

Decomposition of the total stress energy for the generalized Kiselev black hole

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We demonstrate that the anisotropic stress energy supporting the Kiselev black hole can be mimicked by being split into a perfect fluid component plus either an electromagnetic component or a scalar field component, thereby quantifying the precise extent to which the Kiselev black hole fails to represent a perfect fluid spacetime. The perfect fluid component carries either an electric or a scalar charge, which then generates anisotropic electromagnetic or scalar fields. This in turn generates anisotropic contributions to the stress energy. These in turn induce forces that partially (in addition to the fluid pressure gradient) support the matter content against gravity. This decomposition is carried out both for the original one-component Kiselev black hole and for the generalized N -component Kiselev black holes. We also comment on the presence of energy condition violations (specifically for the null energy condition) for certain subclasses of Kiselev black holes.

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I. INTRODUCTION

The Kiselev black hole has proved to be an extremely popular and enduring toy model, with over 200 citations to date [1]. Recently one of the current authors has pointed out that, despite many repeated claims to the contrary, the Kiselev black hole is not a perfect fluid spacetime, nor does it have anything to do with the notion of cosmological quintessence [2]. (Relatively few articles in the follow-up literature are at all careful in this regard; a notable exception is Ref. [3].) In the current article, we extend previous work by three of the current authors [4], wherein it was demonstrated that any arbitrary static anisotropic fluid sphere in general relativity can be mimicked by a decomposition into (perfect fluid) + (electromagnetic field) + (scalar field) components, to the specific case of the Kiselev black hole. Note this is a mimicking procedure, designed to give insight into general features of the underlying physics—we do not claim this is an identity.

We find a trichotomy: Depending on the parameters describing the Kiselev black hole, one has (perfect fluid) + (electromagnetic field) components, or one has a pure cosmological constant, or one has (perfect fluid) + (scalar field)

components. We furthermore generalize this analysis to multicomponent Kiselev black holes where the mass function $m(r)$ is described by a Puiseux series [5].

Furthermore, the Kiselev black holes, being surrounded by matter, can be viewed as examples of “dirty” black holes [6,7]. Depending on the specific parameters of the Kiselev geometry, we shall show that this matter can and often does violate the null energy condition (NEC), with potentially serious implications in terms of instability and other unusual behavior [8–19].

II. STRESS ENERGY FOR GENERIC ANISOTROPIC FLUIDS

In Ref. [4] it was established that one may “mimic” the total stress energy of a general anisotropic fluid sphere by a combination of

$$\begin{aligned} &(\text{perfect fluid}) + (\text{electromagnetic field}) \\ &+ (\text{massless minimally coupled scalar field}). \end{aligned}$$

That is,

$$T_{\text{total}}^{\hat{a}\hat{b}} = T_f^{\hat{a}\hat{b}} + T_{em}^{\hat{a}\hat{b}} + T_s^{\hat{a}\hat{b}}. \quad (1)$$

Here the general forms for the various stress-energy tensors are

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$$T_{\text{total}}^{\hat{a}\hat{b}} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{bmatrix}; \quad T_f^{\hat{a}\hat{b}} = \begin{bmatrix} \rho_f & 0 & 0 & 0 \\ 0 & p_f & 0 & 0 \\ 0 & 0 & p_f & 0 \\ 0 & 0 & 0 & p_f \end{bmatrix};$$

$$T_{em}^{\hat{a}\hat{b}} = \frac{1}{2}E^2 \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}; \quad (2)$$

while

$$T_s^{\hat{a}\hat{b}} = \frac{1}{2}(\nabla\phi)^2 \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (3)$$

The perfect fluid parameters are given by

$$p_f = \frac{1}{2}(p_r + p_t); \quad \rho_f = \rho - \frac{1}{2}|p_r - p_t|. \quad (4)$$

The electromagnetic/scalar field parameters are given by

$$E^2 = \max\{p_t - p_r, 0\}; \quad (\nabla\phi)^2 = \max\{p_r - p_t, 0\}. \quad (5)$$

Already at this level we encounter a trichotomy: At any particular value of the radial coordinate r , one has (perfect fluid) + (electromagnetic field) components, or one has pure perfect fluid, or one has (perfect fluid) + (scalar field) components. Let us now perform this decomposition in more detail for the specific case of the Kiselev black hole.

III. STRESS ENERGY OF THE ONE-COMPONENT KISELEV BLACK HOLE

The (one-component) Kiselev black hole spacetime is defined by the line element [1]:

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}} + r^2 d\Omega_2^2. \quad (6)$$

The Einstein tensor components with respect to an orthonormal basis are

$$G_{\hat{t}\hat{t}} = -G_{\hat{r}\hat{r}} = -\frac{3Kw}{r^{3(1+w)}};$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -\frac{3Kw(1+3w)}{2r^{3(1+w)}}. \quad (7)$$

Consequently the stress-energy tensor components are

$$\rho = -p_r = -\frac{3Kw}{8\pi r^{3(1+w)}}; \quad p_t = -\frac{3Kw(1+3w)}{16\pi r^{3(1+w)}}. \quad (8)$$

Examining

$$p_r - p_t = -(p_t - p_r) = \frac{9Kw(1+w)}{16\pi r^{3(1+w)}}, \quad (9)$$

we see that either $(\nabla\phi)^2 = 0$ or $E^2 = 0$, in a position-independent manner, subject only to the *sign* of the constant $Kw(1+w)$.

We also note

$$\rho + p_r = 0; \quad \rho + p_t = -\frac{9Kw(1+w)}{16\pi r^{3(1+w)}}, \quad (10)$$

so the NEC is either satisfied or violated depending on the *sign* of the constant $Kw(1+w)$. (For background on the classical and semiclassical energy conditions see Refs. [8–19].) We now develop a fully explicit case-by-case argument, based on the *sign* of the constant $Kw(1+w)$, to determine the specific form of the linear decomposition presented in Eq. (1).

A. Case (i): $Kw(1+w) < 0$. NEC satisfied.

In this situation we have $p_r - p_t < 0$. Therefore $(\nabla\phi)^2 = 0$. We obtain

$$p_f = \frac{1}{2}(p_r + p_t) = \frac{3Kw(1-3w)}{32\pi r^{3(1+w)}}; \quad (11)$$

$$\rho_f = \rho - \frac{1}{2}|p_r - p_t| = -\frac{3Kw(1-3w)}{32\pi r^{3(1+w)}}; \quad (12)$$

$$w_f = \frac{p_f}{\rho_f} = -1. \quad (13)$$

For the electromagnetic contribution $T_{em}^{\hat{a}\hat{b}}$ to the stress energy we evaluate E^2 :

$$E^2 = \max\{p_t - p_r, 0\} = -\frac{9Kw(1+w)}{16\pi r^{3(1+w)}}. \quad (14)$$

Therefore

$$T_f^{\hat{a}\hat{b}} = \frac{3Kw(1-3w)}{32\pi r^{3(1+w)}} \text{diag}(-1, +1, +1, +1); \quad (15)$$

$$T_{em}^{\hat{a}\hat{b}} = -\frac{9Kw(1+w)}{32\pi r^{3(1+w)}} \text{diag}(+1, -1, +1, +1), \quad (16)$$

and we have the linear decomposition

$$T_{\text{total}}^{\hat{a}\hat{b}} = T_f^{\hat{a}\hat{b}} + T_{em}^{\hat{a}\hat{b}}$$

$$= -\frac{3Kw}{8\pi r^{3(1+w)}} \text{diag}\left(+1, -1, \frac{1+3w}{2}, \frac{1+3w}{2}\right). \quad (17)$$

Examining our expression for electric field strength $E(r)$ we have

$$E(r) = \pm \frac{3\sqrt{|Kw(1+w)|}}{4\sqrt{\pi}r^{3(1+w)/2}}; \quad \frac{dE}{dr} = -\frac{3E(1+w)}{2r}. \quad (18)$$

Invoking Gauss's law, the charge inside a sphere of radius r is

$$Q(r) = E(r)4\pi r^2 = \pm \frac{3\sqrt{\pi|Kw(1+w)|}}{r^{(3w-1)/2}}. \quad (19)$$

We now compute $\sigma_{em}(r)$, the electric charge density, as

$$\sigma_{em} = \frac{dQ(r)}{dV} = \frac{dQ}{4\pi r^2 \sqrt{g_{rr}} dr} = \frac{\sqrt{g^{rr}}}{4\pi r^2} \frac{d}{dr} [4E\pi r^2]$$

$$= \sqrt{g^{rr}} \left[\frac{dE}{dr} + \frac{2E}{r} \right]. \quad (20)$$

Consequently

$$\sigma_{em}(r) = \sqrt{g^{rr}} \left[\frac{E(1-3w)}{2r} \right]$$

$$= \sqrt{1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}} \left[\frac{E(1-3w)}{2r} \right]. \quad (21)$$

To be fully explicit

$$\sigma_{em}(r) = \pm(1-3w) \sqrt{1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}}$$

$$\times \frac{3\sqrt{|Kw(1+w)|/\pi}}{8r^{1+3(1+w)/2}}. \quad (22)$$

Note that $w = 1/3$ is special in that the (distributed) charge density is zero—indeed setting $w = 1/3$ and $K \rightarrow -Q^2$ reproduces Reissner-Nordström spacetime, with all electric charge concentrated at the origin.

B. Case (ii): $Kw(1+w) = 0$. NEC marginal.

This is the straightforward case: $K = 0$ (standard Schwarzschild spacetime), or $w = 0$ (also Schwarzschild spacetime but with shifted mass $m \rightarrow m + \frac{K}{2}$), or $w = -1$ [Kottler spacetime [20], also called Schwarzschild-(anti-)de Sitter spacetime]. All three possibilities have $p_r - p_t = 0$, i.e., $(\nabla\phi)^2 = E^2 = 0$, so our decomposition of the total stress energy must solely consist of the perfect fluid

component of form $T_f^{\hat{a}\hat{b}}$. (The fact that these three spacetimes model a perfect fluid is, of course, extremely standard.)

By inspection, if either K or $w = 0$, then we have the trivial linear decomposition

$$T_{\text{total}}^{\hat{a}\hat{b}} = T_f^{\hat{a}\hat{b}} = \text{diag}(0, 0, 0, 0), \quad (23)$$

reflective of the fact that Schwarzschild is a vacuum solution to the Einstein equations.

If $w = -1$, then we have

$$\rho_f = \rho = \frac{3K}{8\pi};$$

$$p_f = \frac{1}{2}(p_r + p_t) = -\frac{3K}{8\pi};$$

$$p_r = p_t = -\frac{3K}{8\pi}. \quad (24)$$

So our linear decomposition is (also rather trivially)

$$T_{\text{total}}^{\hat{a}\hat{b}} = T_f^{\hat{a}\hat{b}} = \frac{3K}{8\pi} \text{diag}(+1, -1, -1, -1). \quad (25)$$

C. Case (iii): $Kw(1+w) > 0$. NEC violated.

In this situation we have $p_t - p_r < 0$. Therefore $E^2 = 0$, and we obtain

$$p_f = \frac{1}{2}(p_r + p_t) = \frac{3Kw(1-3w)}{32\pi r^{3(1+w)}}; \quad (26)$$

$$\rho_f = \rho - \frac{1}{2}|p_r - p_t| = -\frac{3Kw(7+3w)}{32\pi r^{3(1+w)}}; \quad (27)$$

$$w_f = \frac{p_f}{\rho_f} = -\frac{1-3w}{7+3w}. \quad (28)$$

Note that in this situation

$$\rho_f + p_f = -\frac{9Kw(1+w)}{16\pi r^{3(1+w)}} < 0, \quad (29)$$

implying NEC violation for the perfect fluid component. While this NEC violation is certainly disturbing, let us nevertheless carry out the case (iii) analysis as far as we can.

For the scalar field contribution $T_s^{\hat{a}\hat{b}}$ we evaluate

$$(\nabla\phi)^2 = \max\{p_r - p_t, 0\} = \frac{9Kw(1+w)}{16\pi r^{3(1+w)}}. \quad (30)$$

Therefore

$$T_f^{\hat{a}\hat{b}} = \frac{3Kw}{32\pi r^{3(1+w)}} \text{diag}(-3w-7, 1-3w, 1-3w, 1-3w); \quad (31)$$

$$T_s^{\hat{a}\hat{b}} = \frac{9Kw(1+w)}{32\pi r^{3(1+w)}} \text{diag}(+1, +1, -1, -1); \quad (32)$$

$$\begin{aligned} T_{\text{total}}^{\hat{a}\hat{b}} &= T_f^{\hat{a}\hat{b}} + T_s^{\hat{a}\hat{b}} \\ &= -\frac{3Kw}{8\pi r^{3(1+w)}} \text{diag}\left(+1, -1, \frac{1+3w}{2}, \frac{1+3w}{2}\right). \end{aligned} \quad (33)$$

Solving explicitly for our scalar field, $\phi(r)$, we have

$$(\nabla\phi)^2 = g^{ab}\partial_a\phi\partial_b\phi = g^{rr}(\partial_r\phi)^2 = \frac{9Kw(1+w)}{16\pi r^{3(1+w)}}. \quad (34)$$

Consequently

$$\begin{aligned} (\partial_r\phi)^2 &= g_{rr}\left(\frac{9Kw(1+w)}{16\pi r^{3(1+w)}}\right) \\ &= \frac{1}{1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}}\left(\frac{9Kw(1+w)}{16\pi r^{3(1+w)}}\right). \end{aligned} \quad (35)$$

Thence the scalar field $\phi(r)$ is given by

$$\phi(r) = \pm \frac{3}{4} \sqrt{Kw(1+w)/\pi} \int \frac{dr}{\sqrt{1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}}} r^{3(1+w)/2}. \quad (36)$$

Evaluating this integral directly is not analytically feasible.

Instead, we shall examine the scalar charge density

$$\sigma_s(r) = \Delta\phi = \nabla^2\phi = \frac{1}{\sqrt{-\det(g)}} \partial_a(\sqrt{-\det(g)} g^{ab}\partial_b\phi). \quad (37)$$

That is,

$$\sigma_s(r) = \frac{1}{r^2} \partial_r(r^2 g^{rr} \partial_r\phi). \quad (38)$$

From this we find an explicit but clumsy formula for the scalar charge density:

$$\begin{aligned} \sigma_s(r) &= \pm \frac{3}{4} \frac{\sqrt{Kw(1+w)/\pi}}{r^2} \partial_r \\ &\quad \times \left(\sqrt{1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}} r^{(1-3w)/2} \right). \end{aligned} \quad (39)$$

The total scalar charge inside a sphere of radius r is then

$$S(r) = \int \sigma_s(r) dV = \int \sigma_s(r) \sqrt{g_{rr}} 4\pi r^2 dr. \quad (40)$$

Explicitly

$$\begin{aligned} S(r) &= \pm 3 \sqrt{Kw(1+w)\pi} \int_0^r \frac{1}{\sqrt{1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}}} \partial_r \\ &\quad \times \left(\sqrt{1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}} r^{(1-3w)/2} \right) dr. \end{aligned} \quad (41)$$

While evaluating this integral directly is not analytically feasible, it is at least a fully explicit formula for the scalar charge $S(r)$. In short, case (iii) shares the two features of being physically dubious: (violating the NEC) and being technically clumsy to work with.

D. Summary (one-component model)

It is worth explicitly pointing out here that an immediate corollary of the above case-by-case analysis is that the one-component Kiselev spacetime is only ever a perfect fluid if we are dealing with case (ii); that is, $Kw(1+w) = 0$. Thus the Kiselev black hole is a perfect fluid spacetime only when it reduces to either Schwarzschild or Kottler spacetime [20]. The fact that the Kiselev solution does not generally model a perfect fluid is stressed in [2], and is an important point to reiterate in view of the quite common historical tendency to misidentify perfect fluid models [21]. (For more discussion of the constraints implied by imposing the perfect fluid condition see also Refs. [22–28].)

Insofar as one is willing to accept the classical energy conditions as a pragmatic guideline [8], case (i) is the most physically interesting situation—it corresponds to an electrically charged fluid supported by both pressure gradients and its own internally generated electric field. Case (ii) is physically more prosaic, representing either Schwarzschild or Kottler spacetime. Case (iii) is physically dubious and technically clumsy, exhibiting null energy condition violations.

IV. MULTICOMPONENT KISELEV DECOMPOSITION

Now consider the N -component generalization of Kiselev spacetime as presented in [1]

$$ds^2 = -\left(1 - \frac{\sum_{i=0}^N K_i r^{-3w_i}}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{\sum_{i=0}^N K_i r^{-3w_i}}{r}} + r^2 d\Omega_2^2. \quad (42)$$

Any Schwarzschild mass term, if present, has now been absorbed into $K_0 = 2m$; while setting the corresponding exponent w_0 to zero. That is, effectively one is defining a position-dependent mass function $m(r)$ by setting [2]

$$2m(r) = \sum_{i=0}^N K_i r^{-3w_i}. \quad (43)$$

Since the exponents w_i can be arbitrary, this is a Puiseux series expansion for the mass function, a generalization of the notion of Taylor series, Laurent series, power series, and Frobenius series [5]. Puiseux expansions, while somewhat uncommon, do have a number of other uses in astrophysical situations [29–32].

The spacetime metric is then written in the form

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega_2^2. \quad (44)$$

Spacetime metrics of this form have very special properties [33]. For instance, the radial coordinate r acts as an affine parameter for radial null curves, the radial null-null components of the Einstein and Ricci tensors vanish [33], and the Einstein and Ricci tensors possess two Lorentz-invariant eigenvalues each of multiplicity two, so that the characteristic polynomial factorizes with two repeated roots, implying a specialized Rainich form for the stress energy [34]. Spacetimes of this general form have also been extensively investigated by Dymnikova [35–39].

It is an utterly standard calculation to show

$$\rho = -p_r = \frac{m'(r)}{4\pi r^2} \quad \text{and} \quad p_t = -\frac{m''(r)}{8\pi r}. \quad (45)$$

Now note

$$\begin{aligned} p_r - p_t &= -(p_t - p_r) = \frac{-2m'(r) + rm''(r)}{8\pi r^2} = \frac{r}{2} \left(\frac{m'}{4\pi r^2} \right)' \\ &= \frac{r}{2} \rho'(r). \end{aligned} \quad (46)$$

Therefore in this multicomponent Kiselev geometry it is the *sign* of the density gradient $\rho'(r)$ that determines whether one is dealing with electromagnetic or scalar fields. In contrast to the one-component Kiselev geometry this may now change sign at various values of the radial coordinate leading to an onionlike layered object. Note that

$$\frac{p_r + p_t}{2} = -\rho - \frac{p_r - p_t}{2} = -\rho - \frac{r}{4} \rho'. \quad (47)$$

Furthermore

$$\rho + p_r = 0; \quad \rho + p_t = -\frac{r}{2} \rho', \quad (48)$$

so the NEC is either satisfied or violated depending on the *sign* of the density gradient $\rho'(r)$. We again perform a case-by-case analysis now conditioned on the sign of the density gradient.

A. Case (i): $\rho'(r) < 0$. NEC satisfied

In this situation $(\nabla\phi)^2 = 0$ while

$$\begin{aligned} \rho_f &= \rho - \frac{r}{4} |\rho'|; & p_f &= -\rho + \frac{r}{4} |\rho'|; \\ w_f &= \frac{p_f}{\rho_f} = -1; & E^2 &= \frac{r}{2} |\rho'|. \end{aligned} \quad (49)$$

Note that, as for the one-component model, $w_f = -1$ in this situation. Furthermore

$$\rho_f + p_f = 0, \quad (50)$$

so the perfect fluid component marginally satisfies the NEC.

The electric charge inside a sphere of radius r is now

$$Q(r) = E(r)4\pi r^2 = \pm 2\sqrt{2\pi} r^{5/2} \sqrt{|\rho'|}. \quad (51)$$

For the electric charge density

$$\sigma_{em}(r) = \frac{dQ(r)}{dV} = \sqrt{1 - 2m(r)}/r \left[\frac{dE}{dr} + \frac{2E}{r} \right]. \quad (52)$$

While we can make these formulas fully explicit in terms of $m(r)$, the discussion above is enough to clarify the basic physics issues.

B. Case (ii): $\rho'(r) = 0$. NEC marginal

The situation $\rho'(r) = 0$ could either arise “instantaneously” at the transition layer between $\rho'(r) < 0$ (electromagnetic mimic) and $\rho'(r) > 0$ (scalar field mimic) or it could hold over some finite interval of r . If $\rho'(r) = 0$ holds over some finite interval, then $\rho(r) = \rho_*$ is a constant over that interval. So $m(r) = m_* + \frac{4\pi}{3} \rho_* r^3$ on that interval, so the spacetime is Kottler over that interval—this corresponds to this region being described by a cosmological constant $\rho_\Lambda = \rho_*$. This is completely compatible with what we saw for the one-component model.

C. Case (iii): $\rho'(r) > 0$. NEC violated

In this situation $E^2 = 0$ while we now have

$$\begin{aligned} \rho_f &= \rho - \frac{r}{4} \rho'; & p_f &= -\rho - \frac{r}{4} \rho'; \\ w_f &= \frac{p_f}{\rho_f} = -\frac{\rho + \frac{r}{4} \rho'}{\rho - \frac{r}{4} \rho'} \neq -1. \end{aligned} \quad (53)$$

Note that, as for the one-component model, $w_f \neq -1$ in this situation. Furthermore

$$\rho_f + p_f = -\frac{r}{2} \rho' < 0, \quad (54)$$

implying NEC violation for the perfect fluid component. (It should also be said that any “dirty” black hole for which the total energy density increases as one moves away from the center is somewhat “odd.”) Despite the NEC violations, let us push this analysis a little farther to see how far we can get.

For the scalar field we have

$$(\nabla\phi)^2 = \frac{r}{2}\rho'. \quad (55)$$

Therefore, we see that

$$\partial_r\phi = \sqrt{g_{rr}}\nabla_{\hat{r}}\phi = \sqrt{\frac{r\rho'/2}{1-2m(r)/r}}, \quad (56)$$

and so

$$\phi(r) = \int \sqrt{\frac{r\rho'/2}{1-2m(r)/r}} dr. \quad (57)$$

The scalar charge density is

$$\begin{aligned} \sigma_s &= \frac{1}{r^2}\partial_r(r^2g^{rr}\partial_r\phi) = \frac{1}{r^2}\partial_r\left(r^2\sqrt{g^{rr}}\nabla_{\hat{r}}\phi\right) \\ &= \frac{1}{r^2}\partial_r\left(r^2\sqrt{1-2m(r)/r}\sqrt{r\rho'/2}\right). \end{aligned} \quad (58)$$

Finally the total scalar charge inside a sphere of radius r is

$$S(r) = \int \sigma_s(r)dV = \int \frac{\sigma_s(r)}{\sqrt{1-2m(r)/r}} 4\pi r^2 dr. \quad (59)$$

While we can make these formulas fully explicit in terms of $m(r)$, the discussion above is enough to clarify the basic physics issues.

D. Summary (multicomponent models)

From the above, the only situation in which the generalized Kiselev spacetime represents a perfect fluid is in case (ii) when $\rho' = 0$. That is, when the generalized Kiselev black hole reduces to Kottler (or Schwarzschild) spacetime. The major difference between simple one-component Kiselev spacetimes and these generalized multicomponent Kiselev spacetimes is that the presence or absence of electromagnetic or scalar fields in the mimicking model can now depend on the radial coordinate r in multicomponent models, whereas in one-component models, either there is only an electromagnetic or a scalar field everywhere or there is nothing (neither electromagnetic nor scalar field).

We again see violations of the NEC in this now generalized case (iii), and insofar as one wishes to be guided by the classical energy conditions, case (iii) should be deprecated. At the very least, if one wishes to work with case (iii) models, one should be aware of the potential risks and drawbacks.

V. DISCUSSION

While the Kiselev black hole [1] is an extremely popular toy model, there are a number of key scientific issues regarding which the published literature is seriously deficient:

- (i) Despite many claims to the contrary, the Kiselev spacetime does not represent a perfect fluid (except for the very special cases where it reduces to Schwarzschild/Kottler/de Sitter spacetime) [2].
- (ii) Despite many claims to the contrary, the word “quintessence” as applied to the Kiselev spacetimes has nothing to do with the word “quintessence” as it is used in the cosmology community [2].

In earlier work three of the current authors showed that it is possible to mimic the matter content of any static spherically symmetric spacetime by a combination of

(perfect fluid) + (electromagnetic field) + (scalar field).

In the current work we apply this decomposition to the specific case of the Kiselev spacetimes, both the original one-component model and the generalized multicomponent models. We find that there is a tight correlation between satisfying the NEC and the type of decomposition that arises.

We find a trichotomy:

- (i) In regions where the Kiselev spacetime strongly satisfies the NEC, its matter content can be mimicked by (electrically charged perfect fluid) + (electromagnetic field).
- (ii) In regions where the Kiselev spacetime marginally satisfies the NEC, its matter content is forced to be a cosmological constant.
- (iii) In regions where the Kiselev spacetime violates the NEC, its matter content can awkwardly be mimicked by (scalar charged perfect fluid) + (scalar field).

Overall we would argue that while the Kiselev spacetime and its generalizations are certainly physically and mathematically interesting, some significant caution should be exercised when interpreting much of the current literature on Kiselev spacetimes (and the Rastallization thereof [40]).

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