State space of a black hole and soft hair

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In this article, we reflect on a problem of a black hole entropy. The main point of the article is that the black hole horizon should be treated as a boundary as well as the boundary at infinity. To make things more concrete, we apply the general ideas to the extremal Oliva-Tempo-Troncoso black hole and construct the corresponding Hilbert space using near-horizon hair. After creating the state space by using the proposed construction, we identify the natural candidates for the microstates responsible for the black hole entropy. The correct value of the black hole entropy is reproduced by counting the number of distinct microstates and applying the Boltzmann formula.

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I. INTRODUCTION

The origin of a black hole entropy is one of the most important open problems in physics. There are many approaches to the problem of black hole entropy, most of which rely heavily on the algebra of asymptotic symmetries [1–9].

The recognition that symmetries near the horizon play an important role in understanding black holes has long been known [9,10]. This idea is further refined for the case of extremal black holes and goes by the name Kerr/CFT (conformal field theory) [8]. New insight, that appeared a few years ago, is that black holes have soft hair [4]; the first specific realization of this idea can be found in Refs. [1–3], in which soft hair microstates, known as fluffs, are constructed.

There is also an approach which suggests that near-horizon Virasoro algebra and 2D CFT are underlining the dynamics of four-dimensional black holes [5,6] similar to asymptotic Virasoro in the three-dimensional case.

These approaches have in common that they all are, in the end, trying to give a better understanding of a black hole entropy. The (definite) solution of the black hole entropy problem is a construction of the Hilbert state space of a black hole. When the Hilbert state space is at our disposal, we can count the number of microstates which correspond to the same macrostate, and after using the Boltzmann formula, we will derive black hole entropy. Because full treatment requires formulation of consistent theory of quantum gravity, we can only hope to obtain semiclassical understanding of microstates. This would be very valuable because it will answer two important questions. The first question can be formulated as follows: is there such thing as quantum gravity? The second question is as follows: are there any microstates underlying black hole entropy, or is something fundamentally wrong with our current understanding of black holes? Indirect affirmation to both of these questions is already obtained through AdS/CFT. Nonetheless, indirect verifications via AdS/CFT cannot substitute direct construction and insights it, possibly, provides. This is, exactly, the problem we will (try to) address in this work.

In the following section, which contains the main ideas of the paper, we will motivate why a black hole horizon should be treated as a true boundary and some related questions. In the next section, we will review the necessary results, which will be used later. After that, we consider the algebras of asymptotic and near-horizon symmetries. The next section is devoted to construction of a state space of an extremal Oliva-Tempo-Troncoso (OTT) black hole, using results of the previous sections and deriving entropy by counting the microstates. In the end, we summarize the results of the paper.

II. TWO BOUNDARIES

In this section, we motivate why we have to acknowledge the black hole horizon as the real boundary and obtain some general conclusions about the factorization of phase space. We will focus on black hole created by collapse of matter, as it is expected to be the only physically realistic situation.

The black hole is distinguished by the presence of the event horizon, which divides space-time into two parts, the interior and exterior of the black hole, the horizon being the dividing surface. Classically, nothing cannot escape from the interior, which means that the event horizon is a null surface. In the standard coordinates, the horizon is located in $r = r_0$, and the fact that horizon is a null surface with normal

$$n^\mu = g^{r\mu}, \tag{2.1}$$

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where $g^{\mu\nu}$ is the inverse metric, gives the constraint on the metric

$$n_\mu n^\mu = g^{rr}(r = r_0) = 0. \tag{2.2}$$

The black hole is created the moment enough matter is inside of the area $r < r_0$. We will focus on pure gravity outside of the horizon; consequently, all matter must be inside of the horizon. The important point is that we know nothing about the distribution of matter inside of the black hole. The standard black hole solutions assume that all mater is compressed in $r = 0$, which is the origin of singularity. Singularities are physically unappealing, but, as stated, their origin lies in the additional assumption that all the matter is compressed into a point. This assumption, implicitly, assumes quite a lot about the short-distance behavior of all interactions including gravity. As is well known, short-distance behavior is governed by quantum gravity, about which we know very little. In light of this, it is better to not get into the details of matter distribution inside of the black hole; it is questionable if this is even possible. This discussion suggests that we should treat the event horizon as a boundary, at least until a satisfactory theory of quantum gravity is established. Immediate implication is that, besides boundary conditions at infinity, we need to specify them, as well, at the horizon.

Now, the question arises as to what features boundary conditions on the horizon must have. We will answer this question in the Hamiltonian formulation. Basic operation in the Hamiltonian formalism is the Poisson bracket

$$[A, Q]; \tag{2.3}$$

as usual, this contains the implicit assumption that functional derivatives of variables are well defined. The procedure for improving variables so that they have well-defined functional derivatives in the case of only a boundary at infinity is very well understood. This procedure consists of adding surface terms which lead to well-defined functional derivatives.

When we introduce a horizon, we have two boundaries. By locality, we can divide all variables into in and out, describing dynamics inside and outside of the black hole, respectively,

$$Q_{\text{full}} = Q_{\text{in}} + Q_{\text{out}}. \tag{2.4}$$

This separation of variables, though, might seem arbitrary but in fact is quite natural. Namely, values of in variables Q_{in} are inaccessible to us due to the presence of a horizon which hides the interior of a black hole. One can object that Q_{full} is found as a solution of (an adequate) (system of) equation(s), and knowing it, we automatically know the Q_{in}. This once again steps into the problem, which we stressed at the beginning, of the matter content and distribution inside of a black hole. With this in mind, we come to the conclusion that for the effective description of a black hole it is very natural to divide variables in the aforementioned manner.

When we focus solely on in variables, the procedure is no different than in the case of only a boundary at infinity; for the sake of completeness, we give the analysis of this case. It is possible that upon functional differentiation of Q_{in} a nonzero surface term, at the horizon, arises and we have to improve it, after which we obtain

$$\tilde{Q}_{\text{in}} = Q_{\text{in}} + \Gamma_{r\to r_0}, \tag{2.5}$$

where $\Gamma_{r\to r_0}$ is a surface term defined at $r = r_0$, which is finite and must assure that variation

$$\delta\tilde{Q}_{\text{in}} \tag{2.6}$$

has no surface term; this is the so-called integrability condition.

The analysis on out space is a bit different due to the presence of two boundaries. At infinity, we come to the same conclusion as in the case of in space. Additionally, functional derivatives can give surface terms at the horizon, and we must add a surface term at the horizon which cancels it. Consequently, we obtain

$$\tilde{Q}_{\text{out}} = Q_{\text{out}} + \Gamma_{r\to\infty} - \Gamma_{r\to r_0}; \tag{2.7}$$

note that

$$\Gamma_{r\to r_0} \tag{2.8}$$

is the same surface term as the one needed for improving Q_{in} but contributes with the opposite sign due to opposite orientation of the boundary at the horizon. Alternatively, the reason for this is easy to see if we look at $Q_{\text{full}} = Q_{\text{in}} + Q_{\text{out}}$ and note that full variables see only one boundary at infinity,

$$\tilde{Q}_{\text{full}} = \tilde{Q}_{\text{out}} + \tilde{Q}_{\text{in}} = Q_{\text{full}} + \Gamma_{r\to\infty}, \tag{2.9}$$

which is a very well-known result in the Hamiltonian approach to conserved charges [11], so surface terms at horizon from $\tilde{Q}_{\text{in}}$ and $\tilde{Q}_{\text{out}}$ must cancel each other.

In the rest of the paper we will be concerned with the case in which Q is a generator of symmetry, in this case we can extract even more information about the structure of surface terms.

The general structure of generators of symmetry is that they are a combination of first-class constraints plus a surface term (charge). Because in quantum, as well as in classical, state space all the constraints must hold, we conclude that the generator of symmetry reduces to a surface charge.

For calculating $\Gamma_{r\to\infty}$, we only need boundary conditions and dynamics at infinity, while for determining $\Gamma_{r\to r_0}$, we need boundary conditions and dynamics in the vicinity out of the horizon. This means that this analysis is insensitive to the concrete matter inside of the black hole. This is a known property of the black hole entropy, and it is tempting to conjecture that this kind of analysis will capture all the aspects relevant for its explanation. In turn, this would mean that we are able to determine semiclassical degrees of freedom of the black hole responsible for the appearance of entropy.

On the quantum level, this discussion, superficially, seems to imply the separation of full Hilbert state space $\mathcal{H}^{\rm full}$ into a tensor product of in and out space,

$$\mathcal{H}^{\rm full} = \mathcal{H}^{\rm in} \otimes \mathcal{H}^{\rm out}. \tag{2.10}$$

The truth is that we need to take care of the condition that charges are continual,

$$\tilde{Q}_{\rm in} = \tilde{Q}_{\rm full} - \tilde{Q}_{\rm out}; \tag{2.11}$$

consequently, the Hilbert space is a tensor product of in and out space modulo the previous constraint

$$\mathcal{H}^{\rm full} = \mathcal{H}^{\rm in} \otimes \mathcal{H}^{\rm out}/\text{Constraint}(\text{Continuity of charges}). \tag{2.12}$$

Now, a few words about the general properties of this construction are in order. In state space, where all constraints hold, we are left only with surface charges. As a manifestation of locality, spacelike separated operators commute, and we have that near-horizon $Q_{nh} = \Gamma_{r\to r_0}$ and asymptotic $Q_{as} = \Gamma_{r\to\infty}$ charges commute,

$$[Q_{nh}, Q_{as}] = 0. \tag{2.13}$$

Asymptotic symmetries are transformations that change boundary data and lead to different field configurations. They act on whole space-time, not only near infinity. The asymptotic form of symmetry is investigated at infinitesimal level; this way, we obtain algebra. For example, in 3D gravity, which is the most investigated and best understood example, with Brown-Henneaux boundary conditions, asymptotic symmetry is Virasoro algebra [12].

Near-horizon symmetries are transformations that make changes near the horizon. As stated, they should contain asymptotic algebra, possibly with different central charges. There can also be additional symmetry not seen from the perspective of infinity. This means that these additional symmetries are small gauge transformations from the infinity viewpoint. The important thing to stress is that at infinity asymptotic conditions capture many different field configurations, while the asymptotic conditions near the horizon describe only black hole with different matter distributions behind the horizon.

The observer very far away from the black hole is well approximated by the observer at infinity, and he will only observe asymptotic charges $\Gamma_{r\to\infty}$. Because at the semiclassical level equations of motion hold, it is to be expected that charges at infinity and near the horizon are not unrelated. The idea for microstates is as follows. Acting with near-horizon algebra, we produce an observable change at infinity which can be measured. But there is more than one transformation that we can apply that leads to the same charges at infinity. Counting different ways W to obtain the same asymptotic charges, we should be able to reproduce black hole entropy using Boltzmann relation

$$S = \ln W. \tag{2.14}$$

This procedure can be consistently applied only for semiclassically well-defined objects; otherwise, we would need quantum gravity, which implies that all measurable nonzero charges are much larger than $\hbar$. This idea identifies black hole microstates as different geometries which differ from each other by small gauge transformations from the infinity perspective but a physical one from the near-horizon viewpoint. This states, in nature, are soft hair on a black hole because a generator which should generate them at infinity is zero.

To extract some general conclusions about what properties of the state space constructed from near-horizon symmetry to expect, we review some known approaches for deriving black hole entropy and draw some conclusion from their success.

Euclidean calculation of black hole entropy in a nutshell is as follows. Cut out the interior from the spacetime and Wick rotate the time outside a black hole and compactify it on the circle of radius $\frac{\beta}{2\pi}$. The partition function calculated using this space-time is identified with appropriate thermodynamical potential from which the entropy is derived. This approach yields viable entropy in all known examples.

Euclidean calculation suggests that the interior of a black hole is not important, at least for the semiclassical properties. This means that $\mathcal{H}^{\rm out}$ contains all the information about the black hole. From this, we expect an isomorphism between the full state space $\mathcal{H}^{\rm full}$ and the state space outside of the black hole $\mathcal{H}^{\rm out}$.

Cardy formula calculation of a black hole entropy is specific for three dimensions only. For three-dimensional asymptotically anti-de Sitter space-times, the algebra of asymptotic symmetries is Virasoro with central charges $c^{\pm}$. Asymptotic charges act in whole Hilbert state space, implying that quantum gravity is 2D CFT. The high-energy, $E \gg 1$, density of states $\rho(E)$ can be calculated using modular invariance, which, after using the Boltzmann formula, leads to the entropy formula

$$S = 2\pi\sqrt{\frac{c^- L_0^-}{6}} + 2\pi\sqrt{\frac{c^+ L_0^+}{6}}. \tag{2.15}$$

Using the Cardy formula, we reproduce black hole entropy in all known cases.

The success of the Cardy formula implies that semiclassical entropy is due to high-energy and angular momentum, if nonzero, states. This is to be expected, because for quantum correction to be negligible, meaning that semiclassical gravity is applicable, the same requirements are needed.

III. REVIEW OF THE NECESSARY RESULTS

In this short section, we review the basic results about the OTT black hole.

A. OTT black hole

The stationary OTT black hole [13], an exact solution of Bergshoeff-Hohm-Townsend gravity [14] and 3D PGT [15], is a three-parameter solution defined by the metric

$$ds^2 = N^2dt^2 - F^{-2}dr^2 - r^2(d\varphi + N_\varphi dt)^2, \tag{3.1a}$$

where

$$\begin{aligned}
F &= \frac{H}{r}\sqrt{\frac{H^2}{\ell^2} + \frac{b}{2}H(1+\eta) + \frac{b^2\ell^2}{16}(1-\eta)^2 - \mu\eta},\\
N &= AF, \qquad A = 1 + \frac{b\ell^2}{4H}(1-\eta),\\
N_\varphi &= \frac{\ell}{2r^2}\sqrt{1-\eta^2}(\mu - bH),\\
H &= \sqrt{r^2 - \frac{\mu\ell^2}{2}(1-\eta) - \frac{b^2\ell^4}{16}(1-\eta)^2}.
\end{aligned} \tag{3.1b}$$

The roots of $N = 0$ are

$$r_\pm = \ell\sqrt{\frac{1+\eta}{2}}\left(-\frac{b\ell}{2}\sqrt{\eta} \pm \sqrt{\mu + \frac{b^2\ell^2}{4}}\right).$$

As we already stated, the metric (3.1) depends on three free parameters, μ, b, and η. For $\eta = 1$, the stationary OTT black hole reduces to the static solution, while for $b = 0$, it reduces to the rotating Bañados-Teitelboim-Zanelli (BTZ) black hole with parameters (m, j), defined by $4Gm \coloneqq \mu$ and $4Gj \coloneqq \mu\ell\sqrt{1-\eta^2}$.

The conserved charges of the rotating black hole take the following form:

$$E = \frac{1}{4G}\left(\mu + \frac{1}{4}b^2\ell^2\right), \tag{3.2a}$$

$$J = \ell\sqrt{1-\eta^2}E. \tag{3.2b}$$

B. Near-horizon of extremal OTT and near-horizon algebra

The extremal limit of a stationary OTT black hole can be achieved in two different ways as shown in Ref. [16]:

(1) The first way imposes the requirement $4\mu + b^2\ell^2 = 0$. This leads to a vanishing of both conserved charges, and consequently the asymptotic symmetry trivializes as we showed in Ref. [15].

(2) The second way to obtain an extremal black hole is to set $\eta = 0$, which is equivalent to the requirement that the angular momentum takes the maximal possible value. This corresponds to the usual procedure for the Kerr black hole.

In the latter case, after imposing consistent asymptotic conditions [15], we get that the Poisson bracket algebra of the well-defined canonical generators takes the form of the semidirect sum of centrally extended Kac-Moody and Virasoro algebra without central extension,

$$\begin{aligned}
\{L_m, L_n\} &= -i(m-n)L_{m+n},\\
\{L_m, J_n\} &= inJ_{m+n},\\
\{J_m, J_n\} &= -i\kappa m\delta_{m+n,0},
\end{aligned} \tag{3.3}$$

where the central charge is given by

$$\kappa = \frac{\ell}{G}. \tag{3.4}$$

IV. SYMMETRY ALGEBRAS

In this section, we will review some results about asymptotic and near-horizon symmetry algebras of an extremal OTT black hole.

A. Algebra of asymptotic symmetries and its reduction on space of extremal geometries

We start with the quick review of the results obtained in Ref. [15], which are necessary for further analysis of this paper. The authors analyzed the asymptotic symmetry of an OTT black hole and derived the form of asymptotic symmetry

$$\xi^t = \ell T + \frac{\ell^5}{2r^2}\partial_t^2 T + \mathcal{O}(r^{-3}), \quad \xi^\varphi = S - \frac{\ell^2}{2r^2}\partial\varphi^2 S + \mathcal{O}(r^{-3}), \tag{4.1}$$

$$\xi^r = -\ell r\partial_t T + \mathcal{O}(1), \tag{4.2}$$

where functions T and S are subject to the constraints

$$T^\pm = T \pm S \qquad \partial_\mp T^\pm = 0, \tag{4.3}$$

meaning that functions satisfy $T^\pm = T^\pm(x^\pm)$ with $x^\pm = \frac{t}{\ell} \pm \varphi$. Also, they constructed the generator of the symmetry in the framework of Poincaré gauge theory, which is given by

$$\tilde{G} = G + \Gamma, \tag{4.4}$$

$$\Gamma = \int_0^{2\pi} (\xi^t \mathcal{E} + \xi^\varphi \mathcal{J}). \tag{4.5}$$

After passing on the Fourier mode of $T^\pm$, it is obtained that Fourier modes of Γ, denoted with $L^\pm$, satisfy the commutation relations of Virasoro algebra with central charges

$$c^+ = c^- = \frac{3\ell}{G}. \tag{4.6}$$

Virasoro algebra is an asymptotic symmetry of an arbitrary geometry with given asymptotic behavior. But we want to describe an extremal OTT black hole and need to further specify boundary conditions which will lead to asymptotic algebra of extremal geometries.

The condition for obtaining an extremal rotating black hole is the equality of energy and angular momentum

$$\ell E = J; \tag{4.7}$$

this is the constraint which we will impose on the general asymptotic algebra of charge. This constraint is realized for all extremal geometries if and only if the generators satisfy the same relation

$$\ell \mathcal{E} = \mathcal{J}; \tag{4.8}$$

this bring us to the conclusion

$$\Gamma(\xi) = \int_0^{2\pi} d\varphi \mathcal{J}\left(\frac{\xi^t}{\ell} + \xi^\varphi\right) = \int_0^{2\pi} d\phi \mathcal{J}\xi^\phi[\phi], \tag{4.9}$$

where

$$\phi = \varphi + \frac{t}{\ell}. \tag{4.10}$$

Consequently, charges are of the form

$$Q[\xi^\phi] = \int_0^{2\pi} d\phi \mathcal{J}\xi^\phi[\phi]. \tag{4.11}$$

This means that asymptotic symmetry of extremal geometries is not full Virasoro algebra but only chiral part L_n^+, which are Fourier modes of $Q[\xi^\phi]$ defined as $L_n^+ = Q[e^{(in\phi)}]$, with commutation relations

$$[L_n^+, L_m^+] = (n-m)L_{n+m}^+ + \frac{c}{12}n^3\delta_{n,-m}, \tag{4.12}$$

where

$$c = \frac{3\ell}{G}, \tag{4.13}$$

while

$$L_n^- = 0. \tag{4.14}$$

B. Near-horizon symmetry algebra

From original coordinates (r, t, φ) to near-horizon extremal geometry (NHEG) (ρ, τ, ϕ) we pass after change of coordinates

$$t = \tau/\epsilon^2, \qquad r = r_0 + \epsilon\rho, \qquad \varphi = \phi - \Omega_H \frac{\tau}{\epsilon^2} = \phi - \frac{\tau}{\ell\epsilon^2}, \tag{4.15}$$

and taking limit $\epsilon \to 0$; see Ref. [16].

Asymptotic symmetry of NHEG of an extremal OTT is studied in Ref. [16], and the following symmetry is derived:

$$\xi^\tau = T(\tau), \qquad \xi^\rho = \rho U(\phi), \qquad \xi^\phi = S(\phi). \tag{4.16}$$

Further construction of the generator revealed that ξ^τ is pure gauge, so we will treat it as zero from now on.

Charges in NHEK are given by

$$Q[\xi^\tau] = 0, \tag{4.17}$$

$$Q[\xi^\rho] = -8a_0 \int_0^{2\pi} U(\phi)e^1_\phi, \tag{4.18}$$

$$Q[\xi^\phi] = -4a_0 \int_0^{2\pi} S(\phi)\omega^i_\phi e_{i\phi}. \tag{4.19}$$

C. Vacuum

Space-time which belongs to the allowed field configurations, is a solution of a field equations and with minimal energy is vacuum of a theory. Bearing in mind that we are interested in state space of the black hole, the first guess in three space-time dimensions would be a massless BTZ black hole. Because a massless BTZ does not have a horizon, it is hard to make sense of a near-horizon limit. For this reason, another more appropriate candidate for a black hole vacuum is a massless OTT, which possesses a horizon even in a massless case because of the presence of a hair parameter.

The metric of a massless OTT is

$$ds^2 = \frac{(r-r_0)^2}{\ell^2}dt^2 - \frac{\ell^2}{(r-r_0)^2}dr^2 - r^2 d\varphi^2. \tag{4.20}$$

The energy and angular momentum of this solution are zero,

$$E = 0, \qquad J = 0, \tag{4.21}$$

as are all other Virasoro charges

$$L_n^+ = L_n^- = 0. \tag{4.22}$$

When the near-horizon limit is taken,

$$t = \tau/\epsilon, \qquad r = r_0 + \epsilon\sqrt{\rho}, \qquad \varphi = \phi - \Omega_H \frac{\tau}{\epsilon} = \phi, \tag{4.23}$$

after the redefinition of τ and ℓ, we obtain

$$ds^2 = \frac{\rho}{\ell} d\tau^2 - \frac{\ell^2}{\rho^2} d\rho^2 - r_0^2 d\phi^2. \tag{4.24}$$

This metric belongs to the allowed metrics analyzed in Ref. [16], and near-horizon values of the charges for this metric are

$$Q[\xi^\tau] = Q[\xi^\rho] = Q[\xi^\phi] = 0. \tag{4.25}$$

This motivates us to treat a massless OTT as the black hole vacuum.

V. CONSTRUCTION OF MICROSTATES

This section is devoted to further development of the idea that state space of the OTT black hole can be constructed from near-horizon and asymptotic symmetry algebra. From now on, we pass from the Poisson bracket to the commutator. Commutation relations are the Poisson bracket multiplied by imaginary unit.

A. Unitary irreducible representations of the in algebra

We will search for the representations of the following algebra:

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad [J_n, J_m] = n\delta_{n+m,0},$$
$$[L_n, J_m] = -mJ_{n+m}. \tag{5.1}$$

Note that we redefined J_n so that the ones we use in the rest of the paper are divided with $\sqrt{\kappa}$; this is the reason for the absence of central charge κ in previous commutation relations.

The reality of charges on the quantum level becomes the condition

$$J_n^\dagger = J_{-n}, L_n^\dagger = L_{-n}. \tag{5.2}$$

We construct irreducible representation starting from the highest state vector $|j, l\rangle$, which satisfies

$$J_0|j, l\rangle = j|j, h\rangle, \qquad L_0|j, h\rangle = h|j, h\rangle. \tag{5.3}$$

The operators with positive n we interpret as annihilation operators

$$J_n|j, h\rangle = L_n|j, h\rangle = 0, \qquad n > 0, \tag{5.4}$$

and operators with negative n we interpret as creation operators. Then, we construct the whole representation by acting with creation operators

$$J_{-n}, L_{-n}, n > 0; \tag{5.5}$$

the arbitrary state is of the form

$$L_{-k_1} \dots L_{-k_i} J_{-n_1} \dots J_{-n_m} |j, l\rangle. \tag{5.6}$$

Because Virasoro algebra has zero central charge norm of the state $L_{-n}|j, l\rangle$ is zero,

$$\| L_{-n}|j, h\rangle \|^2 = \langle j, h|[L_n, L_{-n}]|j, l\rangle = 0. \tag{5.7}$$

Consequently, unitarity implies that the Virasoro part, except L_0, of the near-horizon algebra is trivially represented. The expectation value in the vacuum state of L_0 gives the classical value of $L_0^{\text{classical}}$ in vacuum, which is equal to

$$\langle j, h|L_0|j, h\rangle = L_0^{\text{classical}} = 0; \tag{5.8}$$

this implies

$$h = 0. \tag{5.9}$$

Consequently, the generic vector in irreducible representation is a linear combination of vectors of the form

$$|\{k_i\}\rangle = J_{k_1} \dots J_{-k_i} |j, 0\rangle. \tag{5.10}$$

We assume that in state space is a unitary irreducible representation of in algebra, which we constructed previously. The structure of the in Hilbert space is of the form

$$\mathcal{H}^{\text{in}} = \oplus\, \mathcal{H}^{(n)}; \tag{5.11}$$

this is a structure of the Fock space.

B. Unitary irreducible representations of the out algebra

The same as we do in the case of in state space, we assume that out space is a unitary irreducible representation of out algebra. First, we need to specify what is our out algebra. The $u(1)$ Kac-Moody algebra is certainly present; the problematic part is Virasoro algebra. Comparing charges of asymptotic and near-horizon Virasoro algebra. we see that they share the same structure. This motivates us to interpret them as charges of the same transformation.

Further argument, supporting this claim, is that near-horizon and asymptotic charges have the same value on OTT background. So, we come to the conclusion that Virasoro algebra acting on out space is the sum of asymptotic and near-horizon algebras

$$L_n = L_n^{as} - L_n^{nh}, \tag{5.12}$$

with commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n^3\delta_{n+m,0}, \tag{5.13}$$

where the central charge is the same as the one of asymptotic algebra

$$c = \frac{3\ell}{G}. \tag{5.14}$$

Because near-horizon and asymptotic charges of the Virasoro algebra have the same values on the OTT black hole solution [15,16], recalling the discussion of Sec. II, we derive that values of Virasoro charges on OTT background are zero,

$$L_n|_{\mathrm{OTT}} = 0. \tag{5.15}$$

Classical values of the charges are expectation values of the corresponding operators in state describing the desired geometry, in our case the OTT black hole

$$\langle L_n\rangle_{\mathrm{OTT}} = L_n|_{\mathrm{OTT}}. \tag{5.16}$$

To enforce the previous relations, we are forced to give a further restriction on the relevant representations of out algebra. We demand that Virasoro algebra is trivially represented. Representation of $u(1)$ Kac-Moody algebra is constructed from vacuum $|j\rangle$,

$$J_0|j\rangle = j|j\rangle, \tag{5.17}$$

$$J_k|j\rangle = 0, \qquad k > 0, \tag{5.18}$$

and the rest of states are constructed by acting with creation operators J_{-k}, $k > 0$,

$$|\{n_m\}\rangle = J_{-n_1}\ldots J_{-n_m}|j,0\rangle. \tag{5.19}$$

Now, after we determined relevant irreducible representations of the out algebra, we proceed with construction of the complete state space.

C. Hilbert state space

We construct the state space of the OTT black hole using insight from Sec. II, in which we came to some general conclusions. Hilbert state space is of the form

$$\mathcal{H}^{\mathrm{full}} = \mathcal{H}^{\mathrm{in}} \otimes \mathcal{H}^{\mathrm{out}}/\mathrm{Constraint\,(Continuity\,of\,charges)}, \tag{5.20}$$

states in $\mathcal{H}^{\mathrm{in}}$ we denote with $|\{k_i\}\rangle$, states in $\mathcal{H}^{\mathrm{out}}$ are labeled by $|\{n_i\}\rangle$, and the tensor product state we label with

$$|\{k_i\}\rangle \otimes |\{n_i\}\rangle = |\{k_i\},\{n_i\}\rangle. \tag{5.21}$$

On the classical level, the constraint is that $Q_{as}[\xi^r] = Q_{\mathrm{full}}[\xi^r]$ is small gauge from the infinity perspective, i.e., zero, and that $Q_{\mathrm{in}}[\xi^r] = -Q_{\mathrm{out}}[\xi^r]$. On the quantum level, we enforce this by requiring that J_n^{full} annihilates all states in full state space for every non-negative n. Demanding that J_n^{full} annihilates physical states for every n is too strong a demand that trivializes state space. Our approach is similar to Gupta-Bleuler quantization of electromagnetic field.

Acting on states which are the tensor product of in and out states, this translates into

$$J_n^{\mathrm{full}}|\{k_i\},\{n_i\}\rangle = (J_n^{\mathrm{in}}\otimes I - I\otimes J_n^{\mathrm{out}})|\{k_i\},\{n_i\}\rangle = 0, \quad n\geq 0. \tag{5.22}$$

If we take J_0 in the previous constraint, we derive that both $\mathcal{H}^{\mathrm{in}}$ and $\mathcal{H}^{\mathrm{out}}$ have the same value of j; they are representations with the same highest weight.

Further constraints, for $n > 0$, acting on states of this form gives the following restriction:

$$\{k_i\} = \{n_i\}. \tag{5.23}$$

This has the important consequence that the generic state in $\mathcal{H}^{\mathrm{full}}$ is of the form

$$|\{n_i\},\{n_i\}\rangle. \tag{5.24}$$

We also have

$$\langle\{n_i\}|J_0|\{n_i\}\rangle = J_0^{\mathrm{classical}} = 0, \tag{5.25}$$

from which we conclude that

$$j = 0. \tag{5.26}$$

We introduce creation a_{-n} and annihilation a_n operators acting on $\mathcal{H}^{\mathrm{full}}$, which create and annihilate modes in state space of the black hole

$$a_{-n}|0\rangle = |n,n\rangle. \tag{5.27}$$

These operators satisfy the same commutation relations as J_n,

$$[a_n, a_m] = n\delta_{n+m,0}. \tag{5.28}$$

In the end, we note that we have isomorphism

$$\mathcal{H}^{\text{full}} \cong \mathcal{H}^{\text{in}} \cong \mathcal{H}^{\text{out}}; \tag{5.29}$$

the isomorphism is realized by

$$|0\rangle \cong |0\rangle_{\text{in}} \cong |0\rangle_{\text{out}} \tag{5.30}$$

$$a_n \cong J_n^{\text{in}} \cong J_n^{\text{out}}. \tag{5.31}$$

Interpretation of this result is that the black hole can effectively be described by the scalar near-horizon degree of freedom propagating in two-dimensional $t-\varphi$ spacetime. Carlip [17] arrived at a similar conclusion in his analysis of asymptotic dynamics of radial diffeomorphisms. The presence of a boundary breaks radial diffeomorphisms and leads to the appearance of dynamical degrees of freedom on a boundary.

Recently, Hamiltonian reduction was applied to general relativity in three dimensions [18] in which horizon is treated as a boundary with specific boundary conditions. The authors obtained that, in the set up of their paper, the dynamics of a black hole is effectively described by Floreanini-Jackiw scalar theory on the horizon.

D. Action of asymptotic algebra on state space

Now, we have to specify how the asymptotic algebra, which is Virasoro in our case, acts in state space $\mathcal{H}^{\text{full}}$. Because state space $\mathcal{H}^{\text{full}}$ is constructed solely from the action of $u(1)$ Kac-Moody algebra, this implies that Virasoro algebra can be constructed from Kac-Moody algebra. We will now do this using the well-known Sugawara-Sommerfeld construction [19]. This is the same as the approach taken in Ref. [3].

Operators $L_n^{(1)}$ of the Virasoro algebra with central charge $c=1$ are given as a bilinear combination of Kac-Moody operators

$$L_n^{(1)} = \frac{1}{2}\sum_{p=-\infty}^{\infty} :a_{n-p}a_p:, \tag{5.32}$$

where $::$ stands for normal ordering.

Virasoro algebra with arbitrary integer central charge c can be obtained in the manner [20]

$$L_n^{(c)} = \frac{1}{c}L_{cn}^{(1)} \tag{5.33}$$

or explicitly

$$L_n^{(c)} = \frac{1}{2c}\sum_{p=-\infty}^{\infty} :a_{cn-p}a_p:. \tag{5.34}$$

The representation of the asymptotic Virasoro algebra is identified as

$$L_n^{+} = L_n^{(c)}, \tag{5.35}$$

with the assumption that central charge is an integer, which is supported by the results in Ref. [21]. Because we do not have a full microscopical description, we are not able to deduce the origin of central charge. We, nonetheless, have its value from the asymptotic analysis.

For us, the most important operator is Virasoro zero mode

$$L_0^{(c)} = \frac{1}{c}\sum_{p=0}^{\infty} a_{-p}a_p = \frac{1}{c}N, \tag{5.36}$$

where we introduced the number operator

$$N = \sum_{p=0}^{\infty} a_{-p}a_p. \tag{5.37}$$

Generic state is linear combination of states of the form $\sum_i a_{n_i}^{\dagger}|0\rangle$, for which we define the level as the $\sum_i n_i$. The number operator, as the name suggests, counts to which level state belongs which is obvious from the commutation relation

$$[N, a_n] = -na_n \tag{5.38}$$

and the construction of the unitary irreducible representations of $u(1)$ Kac-Moody algebra.

E. Microstate counting

We start from the well-known observation that the classical value of charge is given by the expectation value of the corresponding quantum generator in the correct microstate. Because states in $\mathcal{H}^{\text{full}}$ are a linear combination of $|\{k_i\}\rangle$, it is natural to interpret them as the underlying states of an OTT black hole. Quantitatively, this discussion is expressed as

$$\langle\{k_i\}|L_n^{+}|\{k_i\}\rangle = \delta_{n,0}L_0^{+} = \delta_{n,0}\frac{r_0^2}{2\ell G}. \tag{5.39}$$

Alternatively, from Sugawara-Sommerfeld construction of Virasoro algebra, we obtain

$$\langle\{k_i\}|L_n^{+}|\{k_i\}\rangle = \delta_{n,0}\frac{1}{c}\langle\{k_i\}|N|\{k_i\}\rangle = \delta_{n,0}\frac{1}{c}\sum k_i. \tag{5.40}$$

From the previous relations, we conclude that

$$k = \sum k_i = cL_0^{+}. \tag{5.41}$$

Assuming $k \gg 1$, we can use the Hardy-Ramanujan formula for the number of partitions $[k]$ of natural number k, which states that the $[k]$ is asymptotically

$$[k] \propto \frac{1}{4\sqrt{3}k} e^{2\pi\sqrt{\frac{k}{6}}}. \tag{5.42}$$

In fact, it is expected for both c and L_0 to be separately much larger than 1, so our assumption is a very reasonable one.

The Boltzmann formula for entropy from the number of microstates W,

$$S = \ln W, \tag{5.43}$$

after identification $W = [k]$ gives

$$S = 2\pi\sqrt{\frac{cL_0^+}{6}}, \tag{5.44}$$

which is the same as the entropy obtained in Refs. [15,16] by different methods.

VI. CONCLUSION

We constructed the state space of an extremal OTT black hole using its asymptotic and near-horizon symmetry algebras. The crucial difference between asymptotic and near-horizon algebra is the presence of $u(1)$ Kac-Moody algebra in the latter, which can be identified as an algebra of creation and annihilation operators. The Virasoro part of asymptotic and near-horizon algebra is recognized as the charges of the same transformation calculated at infinity and at the horizon. We further assumed that state space is constructed from unitary irreducible representations of in and out algebras of symmetry.

The classically observable quantities of the black hole are conserved charges far away from the horizon, meaning that we will differentiate only black holes with different values of asymptotic charges. This way, we identified the microstates which correspond to the same macrostate, and using the Boltzmann formula, we reproduced the black hole entropy.

This construction is essentially quantum, although we did not include any quantum corrections, because we worked with Hilbert spaces, which is in line with our understanding that black hole entropy is quantum in nature.

We also obtained that there is isomorphism of full state space with state spaces *inside* and *outside* of a black hole. This is in agreement with discussion of Sec. II, in which we concluded that knowledge of the exterior of a black hole should be sufficient for a derivation of a black hole entropy in semiclassical approximation.

This approach essentially relies on the existence of $u(1)$ Kac-Moody near-horizon algebra of radial diffeomorphisms and anti-de Sitter asymptotic; consequently, it is expected that this analysis can be applied, possibly with some modifications, in any case which fulfilling previously mentioned requirements. For example, this approach should be applicable to extremal BTZ with(out) torsion [22].

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[1] H. Afshar, S. Detournay, D. Grumiller, W. Merbis, A. Perez, D. Tempo, and R. Troncoso, Soft Heisenberg hair on black holes in three dimensions, Phys. Rev. D **93**, 101503 (2016).
[2] H. Afshar, D. Grumiller, and M. M. Sheikh-Jabbari, Near horizon soft hair as microstates of three dimensional black holes, Phys. Rev. D **96**, 084032 (2017).
[3] M. M. Sheikh-Jabbari and H. Yavartanoo, Horizon fluffs: Near horizon soft hairs as microstates of generic AdS3 black holes, Phys. Rev. D **95**, 044007 (2017).
[4] S. W. Hawking, M. J. Perry, and A. Strominger, Soft Hair on Black Holes, Phys. Rev. Lett. **116**, 231301 (2016).
[5] S. Haco, S. W. Hawking, M. J. Perry, and A. Strominger, Black hole entropy and soft hair, J. High Energy Phys. 12 (2018) 098.
[6] S. Haco, M. J. Perry, and A. Strominger, Kerr-Newman black hole entropy and soft hair, arXiv:1902.02247.
[7] A. Strominger, Black hole entropy from near-horizon microstates, J. High Energy Phys. 02 (1998) 009.
[8] M. Guica, T. Hartman, W. Song, and A. Strominger, The Kerr/CFT correspondence, Phys. Rev. D **80**, 124008 (2009).
[9] S. Carlip, Black Hole Entropy from Conformal Field Theory in any Dimension, Phys. Rev. Lett. **82**, 2828 (1999).
[10] S. Carlip, Near Horizon Conformal Symmetry and Black Hole Entropy, Phys. Rev. Lett. **88**, 241301 (2002).
[11] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Ann. Phys. (N.Y.) **88**, 286 (1974).
[12] J. Brown and M. Henneaux, Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity, Commun. Math. Phys. **104**, 207 (1986).
[13] J. Oliva, D. Tempo, and R. Troncoso, Three-dimensional black holes, gravitational solitons, kinks and wormholes

for BHT massive gravity, J. High Energy Phys. 07 (2009) 011.

[14] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, More on massive 3D gravity, Phys. Rev. D **79**, 124042 (2009).

[15] M. Blagojević and B. Cvetković, Conformally flat black holes in Poincaré gauge theory, Phys. Rev. D **93**, 044018 (2016).

[16] B. Cvetković and D. Simić, Near horizon OTT black hole asymptotic symmetries and soft hair, Chin. Phys. C **43**, 013109 (2019).

[17] S. Carlip, Dynamics of asymptotic diffeomorphisms in $(2+1)$-dimensional gravity, Classical Quantum Gravity **22**, 3055 (2005).

[18] D. Grumiller and W. Merbis, Near horizon dynamics of three dimensional black holes, arXiv:1906.10694.

[19] H. Sugawara, A field theory of currents, Phys. Rev. **170**, 1659 (1968); C. Sommerfeld, Currents as dynamical variables, Phys. Rev. **176**, 2019 (1968).

[20] M. Bañados, Embeddings of the Virasoro Algebra and Black Hole Entropy, Phys. Rev. Lett. **82**, 2030 (1999).

[21] E. Witten, Three-dimensional gravity revisited, arXiv: 0706.3359.

[22] B. Cvetković and D. Simić, Near-horizon geometry with torsion, Phys. Rev. D **99**, 024032 (2019).