{2N}{\sqrt{\gamma}}\left(\frac{\pi}{D-2}s_{kj}{}^j - s_{kij}\pi^{ij}\right)\\ &\quad + \frac{1}{\sqrt{\gamma}}(\pi^{ij}s_{ij}N_k - 2s_{ij}N^i\pi^j_k),\end{aligned} \tag{47}$$

where $s = \gamma^{ij}s_{ij}$. Let us now assume a particular s_{ij} and a particular s_{kij} such that

$$\begin{pmatrix}s_{ij}\\ s_{kij}\end{pmatrix} := \tilde{\mathcal{P}}^*\begin{pmatrix}N\\ N^k\end{pmatrix}, \tag{48}$$

which yields

$$\tilde{\mathcal{P}}\begin{pmatrix}s_{ij}\\ s_{kij}\end{pmatrix} = \tilde{\mathcal{P}}\circ\tilde{\mathcal{P}}^*\begin{pmatrix}N\\ N^k\end{pmatrix}. \tag{49}$$

Then (46) becomes

$$\int_\Sigma dV\begin{pmatrix}N\\ N^k\end{pmatrix}\cdot\tilde{\mathcal{P}}\circ\tilde{\mathcal{P}}^*\begin{pmatrix}N\\ N^k\end{pmatrix} = \mathcal{I}(\mathcal{N}) + \oint_{\partial\Sigma} dSn^k\mathcal{B}_k, \tag{50}$$

where $\mathcal{B}_k$ given in (47) must be evaluated with

$$s_{ij} = N{}^{\Sigma}R_{ij} - D_iD_jN + \gamma_{ij}\triangle N + Q_{ij} \tag{51}$$

and

$$s_{kij} = D_k(2D_{(i}N_{j)} + V_{ij}). \tag{52}$$

Equation (50) shows that generically $\mathcal{I}(\mathcal{N})$ cannot be written as an integral on the boundary of the hypersurface unless $\tilde{\mathcal{P}}\circ\tilde{\mathcal{P}}^*(\mathcal{N}) = 0$. In that case, the invariant reduces to

$$\mathcal{I}(\mathcal{N}) = -\oint_{\partial\Sigma} dSn^k\mathcal{B}_k. \tag{53}$$

Explicit computation shows that one has

$$\begin{aligned}\mathcal{B}_k &= \frac{N^2}{2}D_k{}^{\Sigma}R + N{}^{\Sigma}R_{kj}D^jN - D_kD_jND^jN\\ &\quad - (D-3)D_kN\triangle N + (D-2)ND_k\triangle N\\ &\quad + 4N^i\triangle D_{(k}N_{i)} - 4D_kD_{(i}N_{j)}D^{(i}N^{j)} + b_k,\end{aligned} \tag{54}$$

where

$$\begin{aligned} b_k &\coloneqq Q_{kj}D^jN - ND^jQ_{kj} + ND_kQ - QD_kN + 2N^i\triangle V_{ki} \\ &- 2D_kV_{ij}D^iN^j + \frac{1}{\sqrt{\gamma}}\frac{2N\pi}{D-2}(2D_kD_iN^i + D_kV) \\ &- \frac{2N\pi^{ij}}{\sqrt{\gamma}}(2D_kD_iN_j + D_kV_{ij}) \\ &+ \frac{1}{\sqrt{\gamma}}(\pi^{ij}N_k - 2N^i\pi^j_k) \\ &\times (N^\Sigma R_{ij} - D_iD_jN + \gamma_{ij}\triangle N + Q_{ij}). \end{aligned} \tag{55}$$

In the Gaussian normal coordinates the boundary integral reads

$$\mathcal{I}(\mathcal{N}) = \oint_{\partial\Sigma} dSn^k\left(\left(D-\frac{5}{2}\right)D_kK^2_{ij} + \left(\frac{7}{2}-D\right)D_kK^2 + 2K^{lj}D_jK_{lk}\right). \tag{56}$$

Another physically relevant case is the time symmetric asymptotically flat case for which the boundary integral reduces to

$$\begin{aligned} \mathcal{I}(\mathcal{N}) = \oint_{\partial\Sigma} dSn^k(&D_kD_jND^jN + (D-3)D_kN\triangle N \\ &- (D-2)ND_k\triangle N - 4N^i\triangle D_{(k}N_{i)} \\ &+ 4D_kD_{(i}N_{j)}D^{(i}N^{j)}). \end{aligned}$$

In the most general form N and N^i should satisfy the fourth order equations $\tilde{\mathcal{P}}\circ\tilde{\mathcal{P}}^*(\mathcal{N}) = 0$ which explicitly read

$$\tilde{\mathcal{P}}\circ\tilde{\mathcal{P}}^*\begin{pmatrix} N \\ N^i \end{pmatrix} = \begin{pmatrix} (D-2)\triangle\triangle N - {}^\Sigma R_{ij}D^iD^jN + N(\frac{1}{2}\triangle{}^\Sigma R + {}^\Sigma R^2_{ij}) + 2{}^\Sigma R\triangle N + \frac{3}{2}D_i{}^\Sigma RD^iN + Y \\ 4D^j\triangle D_{(k}N_{j)} + Y_k \end{pmatrix} = 0, \tag{57}$$

where

$$\begin{aligned} Y \coloneqq{}^\Sigma R^{ij}Q_{ij} - D^iD^jQ_{ij} + \triangle Q + \frac{2}{\sqrt{\gamma}}\left(\frac{\pi\gamma^{ij}}{D-2} - \pi^{ij}\right) \\ \times\triangle(2D_iN_j + V_{ij}) + \left(\frac{2}{\gamma}\left(\pi^{ik}\pi^j_k - \frac{\pi\pi^{ij}}{D-2}\right.\right. \\ \left.\left. - \frac{\gamma^{ij}}{\gamma}\left(\pi^2_{kl} - \frac{\pi^2}{D-2}\right)\right)(N^\Sigma R_{ij} - D_iD_jN + \gamma_{ij}\triangle N + Q_{ij}) \end{aligned} \tag{58}$$

and

$$\begin{aligned} Y_k \coloneqq \frac{1}{\sqrt{\gamma}}(\pi^{ij}D_k - 2\delta^i_k\pi^{jl}D_l) \\ \times (N^\Sigma R_{ij} - D_iD_jN + \gamma_{ij}\triangle N + Q_{ij}) + 2D^i\triangle V_{ik}. \end{aligned} \tag{59}$$

B. The approximate KID equation in D dimensions

Following the $D = 4$ discussion of Dain [1] let us now study the approximate KID equation (57) in D dimensions. It is easy to see that it is a fourth order elliptic operator for $D > 2$. This follows by computing the leading symbol: for this purpose let us consider the higher order derivative terms and set $D_i = \zeta_i$ and $|\zeta|^2 = \zeta^i\zeta_i$. Using (57), the leading symbol of operator reads

$$\sigma[\tilde{P}\circ\tilde{\mathcal{P}}^*](\zeta)\begin{pmatrix} N \\ N_i \end{pmatrix} = \begin{pmatrix} (D-2)|\zeta|^4N \\ 4|\zeta|^2\zeta^j\zeta_{(k}N_{j)} \end{pmatrix}. \tag{60}$$

For a nonzero covector ζ, if σ is an isomorphism (here a vector bundle isomorphism), then the operator is elliptic. For the first component, this requires $D \neq 2$ and for the second component contraction with ζ^k yields

$$|\zeta|^4\zeta^kN_k = 0. \tag{61}$$

Assuming $D \neq 2$ one has $\zeta^kN_k = 0$. Inserting it back in the second component one obtains

$$|\zeta|^4N_k = 0, \tag{62}$$

so for $|\zeta|^2 \neq 0$, the leading symbol is injective and the operator $\tilde{P}\circ\tilde{\mathcal{P}}^*$ is elliptic for $D > 2$.

C. Asymptotically flat spaces

Consider the initial data set $(\Sigma, \gamma_{ij}, \pi^{ij})$ for the vacuum Einstein field equations with $n > 1$ asymptotically Euclidean ends: this is to avoid bulk simplicity and allow black holes. There exists a compact set $\mathcal{B}$ such that $\Sigma\backslash\mathcal{B} = \sum_{k=1}^n \Sigma_{(k)}$, where $\Sigma_{(k)}$, $k = 1, \ldots, n$ are open sets diffeomorphic to the complement of a closed ball in $\mathbb{R}^{D-1}$. Each asymptotic end $\Sigma_{(k)}$ admits asymptotically Cartesian coordinates. We consider the following decay assumptions, for $D > 3$, which are consistent with finite ADM mass and momenta:

$$\gamma_{ij} = \delta_{ij} + o(|x|^{(3-D)/2}), \tag{63}$$

$$\pi^{ij} = o(|x|^{(1-D)/2}), \tag{64}$$

where $\delta_{ij} = (+ + \cdots +)$. Note that $\delta_{ij} = \mathcal{O}(1)$ and beware of the small o and the big $\mathcal{O}$ notation. One can compute the following decay behavior for the Christoffel connection

$$ {}^{\Sigma}\Gamma^k_{ij} = o(|x|^{(1-D)/2}), \tag{65} $$

and the curvatures

$$ {}^{\Sigma}R^k{}_{lmn} = o(|x|^{-(1+D)/2}), \qquad {}^{\Sigma}R_{ij} = o(|x|^{-(1+D)/2}), \\ {}^{\Sigma}R = o(|x|^{-(1+D)/2}). \tag{66} $$

D. KIDs in D dimensions

Let $(\Sigma, \gamma_{ij}, \pi^{ij})$ denote a smooth vacuum initial data set satisfying the decay assumptions (63), (64). Let N, N^i be a smooth scalar field and a vector field on Σ satisfying the KID equations. Then generalizing the $D = 4$ result of [9], the behavior of all the possible solutions were given in [10] which we quote here.

(1) There exits an antisymmetric tensor field $\omega_{\mu\nu}$, such that

$$ N - \omega_{0i}x^i = o(|x|^{(5-D)/2}), \\ N^i - \omega^i{}_j x^j = o(|x|^{(5-D)/2}). \tag{67} $$

(2) If $\omega_{\mu\nu} = 0$, then there exists a vector field $\mathcal{U}^\mu$, such that

$$ N - \mathcal{U}^0 = o(|x|^{(3-D)/2}), \\ N^i - \mathcal{U}^i = o(|x|^{(3-D)/2}). \tag{68} $$

(3) If $\omega_{\mu\nu} = 0 = \mathcal{U}^\mu$ then one has the trivial solution $N = 0 = N^i$. Both $\omega_{\mu\nu}$ and $\mathcal{U}^\mu$ are constants in the sense that they are $\mathcal{O}(1)$ whenever they do not vanish.

Case 1 above corresponds to the rotational Killing vectors while case 2 corresponds to the translational ones we shall employ the latter.

We explained in Sec. II that solutions of the $D\Phi^*(N, N^i) = 0$ yield spacetime Killing vectors. It is not difficult to see that the modified equation $\tilde{\mathcal{P}}^*(N, N^i) = 0$ yields only the Killing vectors for the case of translational KIDs (63), (64). Here is the proof: $\tilde{\mathcal{P}}^*(N, N^i) = 0$ implies

$$ N{}^{\Sigma}R_{ij} - D_iD_jN + \gamma_{ij}\Delta N + Q_{ij} = 0, \tag{69} $$

$$ D_m(2D_{(i}N_{j)} + V_{ij}) = 0. \tag{70} $$

If one assumes (N, N^i) decay as in (68) we have $D_{(i}N_{j)} = o(|x|^{(1-D)/2})$; and $V_{ij} = o(|x|^{(1-D)/2})$, then

$$ 2D_{(i}N_{j)} + V_{ij} = o(|x|^{(1-D)/2}) \tag{71} $$

vanishes at infinity; and since it is covariantly constant, it must vanish identically

$$ 2D_{(i}N_{j)} + V_{ij} = 0. \tag{72} $$

Together with the first component of $\tilde{\mathcal{P}}^*(N, N^i) = 0$ we get the formal adjoint of the linearized constraint map, namely $D\Phi^*(N, N^i) = 0$. We can conclude that if $\tilde{\mathcal{P}}^*(N, N^i) = 0$ then (N, N^i) solve the KID equations.

E. Approximate KIDs in D dimensions

Generalizing Dain's $D = 4$ result, let us search for translational solutions of the approximate Killing equation[2]

$$ \tilde{\mathcal{P}}\circ\tilde{\mathcal{P}}^* \begin{pmatrix} N \\ N^i \end{pmatrix} = 0 \tag{73} $$

as a deformation of the KIDs (X, N^i) in the following form:

$$ N = \lambda\varphi + X, \qquad N^i = N^i, \tag{74} $$

where the function φ is to be found, λ is a constant. KIDs decay as

$$ X - \mathcal{U}^0 = o(|x|^{(3-D)/2}), \tag{75} $$

$$ N^i - \mathcal{U}^i = o(|x|^{(3-D)/2}). \tag{76} $$

Inserting the ansatz (74) into the approximate KID equation (73), one gets

$$ \tilde{\mathcal{P}}\circ\tilde{\mathcal{P}}^* \begin{pmatrix} \varphi \\ 0 \end{pmatrix} = -\tilde{\mathcal{P}}\circ\tilde{\mathcal{P}}^* \begin{pmatrix} X \\ N^i \end{pmatrix} = 0, \tag{77} $$

or more explicitly

$$ \tilde{\mathcal{P}}\circ\tilde{\mathcal{P}}^* \begin{pmatrix} \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} (D-2)\Delta\Delta\varphi - {}^{\Sigma}R_{ij}D^iD^j\varphi + \varphi(\frac{1}{2}\Delta{}^{\Sigma}R + {}^{\Sigma}R^2_{ij}) + 2{}^{\Sigma}R\Delta\varphi + \frac{3}{2}D_i{}^{\Sigma}RD^i\varphi + Y \\ Y_k \end{pmatrix} = 0. \tag{78} $$

[2]We work in a given asymptotic end and not the clutter the notation we do not denote the corresponding index referring to the asymptotic end.

For such a φ, the bulk integral (28) becomes

$$\begin{aligned}\mathcal{I}(\mathcal{N}) = \lambda^2 \int_\Sigma dV\{&|D_m V^{ij}|^2 + {}^\Sigma R_{ij}^2\varphi^2 + (D_i D_j\varphi)^2 \\ &- 2{}^\Sigma R^{ij} N D_i D_j \varphi + 2{}^\Sigma R\varphi\Delta\varphi \\ &+ (D-3)\Delta\varphi\Delta\varphi + 2Q\Delta\varphi + Q_{ij}^2 \\ &+ 2{}^\Sigma R_{ij}\varphi Q^{ij} - 2Q^{ij}D_i D_j\varphi\},\end{aligned} \tag{79}$$

where

$$V^{ij} = 2\varphi K^{ij}, \tag{80}$$

and

$$Q^{ij} = 2\varphi(K^i_k K^{kj} - KK^{ij}) - \varphi\gamma^{ij}(K_{kl}^2 - K^2). \tag{81}$$

The boundary form for the asymptotically flat case follows similarly

$$\begin{aligned}\mathcal{I}(\mathcal{N}) = -\lambda^2 \oint_{\partial\Sigma} dSn^k\{&-D_k D_j\varphi D^j\varphi - (D-3)D_k\varphi\Delta\varphi \\ &+ (D-2)\varphi D_k\Delta\varphi + Q_{kj}D^j\varphi - \varphi D^j Q_{kj} \\ &- 2\varphi K^{ij} D_k V_{ij}\},\end{aligned} \tag{82}$$

where we used $K_{kl}^2 - K^2 = {}^\Sigma R = 0$ on the boundary.

IV. CONCLUSIONS

Using the Hamiltonian form of the Einstein evolution equations as given by Fischer and Marsden [8], we constructed an integral that measures the nonstationary energy contained in a spacelike hypersurface in D dimensional general relativity with or without a cosmological constant. This integral was previously studied by Dain [1] who used the Einstein constraints but not the evolution equations. The crucial observation is the following: the critical points of the first order Hamiltonian form of Einstein equations correspond to the initial data which possess Killing symmetries, a result first observed by Moncrief [5]. Hence, our vantage point is that the failure of an initial data to possess Killing symmetries is given by the evolution equations, namely nonvanishing of the time derivatives of the spatial metric and the canonical momenta. Then manipulating the evolution equations, one arrives at the integral (28). Once an initial data is given, one can compute this integral, which by construction, vanishes for stationary spacetimes.

APPENDIX: ADM SPLIT OF EINSTEIN'S EQUATIONS IN *D* DIMENSIONS

For the sake of completeness let us give here the ADM split of Einstein's equations and all the relevant tensors. Using the $(D-1)+1$ dimensional decomposition of the metric given as (1) we have:

$$g_{00} = -N^2 + N_i N^i, \qquad g_{0i} = N_i, \qquad g_{ij} = \gamma_{ij}, \tag{A1}$$

and

$$g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = \frac{1}{N^2}N^i, \qquad g^{ij} = \gamma^{ij} - \frac{1}{N^2}N^i N^j. \tag{A2}$$

Let $\Gamma^\mu_{\nu\rho}$ denote the Christoffel symbol of the D dimensional spacetime

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}) \tag{A3}$$

and let ${}^\Sigma\Gamma^k_{ij}$ denote the Christoffel symbol of the $D-1$ dimensional hypersurface, which is compatible with the spatial metric γ_{ij} as

$${}^\Sigma\Gamma^k_{ij} = \frac{1}{2}\gamma^{kp}(\partial_i\gamma_{jp} + \partial_j\gamma_{ip} - \partial_p\gamma_{ij}). \tag{A4}$$

Then a simple computation shows that

$$\Gamma^0_{00} = \frac{1}{N}(\dot{N} + N^k(\partial_k N + N^i K_{ik})) \tag{A5}$$

and

$$\begin{aligned}&\Gamma^0_{0i} = \frac{1}{N}(\partial_i N + N^k K_{ik}), \qquad \Gamma^0_{ij} = \frac{1}{N}K_{ij}, \\ &\Gamma^k_{ij} = {}^\Sigma\Gamma^k_{ij} - \frac{N^k}{N}K_{ij}\end{aligned} \tag{A6}$$

and

$$\Gamma^i_{0j} = -\frac{1}{N}N^i(\partial_j N + K_{kj}N^k) + NK_j{}^i + D_j N^i \tag{A7}$$

and also

$$\begin{aligned}\Gamma^i_{00} = &-\frac{N^i}{N}(\dot{N} + N^k(\partial_k N + N^l K_{kl})) + N(\partial^i N + 2N^k K_k{}^i) \\ &+ \dot{N}^i + N^k D_k N^i.\end{aligned} \tag{A8}$$

Starting with the definition of the D dimensional Ricci tensor

$$R_{\rho\sigma} = \partial_\mu\Gamma^\mu_{\rho\sigma} - \partial_\rho\Gamma^\mu_{\mu\sigma} + \Gamma^\mu_{\mu\nu}\Gamma^\nu_{\rho\sigma} - \Gamma^\mu_{\sigma\nu}\Gamma^\nu_{\mu\rho} \tag{A9}$$

one arrives at

$$\begin{aligned}R_{ij} = {}^\Sigma R_{ij} + KK_{ij} - 2K_{ik}K^k_j + \frac{1}{N}(&\dot{K}_{ij} - N^k D_k K_{ij} \\ &- D_i D_j N - K_{ki}D_j N^k - K_{kj}D_i N^k),\end{aligned} \tag{A10}$$

where ${}^{\Sigma}R_{ij}$ denotes the Ricci tensor of the hypersurface. The remaining components can also be found to be

$$R_{00} = N^i N^j R_{ij} - N^2 K_{ij} K^{ij} + N(D_k D^k N - \dot{K} - N^k D_k K + 2N^k D_m K^m_k) \tag{A11}$$

and

$$R_{0i} = N^j R_{ij} + N(D_m K^m_i - D_i K). \tag{A12}$$

The scalar curvature can be found as

$$R = {}^{\Sigma}R + K^2 + K_{ij} K^{ij} + \frac{2}{N}(\dot{K} - D_k D^k N - N^k D_k K). \tag{A13}$$

Under the above splitting the cosmological Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{A14}$$

split in to constraints and evolution equations in local coordinates. The momentum constraints read

$$N(D_k K^k_i - D_i K) - \kappa(T_{0i} - N^j T_{ij}) = 0, \tag{A15}$$

via the Hamiltonian constraint becomes

$$N^2({}^{\Sigma}R + K^2 - K^2_{ij} - 2\Lambda) - 2\kappa(T_{00} + N^i N^j T_{ij} - 2N^i T_{0i}) = 0. \tag{A16}$$

On the other hand the evolution equations for the metric and the extrinsic curvature become

$$\frac{\partial}{\partial t}\gamma_{ij} = 2NK_{ij} + D_i N_j + D_j N_i, \tag{A17}$$

$$\frac{\partial}{\partial t}K_{ij} = N(R_{ij} - {}^{\Sigma}R_{ij} - KK_{ij} + 2K_{ik}K^k_j) + \mathcal{L}_{\vec{N}} K_{ij} + D_i D_j N, \tag{A18}$$

where $\mathcal{L}_{\vec{N}}$ is the Lie derivative along the shift vector.

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