Black hole nonmodal linear stability: Even perturbations in the Reissner-Nordström case

Gustavo Dotti and Julián M. Fernández Tío
Facultad de Matemática, Astronomía y Física (FaMAF), Universidad Nacional de Córdoba and Instituto de Física Enrique Gaviola, CONICET. Ciudad Universitaria, (5000) Córdoba, Argentina

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This paper is a companion of J. M. Fernández Tío and G. Dotti, Phys. Rev. D **95**, 124041 (2017), in which, following a program on black hole nonmodal linear stability initiated in G. Dotti, Phys. Rev. Lett. **112**, 191101 (2013), odd perturbations of the Einstein-Maxwell equations around a Reissner-Nordström (A)dS black hole were analyzed. Here we complete the proof of the nonmodal linear stability of this spacetime by analyzing the even sector of the linear perturbations. We show that all the gauge invariant information in the metric and Maxwell field even perturbations is encoded in two spacetime scalars: $\mathcal{S}$, which is a gauge invariant combination of $\delta(C_{\alpha\beta\gamma\epsilon}C^{\alpha\beta\gamma\epsilon})$ and $\delta(C_{\alpha\beta\gamma\delta}F_{\alpha\beta}F^{\gamma\delta})$, and $\mathcal{T}$, a gauge invariant combination of $\delta(\nabla_\mu F_{\alpha\beta}\nabla^\mu F^{\alpha\beta})$ and $\delta(\nabla_\mu C_{\alpha\beta\gamma\delta}\nabla^\mu C^{\alpha\beta\gamma\delta})$. Here $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, $F_{\alpha\beta}$ the Maxwell field, and δ means first order variation. We prove that $\mathcal{S}$ and $\mathcal{T}$ are in one-one correspondence with gauge classes of even linear perturbations, and that the linearized Einstein-Maxwell equations imply that these scalar fields are pointwise bounded on the outer static region.

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I. INTRODUCTION

The Einstein-Maxwell field equations with cosmological constant Λ

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{1}$$

$$T_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}\right), \tag{2}$$

$$\nabla_{[\alpha}F_{\beta\gamma]} = 0, \tag{3}$$

$$\nabla^\beta F_{\alpha\beta} = 0, \tag{4}$$

admit the solution

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

$$F = E_0 dt \wedge dr, \qquad E_0 = \frac{Q}{r^2}, \tag{6}$$

where the norm $f(r)$ of the Killing vector $\partial/\partial t$ in (5) is

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2. \tag{7}$$

Note that r has geometrical meaning: it is (the square root of one-fourth of) the areal radius of the spheres of symmetry under $SO(3)$. Note also that $f = g^{\alpha\beta}\nabla_\alpha r \nabla_\beta r$.

We assume $\Lambda \geq 0$. $M > 0$ and Q are constants of integration; they correspond to mass and charge respectively and we assume that their values are such that (5) is a nonextremal black hole, that is

$$f = -\frac{\Lambda}{3r^2}(r - r_i)(r - r_h)(r - r_c)(r + r_i + r_h + r_c), \tag{8}$$

where $0 < r_i < r_h < r_c$ are the inner, event, and cosmological horizons respectively.

We are interested in proving the *nonmodal linear stability* of the outer static region $r_h < r < r_c$ of the solution (5)–(6) of the field equations (1)–(4). This concept of stability was defined in [1,2] and implies proving the following:

(i) There are gauge invariant (both in the Maxwell and infinitesimal diffeomorphism senses) scalar fields from the spacetime $\mathcal{M}$ into $\mathbb{R}$ that contain the same information as the gauge class $[(\mathcal{F}_{\alpha\beta}, h_{\alpha\beta})]$ of the perturbation $(\mathcal{F}_{\alpha\beta}, h_{\alpha\beta})$. Here $\mathcal{F}_{\alpha\beta} = \delta F_{\alpha\beta}$ is the first order perturbation of the electromagnetic field and $h_{\alpha\beta} = \delta g_{\alpha\beta}$ is the metric perturbation. These scalar fields then measure the distortion of the geometry and the Maxwell field and the perturbation fields $h_{\alpha\beta}$ and $\mathcal{F}_{\alpha\beta}$ in a given gauge can be obtained by applying a linear functional on them.

(ii) The gauge invariant curvature fields are pointwise bounded on the outer static region by constants that depends on the initial data of the perturbation on a Cauchy surface for that region.

For odd perturbations (i) and (ii) were proved in the companion paper [3]. In this paper we complete the proof of nonmodal linear stability of the charged black hole by considering the even sector of the linear perturbation fields.

A discussion of the relevance of the nonmodal linear stability concept above, which was introduced in [1], can be found in Sec. I of [2]. For the Schwarzschild spacetime, the strategy behind the proof of nonmodal linear stability in [1] was using the supersymmetric even/odd duality to show that *both odd and even* linear gravity perturbation equations are equivalent to (independent) four dimensional Regge-Wheeler equations. This also holds for Schwarzschild de Sitter. (A detailed proof covering the $\Lambda \geq 0$ cases is given in Lemma 7 in [2].) Once the linearized gravity problem is reduced to uncoupled four dimensional scalar wave equations with a time independent potential, it is possible to place pointwise bounds on the geometric scalar fields mentioned above, and to analyze their decay along future causal directions. This duality is of no use in the charged black hole case because the odd sector equations have the same level of complexity of those of the even sector and, contrary to what happens in the $Q = 0$ case, the set of odd mode equations is *not* equivalent to a four dimensional scalar field equation with a time independent potential.

Our emphasis in this series of papers is on finding the appropriate set of gauge invariant, curvature related scalar fields encoding the information of the gauge class of the perturbation; we do not analyze their decay.

We leave aside the asymptotically AdS $\Lambda < 0$ case. We do so because the dynamics of perturbations is nonunique in this case—in particular, the notion of stability is ambiguous—due to the conformal timelike boundary. In this case also, a choice of boundary conditions at the conformal boundary generically breaks the even/odd duality, so that the even sector perturbation equations are *not* equivalent (even in the uncharged case) to a four dimensional Regge-Wheeler equation, as happens for $\Lambda \geq 0$ (for further details see Sec. IV in [4]).

As in [5], the warped structure of the spacetime (5) $\mathcal{M} = \mathcal{N} \times_{r^2} \sigma$

$$g_{\alpha\beta} dz^\alpha dz^\beta = \tilde{g}_{ab}(y) dy^a dy^b + r^2(y) \hat{g}_{AB}(x) dx^A dx^B \quad (9)$$

is used to simplify the linearized Einstein Maxwell equations (LEME). (We also use the acronyms LEE for linearized Einstein equations and LME for linearized Maxwell equations). The "orbit manifold" $\mathcal{N}$ is two dimensional and Lorentzian, with line element $\tilde{g}_{ab}(y) dy^a dy^b$ $(= -f dt^2 + \frac{dr^2}{f}$ in Schwarzschild coordinates); the "horizon manifold" σ with metric $\hat{g}_{AB}(x) \times dx^A dx^B$ is the unit two sphere (for a treatment of linearization around warped metrics in arbitrary dimensions and with constant curvature horizon manifolds see [5] and references therein). In (5), (t, r) coordinates are used for $\mathcal{N}$ and the standard angular coordinates $\hat{g}_{AB}(x) dx^A dx^B = d\theta^2 + \sin^2\theta d\phi^2$ are used for the unit sphere. In what follows our treatment is "2D-covariant," that is, it allows independent coordinate changes in $\mathcal{N}$ and the unit sphere.

Equation (9) illustrates our notation, which we adopted from [6]; we use lower case indexes a, b, c, d, e for tensors on the orbit manifold $\mathcal{N}$, upper case indexes $A, B, C, D, \ldots$ for tensors on S^2, and Greek indexes for space-time tensors. We follow the additional convention in [2] that

$$\alpha = (a, A), \quad \beta = (b, B), \quad \gamma = (c, C), \quad \delta = (d, D). \quad (10)$$

Tensor fields *introduced* with a lower S^2 index (say Z_A) and *then shown with an upper S^2 index* are assumed to have been acted upon *with the unit S^2 metric inverse* $\hat{g}^{AB}$ (i.e., in our example, $Z^A \equiv \hat{g}^{AB} Z_B$), and similarly with upper S^2 indexes moving down. This has to be kept in mind to avoid wrong $r^{\pm 2}$ factors in the equations. $\tilde{D}_a$, $\tilde{\epsilon}_{ab}$ and $\tilde{g}^{ab}$ are the covariant derivative, volume form (any chosen orientation), and metric inverse for the $\mathcal{N}$ orbit space; $\hat{D}_A$ and $\hat{\epsilon}_{AB}$ are the covariant derivative and volume form $\sin(\theta) d\theta \wedge d\phi$ on the unit sphere.

The metric and Maxwell field perturbations $h_{\alpha\beta}$ and $\mathcal{F}_{\alpha\beta}$ admit a series expansion in rank 0, 1, and 2 eigentensor fields of the horizon manifold Laplace-Beltrami (LB) operator, with "coefficients" that are tensor fields on the orbit space $\mathcal{N}$ [5]. Individual terms of this series are called "modes"; they are not mixed by the LEME. In the standard modal approach a master scalar field $\mathcal{N} \to \mathbb{R}$ is extracted for each mode and the LEME is reduced to an infinite set of *scalar wave equations on* $\mathcal{N}$ (that is, $1 + 1$ wave equations), one for each master mode. Modal stability consists in proving the boundedness/decay of these master fields. This was proved in four dimensional general relativity in the seminal black hole stability papers [7–9] and in higher dimensions more recently by Kodama and Ishibashi (see, e.g., [5,10]). All notions of linear stability prior to [1] were *modal*, that is, restricted to the boundedness of the $1 + 1$ master fields. For four dimensional charged black holes the modal linear stability in the case $\Lambda = 0$ was proved by Zerilli and Moncrief in the series of articles [9,11–13] (see also [14]).

The limitations of the modal linear stability are explained in [1,2] (see the Introduction of [2] for a detailed explanation). These two papers are devoted to the nonmodal linear stability of the Schwarzschild and Schwarzschild de Sitter black hole. The nonmodal linear stability of the Reissner-Nordström black hole with $\Lambda \geq 0$, under odd perturbations, was established in [3]. In the following sections we complete the proof of nonmodal linear stability of this black hole by proving its stability under even perturbations. We do so by showing that there are two fields made out of gauge invariant first order perturbations of curvature scalars (for details refer to Sec. III A). These fields encode all the gauge invariant information of arbitrary even perturbations, allow one to reconstruct the metric and Maxwell field perturbations in a given gauge, and are pointwise bounded.

II. LINEARIZED EINSTEIN-MAXWELL EQUATIONS

The LEME are obtained by linearizing equations (1)–(4), that is, we assume that there is a smooth one-parameter set of solutions $(g(\varepsilon)_{\alpha\beta}, F(\varepsilon)_{\alpha\beta})$ of the Einstein-Maxwell equations (1)–(4) such that $(g(\varepsilon=0)_{\alpha\beta}, F(\varepsilon=0)_{\alpha\beta})$ are the Reissner-Nordström fields (5)–(6), take the derivative with respect to ε and evaluate it at $\varepsilon = 0$. The resulting equations are linear in the *perturbation fields* $h_{\alpha\beta} \coloneqq dg_{\alpha\beta}/d\varepsilon|_0$ and $\mathcal{F}_{\alpha\beta} \coloneqq dF_{\alpha\beta}/d\varepsilon|_0$.

The linearization of Eq. (3) gives $d\mathcal{F} = 0$. Since the region we are interested in (see Theorem 2 for details) is homeomorphic to $\mathbf{R}^2 \times S^2$, then of the same homotopy type of S^2, $d\mathcal{F} = 0$ implies that there exists A_α such that

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + p\hat{\epsilon}_{\alpha\beta}, \tag{11}$$

where p is a constant and $\hat{\epsilon}_{\alpha\beta}$ is the pullback to $\mathcal{M}$ of the S^2 volume form $\hat{\epsilon}_{AB}$. Under the index convention (10) the covector field A_α is written as

$$A_\alpha = (A_a, A_A) \tag{12}$$

and, as explained in [2,15], admits a decomposition in a set of even (+) and odd (−) fields:

$$A_\alpha = (A^+_a, \hat{D}_A A^+ + \hat{\epsilon}_A{}^C \hat{D}_C A^-). \tag{13}$$

Even and odd fields are characterized by the way they transform when pulled back by the antipodal map P on S^2 [1]. Note that $\hat{\epsilon}_{\alpha\beta}$ is an odd field, and that equations (11)–(13) imply that we can replace

$$\mathcal{F}_{\alpha\beta} \quad \text{with} \quad \{A^+_a, A^+\} \cup \{A^-, p\}. \tag{14}$$

The constant p associated with the odd Maxwell field perturbation $p\hat{\epsilon}_{\alpha\beta}$ corresponds to turning on a magnetic charge [16]. The scalar fields $A^\pm$ are unique if they are required to belong to $L^2(S^2)_{>0}$ [2]. Here $L^2(S^2)_{>\ell_o}$ is the space of square integrable functions on S^2 orthogonal to the $\ell = 0, 1, \ldots, \ell_o$ eigenspaces of the Laplace-Beltrami operator, and ℓ labels the LB scalar field eigenvalue $-\ell(\ell+1)$.

Similarly, a symmetric tensor field $S_{\alpha\beta} = S_{(\alpha\beta)}$, such as $h_{\alpha\beta}$, $\mathcal{G}_{\alpha\beta} \coloneqq dG_{\alpha\beta}/d\varepsilon|_0$ and $\mathcal{T}_{\alpha\beta} \coloneqq dT_{\alpha\beta}/d\varepsilon|_0$, decomposes as [2,3,15]

$$S_{\alpha\beta} = \begin{pmatrix} S_{ab} & S_{aB} \\ S_{Ab} & S_{AB} \end{pmatrix}, \tag{15}$$

with

$$S_{aB} = \hat{D}_B S^+_a + \hat{\epsilon}_B{}^C \hat{D}_C S^-_a. \tag{16}$$

Assuming that $S^\pm_a \in L^2(S^2)_{>0}$, they are unique [2,15]. $S_{AB} = S_{(AB)}$ further decomposes as

$$S_{AB} = \hat{D}_{(A}(\epsilon_{B)C}\hat{D}^C S^-) + \left(\hat{D}_A\hat{D}_B - \frac{1}{2}\hat{g}_{AB}\hat{D}^C\hat{D}_C\right)S^+ + \frac{1}{2}S^+_T \hat{g}_{AB}, \tag{17}$$

where $S^+_T = S_C{}^C$ and the fields $S^\pm \in L^2(S^2)_{>1}$ are unique.

In this way, as happens for covector fields [Eq. (14)], the symmetric tensor field $S_{\alpha\beta}$ is replaced by a set of even and odd fields

$$\{S^+_{ab} = S_{ab}, S^+_a, S^+, S^+_T\} \cup \{S^-_a, S^-\}. \tag{18}$$

In particular, the perturbed metric, Einstein tensor, and energy momentum tensors contain the fields

$$h_{\alpha\beta} \sim \{h^+_{ab}, h^+_a, h^+, h^+_T\} \cup \{h^-_a, h^-\}, \tag{19}$$

$$\mathcal{G}_{\alpha\beta} \sim \{G^+_{ab}, G^+_a, G^+, G^+_T\} \cup \{G^-_a, G^-\}, \tag{20}$$

$$\mathcal{T}_{\alpha\beta} \sim \{T^+_{ab}, T^+_a, T^+, T^+_T\} \cup \{T^-_a, T^-\}. \tag{21}$$

Even and odd fields are not mixed by the LEME. The restriction of the LEME to the odd sector was the subject of [3]; even perturbations are studied in the following sections.

Let $J_{(1)}$, $J_{(2)}$, and $J_{(3)}$ be S^2 (and therefore spacetime) Killing vector fields corresponding to rotations around orthogonal axis in $\mathbb{R}^3 \supset S^2$, normalized such that the length of their closed orbits in the unit sphere is 2π (e.g., $J_{(3)} = \partial/\partial_\phi$). The square angular momentum operator

$$\mathbf{J}^2 \equiv (\pounds_{J_{(1)}})^2 + (\pounds_{J_{(2)}})^2 + (\pounds_{J_{(3)}})^2 \tag{22}$$

is defined both in S^2 and the spacetime. This operator commutes with the LEME and preserves parity. It thus allows a further decomposition of even and odd fields into modes (eigenfields of $\mathbf{J}^2$). On S^2 scalars the operator $\mathbf{J}^2$ agrees with the LB operator of S^2, $\hat{D}^A\hat{D}_A$; however, on higher rank tensors these two operators act differently. Since $[\nabla_a, \pounds_{J_k}] = 0 = [\hat{D}_A, \pounds_{J_k}] = [\tilde{D}_a, \pounds_{J_k}]$, it follows that $\mathbf{J}^2$ commutes with ∇_α, $\tilde{D}_a$, and $\hat{D}_A$. In a modal decomposition approach the tensor fields on the right sides of (19)–(21) into eigenfields of $\mathbf{J}^2$.

In the following sections we restrict ourselves to even perturbations and assume the restrictions above: A^+, $S^+_a \in L^2(S^2)_{>0}$, $S^+ \in L^2(S^2)_{>1}$. These conditions guarantee that the linear operators $(A^+_a, A^+) \to A_\alpha$ in (13), and $\{S^+_{ab}, S^+_a, S^+, S^+_T\} \to S_{\alpha\beta}$ in (15)–(17) are injective

(Lemma 2 in [2]). Since we restrict our discussion to even perturbations, there is no risk of confusion and + superscripts will be suppressed from now on.

A. Even sector perturbations

Even perturbations are those for which the minus fields in (13) and (19) are zero. Rescaling and dropping the + superscripts, $h_T^+ =: r^2 h_T$, $h^+ =: 2r^2 h$, gives

$$h_{\alpha\beta} = \begin{pmatrix} h_{ab} & \hat{D}_B h_a \\ \hat{D}_A h_b & r^2[(2\hat{D}_A\hat{D}_B - \hat{g}_{AB}\hat{D}^C\hat{D}_C)h + \frac{1}{2}h_T\hat{g}_{AB}] \end{pmatrix} \tag{23}$$

with the restrictions $h_a \in L^2(S^2)_{>0}$, $h \in L^2(S^2)_{>1}$.

Similarly, Eqs. (11) and the even piece of (13) give (dropping superscripts)

$$\mathcal{F}_{\alpha\beta} = \begin{pmatrix} \tilde{D}_a A_b - \tilde{D}_b A_a & \tilde{D}_a \hat{D}_B A - \hat{D}_B A_a \\ \hat{D}_A A_b - \tilde{D}_b \hat{D}_A A & 0 \end{pmatrix} \tag{24}$$

with $A \in L^2(S^2)_{>0}$.

$U(1)$ gauge transformations of the Maxwell field leave $\mathcal{F}_{\alpha\beta}$ invariant while changing the potential as $A_\alpha \to A_\alpha + \partial_\alpha B$. The even piece of the vector potential (13) then changes as $A_a \to A_a + \partial_a B$ and $A \to A + B_{(>0)}$, where $B_{>0}$ is the projection of B onto $L^2(S^2)_{>0}$.

Under a *coordinate* gauge transformation (infinitesimal diffeomorphism) along the even vector field defined by

$$X_\alpha = (X_a, r^2\hat{D}_A X), \qquad X \in L^2(S^2)_{>0}, \tag{25}$$

$h_{\alpha\beta}$ and $\mathcal{F}_{\alpha\beta}$ transform into the physically equivalent fields:

$$h'_{\alpha\beta} = h_{\alpha\beta} + \pounds_X g_{\alpha\beta}, \qquad \mathcal{F}'_{\alpha\beta} = \mathcal{F}_{\alpha\beta} + \pounds_X F_{\alpha\beta}. \tag{26}$$

From (5), (6), (23), (24), (25), and (26) we find that (26) is equivalent to

$$\left.\begin{aligned} h_{ab} &\to h'_{ab} = h_{ab} + \tilde{D}_a X_b + \tilde{D}_b X_a, \\ h_T &\to h'_T = h_T + \tfrac{4}{r}\tilde{D}^a r X_a + 2\hat{D}^C\hat{D}_C X \\ A_b &\to A'_b = A_b - \tilde{\epsilon}_{bc} X^c E_0 \end{aligned}\right\} \text{all } \ell \tag{27}$$

where the legend "all ℓ" reminds us that these fields have projections on all the ℓ subspaces, whereas

$$\begin{aligned} A &\to A' = A, && (\ell > 0 \text{ only}), \\ h_a &\to h'_a = h_a + X_a^{>0} + r^2\tilde{D}_a X, && (\ell > 0 \text{ only}), \\ h &\to h' = h + X_{>1}, && (\ell > 1 \text{ only}). \end{aligned} \tag{28}$$

1. $\ell = 0$: Solution of the LEME and linearized Birkoff theorem

Given that $\ell = 0$ corresponds to the spherically symmetric part of the perturbation, $\ell = 0$ perturbations to the spherically symmetric Reissner-Nordström background that solve the LEME should amount, in view of Birkhoff's theorem, to a modification of the parameters Q and M in (5)–(7). In this section we prove that this is the case.

On $\ell = 0$, the fields h_a, h, A, and X have trivial projections, and we can use (27) as in [2], choosing $\tilde{D}^a r X_a^{(\ell=0)} = -\frac{r}{4}h_T^{(\ell=0)}$ to set $h'_T = 0$ and then $2\tilde{D}^a X_a^{(\ell=0)} = -g^{ab}h_{ab}^{(\ell=0)}$ to get a traceless h'_{ab}. Dropping primes, the resulting metric perturbation is of the form

$$h_{\alpha\beta}^{(\ell=0)} = \begin{pmatrix} h_{ab}^{T,(\ell=0)} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{g}^{ab}h_{ab}^{T,(\ell=0)} = 0. \tag{29}$$

This gauge choice admits a residual freedom $X_\alpha = (X_a, 0)$ preserving the conditions (29), for which X_a must satisfy

$$(\tilde{D}^a r)X_a = 0, \qquad \tilde{D}^a X_a = 0, \tag{30}$$

whose solution is

$$X_a = \tilde{\epsilon}_{ab}\tilde{D}^b X(r). \tag{31}$$

Since the $\ell = 0$ piece of A is trivial, the $\ell = 0$ Maxwell field is

$$\mathcal{F}_{\alpha\beta}^{(\ell=0)} = \begin{pmatrix} \tilde{D}_a A_b^{(\ell=0)} - \tilde{D}_b A_a^{(\ell=0)} & 0 \\ 0 & 0 \end{pmatrix} \tag{32}$$

and the linearization of (4) reduces to

$$\tilde{D}^b(r^2(\tilde{D}_a A_b^{(\ell=0)} - \tilde{D}_b A_a^{(\ell=0)})) = 0. \tag{33}$$

Defining $\tilde{D}_a A_b^{(\ell=0)} - \tilde{D}_b A_a^{(\ell=0)} =: \tilde{\epsilon}_{ab}\mathcal{E}^{(\ell=0)}$ the above equation reads

$$\tilde{\epsilon}_{ab}\tilde{D}^b(r^2\mathcal{E}^{(\ell=0)}) = 0. \tag{34}$$

Its solution,

$$\mathcal{E}^{(\ell=0)} = \frac{q}{r^2}, \tag{35}$$

corresponds to a change in charge $Q \to Q + \epsilon q$, as anticipated.

To complete our proof of the "linearized Birkoff theorem" we choose coordinates (t, r) in orbit space, work in the transverse gauge (29) and use the residual gauge freedom (31) to set $h_{\text{tr}} = 0$ [this fixes $X(r)$ in (31) up to a linear function of r]. Using this additional condition together with the trace-free condition $h_{tt} = f^2 h_{rr}$ and $A_t = q/r$, $A_r = 0$, the $t-r$ component of the LEE

$$\mathcal{G}_{\alpha\beta}{}^{(\ell=0)} + \Lambda h_{\alpha\beta}{}^{(\ell=0)} = 8\pi \mathcal{T}_{\alpha\beta}^{(\ell=0)} \tag{36}$$

gives $\partial_t h_{rr} = 0$, so that h_{rr} and $h_{tt} = f^2 h_{rr}$ depend only on r. Inserting this condition in the $r-r$ LEE (36) gives

$$h_{rr} = -\frac{2r^2(qQ - mr)}{r^4 f^2}, \tag{37}$$

where m is a constant of integration. We conclude that

$$h_{tt} = f^2 h_{rr} = -\frac{2qQ}{r^2} + \frac{2m}{r}. \tag{38}$$

Note that (37) and (38) correspond precisely to, respectively, $(m\partial/\partial_M + q\partial/\partial_Q)f^{-1}$ and $(m\partial/\partial_M + q\partial/\partial_Q)(-f)$, so we recognize that m and q correspond respectively to first order variations δM and δQ of the mass and charge in the background Reissner-Nordström metric.

2. $\ell = 1$ modes: Gauge choice

Using the gauge freedom (27) we can put the metric perturbation in Regge-Wheeler (RW) form:

$${}^{\text{RW}}h_{\alpha\beta}^{(\ell=1)} = \begin{pmatrix} h_{ab}^{(\ell=1)} & 0 \\ 0 & \frac{r^2}{2}\hat{g}_{AB} h_T^{(\ell=1)} \end{pmatrix}. \tag{39}$$

Contrary to what happens for $\ell > 1$, for $\ell = 1$ there is no unique RW gauge: once the metric is put in RW form (39), we can gauge transform it into a *different* RW gauge using a gauge vector of the form $X_\alpha = (X_a, \hat{D}_A X)$ with

$$X_c^{(\ell=1)} = -r^2 \tilde{D}_c X^{(\ell=1)}. \tag{40}$$

We will use this gauge freedom to further set

$$\tilde{h}^{(\ell=1)} := \tilde{g}^{ab} h_{ab}^{(\ell=1)} = 0. \tag{41}$$

We will assume the RW traceless gauge conditions (39) and (41) when solving the LEME. Note that this does not exhaust the gauge transformations (40): a residual gauge freedom keeping these conditions is one for which the gauge vector satisfies (40) together with

$$\tilde{D}^c (r^2 \tilde{D}_c X^{(\ell=1)}) = 0. \tag{42}$$

3. $\ell \geq 2$ modes: Gauge choice and gauge invariants

For $\ell \geq 2$ the field

$$p_a = h_a^{(\geq 2)} - r^2 \tilde{D}_a h \tag{43}$$

transforms as $p_a \to p'_a = p_a + X_a^{(\geq 2)}$. This allows us to construct the following ($\ell \geq 2$) gauge invariant fields [we use (24)–(28)]:

$$\left.\begin{aligned} H_{ab} &:= h_{ab}^{(\geq 2)} - \tilde{D}_a p_b - \tilde{D}_b p_a \\ H_T &:= h_T^{(\geq 2)} - \tfrac{4}{r} p_a \tilde{D}^a r - 2\hat{D}^C \hat{D}_C h^{(\geq 2)} \\ \mathcal{E}\tilde{\epsilon}_{ab} &:= \mathcal{F}_{ab}^{\geq 2} - \tilde{\epsilon}_{ab} \tilde{D}_c (E_0 p^c) \\ r\tilde{\epsilon}_{ab} \hat{D}_B \mathcal{E}^b &:= \mathcal{F}_{aB}^{\geq 2} - \tilde{\epsilon}_{ab} E_0 \tilde{D}_B p^b \end{aligned}\right\} \quad \ell \geq 2 \text{ gauge invariant fields.} \tag{44}$$

The RW gauge is defined by the condition $p_a = 0$. It is unique, since any nontrivial gauge transformation (27)–(28) requires $X \neq 0$ to keep $h_a = 0$, and this spoils the condition $h = 0$,

$${}^{\text{RW}}h_{\alpha\beta}^{\geq 2} = \begin{pmatrix} H_{ab} & 0 \\ 0 & \frac{r^2}{2}\hat{g}_{AB} H_T \end{pmatrix}. \tag{45}$$

Note that this is formally identical to (39).

4. Recasting the linearized $\ell \geq 2$ equations

In what follows we will decompose S^2 and orbit space symmetric 2-tensors into their traceless pure trace pieces as

$$S_{ab} = S_{ab}^T + \frac{1}{2} g_{ab} \tilde{S}, \qquad \tilde{S} := S_{ab} g^{ab}, \tag{46}$$

$$S_{AB} = S_{AB}^T + \frac{1}{2} \hat{g}_{AB} \hat{S}, \qquad \hat{S} := S_{AB} \hat{g}^{AB}. \tag{47}$$

We will assume the linearized $\ell \geq 2$ Maxwell field is given by (24) and that the linearized $\ell \geq 2$ metric is in RW form (45).

Consider first the linearized Maxwell equations. Equations (24) and (45) imply that the $\beta = B$ components of the linearization of the Maxwell equation $\nabla^\alpha F_{\alpha\beta}$ are equivalent to the condition

$$\hat{D}_B (\tilde{D}^d A_d - \tilde{D}^d \tilde{D}_d A) = 0, \tag{48}$$

which can be written as

$$\tilde{\epsilon}^{ab} \tilde{D}_a (r \mathcal{E}_b) = 0, \qquad \mathcal{E}^b := r^{-1} \tilde{\epsilon}^{bc} (\tilde{D}_c A - A_c). \tag{49}$$

This implies that

$$\mathcal{E}_b = -\frac{1}{r} \tilde{D}_b \mathcal{A}, \tag{50}$$

for some scalar $\mathcal{A}$, and simplifies (24) to

$$\mathcal{F}_{\alpha\beta} = \begin{pmatrix} -\tilde{\epsilon}_{ab}\tilde{D}_c\tilde{D}^c\mathcal{A} & -\tilde{\epsilon}_a{}^c\tilde{D}_c\hat{D}_B\mathcal{A} \\ \tilde{\epsilon}_b{}^c\tilde{D}_c\hat{D}_A\mathcal{A} & 0 \end{pmatrix}. \tag{51}$$

The $\beta = b$ components of the linearization of $\nabla^\alpha F_{\alpha\beta}$ then gives $E_0\tilde{\epsilon}_a{}^b\partial_b z = 0$, where $z \coloneqq -\frac{r^2}{Q}\tilde{D}_a\tilde{D}^a\mathcal{A} - \frac{1}{Q}\hat{D}_A\hat{D}^A\mathcal{A} - \frac{1}{2}(H_{ab}g^{ab} - h_T)$. This gives $z = z(\theta,\phi)$. However, in view of the $U(1)$ gauge freedom freedom $\mathcal{A} \to \mathcal{A}' = \mathcal{A} + p(\theta,\phi)$ implicit in the definition (50) of $\mathcal{A}$, and given that z has no $\ell = 0$ component, we can choose p such that $\hat{D}^B\hat{D}_B p = Qz$, then for $\mathcal{A}'$ we find $z' = 0$ and (dropping the prime on $\mathcal{A}$)

$$\tilde{D}_a\tilde{D}^a\mathcal{A} + \frac{1}{r^2}\hat{D}_A\hat{D}^A\mathcal{A} = \frac{Q}{2r^2}(h_T - \tilde{H}), \tag{52}$$

where $\tilde{H}$ denotes the trace part of H_{ab} according to (46).

From now on we switch from H^T_{ab} to the one form $C_a = H^T_{ab}\tilde{D}^b r$, which contains the same information, in view of the equality

$$H^T_{ab} = \frac{1}{f}(\tilde{D}_a r C_b + C_a\tilde{D}_b r - g_{ab}\tilde{D}^d r C_d). \tag{53}$$

Having solved the LME we proceed with the LEE. The traceless S^2 piece

$$\mathcal{G}^T_{AB} + \Lambda h^T_{AB} = 8\pi\mathcal{T}^T_{AB} \tag{54}$$

gives

$$\tilde{H} = 0. \tag{55}$$

The off-diagonal piece

$$\mathcal{G}_{Ab} + \Lambda h_{Ab} = 8\pi\mathcal{T}_{Ab}, \tag{56}$$

combined with the condition (55) (and $h_{Ab} = 0$), gives

$$q_b \coloneqq \tilde{D}_a C^a\tilde{D}_b r + \tilde{\epsilon}^{ec}\tilde{D}_e C_c\tilde{\epsilon}_b{}^a\tilde{D}_a r - \frac{f}{2}\tilde{D}_b h_T - 4fE_0\tilde{D}_b\mathcal{A} = 0. \tag{57}$$

Contracting (57) with $\tilde{\epsilon}^{bd}\tilde{D}_d r$ gives

$$\tilde{\epsilon}^{bd}\left[\tilde{D}_b C_d + \frac{1}{2}\tilde{D}_b h_T\tilde{D}_d r + 4E_0\tilde{D}_b\mathcal{A}\tilde{D}_d r\right] = 0. \tag{58}$$

This allows us to introduce the field ξ, defined by

$$\tilde{D}_d\xi = Z_d \coloneqq C_d - \frac{1}{2}r\tilde{D}_d h_T + 4E_0\mathcal{A}\tilde{D}_d r. \tag{59}$$

Contracting (57) with $\tilde{D}^b r$ and using the above equation then gives

$$\tilde{D}^a\tilde{D}_a\xi + \frac{r}{2}\tilde{D}^a\tilde{D}_a h_T - 8E_0(\tilde{D}^a r)\tilde{D}_a\mathcal{A} - 4E_0\mathcal{A}\tilde{D}^a\tilde{D}_a r + \frac{8E_0}{r}(\tilde{D}^a r)(\tilde{D}_a r)\mathcal{A} = 0. \tag{60}$$

Using Eqs. (52), (55), (57) and (60) in the LEE

$$\tilde{\mathcal{G}} + \Lambda\tilde{H} = 8\pi\tilde{\mathcal{T}} \tag{61}$$

gives

$$\frac{2}{r}\tilde{D}^a\tilde{D}_a\xi - 8E_0\left(\frac{(\tilde{D}^a\tilde{D}_a r)\mathcal{A}}{r} + \frac{\hat{D}^A\hat{D}_A\mathcal{A}}{r^2}\right) + \frac{4}{r^2}\tilde{D}^a\xi\tilde{D}_a r = \left[\left(\frac{\hat{D}^A\hat{D}_A + 2}{r^2}\right) - 4E_0^2\right]h_T. \tag{62}$$

Note that the operator on the right side above is invertible; this proves that all components of $\mathcal{F}_{\alpha\beta}$ and $h_{\alpha\beta}$ can be written in terms of ξ and $\mathcal{A}$. If we do so and use the remaining LEE, we arrive, after some work, to the following system of partial differential equations for ξ and $\mathcal{A}$:

$$(\hat{D}_A\hat{D}^A + 2f - r\tilde{D}_a\tilde{D}^a r)\left[\tilde{D}_a\tilde{D}^a\xi + \frac{2}{r}\tilde{D}_a\xi\tilde{D}^a r - (\hat{D}_A\hat{D}^A + r\tilde{D}^a\tilde{D}_a r)\left(\frac{4Q\mathcal{A}}{r^3}\right)\right] + \left(\hat{D}_A\hat{D}^A + 2 - \frac{4Q^2}{r^2}\right) \times \left(\frac{1}{r^2}\hat{D}_A\hat{D}^A\xi - \frac{2}{r}\tilde{D}_a\xi\tilde{D}^a r + \frac{8fQ}{r^3}\mathcal{A}\right) = 0 \tag{63}$$

and

$$(\hat{D}_A\hat{D}^A + 2f - r\tilde{D}_a\tilde{D}^a r)\left(\tilde{D}_a\tilde{D}^a\mathcal{A} + \frac{1}{r^2}\hat{D}_A\hat{D}^A\mathcal{A}\right) + \frac{8fQ^2}{r^4}\mathcal{A} + \frac{Q}{r}\left(\frac{1}{r^2}\hat{D}_A\hat{D}^A\xi - \frac{2}{r}\tilde{D}_a\xi\tilde{D}^a r\right) = 0. \tag{64}$$

Note that, since they are derived from gauge invariant fields, $\mathcal{A}$ and ξ above are gauge invariant.

5. Solution of the $\ell = 1$ LEME

Equation (39) is formally identical to (45), the difference being that the latter is given in terms of gauge invariant fields. Thus, the steps (48) to (52) from the previous section hold for $\ell = 1$ with the replacements $H_{ab} \to h^{(\ell=1)}_{ab}$, etc. Now, in view of Eq. (17), Eq. (54) is void for $\ell = 1$. However, the trace free condition (55) to where this equation leads corresponds to the traceless *gauge choice* (41) for $\ell = 1$. This implies that the reasoning following (55) can also be taken without change for $\ell = 1$. As a result, we obtain the system (63)–(64) with $\mathcal{A} \to \mathcal{A}^{(\ell=1)}$,

$\xi \to \xi^{(\ell=1)}$ [defined in a way analogous to (59)], and $\hat{D}^A \hat{D}_A \to -2$.

A conceptual difference between the $\ell = 1$ and $\ell > 1$ cases is that the fields $\mathcal{A}$ and ξ, being defined from the gauge invariant fields, are themselves gauge invariant, whereas $\mathcal{A}^{(\ell=1)}$ and $\xi^{(\ell=1)}$ are *not*. Tracing back the gauge transformations of the fields involved in their definition we find that, under the residual gauge freedom (40)–(42) [note that $(2 - 2f + r\tilde{D}_a \tilde{D}^a r)r = 6M - 4Q^2/r$],

$$\begin{aligned} Z_a^{(\ell=1)} &= C_a^{(\ell=1)} - \frac{r}{2}\tilde{D}_a h_T^{(\ell=1)} + 4E_0 A^{(\ell=1)} \tilde{D}_a r \\ &\to Z_a^{(\ell=1)} + \left(6M - \frac{4Q^2}{r}\right)\tilde{D}_a X^{(\ell=1)} \\ &\quad + 4E_0 (Q X^{(\ell=1)}) \tilde{D}_a r \\ &= Z_a^{(\ell=1)} + \tilde{D}_a \left[\left(6M - \frac{4Q^2}{r}\right) X^{(\ell=1)}\right]. \end{aligned} \tag{65}$$

And then

$$\xi^{(\ell=1)} \to \xi'^{(\ell=1)} = \xi^{(\ell=1)} + \left(6M - \frac{4Q^2}{r}\right) X^{(\ell=1)}. \tag{66}$$

Also

$$\mathcal{A}^{(\ell=1)} \to \mathcal{A}^{(\ell=1)\prime} = \mathcal{A}^{(\ell=1)} + Q X^{(\ell=1)}. \tag{67}$$

A priori, Eq. (67) does not imply that $\mathcal{A}$ is pure gauge, since the $X^{(\ell=1)}$ field is not arbitrary but restricted to the condition (42). As we will see, the situation is quite subtle.

Equations (66)–(67) suggest that, for $\ell = 1$, we replace in the LEME (63)–(64) $\xi^{(\ell=1)}$ and $A^{(\ell=1)}$ by the gauge invariant field

$$\begin{aligned} \varphi &:= \frac{Q}{r(2 - 2f + r\tilde{D}_a \tilde{D}^a r)} \xi^{(\ell=1)} - \mathcal{A}^{(\ell=1)} \\ &=: \sum_m \varphi^{(m)} S_{(\ell=1,m)}. \end{aligned} \tag{68}$$

If we rewrite (64)–(63) in terms of φ and $\mathcal{A}$, eliminate second order $\mathcal{A}$ derivatives from (64) using (63), we get a decoupled equation for φ:

$$[-f \tilde{D}^a \tilde{D}_a + V^{(\ell=1)}]\varphi = 0, \tag{69}$$

where

$$\begin{aligned} V^{(\ell=1)} = &-\frac{2f}{3r^4(3Mr - 2Q^2)^2}((4Q^4\Lambda - 27M^2)r^4 \\ &+ 54M^2Q^2r^2 - 48MQ^4 r + 12Q^6). \end{aligned} \tag{70}$$

For $\mathcal{A}^{(\ell=1,m)}$ we obtain

$$r^{-2}\tilde{D}^c(r^2 \tilde{D}_c \mathcal{A}^{(\ell=1,m)}) = 2r^{-1}\tilde{D}^c r \tilde{D}_c \varphi^{(m)} + Z(r)\varphi^{(m)} \tag{71}$$

with

$$Z(r) = \frac{-4Q^2\Lambda r^4 + 18Mr^3 - 24Q^2Mr + 12Q^4}{3r^4(3Mr - 2Q^2)}. \tag{72}$$

φ is a physical (gauge invariant) degree of freedom (d.o.f.) obeying (69). Once a solution of this equation is picked, the source on the right side of (71) is defined, and the solution of (71) will be unique up to a solution of the homogeneous equation. However, since the homogeneous equation agrees with (42), in view of (67), any two solutions of (71) are gauge related and therefore equivalent. This implies that *the gauge class* of $\mathcal{A}^{(\ell=1,m)}$ is uniquely determined once the three gauge invariant functions on the orbit space $\varphi(m)$ are given, and then φ contains the only d.o.f. in the $\ell = 1$ subspace (three functions defined on the orbit space).

This situation should be contrasted with that of the projections on the higher harmonic subspaces $\ell \geq 2$, for which the number of d.o.f. is *two* (instead of one) functions on the orbit space for every (ℓ, m): the harmonic components solutions $\Phi_n^{(\ell,m)}$ of the Zerilli fields Φ_n, $n = 1, 2$ (see next section). It should also be contrasted with the Schwarzschild black hole case, for which the even $\ell = 1$ mode is pure gauge [2].

6. Solution of the $\ell > 1$ LEME

To decouple the system (63)–(64) we introduce

$$\kappa_1 = -\left(\frac{1}{\mathbf{J}^2 + 2f - r\tilde{D}_a \tilde{D}^a r}\right)\sqrt{-(\mathbf{J}^2 + 2)}\xi, \tag{73}$$

$$\kappa_2 = -\frac{2Q}{r}\left(\frac{1}{\mathbf{J}^2 + 2f - r\tilde{D}_a \tilde{D}^a r}\right)\xi^{(\ell\geq 2)} - 2\mathcal{A}. \tag{74}$$

$\mathbf{J}^2$ acts as a $-\ell(\ell+1)$ factor on the ℓ subspace of $L^2(S^2)$ then, e.g., if $\xi = \sum_{(\ell,m)} \xi^{(\ell,m)} S_{(\ell,m)}$ is the expansion of ξ in spherical harmonics $S_{(\ell,m)}$, the linear operator in the definition of κ_1 above acts as

$$\begin{aligned} &-\left(\frac{1}{\mathbf{J}^2 + 2f - r\tilde{D}_a \tilde{D}^a r}\right)\sqrt{-(\mathbf{J}^2 + 2)}\sum_{(\ell,m)} \xi^{(\ell,m)} S_{(\ell,m)} \\ &\quad = \sum_{(\ell,m)} \left(\frac{\sqrt{(\ell+2)(\ell-1)}}{\ell(\ell+1) - 2f + r\tilde{D}_a \tilde{D}^a r}\right)\xi^{(\ell,m)} S_{(\ell,m)}. \end{aligned} \tag{75}$$

Using the fact that on scalar fields $\hat{D}^A \hat{D}_A = \mathbf{J}^2$, the projection of equations (63)–(64) on the $\ell > 1$ space can then be written as

$$\begin{pmatrix} -\tilde{D}^a\tilde{D}_a + \mathbf{U} - 3M\mathbf{W} & 2Q\sqrt{-(\mathbf{J}^2+2)}\mathbf{W} \\ 2Q\sqrt{(-\mathbf{J}^2+2)}\mathbf{W} & -\tilde{D}^a\tilde{D}_a + \mathbf{U} + 3M\mathbf{W} \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = 0 \tag{76}$$

where $\mathbf{U}$ and $\mathbf{W}$ entering the symmetric matrix operator $\mathcal{O}$ in the above are defined by

$$\begin{aligned} &[r^2(\mathbf{J}^2+2f-r\tilde{D}_a\tilde{D}^a r+r^2\Lambda)^2]\mathbf{U} \\ &= -(\mathbf{J}^2+2)^3 + \left(2+\frac{9M}{r}-\frac{4Q^2}{r^2}\right)(\mathbf{J}^2+2)^2 \\ &\quad + \left(\frac{3M}{r}+\frac{9M^2+2Q^2}{r^2}-\frac{16Q^2M}{r^3}+\frac{6Q^4}{r^4}+\frac{2\Lambda Q^2}{3}\right)(\mathbf{J}^2+2) \\ &\quad + 4\left(\frac{9M^2}{r^2}+\frac{9M^3}{r^3}-\frac{39Q^2M^2}{r^4}+\frac{32Q^4M}{r^5}-\frac{8Q^6}{r^6}\right) \\ &\quad - \frac{4r^2\Lambda}{3}\left(\frac{9M^2}{r^2}-\frac{12Q^2M}{r^3}+\frac{8Q^4}{r^4}\right) \end{aligned} \tag{77}$$

and

$$\begin{aligned} &[r^3(\mathbf{J}^2+2f-r\tilde{D}_a\tilde{D}^a r+r^2\Lambda)^2]\mathbf{W} \\ &\quad = (\mathbf{J}^2+2)^2 - 4(\mathbf{J}^2+2) + \frac{4M}{r}\left(3-\frac{3M}{r}+\frac{Q^2}{r^2}\right) \\ &\qquad + \frac{4\Lambda}{3}(3Mr-4Q^2). \end{aligned} \tag{78}$$

The matrix $\mathcal{O}$ can be diagonalized by introducing $\Xi = \sqrt{9M^2-4Q^2(\mathbf{J}^2+2)}$, $\beta_n = 3M + (-1)^n\Xi$, $n = 1, 2$ and

$$P = \begin{pmatrix} -\beta_1 & \beta_2 \\ 2Q\sqrt{-(\mathbf{J}^2+2)} & -2Q\sqrt{-(\mathbf{J}^2+2)} \end{pmatrix}. \tag{79}$$

We find that

$$P^{-1}\mathcal{O}P = \begin{pmatrix} -\tilde{D}^a\tilde{D}_a + U_1 & 0 \\ 0 & -\tilde{D}^a\tilde{D}_a + U_2 \end{pmatrix}, \tag{80}$$

where

$$U_n = \mathbf{U} + \frac{(-1)^{n+1}}{2}(\beta_2-\beta_1)\mathbf{W}. \tag{81}$$

In view of (80), the Zerilli fields

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = P^{-1}\begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} \tag{82}$$

satisfy the equations

$$[-f\tilde{D}^a\tilde{D}_a + V_n]\Phi_n = 0, \quad n = 1, 2, \tag{83}$$

where $V_n = U_n/f$ can be written in Ricatti form

$$V_n = f\beta_n\partial_r f_n + \beta_n^2 f_n^2 + \mathbf{J}^2(\mathbf{J}^2+2)f_n, \tag{84}$$

with

$$f_n = \frac{f}{(r\beta_n - r^2(\mathbf{J}^2+2))}. \tag{85}$$

In $t-r$ coordinates (83) reads

$$\partial_t^2\Phi_n + A_n\Phi_n = 0 \tag{86}$$

where

$$A_n = -\partial_{r^*}^2 + V_n, \tag{87}$$

and r^* is a tortoise coordinate, defined by $dr^*/dr = 1/f$.

Since ξ and $\mathcal{A}$ are gauge invariant fields, so are κ_1, κ_2, and the Zerilli fields Φ_1 and Φ_2.

Tracing our definitions back we find that

$$\mathcal{A} = -\frac{Q}{r^2}[(r\beta_1 - r^2(\mathbf{J}^2+2))\Phi_1 - (r\beta_2 - r^2(\mathbf{J}^2+2))\Phi_2], \tag{88}$$

$$\xi = (\mathbf{J}^2 + 2f - r\tilde{D}_a\tilde{D}^a r)(\beta_1\Phi_1 - \beta_2\Phi_2) \tag{89}$$

and that the LEME (63), (64) reduce to the decoupled Zerilli equations (83). If we replace $\mathbf{J}^2 \to -\ell(\ell+1)$ and use (t, r) coordinates on the orbit space, $[-f\tilde{D}^a\tilde{D}_a]\Phi$ reads $\partial_t^2\Phi + f\partial_r(f\partial_r\Phi)$ and (83) gives the Zerilli equation for the $\Phi_n^{(\ell,m)}$ harmonic components of Φ_n, $n = 1, 2$, as found for $\Lambda = 0$ by Zerilli in [9] and Moncrief in [12] and for $\Lambda \neq 0$ in [5].

III. NONMODAL LINEAR STABILITY FOR EVEN PERTURBATIONS

From the results of the previous section follows that the set $\mathcal{L}_+$ of equivalent classes $[(h_{\alpha\beta}, F_{\alpha\beta})]$ of even solutions $(h_{\alpha\beta}, \mathcal{F}_{\alpha\beta})$ of the LEME mod the Maxwell and the diffeomorphism gauge equivalence relation (26) can be parametrized by the first order variation δM and δQ of the mass M and charge Q ($\ell = 0$ modes), the $\ell = 1$ field $\varphi = \sum_m \varphi_m S^{(\ell=1,m)}$, $\varphi_m\colon \mathcal{N} \to \mathbb{R}$, $m = 1, 2, 3$ satisfying (69), and the Zerilli fields $\Phi_n\colon \sum_{(\ell\geq 2,m)} \Phi_n^{(\ell,m)} S^{(\ell,m)}\colon \mathcal{M} \to \mathbb{R}$, $n = 1, 2$ (alternatively $\Phi_n^{(\ell,m)}\colon \mathcal{N} \to \mathbb{R}$) obeying (83) ($\ell \geq 2$ modes):

$$\mathcal{L}_+ = \{[(h_{\alpha\beta}, F_{\alpha\beta})]|(h_{\alpha\beta}, F_{\alpha\beta}) \text{ is a solution of the LEME}\} = \{(\delta M, \delta Q, \varphi, \Phi_n)|\varphi \text{ satisfies (69) and } \Phi_n,\ n = 1,\ 2 \text{ satisfy (83)}\}. \tag{90}$$

Although these fields and constants measure the effects of the perturbation, there is a distinction between the $\ell = 0$ constants δM and δQ, which have a clear physical meaning as mass and charge shifts within the Kerr-Newman (A)dS family, and the $\ell \geq 1$ fields φ, Φ_1 and Φ_2. The latter are convenient to disentangle the $\ell \geq 1$ LEME but have, *a priori*, no direct physical interpretation. In the following section we will find scalar fields that substitute these and have a direct geometrical meaning.

A. Measurable effects of the perturbations

There are 16 real algebraically independent basic sets of scalars made out of the Riemann tensor in the Carminati-McLenaghan [17] basis. Any other scalar field made out of contractions of the tensor product of any number of Riemann tensors, volume form, and metric tensor can be written as a polynomial on these basic scalars. Among these there are six real fields (we follow the notation in [17]):

$$\{R, r_1, r_2, r_3, m_3, m_4\} \tag{91}$$

and the five complex fields

$$\{w_1, w_2, m_1, m_2, m_5\}. \tag{92}$$

In the electro-vacuum case, they are constrained by the following (seven real) syzygies [17]:

$$R = 0, \qquad r_2 = 0, \qquad 4r_3 - r_1^2, \qquad m_4 = 0,$$
$$m_1\bar{m}_2 - r_1\bar{m}_5 = 0, \qquad m_2\bar{m}_2 m_3 - r_1 m_5\bar{m}_5 = 0, \tag{93}$$

which leave r_1, w_1, w_2, m_1, m_2 as independent fields in the general electro-vacuum case. Note that these constraints do not define a manifold but an algebraic variety: the dimension of the tangent space (defined by the linearization of the constraints) may change at different points.

We may also consider invariants involving the Maxwell fields, as well as mixed invariants such as ($C_{\alpha\beta\gamma\delta}$ the Weyl tensor)

$$F = F_{\alpha\beta}F^{\alpha\beta}, \qquad C = C_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta}. \tag{94}$$

Due to the symmetries of the background, it can be proved that the imaginary part of first order variations of the complex scalars $\delta w_1, \ldots, \delta m_5$ vanish trivially under even perturbations, so we will focus our attention on the first order variations

$$\delta r_1, \qquad \delta\Re w_1 = \delta w_1, \qquad \delta\Re w_2 = \delta w_2,$$
$$\delta\Re m_1 = \delta m_1, \qquad \delta\Re m_2 = \delta m_2, \quad \delta F, \quad \delta C. \tag{95}$$

Note that [17]

$$w_1 = \Re\omega_1 = \frac{1}{8}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}. \tag{96}$$

Since the background values of these fields

$$r_{1o} = \frac{Q^2}{r^8}, \quad w_{1o} = \frac{6(Q^2 - Mr)^2}{r^8}, \quad w_{2o} = \frac{6(Q^2 - Mr)^3}{r^{12}},$$
$$F_o = -\frac{2Q^2}{r^4}, \qquad m_{1o} = \frac{2Q^4(Q^2 - Mr)}{r^{12}},$$
$$m_{2o} = \frac{4Q^4(Q^2 - Mr)^2}{r^{16}}, \qquad C_o = \frac{8Q^2(Q^2 - Mr)}{r^8} \tag{97}$$

do not vanish, none of the fields in (95) is gauge invariant. However, it is possible to construct gauge invariant fields out of them, such as

$$\mathcal{S} := w_1{}'_o\delta C - C'_o\delta w_1, \tag{98}$$

etc., where the prime denotes derivative with respect to r. Under a gauge transformation along X^α

$$\mathcal{S} \to \mathcal{S} + w_1{}'_o X^\alpha\partial_\alpha C_o - C'_o X^\alpha\partial_\alpha w_{1o} = \mathcal{S}, \tag{99}$$

since w_{1o} and C_o (as every curvature scalar) depend only on r. This idea generalizes as follows: Let $I_{(1)}, \ldots, I_{(s)}$ be a set of scalar curvature. Then $S = \sum_k f_k\delta I_{(k)}$ is gauge invariant as long as the f_k satisfy $\sum_k f_k I_{(k)}{}'_o = 0$, since for a gauge transformation along X^α, $X^\alpha, \delta I_{(j)} = X^r I_{(J)}{}'_o$ and $\delta S = 0$. For $s = 2$ this reduces to

$$S = K(I_{(1)}{}'_o\delta I_{(2)} - I_{(2)}{}'_o\delta I_{(1)}). \tag{100}$$

When calculating the $\ell \geq 2$ projection of the first order even perturbation of the fields (95) *in the RW gauge* (45) in terms of the Zerilli fields, we get expressions involving up to five derivatives of the Φ_n. On shell, that is, assuming the LEME, we can use the Zerilli equation (83) and its r derivatives repeatedly and, after lengthy manipulations, obtain simpler expressions involving only the Φ_n and their first r derivatives. We proved that, on shell, all the gauge invariant combinations of the first order variation of the fields (95) are proportional to each other. In other words, there is a single independent gauge invariant combination of first order variation of curvature scalars. This certainly could not carry the same information as the *two* fields Φ_n,

$n = 1, 2$. Since all the algebraic gauge invariant curvature variation scalars are proportional on shell, it is irrelevant to our purposes of a nonmodal approach which one we choose. For the field $\mathcal{S}$ in (98) we found, after lengthy calculations with the help of symbolic manipulation programs,

$$\mathcal{S}^{(\ell>1)} = \frac{96Q^2\mathbf{J}^2(\mathbf{J}^2+2)(Mr-Q^2)^2}{r^{17}} \times ((r\beta_2 - 4Q^2)\Phi_1 - (r\beta_1 - 4Q^2)\Phi_2), \quad (101)$$

$$\mathcal{S}^{(\ell=1)} = \frac{384Q(3Mr-2Q^2)(Mr-Q^2)^2}{r^{17}}\varphi, \quad (102)$$

and

$$\begin{aligned}\mathcal{S}^{(\ell=0)} &= \delta M({w_1}'_o \partial_M C_o - C'_o \partial_M w_{1o}) \\ &\quad + \delta Q({w_1}'_o \partial_Q C_o - C'_o \partial_Q w_{1o}) \\ &= \frac{192Q(Q^2-Mr)^2}{r^{16}}(3M\delta Q - 2Q\delta M). \end{aligned} \quad (103)$$

It is an interesting fact that first order r derivatives of the Φ_n, which are present in δw_1 and δC, cancel out in (101). Note also that Eq. (102) gives a geometrical interpretation for the gauge invariant $\ell = 1$ field φ.

To construct a second curvature related gauge invariant field that, together with (98), allows us to recover the Zerilli fields, we need to consider differential invariants. These will give (at least) one more derivative of the Zerilli fields. When simplifying their on shell form we find that first order derivatives do not cancel out (at least, in the many examples that we have worked out).

The field we chose is constructed as follows: define

$$I = \frac{1}{720}(\nabla_\alpha C_{\beta\gamma\delta\tau})(\nabla^\alpha C^{\beta\gamma\delta\tau}), \qquad J = (\nabla_\alpha F_{\beta\delta})(\nabla^\alpha F^{\beta\delta}), \quad (104)$$

whose background values are

$$\begin{aligned} I_o &= \frac{f(r)}{15r^{10}}(15M^2r^2 - 36MQ^2r + 22Q^4), \\ J_o &= -\frac{12Q^2 f(r)}{r^6}, \end{aligned} \quad (105)$$

then the gauge invariant field

$$\mathcal{T} = I'_o \delta J - J'_o \delta I \quad (106)$$

has an on shell expression with

$$\mathcal{T}^{(\ell>1)} = \Upsilon_1 \Phi'_1 + \Upsilon_2 \Phi'_2 + \Omega_1 \Phi_1 + \Omega_2 \Phi_2. \quad (107)$$

The operators Υ_n and Ω_n in (107) do not admit a simple expression. In any case, all we need know about them is that, for $\Lambda > 0$ and $r_h \leq r \leq r_c$ they are bounded, whereas for $\Lambda = 0$ and $r \leq r_h$ they are bounded and, as $r \to \infty$, behave as

$$\begin{aligned} \Upsilon_n &= 12MQ^2r^{-14}\mathbf{J}^2(\mathbf{J}^2+2)\Big[(-1)^n 5M \\ &\quad - \sqrt{9M^2 - 4Q^2(\mathbf{J}^2+2)}\Big] + \mathcal{O}(r^{-15}), \\ \Omega_n &= 12MQ^2r^{-15}\mathbf{J}^2(\mathbf{J}^2+2)\Big[(-1)^{n+1} 7M \\ &\quad + \sqrt{9M^2 - 4Q^2(\mathbf{J}^2+2)}\Big] + \mathcal{O}(r^{-16}). \end{aligned} \quad (108)$$

For $\mathcal{T}^{(\ell=1)}$ we find

$$\mathcal{T}^{(\ell=1)} = \frac{16Qf(r)}{15r^{18}}C(r)\partial_r\varphi + \frac{Qf(r)}{15r^{19}(2Q^2-3Mr)}D(r)\varphi, \quad (109)$$

with

$$\begin{aligned} C(r) &= 45M^2\Lambda r^6 - 110MQ^2\Lambda r^5 + (68Q^4\Lambda - 180M^2)r^4 \\ &\quad + 9M(45M^2+46Q^2)r^3 - 3Q^2(379M^2+80Q^2)r^2 \\ &\quad + 1014MQ^4r - 276Q^6, \end{aligned} \quad (110)$$

$$\begin{aligned} D(r) &= 135M^3\Lambda r^7 - 387M^2Q^2\Lambda r^6 - 2M(-193Q^4\Lambda \\ &\quad + 270M^2)r^5 + (-140Q^6\Lambda + 1215M^4 \\ &\quad + 1431M^2Q^2)r^4 - 81MQ^2(47M^2+16Q^2)r^3 \\ &\quad + 3Q^4(1477M^2+144Q^2)r^2 \\ &\quad - 2310MQ^6r + 444Q^8. \end{aligned} \quad (111)$$

For the $\ell = 0$ piece of $\mathcal{T}$:

$$\begin{aligned} \mathcal{T}^{(\ell=0)} &= \delta M(I'_o\partial_M J_o - J'_o\partial_M I_o) + \delta Q(I'_o\partial_Q J_o - J'_o\partial_Q I_o) \\ &= \frac{16f(r)Q}{5r^{18}}(Q\mathcal{T}_M\delta M - \mathcal{T}_Q\delta Q). \end{aligned} \quad (112)$$

Here

$$\begin{aligned} \mathcal{T}_Q &= 15M^2\Lambda r^5 - 18MQ^2\Lambda r^4 - 60M^2r^3 \\ &\quad + 27M(5M^2+2Q^2)r^2 + 22Q^2(Q^2-9M^2)r \\ &\quad + 42MQ^4 \end{aligned} \quad (113)$$

and

$$\begin{aligned} \mathcal{T}_M &= 10M\Lambda r^5 - 12Q^2\Lambda r^4 - 45Mr^3 + (90M^2+54Q^2)r^2 \\ &\quad - 132MQ^2r + 28Q^4. \end{aligned} \quad (114)$$

Equations (101)–(103) and (107)–(112) allow us to prove that $\mathcal{S}$ and $\mathcal{T}$ contain all the gauge invariant information of a given perturbation.

Theorem 1. Consider the set of gauge classes of even solutions $[(h_{\alpha\beta}, \mathcal{F}_{\alpha\beta})]$ of the LEME around a Reissner-Nordström (A)dS black hole background, and the perturbed fields $\mathcal{S}$ and $\mathcal{T}$ defined above. The map $[(h_{\alpha\beta}, \mathcal{F}_{\alpha\beta})] \to (\mathcal{S}, \mathcal{T})$ is injective: it is possible to reconstruct a representative of $[h_{\alpha\beta}]$ and $[\mathcal{F}_{\alpha\beta}]$ from $(\mathcal{S}, \mathcal{T})$.

Proof. Assume $\mathcal{S} = 0 = \mathcal{T}$, then Eqs. (103) and (112)–(114) imply $\delta M = 0 = \delta Q$. Equation (102) implies $\varphi = 0$, and the combination of (101) and (107) gives $\Phi_1 = 0 = \Phi_2$. This last assertion follows from a reasoning on the line of the proof of Theorem 5 in [2]: from $\mathcal{S}^{(\ell>1)} = 0$ and (101) we may write Φ_2 in terms of Φ_1 which, inserted in the equation $\mathcal{T}^{>1} = 0$ using (107), gives an equation for Φ_1 whose only solution compatible with (83) is the trivial one. Thus, an electro-gravitational perturbation must be trivial if $\mathcal{S} = 0 = \mathcal{T}$.

To reconstruct the perturbation from $\mathcal{S}$ and $\mathcal{T}$ we proceed as in Theorem 1.i in [3]. ■

The fact that the $\ell = 0$ d.o.f. are δQ and δM explains why these quantities can be obtained from $\mathcal{S}_0$ and $\mathcal{T}_0$ by inverting (103) and (112). In [18], a characterization of subclasses of type-D spacetimes is made in terms of equations involving curvature tensors and scalars. In particular, two curvature scalars are given such that, when evaluated on a Reissner-Nordström spacetime, they give the mass and charge (see Theorem 5). Since these scalar fields are constant on Reissner-Nordström backgrounds, their first order perturbations are gauge invariant. For $\ell = 0$ perturbations they agree exactly with δM and δQ, because these are perturbations along the Reissner-Nordström family. The scalar fields in [18] are made out of rational functions of rational powers of the basic polynomial invariants (91), (92) and (104). To illustrate the relation between these (*a priori*) more general scalar gauge invariants and the ones we constructed above, we consider the case of scalar perturbations of an uncharged (Schwarzschild) black hole. In this case we get from (97) and (105)

$$M^2 = \frac{9}{2} \frac{{w_1}_o^4}{\Lambda {w_1}_o + \sqrt{6}{w_1}_o^{3/2} + 3I_o}. \tag{115}$$

Thus, for

$$Z = \frac{9}{2} \frac{{w_1}^4}{\Lambda w_1 + \sqrt{6}{w_1}^{3/2} + 3I}, \tag{116}$$

δZ is gauge invariant, and so is

$$\begin{aligned} -r^{-5}\delta Z &= (9M - 4r + \lambda r^3)\frac{\delta\omega_1}{6} + 3r^3\delta I \\ &= \frac{r^{10}}{12M^2}(I_o'\delta w_1 - {w_1}_o'\delta I), \end{aligned} \tag{117}$$

which is of the form (100) and agrees with the gauge invariant field G_+ used in the analysis of the even Schwarzschild perturbations in [2] [Eq. (202)].

B. Pointwise boundedness of Φ_n and $\partial_{r^*}\Phi_n$

In this section we restrict our attention to black hole solutions with horizons $0 < r_i < r_h < r_c$. The relation between the radii of the horizons and Λ, M, and Q can be easily obtained from (7) and (8). For $\Lambda > 0$ $[\Lambda = 0]$ we are interested in the range $r_h < r < r_c$ $[r > r_h]$. In both cases the tortoise radial coordinate satisfies $-\infty < r^* < \infty$.

Theorem 2. Assume Φ_n is a smooth solution of Eq. (86) on the union of regions II, II', III, and III' of the extended Reissner-Nordström (Fig. 1) or Reissner-Nordström de Sitter (Fig. 2) spacetimes, with compact support on Cauchy surfaces. There exist constants C_o, L_o that depend on the datum of this field at a Cauchy surface, such that $|\Phi_n| < C_o$ and $|\partial_{r^*}\Phi_n| < L_o$ for all points in the outer static region III.

Proof. As in the proof of Theorem 2 in [3], and following [19], we may restrict our attention, without loss of generality, to fields that vanish on the bifurcation sphere together with their Kruskal time derivatives (for details, see [19]).

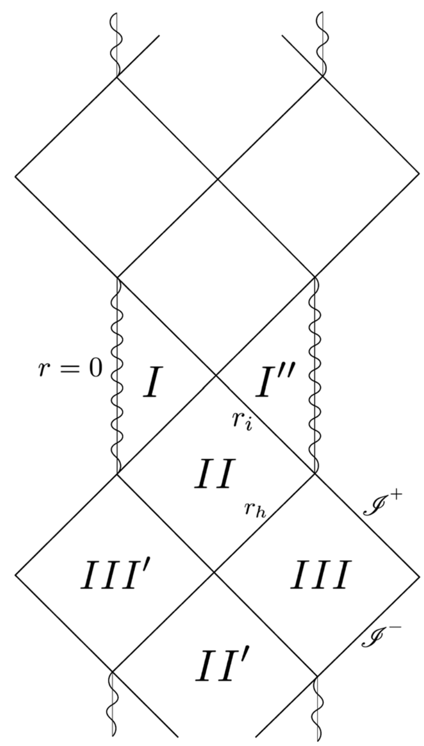

FIG. 1. The Carter-Penrose diagram of (part of) the maximal analytic extension of the $|Q| < M$ Reissner-Nordström black hole. The union of II, II', III, and III' is globally hyperbolic; its boundary at r_i is a Cauchy horizon.

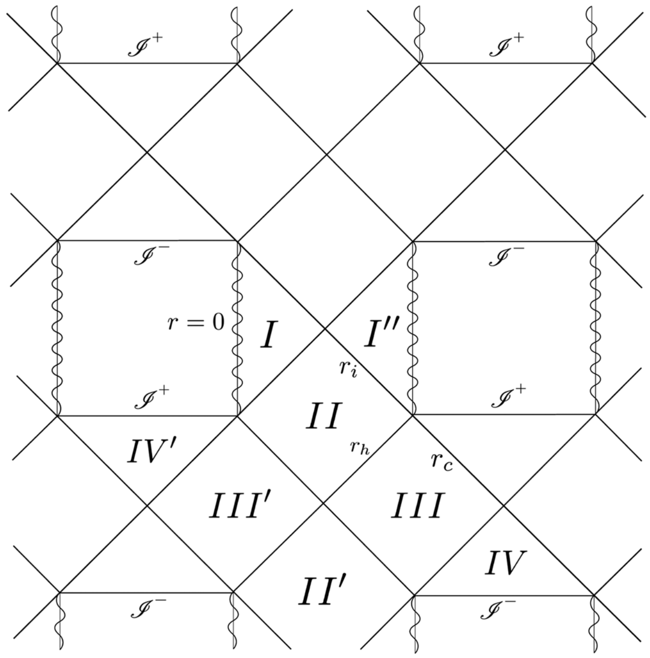

FIG. 2. The Carter-Penrose diagram of (part of) the maximal analytic extension of a nonextremal (three different horizons) Reissner-Nordström de Sitter black hole.

On a t slice of region III, define the L^2 norm of a real field G as

$$\|G\|^2 = \langle G|G\rangle = \int_{\mathbb{R}\times S^2} G^2 dr^* \sin(\theta) d\theta d\phi, \qquad dr^* = \frac{dr}{f}. \tag{118}$$

Using a Sobolev type inequality (Eq. (5.27) in [20]) on the Zerilli fields Φ_n at a fixed time t gives

$$|\Phi_n(t,r^*,\theta,\phi)| \le C(\|\Phi_n|_t\| + \|\partial^2_{r^*}\Phi_n|_t\| + \|\mathbf{J}^2\Phi_n|_t\|), \tag{119}$$

where C is a constant. We will follow the strategy in [19] of proving that the L^2 norms on the right-hand side of (119) can be bounded by the energies of related field configurations. Since energy is conserved for solutions of (86), we get in this way a t-independent upper bound of the right side of (119) and therefore, a global bound of $|\Phi_n(t,r^*,\theta,\phi)|$ for all (t,r^*,θ,ϕ), i.e., of Φ_n in the outer static region III.

The inner product defined by the norm (118), simplifies, after introducing an expansion in real orthonormal spherical harmonics (e.g., tesseral spherical harmonics) $S_{(\ell,m)}$. If $G=\sum_{(\ell,m)} g_{(\ell,m)}S_{(\ell,m)}$ and $K=\sum_{(\ell,m)} k_{(\ell,m)}S_{(\ell,m)}$ then

$$\langle G|K\rangle = \sum_{(\ell,m)} \int_{\mathbb{R}} g_{(\ell,m)} k_{(\ell,m)} dr^*. \tag{120}$$

From (87) we get

$$\|\partial^2_{r^*}\Phi_n|_t\| \le \|A_n\Phi_n|_t\| + \|V_n\Phi_n|_t\| \tag{121}$$

where V_n, given in (83), can be written as

$$V_n = \frac{{}_nZ_1}{D_n}\beta_n + \frac{{}_nZ_2}{D_n^2}\beta_n^2 + \frac{{}_nZ_3}{D_n}\mathbf{J}^2(\mathbf{J}^2+2) \tag{122}$$

being

$$\begin{aligned} {}_nZ_1 &= f(f/r^2)' = -\frac{2f}{r^4}\left(r - 3M + \frac{2Q^2}{r}\right), \\ {}_nZ_2 &= \frac{2f^2}{r^4}, \qquad {}_nZ_3 = \frac{f}{r^2}, \qquad D_n = \beta_n/r - (\mathbf{J}^2+2). \end{aligned} \tag{123}$$

Note the following:

(i) The ${}_nZ_j$, $j=1, 2, 3$, depend only on r and are bounded in the domain of interest $r_h \le r \le r_c$ if $\Lambda > 0$ [$r > r_h$ if $\Lambda = 0$] by constants ${}_nz_j > |{}_nZ_j(r)|$ that depend on M, Q and Λ.

(ii) $\Phi_n^{(1)} := \beta_n\Phi_n$, $\Phi_n^{(2)} := \beta_n^2\Phi_n$ and $\Phi_n^{(3)} := \mathbf{J}^2(\mathbf{J}^2+2)\Phi_n$ are solutions of the Zerilli equation (83) if Φ_n is a solution; also $\Phi_n^{(4)} := A_n\Phi_n = -\partial_t^2\Phi_n$ is a solution. This is so because any operator that is a function of $\mathbf{J}^2$ and ∂_t, commutes with the operator (83).

(iii) On the ℓ eigenspace of $\mathbf{J}^2$, $\ell = 2,3,4,\ldots$, D_n acts multiplicatively as

$$D_n^\ell(r) = \beta_n/r + (\ell-1)(\ell+2). \tag{124}$$

For $r_h \le r \le r_c$ [$r > r_h$ if $\Lambda = 0$] and $\ell \ge 2$, $|D_2^\ell(r)|$ has an absolute minimum $D_2^* > 0$ at $r = r_h$ and $\ell = 2$, whereas $|D_1^\ell(r)|$ also has a nonzero absolute minimum D_1^* (possibly at an $\ell > 2$). Then the first term of $V_n\Phi_n$ [see (122)] can be bounded as follows:

$$\begin{aligned} \left\|\frac{{}_nZ_1}{D_n}\beta_n\Phi_n|_t\right\|^2 &= \int_{\mathbb{R}\times S^2}\left[\sum\nolimits_{\ell m}\frac{{}_nZ_1}{D_n}(\Phi_n^{(1)})^{(\ell,m)}|_t S_{(\ell,m)}\right]^2 dr^* \\ &\quad\times \sin(\theta)d\theta d\phi \end{aligned} \tag{125}$$

$$\le \left(\frac{{}_nz_1}{D_n^*}\right)^2 \int_{\mathbb{R}}\sum_{\ell m}[(\Phi_n^{(1)})^{(\ell,m)}|_t]^2 dr^* \tag{126}$$

$$= \left(\frac{{}_nz_1}{D_n^*}\right)^2 \|\Phi_n^{(1)}|_t\|^2. \tag{127}$$

Proceeding similarly with the other terms in (121)–(122) and using the triangle inequality gives

$$\|\partial_{r^*}^2\Phi_n|_t\| \leq \left(\frac{{}_nz_1}{D_n^*}\right)\|\Phi_n^{(1)}|_t\| + \left(\frac{{}_nz_2}{D_n^*}\right)\|\Phi_n^{(2)}|_t\| + \left(\frac{{}_nz_3}{D_n^*}\right)\|\Phi_n^{(3)}|_t\| + \|\Phi_n^{(4)}|_t\|. \quad (128)$$

Inserting this in (135) gives

$$|\Phi_n(t,r^*,\theta,\phi)| \leq K'(\|\Phi_n|_t\| + \|\Phi_n^{(1)}|_t\| + \|\Phi_n^{(2)}|_t\| + \|\Phi_n^{(3)}|_t\| + \|\Phi_n^{(4)}|_t\| + \|\Phi_n^{(5)}|_t\|) \quad (129)$$

where K' is the maximum over j and n of the constants ${}_nz_jK/D_n^*$, and $\Phi_n^{(5)} = \mathbf{J}^2\Phi_n$.

The conserved (i.e., t-independent) energy associated with Eq. (86) is

$$E = \frac{1}{2}\int_{\mathbb{R}\times S^2}((\partial_t\Phi_n)^2 + \Phi_n A_n\Phi_n)dr^*\sin(\theta)d\theta d\phi. \quad (130)$$

Since E does not depend on t, we may regard it as a functional on the initial datum: $E = E(\Phi_n^o, \dot{\Phi}_n^o)$, where $\Phi_n^o = \Phi_n|_{t_o}$ and $\dot{\Phi}_n^o = (\partial_t\Phi_n)|_{t_o}$:

$$E(\Phi_n^o, \dot{\Phi}_n^o) = \frac{1}{2}\int_{\mathbb{R}\times S^2}((\dot{\Phi}_n^o)^2 + \Phi_n^o A_n\Phi_n^o)dr^*\sin(\theta)d\theta d\phi. \quad (131)$$

Using the facts that (i) A_n, $n = 1, 2$, is positive definite in the cases we are interested in (proved by means of an S-deformation in [5]) and so $A_n^{\pm 1/2}$ can be defined by means of the spectral theorem (as well as any other power of A_n); (ii) for a solution Φ_n of (86), $A_n^p\Phi_n$ is a solution of (86); and (iii) Eq. (131), follows that for a solution of (86)

$$\|\Phi_n|_t\|^2 \leq 2E(A_n^{-1/2}\Phi_n^o, A_n^{-1/2}\dot{\Phi}_n^o). \quad (132)$$

We now use the fact that applying to a Cauchy datum $(\Phi_n^o, \dot{\Phi}_n^o)$ an operator that is a function of $\mathbf{J}^2$ or A_n commutes with time evolution. This allows us to estimate each term on the right-hand side of (129) with the energy of field configurations related to the one with initial datum $(\Phi_n^o, \dot{\Phi}_n^o)$. Let $B^{(j)}\Phi_n \coloneqq \Phi_n^{(j)}$, $j = 1, \ldots, 5$ [that is, for $j = 1, \ldots, 5$ these operator are respectively β_n, β_n^2, $\mathbf{J}^2(\mathbf{J}^2+2)$, A_n and $\mathbf{J}^2$]. From (132)

$$\|\Phi_n^{(j)}|_t\|^2 \leq 2E\left(A_n^{-\frac{1}{2}}B^{(j)}\Phi_n^o, A_n^{-\frac{1}{2}}B^{(j)}\dot{\Phi}_n^o\right). \quad (133)$$

Thus, we can replace the right-hand side of (129) by time independent constant C_o made out of the initial data $(\Phi_n^o, \dot{\Phi}_n^o)$ (from which the energies of the related fields $B^{(j)}\Phi_n$ can be computed)

$$|\Phi_n| < C_o. \quad (134)$$

It is interesting to note why the fields $B^{(j)}\Phi_n$ have finite energy (a fact tacitly used above): we are assuming smooth solutions of the LEME; therefore the Φ_n are C^∞ on the sphere, and the series $\sum_{\ell m}[\ell(\ell+1)]^k\Phi_n^{(\ell,m)} = (-\mathbf{J}^2)^k\Phi_n$ converge for any k. In particular, the $\Phi_n^{(\ell,m)}$ decay faster than any power of ℓ.

We will also need a t-independent bound for $|\partial_{r^*}\Phi_n|$. This can be obtained following the same ideas in [2], taken from [21]. Starting from the Sobolev inequality [cf. Eq. (135)] applied to $|\partial_r^*\Phi_n|$

$$|\partial_{r^*}\Phi_n(t,r^*,\theta,\phi)| \leq L(\|\partial_{r^*}\Phi_n|_t\| + \|\partial_{r^*}^3\Phi_n|_t\| + \|\mathbf{J}^2\partial_{r^*}\Phi_n|_t\|). \quad (135)$$

Now, using the fact that, for $\Lambda \geq 0$ and four dimensions, the V_n are non-negative in the interval of interest [5]

$$\|\partial_{r^*}\Phi_n|_t\|^2 \leq \langle\Phi_n|A_n\Phi_n\rangle \leq 2E(\Phi_n^o, \dot{\Phi}_n^o). \quad (136)$$

This places a t-independent bound on the first term on the right of Eq. (135). The third term can be similarly bounded with $E(\mathbf{J}^2\Phi_n^o, \mathbf{J}^2\dot{\Phi}_n^o)$. For the second term we use

$$\partial_{r^*}^3\Phi_n = -\partial_{r^*}(A_n\Phi_n) + \partial_{r^*}V_n\Phi_n + V_n\partial_{r^*}\Phi_n, \quad (137)$$

$\|\partial_{r^*}(A_n\Phi_n)\|^2 \leq 2E(A_n\Phi_n^o, \dot{A}_n\Phi_n^o)$ and the boundedness of the operator $\partial_{r^*}V_n$.

Proceeding as above, we arrive at the desired pointwise bound:

$$|\partial_{r^*}\Phi_n| < L_o. \quad (138)$$

■

C. Pointwise boundedness of $\mathcal{S}$ and $\mathcal{T}$

We can now proceed to complete the proof of nonmodal stability by showing that the scalar fields $\mathcal{S}$ and $\mathcal{T}$ are pointwise bounded in the region of interest by constants that depend on the initial conditions.

Theorem 3. Under the assumptions of the Theorem 2, in the outer static region III of a $\Lambda \geq 0$ Reissner-Nordström black hole there holds

$$\mathcal{S} < \frac{A_o}{r^{14}}, \qquad \mathcal{T} < \frac{B_o}{r^{14}}, \quad (139)$$

where A_o and B_o are constants that depend on the Cauchy datum $(\Phi_n^o, \dot{\Phi}_n^o)$ of the perturbation.

Proof. Let us consider the first inequality. Using the facts that $\mathbf{J}^2(\mathbf{J}^2+2)\Phi_n$ and $\mathbf{J}^2(\mathbf{J}^2+2)\beta_k\Phi_n$ are solutions of the Zerilli equation (83) with an energy that is a function of $(\Phi_n^o, \dot{\Phi}_n^o)$, Theorem 2 and Eq. (101), we find for $\Lambda = 0$ that $|\mathcal{S}^{(\ell>1)}| < \frac{A_o^{\ell>1}}{r^{14}}$ with $A_o^{\ell>1}$ a constant that depends on the $\ell > 1$ piece of the initial datum (the inequality holds trivially for $\Lambda > 0$ and $r_h < r < r_c$). For the $\ell = 1$ piece we use Eq. (19) in the erratum in [14], applied to the harmonic components of φ. This gives $|\varphi|$ less than a constant that depends on the $\ell = 1$ piece of the datum. Then, from (102) follows $|\mathcal{S}^{(\ell=1)}| < \frac{A_o^{(\ell=1)}}{r^{14}}$ with $A_o^{(\ell=1)}$ a constant that depends on the $\ell = 1$ piece of the initial datum (once again, the equality holds trivially for $\Lambda > 0$ and $r_h < r < r_c$). Finally, from Eq. (103) follows trivially that $|\mathcal{S}^{(\ell=0)}| < \frac{A_o^{(\ell=0)}}{r^{14}}$ where $A_o^{(\ell=0)}$ is a constant made related to the $\ell = 0$ initial data $(\delta M, \delta Q)$.

To prove the second inequality in (139) we proceed exactly as above. We only need a proof of the pointwise boundedness for $f(r)\partial_r\varphi$, for which we proceed as in [21] [see the paragraph starting at Eq. (83)]. ■

IV. DISCUSSION

We have shown in Theorem 1 that the gauge invariant curvature related perturbation fields $\mathcal{S}$ and $\mathcal{T}$, defined in Eqs. (98) and (106), contain all the gauge invariant information of an even perturbation class $[(h_{\alpha\beta}, \mathcal{F}_{\mu\nu})]$ around a Reisner-Nordström (dS) black hole. From these fields, a representative $(h_{\alpha\beta}, \mathcal{F}_{\mu\nu})$ of the perturbation in, say, the Regge-Wheeler gauge, can be reconstructed (Theorem 1). For smooth perturbations with compact support on a Cauchy surface of (a copy of) the union of regions II, II', IIII, and III' (see Figs. 1 and 2), these fields are pointwise bounded on the outer region [Eq. (139) in Theorem 3]. These results, together with those in [3], complete the proof of nonmodal linear stability of the outer region of a (dS) Reissner-Nordström black hole.

The large $|t|$ decay of the Zerilli fields (see [22,23] and the recent decay results by Georgi in [24] and references therein) and the similarly expected behavior of φ, together with Eqs. (101)–(103) and (107)–(112) give

$$\mathcal{S} \simeq \frac{192Q(Q^2 - Mr)^2}{r^{16}}(3M\,\delta Q - 2Q\,\delta M), \tag{140}$$

and

$$\mathcal{T} \simeq \frac{16f(r)Q}{5r^{18}}(Q\mathcal{T}_M\,\delta M - \mathcal{T}_Q\,\delta Q) \tag{141}$$

as $t \to \infty$, within a bounded range of r (that grows toward the future) in region III [the quantities $\mathcal{T}_M$ and $\mathcal{T}_Q$ were defined in Eqs. (112)–(114)]. The inequalities (139), instead, hold on the entire region III.

Together with Eqs. (122) and (123) in [3], Eqs. (140) and (141) indicate that, for large t, the perturbed black hole settles into a Kerr-Newman black hole with parameters $M + \delta M$, $Q + \delta Q$ and $\vec{J} + \delta\vec{J}$.

The importance of the result in Theorem 2 lies in the possibility of analyzing stability and instability effects in terms of the fields $\mathcal{S}$, $\mathcal{T}$ (and $\mathcal{Q}$ and $\mathcal{F}$ in [3]). The divergence of $\frac{d}{d\tau}\mathcal{S}$ and $\frac{d}{d\tau}\mathcal{T}$ for observers crossing the Cauchy horizon r_i can be proved in the same way the divergence of $\frac{d}{d\tau}\mathcal{Q}$ and $\frac{d}{d\tau}\mathcal{F}$ was proved for the odd sector scalars in Sec. IV in [3]. Using these four fields, statements such as the Cauchy horizon instability or the event horizon transverse derivative instabilities [25–32] acquire a clear geometrical meaning.

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[1] G. Dotti, Nonmodal Linear Stability of the Schwarzschild Black Hole, Phys. Rev. Lett. **112**, 191101 (2014).

[2] G. Dotti, Black hole nonmodal linear stability: The Schwarzschild (A)dS cases, Classical Quantum Gravity **33**, 205005 (2016).

[3] J. M. Fernández Tío and G. Dotti, Black hole nonmodal linear stability under odd perturbations: The Reissner-Nordstrm case, Phys. Rev. D **95**, 124041 (2017).

[4] B. Araneda and G. Dotti, Instability of asymptotically anti de Sitter black holes under Robin conditions at the timelike boundary, Phys. Rev. D **96**, 104020 (2017).

[5] H. Kodama and A. Ishibashi, Master equations for perturbations of generalized static black holes with charge in higher dimensions, Prog. Theor. Phys. **111**, 29 (2004).

[6] E. Chaverra, N. Ortiz, and O. Sarbach, Linear perturbations of self-gravitating spherically symmetric configurations, Phys. Rev. D **87**, 044015 (2013).

[7] T. Regge and J. A. Wheeler, Stability of a Schwarzschild singularity, Phys. Rev. **108**, 1063 (1957).
[8] F. J. Zerilli, Effective Potential for Even Parity Regge-Wheeler Gravitational Perturbation Equations, Phys. Rev. Lett. **24**, 737 (1970).
[9] F. J. Zerilli, Perturbation analysis for gravitational and electromagnetic radiation in a Reissner-Nordström geometry, Phys. Rev. D **9**, 860 (1974).
[10] A. Ishibashi and H. Kodama, Perturbations and stability of static black holes in higher dimensions, Prog. Theor. Phys. Suppl. **189**, 165 (2011).
[11] V. Moncrief, Odd-parity stability of a Reissner-Nordstrom black hole, Phys. Rev. D **9**, 2707 (1974).
[12] V. Moncrief, Stability of Reissner-Nordstrom black holes, Phys. Rev. D **10**, 1057 (1974).
[13] V. Moncrief, Gauge-invariant perturbations of Reissner-Nordstrom black holes, Phys. Rev. D **12**, 1526 (1975).
[14] R. Wald, Note on the stability of the Schwarzschild metric, J. Math. Phys. (N.Y.) **20**, 1056 (1979); Erratum, J. Math. Phys. (N.Y.)**21**, 218(E) (1980).
[15] A. Ishibashi and R. M. Wald, Dynamics in nonglobally hyperbolic static space-times. 3. Anti-de Sitter space-time, Classical Quantum Gravity **21**, 2981 (2004).
[16] This odd perturbation was disregarded in the companion paper [3]. It produces no backreaction *at first order*: Q^2 in (7) has to be replaced with $Q^2 + p^2$.
[17] J. Carminati and R. G. McLenaghan, Algebraic invariants of the Riemann tensor in a fourdimensional Lorentzian space, J. Math. Phys. (N.Y.) **32**, 3135 (1991).
[18] J. J. Ferrando and J. A. Saez, On the classification of type D space-times, J. Math. Phys. (N.Y.) **45**, 652 (2004).
[19] B. S. Kay and R. M. Wald, Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation two sphere, Classical Quantum Gravity **4**, 893 (1987).
[20] J. Dimock and B. S. Kay, Classical and quantum scattering theory for linear scalar fields on the Schwarzschild metric I, Ann. Phys. (NY) **175**, 366 (1987).
[21] S. Dain and G. Dotti, The wave equation on the extreme Reissner-Nordstróm black hole, Classical Quantum Gravity **30**, 055011 (2013).
[22] R. H. Price, Nonspherical perturbations of relativistic gravitational collapse. 1. Scalar and gravitational perturbations, Phys. Rev. D **5**, 2419 (1972).
[23] P. R. Brady, C. M. Chambers, W. Krivan, and P. Laguna, Telling tails in the presence of a cosmological constant, Phys. Rev. D **55**, 7538 (1997).
[24] E. Giorgi, The linear stability of Reissner-Nordström spacetime: The full subextremal range, arXiv:1910.05630.
[25] R. Penrose, in *Battelle Rencontres*, edited by C. de Witt and J. A. Wheeler (W. A. Benjamin, New York, 1968), p. 222.
[26] S. Chandrasekhar and J. B. Hartle, On crossing the cauchy horizon of a Reissner-Nordstrom black-hole, Proc. R. Soc. A **384**, 301 (1982).
[27] C. M. Chambers, The Cauchy horizon in black hole de sitter space-times, in *Internal Structure of Black Holes and Spacetime Singularities*, edited by L. M. Burko and A. Ori (Institute of Physics Publishing and The Israel Physical Society, Bristol, Philadelphia and Jerusalem, 1997).
[28] P. R. Brady, I. G. Moss, and R. C. Myers, Cosmic Censorship: As Strong as Ever, Phys. Rev. Lett. **80**, 3432 (1998).
[29] E. Poisson and W. Israel, Internal structure of black holes, Phys. Rev. D **41**, 1796 (1990).
[30] H. Maeda, T. Torii, and T. Harada, Novel Cauchy-horizon instability, Phys. Rev. D **71**, 064015 (2005).
[31] J. Lucietti, K. Murata, H. S. Reall, and N. Tanahashi, On the horizon instability of an extreme Reissner-Nordstróm black hole, J. High Energy Phys. 03 (2013) 035.
[32] P. Bizon and H. Friedrich, A remark about wave equations on the extreme Reissner-Nordstróm black hole exterior, Classical Quantum Gravity **30**, 065001 (2013).
[33] See https://grtensor.phy.queensu.ca/.