Neutrino oscillations in extended theories of gravity

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We study neutrino oscillations within the framework of extended theories of gravity. Based on the covariant reformulation of Pontecorvo's formalism, we evaluate the oscillation probability of neutrinos propagating in static spacetimes described by gravitational actions quadratic in the curvature invariants. Calculations are carried out in the two-flavor approximation, for oscillations both in vacuum and matter. It is shown that the neutrino phase is sensitive to the violation of the strong equivalence principle. By way of illustration, we specialize our analysis to various extended models of gravity in order both to quantify such a violation and to understand how the characteristic free parameters of these models affect the neutrino phase. The possibility to fix new bounds on these parameters and to constrain extended theories of gravity is finally discussed.

I. INTRODUCTION

Neutrinos are among the most enigmatic entities in particle physics. Because of their zero charge and extremely small masses, they impinge on matter (almost) only by the weak interaction. Such an elusive nature justifies the nearly three-decades delay between Pauli's prediction of the existence of the (anti)neutrino in 1930 and its real detection by Reines and Cowan, Jr., [1] in 1956. Since then, neutrino physics has been largely addressed, drawing even more attention after Pontecorvo's pioneering idea of flavor mixing and oscillations [2]. Although a firm treatment of these phenomena has now been set up both at the theoretical [3] and experimental [4] levels, in vacuum [5] and in matter [6], puzzling questions such as the correct quantum field theoretical definition of flavor states [7,8], the nature of neutrino masses (Dirac or Majorana) [9], and the dynamical origin of the nonvanishing neutrino masses and mixings [10,11] are still under investigation.

All of the above is set in flat spacetime. Neutrino oscillations in the presence of gravity were first studied by Stodolsky [12], and their relevance in cosmology and astrophysics was later pointed out in Refs. [13,14]. Recently, a similar analysis in accelerated frames burst into the spotlight [15–17] in connection with the controversy on the asymptotic nature of mixed neutrinos in the decay of accelerated protons [18]. These studies, however, have been carried out within the framework of Einstein's

General Relativity (GR). Despite providing the most successful description of gravitational interaction [19], it is nowadays commonly thought that GR might not be the ultimate theory, because of its incompleteness at short distances or, in other words, at high energies (think of classical singularities and lack of renormalizability), and its failure to explain issues such as the cosmic inflation or the possible existence of dark matter and dark energy. This paves the way for a strenuous search of new models [20] that may encompass these problems in a self-consistent scheme, preserving at the same time the positive results of GR.

Among all the extended theories of gravity formulated throughout the years, the most straightforward approaches are the so-called *quadratic theories*, which consist of generalizing the Einstein-Hilbert gravitational action by including contributions quadratic in the curvature invariants. In this context, worthy of note are the results achieved by Stelle [21], who realized that a description of gravity arising from the Einstein-Hilbert action containing the squared scalar curvature and squared Ricci tensor, $\mathcal{R}^2$ and $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$, is power-counting renormalizable. However, such a theory lacks of predictability above a certain cutoff, which is given by the mass of a spin-2 ghost degree of freedom (d.o.f.) appearing in the theory when the standard quantization is adopted. Developments have been subsequently gained in the Starobinsky model of cosmic inflation [22], which only involves the $\mathcal{R}^2$ term, and also in other scenarios [23]. Interesting results have been also highlighted in *nonlocal* quadratic theories [24–31].

Understanding which of the above extended theories may be considered as the best candidate to generalize GR

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and, as a consequence, how it affects physical phenomena is certainly a crucial task [19,20]. For instance, a recent attempt to fulfill this aim has been made in the context of the Casimir effect in Refs. [32,33], in which nontrivial bounds on the free parameters appearing in such theories have been inferred by the evaluation of Casimir energy density and pressure. In the present paper, we will face this issue by analyzing neutrino flavor oscillations both in matter and vacuum and computing the correction to the quantum mechanical phase arising from the extra terms in the gravitational action. In this regard, we remark that a similar analysis has been carried out in Brans-Dicke theory in Ref. [34] and in other extended models in Ref. [20].

On the other hand, differently from the previous approaches, we will also discuss the possibility to pinpoint phenomenological implications of the strong equivalence principle (SEP) on neutrino propagation, as already investigated in Refs. [35]. Indeed, there is a common agreement on the SEP violation occurrence in some extended models of gravity [19]. For instance, in the context of the aforementioned Brans-Dicke theory, one can evaluate the inertial and gravitational mass of the source of gravity and notice that the presence of the dynamical scalar field is the responsible for the discrepancy between the two terms [36,37]. Such a violation can be also extended to $f(\mathcal{R})$ models, in light of the close bond they share with scalar-tensor theories, which has been the subject of an intense line of research (i.e., see, for example, Refs. [20,38]). Motivated by these ideas, one of our primary aims is to seek the contribution to the neutrino oscillation phase that can be associated to SEP violation.

The layout of the paper is the following. In Sec. II, we analyze the standard formalism of vacuum neutrino oscillations in the Minkowski framework. We also introduce the covariant formulation of Ref. [14]. The same considerations are then extended to the case of oscillations in matter. Section III is devoted to a review of the most important features of some quadratic theories of gravity. Corrections to the neutrino quantum mechanical phase and to the related oscillation probability are explicitly calculated in Sec. IV. Aside from this, we discuss the possibility of fixing constraints on the free parameters appearing in the considered theories. Section V contains a thorough application of the aforementioned general notions to several quadratic models of gravity of which the relevance has been proven in a quantum field theoretical framework. Moreover, we explicitly point out the contribution to the covariant oscillation phase that is directly related to the SEP. Concluding remarks can be found in Sec. VI.

Throughout the work, we assume natural units $\hbar = c = 1$ and the mostly negative metric convention, $\eta_{\mu\nu} = \text{diag}[1 - 1, -1, -1]$. Furthermore, we consider a simplified two-flavor model for neutrinos: the obtained results can be easily extended to a more general three-flavor description with *CP* violation.

II. NEUTRINO OSCILLATIONS IN CURVED SPACETIME

A. Vacuum oscillations

Let us consider a flavor neutrino emitted via weak interaction at a generic spacetime point. According to Pontecorvo's quantum mechanical formalism [2], the flavor state $|\nu_\alpha\rangle$ ($\alpha = e, \mu$) can be expressed as a superposition of the mass eigenstates $|\nu_k\rangle$ ($k = 1, 2$) as[1]

$$|\nu_\alpha\rangle = \sum_{k=1,2} U_{\alpha k}(\theta)|\nu_k\rangle, \tag{1}$$

where $U_{\alpha k}(\theta)$ is the generic element of the Pontecorvo mixing matrix

$$U(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2}$$

The states that indeed propagate are the mass ones, the energy E_k and 3-momentum $\vec{p}_k$ of which are related by the usual mass-shell condition $E_k^2 = m_k^2 + |\vec{p}_k|^2$.

In Minkowski spacetime, the propagation of the state $|\nu_k\rangle$ from a point $A(t_A, \vec{x}_A)$ to a point $B(t_B, \vec{x}_B)$ can be described by a plane wave as

$$|\nu_k(x)\rangle = \exp[-i\varphi_k(x)]|\nu_k\rangle, \tag{3}$$

where the phase Φ_k is defined as

$$\varphi_k = E_k(t_B - t_A) - \vec{p}_k \cdot (\vec{x}_B - \vec{x}_A). \tag{4}$$

Therefore, by using Eqs. (1) and (4), the probability that a neutrino produced with flavor α at the point A is detected with flavor β at the point B takes the form

$$\mathcal{P}_{\alpha\to\beta} = |\langle \nu_\beta(t_B, \vec{x}_B)|\nu_\alpha(t_A, \vec{x}_A)\rangle|^2$$
$$= \sin^2(2\theta)\sin^2\left(\frac{\varphi_{12}}{2}\right), \tag{5}$$

where the phase shift is given by $\varphi_{12} = \varphi_1 - \varphi_2$.

For relativistic neutrinos, by assuming the mass eigenstates to be energy eigenstates with a common energy E, one can show that

$$\varphi_{12} \simeq \frac{\Delta m^2}{2E} L_p, \tag{6}$$

where $\Delta m^2 \equiv |m_2^2 - m_1^2|$ is the mass-squared difference and $L_p = |\vec{x}_B - \vec{x}_A|$ is the distance traveled by neutrinos.

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[1]In what follows, we denote flavor (mass) indices by greek (latin) indices.

The above formalism can be generalized in a straightforward way to curved spacetime by rewriting the phase (4) as the eigenvalue of the covariant operator [14]

$$\Phi = \int_{\lambda_A}^{\lambda_B} P_\mu \frac{dx_{\text{null}}^\mu}{d\lambda} d\lambda, \tag{7}$$

where P_μ is the generator of spacetime translations of neutrino mass eigenstates and $dx_{\text{null}}^\mu/d\lambda$ is the null tangent vector to the neutrino worldline parametrized by λ. For neutrino propagating in flat spacetime, the above relation recovers Eq. (4), as it should.

The quantity $P_\mu dx_{\text{null}}^\mu/d\lambda$ in Eq. (7) can be calculated starting from the covariant Dirac equation for a doublet of spinors ν of different masses [39]

$$[i\gamma^{\hat{a}} e_{\hat{a}}^\mu (\partial_\mu + \Gamma_\mu) - M]\nu = 0, \tag{8}$$

where $M = \text{diag}[m_1, m_2]$ and $\gamma^{\hat{a}}$ are the Dirac matrices. The general curvilinear and locally inertial sets of coordinates are denoted without and with a hat, respectively, and they are related by the vierbein field $e_{\hat{a}}^\mu$. The explicit expression for the Fock-Kondratenko connection is $\Gamma_\mu = \frac{1}{8} [\gamma^{\hat{b}}, \gamma^{\hat{c}}] e_{\hat{b}}^\nu e_{\hat{c}\nu;\mu}$, where the semicolon stands for the covariant derivative.

Note that the Dirac equation (8) can be simplified by means of the relation [14]

$$\gamma^{\hat{a}} e_{\hat{a}}^\mu \Gamma_\mu = \gamma^{\hat{a}} e_{\hat{a}}^\mu \left\{ i A_{G\mu} \left[-g^{-1/2} \frac{\gamma^5}{2} \right] \right\}, \tag{9}$$

where $g \equiv |\det g_{\mu\nu}|$, $\gamma^5 = i\gamma^{\hat{0}}\gamma^{\hat{1}}\gamma^{\hat{2}}\gamma^{\hat{3}}$ and the vector potential A^μ is given by

$$A_G^\mu = \frac{1}{4} g^{1/2} e_{\hat{a}}^\mu \epsilon^{\hat{a}\hat{b}\hat{c}\hat{d}} (e_{\hat{b}\nu,\sigma} - e_{\hat{b}\sigma,\nu}) e_{\hat{c}}^\nu e_{\hat{d}}^\sigma. \tag{10}$$

Here, $\epsilon^{\hat{a}\hat{b}\hat{c}\hat{d}}$ is the totally antisymmetric Levi-Civita symbol with component $\epsilon^{\hat{0}\hat{1}\hat{2}\hat{3}} = +1$.

In the above setting, the momentum operator P_μ used to calculate the neutrino oscillation phase can be derived from the generalized mass-shell relation

$$(P^\mu + A_G^\mu P_L)(P_\mu + A_{G\mu} P_L) = M^2, \tag{11}$$

where $P_L \equiv (1 - \gamma_5)/2$ is the left-hand projector and we have added a term proportional to the identity without any physical consequences [14]. By neglecting terms of order $\mathcal{O}(A_G^2)$ and $\mathcal{O}(A_G M^2)$ and considering relativistic neutrinos, we then obtain

$$P_\mu \frac{dx_{\text{null}}^\mu}{d\lambda} = \left(\frac{M^2}{2} - \frac{dx_{\text{null}}^\mu}{d\lambda} A_{G\mu} P_L \right), \tag{12}$$

where we have required $P^i \approx p^i$ and $P^0 = p^0$ [14]. In this regard, we emphasize that $E \equiv P_0 = g_{0\nu} P^\nu$.

Finally, by denoting the differential proper distance at constant t by $d\ell$, we can write

$$d\lambda = d\ell \left(-g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right)^{-\frac{1}{2}}$$
$$= d\ell \left[g_{00} \left(\frac{dx^0}{d\lambda} \right)^2 + 2g_{0i} \frac{dx^0}{d\lambda} \frac{dx^i}{d\lambda} \right]^{-\frac{1}{2}}, \tag{13}$$

where we have exploited the condition of null trajectory $ds^2 = 0$.

B. Matter effects

The weak-field approximation, which we use throughout this paper, is suitable for the description of solar and supernovae neutrinos [13,34,40]. However, in these cases, a matter effect such as the Mikheyev-Smirnov-Wolfenstein effect [6] cannot be disregarded. Following the treatment of Ref. [14], we shall treat these extra contributions in a fashion similar to the gravity-induced corrections computed above.

Let us assume that only electron neutrinos weakly interact with an electron background fluid. In this case, the generalized mass-shell relation takes the form

$$(P^\mu + A_f^\mu P_L)(P_\mu + A_{f\mu} P_L) = M_f^2, \tag{14}$$

where $M_f^2 \equiv U(\theta) M^2 U^\dagger(\theta)$ and

$$A_f^\mu \equiv \begin{pmatrix} -\sqrt{2} G_F N_e^\mu & 0 \\ 0 & 0 \end{pmatrix} \tag{15}$$

is the interaction term in the flavor basis, G_F is the Fermi constant, and $N_e^\mu = n_e u^\mu$ is the number current of the electron fluid. Here, n_e and u^μ are the electron density in the fluid rest frame and the fluid's 4-velocity, respectively.

Finally, by taking into account both geometric and matter effects, the generalization of Eq. (12) reads

$$P_\mu \frac{dx_{\text{null}}^\mu}{d\lambda} = \left(\frac{M^2}{2} - \frac{dx_{\text{null}}^\mu}{d\lambda} A_\mu P_L \right). \tag{16}$$

III. QUADRATIC THEORIES OF GRAVITY

In this section, we introduce a wide class of extended theories of gravity for which the neutrino oscillation phenomenon will be studied.

Let us consider the gravitational action, which is the most general parity-invariant and torsion-free action around maximally symmetric backgrounds [24,28],

$$S = \frac{1}{2\kappa^2} \int \left\{ \mathcal{R} + \frac{1}{2}[\mathcal{R}\mathcal{F}_1(\Box)\mathcal{R} + \mathcal{R}_{\mu\nu}\mathcal{F}_2(\Box)\mathcal{R}^{\mu\nu} + \right.$$

$$\left. + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{F}_3(\Box)\mathcal{R}^{\mu\nu\rho\sigma}] \right\} \sqrt{-g} d^4x, \qquad (17)$$

where $\kappa \equiv \sqrt{8\pi G} = 1/M_p$ is the inverse of the reduced Planck mass, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the curved d'Alembertian, and the three differential operators $\mathcal{F}_i(\Box)$ are generic functions of $\Box$:

$$\mathcal{F}_i(\Box) = \sum_{n=0}^{N} f_{i,n}\Box^n, \qquad i = 1, 2, 3. \qquad (18)$$

Here, we deal with both positive $(n > 0)$ and negative $(n < 0)$ powers of the d'Alembertian; namely, we analyze both ultraviolet and infrared modifications of Einstein's GR. When N is finite $(N < \infty)$ and $n > 0$, we have a local theory of gravity of which the derivative order is $2N + 4$, while if $N = \infty$ and/or $n < 0$, the corresponding gravitational theory is nonlocal, and the form factors $\mathcal{F}_i(\Box)$ are nonpolynomial differential operators of $\Box$.

Since we are interested in computing and studying the neutrino oscillation phase in presence of a weak gravitational field, we can work in the linear regime by expanding the action in Eq. (17) around the Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \qquad (19)$$

where $h_{\mu\nu}$ is the metric perturbation.

In our perturbative approach, we truncate the action in Eq. (17) at order[2] $\mathcal{O}(h^2)$ [26],

$$S = \frac{1}{4} \int \left\{ \frac{1}{2} h_{\mu\nu} f(\Box)\Box h^{\mu\nu} - h_\mu^\sigma f(\Box)\partial_\sigma\partial_\nu h^{\mu\nu} \right.$$

$$+ hg(\Box)\partial_\mu\partial_\nu h^{\mu\nu} - \frac{1}{2} hg(\Box)\Box h$$

$$\left. + \frac{1}{2} h^{\lambda\sigma}\frac{f(\Box) - g(\Box)}{\Box}\partial_\lambda\partial_\sigma\partial_\mu\partial_\nu h^{\mu\nu} \right\} d^4x, \qquad (20)$$

where $h \equiv \eta_{\mu\nu}h^{\mu\nu}$ is the trace of the metric perturbation and we have defined

$$f(\Box) = 1 + \frac{1}{2}\mathcal{F}_2(\Box)\Box, \qquad (21)$$

$$g(\Box) = 1 - 2\mathcal{F}_1(\Box)\Box - \frac{1}{2}\mathcal{F}_2(\Box)\Box. \qquad (22)$$

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[2]In this regime, the term $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{F}_3(\Box)\mathcal{R}^{\mu\nu\rho\sigma}$ in Eq. (17) can be neglected. Indeed, the identity $\mathcal{R}_{\mu\nu\rho\sigma}\Box^n\mathcal{R}^{\mu\nu\rho\sigma} = 4\mathcal{R}_{\mu\nu}\Box^n\mathcal{R}^{\mu\nu} - \mathcal{R}\Box^n\mathcal{R} + \mathcal{O}(\mathcal{R}^3) + \text{div}$, where div takes into account total derivatives and $\mathcal{O}(\mathcal{R}^3)$ only contributes at order $\mathcal{O}(h^3)$, holds true. Hence, in the linearized regime, we can set $\mathcal{F}_3(\Box) = 0$ without loss of generality.

The corresponding linearized field equations are given by

$$2\kappa^2 T_{\mu\nu} = f(\Box)(\Box h_{\mu\nu} - \partial_\sigma\partial_\nu h_\mu^\sigma - \partial_\sigma\partial_\mu h_\nu^\sigma)$$

$$+ g(\Box)(\eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h)$$

$$+ \frac{f(\Box) - g(\Box)}{\Box}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma h^{\rho\sigma}, \qquad (23)$$

where the stress-energy tensor sourcing the gravitational field is defined by

$$T_{\mu\nu} \simeq -2\frac{\delta S_m}{\delta h^{\mu\nu}}, \qquad (24)$$

with S_m being the matter action.

We are interested in finding the expression for the linearized spacetime metric in the presence of a static pointlike source,

$$ds^2 = (1 + 2\phi)dt^2 - (1 - 2\psi)(dr^2 + r^2d\Omega^2), \qquad (25)$$

where $d\Omega = d\theta^2 + \sin^2\theta d\varphi^2$ and ϕ and ψ are the two metric potentials, while the matter sector is described by

$$T_{\mu\nu} = m\delta_\mu^0\delta_\nu^0\delta^{(3)}(\vec{r}). \qquad (26)$$

By setting $\kappa h_{00} = 2\phi$, $\kappa h_{ij} = 2\psi\delta_{ij}$, $\kappa h = 2(\phi - 3\psi)$, and using the assumption of static pressureless source, i.e., $\Box \simeq -\nabla^2$ and $T = \eta_{\rho\sigma}T^{\rho\sigma} \simeq T_{00}$, the modified Poisson equations for the two metric potentials read

$$\frac{f(f - 3g)}{f - 2g}\nabla^2\phi(r) = 8\pi Gm\delta^{(3)}(\vec{r}), \qquad (27)$$

$$\frac{f(f - 3g)}{g}\nabla^2\psi(r) = -8\pi Gm\delta^{(3)}(\vec{r}), \qquad (28)$$

where $f \equiv f(\nabla^2)$, $g \equiv g(\nabla^2)$ are now functions of the Laplace operator.

The two modified Poisson equations (27) and (28) can be solved with the use of the Fourier transform method, by going to momentum space and then antitransforming back to coordinate space. Thus, we obtain

$$\phi(r) = -8\pi Gm \int \frac{1}{k^2}\frac{f - 2g}{f(f - 3g)}e^{i\vec{k}\cdot\vec{r}}\frac{d^3k}{(2\pi)^3}$$

$$= -\frac{4Gm}{\pi r}\int_0^\infty \frac{f - 2g}{f(f - 3g)}\frac{\sin(kr)}{k}dk, \qquad (29)$$

$$\psi(r) = 8\pi Gm \int \frac{1}{k^2}\frac{g}{f(f - 3g)}e^{i\vec{k}\cdot\vec{r}}\frac{d^3k}{(2\pi)^3}$$

$$= \frac{4Gm}{\pi r}\int_0^\infty \frac{g}{f(f - 3g)}\frac{\sin(kr)}{k}dk, \qquad (30)$$

where $f \equiv f(-k^2)$ and $g \equiv g(-k^2)$ are now functions of the Fourier momentum squared.

For a first check, we can notice that in the case $f = g$ we recover the weak-field limit of Einstein's GR,

$$f = g = 1 \Rightarrow \phi(r) = \psi(r) = -\frac{Gm}{r}, \qquad (31)$$

as expected.

IV. OSCILLATION PHASE EXPRESSION

In this Section, we want to study the form that the covariant oscillation phase acquires when the spacetime is described by several quadratic models of gravity of which the action is given by Eq. (17). Specifically, we refer to the phase that appears in the expression of the flavor transition probability (5)

$$\mathcal{P}_{\alpha \to \beta} = \sin^2(2\theta) \sin^2\left(\frac{\varphi_{12}}{2}\right), \qquad (32)$$

where now $\varphi_{12} \equiv \varphi_1 - \varphi_2$ denotes the oscillation phase in curved background, i.e.,

$$\Phi|\nu_k\rangle = \varphi_k|\nu_k\rangle, \qquad (33)$$

with Φ given by Eq. (7).

A. Vacuum oscillations

Let us start by considering only geometric effects (matter effects will be accommodated later). Since our attention is focused on the analysis of a radial propagation, it is possible to prove that, in all the upcoming discussions, we have $A_{G\mu} = 0$. Actually, this is always true for such diagonal metrics as the one in Eq. (25) (see, e.g., Refs. [14,41,42]). In fact, a brief analysis of Eq. (10) shows that a nonvanishing $A_{G\mu}$ requires nonzero off-diagonal components of the tetrads. It is immediate to verify that, in our case,

$$e^0_0 = 1 - \phi, \qquad e^i_j = (1 + \psi)\delta^i_j. \qquad (34)$$

At this point, the phase φ_{12} takes the form [14,42]

$$\varphi_{12} = \frac{\Delta m^2}{2} \int_{\lambda_A}^{\lambda_B} d\lambda = \frac{\Delta m^2}{2} \int_{\ell_A}^{\ell_B} \frac{d\ell}{E_\ell}, \qquad (35)$$

where we have made use of Eq. (13) in the second step and $E_\ell = e^0_0 E$ is the energy measured by a *locally* inertial observer momentarily at rest in the curved spacetime and E represents the energy measured by an inertial observer at rest at infinity. Since we are assuming to work with a stationary metric, it is worth emphasizing that $E \equiv P_0$ is a conserved quantity.

By use of Eq. (34), Eq. (35) can be rephrased as

$$\varphi_{12} = \frac{\Delta m^2}{2E} \int_{r_A}^{r_B} [1 + \phi(r) - \psi(r)]dr, \qquad (36)$$

given that $d\ell^2 = (1 - 2\psi)dr^2$ for radial motion.

In accordance with the reasoning exhibited so far, the flavor oscillation probability can be rewritten as

$$\begin{aligned}
\mathcal{P}_{\alpha \to \beta} &= \sin^2(2\theta)\sin^2\left\{\frac{\Delta m^2}{4E} \int_{r_A}^{r_B} [1 + \phi(r) - \psi(r)]dr\right\} \\
&= \sin^2(2\theta)\sin^2\left[\frac{\Delta m^2}{4E}(r_B - r_A) + \frac{\varphi_{\text{SEP}}}{2}\right], \qquad (37)
\end{aligned}$$

where we have introduced the shorthand notation

$$\varphi_{\text{SEP}} = \frac{\Delta m^2}{2E} \int_{r_A}^{r_B} [\phi(r) - \psi(r)]dr, \qquad (38)$$

the meaning of which will be clarified in the next subsection.

Depending on the choice of the form factors in Eq. (17), we expect $\mathcal{P}_{\alpha \to \beta}$ to be a function of the free parameters of the selected quadratic theory of gravity. In turn, this implies that the neutrino oscillation probability is strictly related to the model used to investigate the geometric features of the curved background.

In addition, it is possible to show that the covariant oscillation phase can always be split in three different contributions. Guided by this idea, one can check that Eq. (36) always includes the terms

$$\varphi_{12} = \varphi_0 + \varphi_{\text{GR}} + \varphi_Q, \qquad (39)$$

where φ_0 is formally the same as the usual "flat" phase when $m = 0$ (6), φ_{GR} is the quantity associated to GR, whereas φ_Q includes all the corrections due to the quadratic models of gravity. The feasibility of such a procedure is guaranteed by the fact that the two metric potentials ϕ and ψ can be always recast as $\phi = \phi_{\text{GR}} + \phi_Q$ and $\psi = \psi_{\text{GR}} + \psi_Q = \phi_{\text{GR}} + \psi_Q$, respectively (since $\psi_{\text{GR}} = \phi_{\text{GR}}$, as seen in the previous section). At this point, the appearance of φ_0 ensues from a simple consideration: starting from Eq. (36), indeed, we can cast E in terms of the local energy by using $E_\ell = (1 - \phi)E$ and then introduce the proper distance covered by the neutrino propagating on a curved background:

$$L_p = \int_{r_A}^{r_B} \sqrt{-g_{rr}}dr = \int_{r_A}^{r_B} [1 - \psi(r)]dr. \qquad (40)$$

In view of these notions, the covariant phase (36) can be expressed as

$$\varphi_{12} = \frac{\Delta m^2 L_p}{2E_\ell}\left[1 - \phi(r_B) + \frac{1}{L_p}\int_{r_A}^{r_B}\phi(r)dr\right]. \quad (41)$$

Hence, the first term on the rhs precisely returns the phase in Eq. (6), with the difference that here it is written as a function of the local energy and the proper propagation distance. Since we are interested in a slightly curved background (i.e., in the weak-field regime), we will now report the explicit linearized expressions for φ_{GR} [42],

$$\varphi_{GR} = \frac{\Delta m^2 L_p}{2E_\ell}\left[\frac{Gm}{r_B} - \frac{Gm}{L_p}\ln\left(\frac{r_B}{r_A}\right)\right], \quad (42)$$

while the contribution to the phase only due to the quadratic theories correction is

$$\varphi_Q = \frac{\Delta m^2 L_p}{2E_\ell}\left[\frac{1}{L_p}\int_{r_A}^{r_B}\phi_Q(r)dr - \phi_Q(r_B)\right]. \quad (43)$$

Let us remark that we are working in the linear regime, where we can always perform analytical computations. Interesting physical scenarios in which our analysis and outcomes may be tested are the ones of solar and supernova neutrinos, the latter being relevant due to the extremely large fluxes of particles produced in a wide range of energies. A detailed study of these aspects, however, goes beyond the scope of the present manuscript and will be treated elsewhere. We refer back to the existing literature for more specific discussions on this (see, for instance, Refs. [13,34,40] and references therein). Furthermore, we emphasize that, consistently with our work assumption, the term $|\varphi_0|$ will be always larger than the gravitational corrections; indeed, we can have the two cases $|\varphi_0| > |\varphi_{GR}| \gtrsim |\varphi_Q|$ and $|\varphi_0| > |\varphi_Q| \gtrsim |\varphi_{GR}|$, which are both compatible with the linearized approximation. Given such inequalities and by making a comparison with experiments, one can eventually put constraints on the free parameters of the given gravitational theory.

B. Link with the equivalence principle violation

Before applying the aforementioned considerations to several quadratic theories of gravity, it is worth focusing attention on a possible connection between the covariant phase (37) and the violation of the strong equivalence principle [19]. In particular, by looking at Eq. (37), we refer to the term proportional to $\phi - \psi$, which in the case of pure GR would be identically zero. However, this difference can be recognized as a clear signal for SEP violation, since the two metric potentials are not equal [19,43].

In view of the last consideration, one can indeed evaluate the Eddington-Robertson-Schiff parameter γ that arises from a post-Newtonian limit and which is related to how much space curvature is produced by the unit rest mass of the gravitational source (for a detailed review of this topic,

see Refs. [19,44]). If we adopt the metric (25), one can show that

$$\gamma = \frac{\psi}{\phi}, \quad (44)$$

but since we have already pointed out that both metric potentials can be decomposed in a term related to GR and a correction due to the presence of quadratic contributions in the gravitational action, the previous equation can be also cast into the (more convenient) form

$$\gamma - 1 = \frac{\psi_Q - \phi_Q}{\phi}. \quad (45)$$

As expected, if the gravitational action is the Einstein-Hilbert one, we have $\phi_Q = \psi_Q = 0$, which means $\gamma = 1$, that is the known value of such a parameter in the case of GR.

At this point, in order to properly quantify the violation of SEP, it is customary to analyze the so-called Nordtvedt parameter η [45,46], defined as

$$\eta = 4(\beta - 1) - (\gamma - 1), \quad (46)$$

where the post-Newtonian parameter β quantifies nonlinear gravitational effects. The strong equivalence principle is violated as long as $\eta \neq 0$ [46].

In reporting the expression (46), we have tacitly required the absence of anisotropies and preferred-frame effects [19,44], which should have been described by other post-Newtonian parameters that have been set to zero in the current analysis (see Ref. [46] for more details). If we perform the further assumption that nonlinear effects are essentially described by the contributions coming from GR, then $\beta = 1$ [19], which in turn entails

$$\eta = 1 - \gamma. \quad (47)$$

In principle, for a given metric as in Eq. (25), the quantity γ depends on the position, namely, $\gamma \equiv \gamma(r)$. However, we can assume to investigate the scenario in which γ is slowly varying with respect to the spatial coordinates.[3] Therefore, we may treat it as a constant, in such a way as to render all the considerations centered around η enforceable. Indeed, the analysis performed on the SEP violation with the aid of the Nordtvedt parameter has been developed by taking the post-Newtonian expansion coefficients to be constant.

Now, we can observe from Eq. (45) that the deviation from the GR prediction is strictly related to the difference of the metric potentials associated to the quadratic part of the gravitational action. By means of Eq. (47), such a discrepancy is an evident indication of the SEP violation. Hence,

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[3]In other words, we can restrict attention to the spatial region in which variations of γ are negligible.

from Eq. (37), it follows that the neutrino oscillation phase does *discriminate* whether the particle propagates in the conditions in which SEP is satisfied or not. This explains the meaning of φ_{SEP} in Eq. (38).

The reason for the occurrence of SEP violation could be readily attributed to the emergence of a nonstandard term in the Dirac Hamiltonian that explicitly depends upon the difference $\phi - \psi$; this aspect will be investigated in future works.

C. Matter effects

Let us now discuss how to generalize our previous considerations when effects of background matter are taken into account as shown in Sec. II B.

Neutrino flavor states evolution can be described as [14]

$$|\nu_f(\lambda_B)\rangle = \exp\left[\int_{\lambda_A}^{\lambda_B}\left(\frac{M_f^2}{2} - \frac{dx_{\text{null}}^\mu}{d\lambda}A_{f\mu}P_L\right)d\lambda\right]|\nu_f\rangle, \quad (48)$$

where

$$|\nu_f(\lambda)\rangle = \begin{pmatrix} |\nu_e(\lambda)\rangle \\ |\nu_\mu(\lambda)\rangle \end{pmatrix} \quad (49)$$

is the doublet of spinors in the flavor basis and $|\nu_\alpha\rangle = |\nu_\alpha(\lambda_A)\rangle$ ($\alpha = e, \mu$) is defined as in Eq. (1). Here, we have taken into account that the gravitational corrections in the present case are vanishing, i.e., $A_{G\mu} = 0$.

By assuming that the experimental setup is at rest with respect to the electron background, we get

$$|\nu_f(\lambda)\rangle = \exp\left[\frac{1}{2E}\int_{r_A}^{r_B}\frac{\tilde{M}_f^2}{2}dr\right]|\nu_f\rangle, \quad (50)$$

with

$$\tilde{M}_f^2 \equiv M_f^2(1 + \phi - \psi) + V_f(1 - \psi), \quad (51)$$

where V_f is defined by

$$V_f \equiv \begin{pmatrix} v(r) & 0 \\ 0 & 0 \end{pmatrix}, \quad (52)$$

and $v(r) = 2\sqrt{2}EG_F n_e(r)P_L$.

We now diagonalize the matrix (51) via the transformation

$$\tilde{M}^2 = \begin{bmatrix} \tilde{m}_1^2 & 0 \\ 0 & \tilde{m}_2^2 \end{bmatrix} \equiv U^\dagger(\tilde{\theta})\tilde{M}_f^2 U(\tilde{\theta}), \quad (53)$$

where

$$\tan 2\tilde{\theta} = \frac{\Delta m^2[\Delta m^2\cos 2\theta - v(1 + \phi)]\sin 2\theta}{(\Delta m^2\cos 2\theta - v)^2}. \quad (54)$$

By following Ref. [14], we assume the resonance condition $\tilde{\theta} = \pi/4$, which leads to

$$v(r) \equiv v = \Delta m^2\cos 2\theta. \quad (55)$$

This means that the interaction between the electron background and gravity is negligible in the weak field approximation, thus yielding a constant value of v.

Equation (50) can be rewritten as

$$|\nu_f(\lambda)\rangle = U(\tilde{\theta})\exp\left[\frac{1}{2E}\int_{r_A}^{r_B}\frac{\tilde{M}^2}{2}dr\right]|\tilde{\nu}_m\rangle, \quad (56)$$

where

$$|\tilde{\nu}_m\rangle = \begin{pmatrix} |\tilde{\nu}_1\rangle \\ |\tilde{\nu}_2\rangle \end{pmatrix} \equiv U^\dagger(\tilde{\theta})|\nu_f\rangle. \quad (57)$$

Therefore, the flavor oscillation probability takes the form

$$\mathcal{P}_{\alpha\to\beta} = \sin^2\left(\frac{\tilde{\varphi}_{12}}{2}\right), \quad (58)$$

where we have exploited the condition $\sin^2(2\tilde{\theta}) = 1$ and

$$\tilde{\varphi}_{12} = \frac{\Delta\mu^2}{2E}\int_{r_A}^{r_B}[1 + \phi(r) - \psi(r) + \chi\phi(r)]dr. \quad (59)$$

Here, we have used the following notation:

$$\chi \equiv \frac{\Delta m^2(\Delta m^2 + v\cos 2\theta)}{(\Delta\mu^2)^2} - 1, \quad (60)$$

$$\Delta\mu^2 \equiv (\Delta\tilde{m}^2)_{\phi=\psi=0} = \sqrt{(\Delta m^2)^2 + v^2 - 2\Delta m^2 v\cos 2\theta}. \quad (61)$$

Now, as in the case of vacuum oscillations, $\tilde{\varphi}_{12}$ can be expanded as

$$\tilde{\varphi}_{12} = \frac{\Delta\mu^2}{2E}(r_B - r_A) + \frac{\chi\Delta\mu^2}{2E}\int_{r_A}^{r_B}\phi(r)dr + \tilde{\varphi}_{\text{SEP}}, \quad (62)$$

where we have defined

$$\tilde{\varphi}_{\text{SEP}} \equiv \frac{\Delta\mu^2}{2E}\int_{r_A}^{r_B}[\phi(r) - \psi(r)]dr. \quad (63)$$

In terms of local quantities, Eq. (62) becomes

$$\tilde{\varphi}_{12} = \frac{\Delta\mu^2 L_p}{2E_\ell}\left[1 - \phi(r_B) + \frac{(1 + \chi)}{L_p}\int_{r_A}^{r_B}\phi(r)dr\right]. \quad (64)$$

Note that, when matter effects are negligible or absent $(v \to 0)$, then $\Delta\mu^2 \to \Delta m^2$ and $\chi \to 0$, thus recovering Eq. (41). Let us also remark that considerations on the possibility of constraining the free parameters of extended theories can be easily repeated as in the case of vacuum oscillations.

V. APPLICATIONS

To better explore the above scenario, in the following, we determine φ_Q and $\varphi_{\rm SEP}$ appearing in the oscillation formula for several quadratic theories of which the relevance has been pointed out in the recent literature. For simplicity, we only compute the oscillation phases in vacuum as their generalization to the presence of matter is straightforward. Indeed, for each gravitational theory we consider below, once the vacuum formula (41) for the phase is known, one can obtain the corresponding expression including matter effects by making the following substitutions[4]:

$$\Delta m^2 \to \Delta\mu^2, \tag{65}$$

$$\frac{1}{L_p}\int_{r_A}^{r_B}\phi(r)dr \to \frac{1+\chi}{L_p}\int_{r_A}^{r_B}\phi(r)dr. \tag{66}$$

Before going any further, it is worth mentioning that all the theories we will study are characterized by new physical scales which are described by free parameters. The best constraints on such parameters come from torsion balance experiments [47] with which Newton's law has been tested up to roughly 10 mm.

A. $f(\mathcal{R})$ gravity

We first address the easiest extension of the Einstein-Hilbert action by including a Ricci squared contribution with a constant form factor α

$$\mathcal{F}_1 = \alpha, \mathcal{F}_2 = 0 \Rightarrow f = 1, g = 1 - 2\alpha\Box. \tag{67}$$

This choice belongs to the class of $f(\mathcal{R})$ theories, in which the Lagrangian is truncated up to the order $\mathcal{O}(\mathcal{R}^2)$,

$$f(\mathcal{R}) \simeq \mathcal{R} + \frac{\alpha}{2}\mathcal{R}^2, \tag{68}$$

and the cosmological constant is set to zero.

For the above selection of the form factors, the two metric potentials in Eqs. (29) and (30) become

$$\phi(r) = -\frac{Gm}{r}\left(1 + \frac{1}{3}e^{-m_0 r}\right), \tag{69}$$

$$\psi(r) = -\frac{Gm}{r}\left(1 - \frac{1}{3}e^{-m_0 r}\right), \tag{70}$$

where $m_0 = 1/\sqrt{3\alpha}$ is the mass of the spin-0 massive d.o.f. coming from the Ricci scalar squared contribution.

The Eddington-Robertson-Schiff parameter γ for this model turns out to be

$$\gamma = \frac{1 - \frac{1}{3}e^{-m_0 r}}{1 + \frac{1}{3}e^{-m_0 r}} \simeq 1 - \frac{2}{3}e^{-m_0 r}, \tag{71}$$

where after the second equality we have performed an expansion for small values of the exponential function correction. Such a limit is feasible because we expect m_0 to be large. Note that the GR limit (and therefore $\gamma = 1$) is restored for $m_0 \to \infty$.

By using $\phi = \phi_{\rm GR} + \phi_Q$, with

$$\phi_Q(r) = -\frac{1}{3}\frac{Gm}{r}e^{-m_0 r}, \tag{72}$$

and relying on Eq. (43), we obtain

$$\varphi_Q = \frac{\Delta m^2 L_p}{2E_\ell}\left\{\frac{Gme^{-m_0 r_B}}{3r_B} - \frac{Gm}{3L_p}[\text{Ei}(-m_0 r)]_{r_A}^{r_B}\right\}, \tag{73}$$

where the special function

$$\text{Ei}(x) = -\int_{-x}^{\infty}\frac{e^{-\xi}}{\xi}d\xi \tag{74}$$

is known as the exponential integral function [48] and we have introduced the shorthand notation

$$[f(x)]_{x_A}^{x_B} \equiv f(x_B) - f(x_A). \tag{75}$$

Moreover, from Eq. (38), one can evaluate the SEP violating phase as follows:

$$\varphi_{\rm SEP} = \frac{\Delta m^2 Gm}{3E_\ell}[\text{Ei}(-m_0 r)]_{r_A}^{r_B}. \tag{76}$$

This term can be identified with the second contribution in the rhs of Eq. (73).

B. Stelle's fourth-order gravity

Let us now consider Stelle's fourth-order gravity [21], which is achieved with the following form factors:

$$\mathcal{F}_1 = \alpha, \qquad \mathcal{F}_2 = \beta \Rightarrow f = 1 + \frac{1}{2}\beta\Box,$$

$$g = 1 - 2\alpha\Box - \frac{1}{2}\beta\Box. \tag{77}$$

Unlike the $f(\mathcal{R})$ case, the Ricci tensor squared contribution in the action is clearly recognizable through a constant,

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[4]We recall that the following analysis is carried out by assuming the resonance condition (55).

nonvanishing form factor. It is possible to check that the gravitational action related to this model turns out to be renormalizable [21].

For the above choice of the form factors, the two metric potentials in Eqs. (29) and (30) now read

$$\phi(r) = -\frac{Gm}{r}\left(1 + \frac{1}{3}e^{-m_0 r} - \frac{4}{3}e^{-m_2 r}\right), \quad (78)$$

$$\psi(r) = -\frac{Gm}{r}\left(1 - \frac{1}{3}e^{-m_0 r} - \frac{2}{3}e^{-m_2 r}\right), \quad (79)$$

where $m_0 = 2/\sqrt{12\alpha + \beta}$ and $m_2 = \sqrt{2/(-\beta)}$ correspond to the masses of the spin-0 and of the spin-2 massive mode, respectively. To avoid tachyonic solutions, we need to require $\beta < 0$. Additionally, the spin-2 mode is a ghostlike d.o.f. Such an outcome is not surprising, since it is known that, for any local higher derivative theory of gravity, ghostlike d.o.f. always appear.[5]

The factor γ appearing in Eq. (44) for Stelle's fourth-order gravity is given by

$$\gamma = \frac{1 - \frac{1}{3}e^{-m_0 r} - \frac{2}{3}e^{-m_2 r}}{1 + \frac{1}{3}e^{-m_0 r} - \frac{4}{3}e^{-m_2 r}} \simeq 1 - \frac{2}{3}e^{-m_0 r} + \frac{2}{3}e^{-m_2 r}. \quad (80)$$

As for the previous case, the limit of large masses m_0, $m_2 \to \infty$ returns GR.

If we single out the contribution of this quadratic model to the potential ϕ, we note that

$$\phi_Q(r) = -\frac{1}{3}\frac{Gm}{r}e^{-m_0 r} + \frac{4}{3}\frac{Gm}{r}e^{-m_2 r}. \quad (81)$$

Hence, the phase φ_Q turns out to be

$$\varphi_Q = \frac{\Delta m^2 L_p}{2E_\ell}\left\{\frac{Gme^{-m_0 r_B}}{3r_B} - \frac{4Gme^{-m_2 r_B}}{3r_B}\right.$$
$$\left. - \frac{Gm}{3L_p}[\text{Ei}(-m_0 r)]_{r_A}^{r_B} + \frac{4Gm}{3L_p}[\text{Ei}(-m_2 r)]_{r_A}^{r_B}\right\}. \quad (82)$$

The SEP violating phase (38) is now

$$\varphi_{\text{SEP}} = \frac{\Delta m^2 Gm}{3E_\ell}[\text{Ei}(-m_0 r) - \text{Ei}(-m_2 r)]_{r_A}^{r_B}. \quad (83)$$

C. Sixth-order gravity

Let us now deal with a sixth-order gravity model, which is an example of super-renormalizable theory [51,52],

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[5]See Refs. [49,50] for recent works in which the authors have shown that both renormalizability and unitarity can be made to coexist by implementing a new quantization prescription.

$$\mathcal{F}_1 = \alpha\Box, \qquad \mathcal{F}_2 = \beta\Box$$

$$\Rightarrow f = 1 + \frac{1}{2}\beta\Box^2, \qquad g = 1 - 2\alpha\Box^2 - \frac{1}{2}\beta\Box^2. \quad (84)$$

It is possible to show that the two metric potentials in Eqs. (29) and (30) assume the expressions

$$\phi = -\frac{Gm}{r}\left(1 + \frac{1}{3}e^{-m_0 r}\cos(m_0 r) - \frac{4}{3}e^{-m_2 r}\cos(m_2 r)\right), \quad (85)$$

$$\psi = -\frac{Gm}{r}\left(1 - \frac{1}{3}e^{-m_0 r}\cos(m_0 r) - \frac{2}{3}e^{-m_2 r}\cos(m_2 r)\right), \quad (86)$$

where the masses of the spin-0 and spin-2 d.o.f. are now given by $m_0 = 2^{-1/2}(-3\alpha - \beta)^{-1/4}$ and $m_2 = (2\beta)^{-1/4}$, respectively. Note that, in this case, tachyonic solutions are avoided for $-3\alpha - \beta > 0$, which can be satisfied by the requirement $\alpha < 0$ and $-3\alpha > \beta$, with $\beta > 0$. The current higher-derivative theory of gravity has no real ghost modes around the Minkowski background, but a pair of complex conjugate poles with equal real and imaginary parts [52], and corresponds to the so called Lee-Wick gravity [53]. It is worth noting that in this model the unitarity condition is not violated; indeed, the optical theorem still holds [49,54,55].

The parameter γ related to SEP violation now reads

$$\gamma = \frac{1 - \frac{1}{3}e^{-m_0 r}\cos(m_0 r) - \frac{2}{3}e^{-m_2 r}\cos(m_2 r)}{1 + \frac{1}{3}e^{-m_0 r}\cos(m_0 r) - \frac{4}{3}e^{-m_2 r}\cos(m_2 r)}$$
$$\simeq 1 - \frac{2}{3}e^{-m_0 r}\cos^2(m_0 r) + \frac{2}{3}e^{-m_2 r}\cos^2(m_2 r). \quad (87)$$

For this model, we have

$$\phi_Q(r) = -\frac{1}{3}\frac{Gm}{r}e^{-m_0 r}\cos(m_0 r) + \frac{4}{3}\frac{Gm}{r}e^{-m_2 r}\cos(m_2 r). \quad (88)$$

Accordingly, the gravitational phase due to the quadratic part of the action reads

$$\varphi_Q = \frac{\Delta m^2 L_p}{2E_\ell}\left\{\frac{Gme^{-m_0 r_A}}{3r_B}\cos[m_0 r_B]\right.$$
$$- \frac{4Gme^{-m_2 r_B}}{3r_B}\cos[m_2 r_B]$$
$$- \frac{Gm}{6L_p}[\text{Ei}(k_1 m_0 r) + \text{Ei}(k_2 m_0 r)]_{r_A}^{r_B}$$
$$\left. + \frac{2Gm}{3L_p}[\text{Ei}(k_1 m_2 r) + \text{Ei}(k_2 m_2 r)]_{r_A}^{r_B}\right\}, \quad (89)$$

with

$$k_1 = -1 - i, \qquad k_2 = -1 + i. \qquad (90)$$

The SEP violating phase is now

$$\varphi_{\text{SEP}} = \frac{\Delta m^2 Gm}{3E_\ell} \{[\text{Ei}(k_1 m_2 r) + \text{Ei}(k_2 m_2 r)]_{r_A}^{r_B} - [\text{Ei}(k_1 m_0 r) + \text{Ei}(k_2 m_0 r)]_{r_A}^{r_B}\}. \qquad (91)$$

D. Ghost-free infinite derivative gravity

We now consider an example of ghost-free nonlocal theory of gravity [24–31,56–62]. For the sake of clarity, we adopt the simplest ghost-free choice for the nonlocal form factors [26]

$$\mathcal{F}_1 = -\frac{1}{2}\mathcal{F}_2 = \frac{1 - e^{\Box/M_s^2}}{2\Box} \Rightarrow f = g = e^{\Box/M_s^2}, \qquad (92)$$

where M_s is the scale at which the nonlocality of the gravitational interaction should become manifest. Note that, for the special ghost-free choice in Eq. (92), *no* extra d.o.f. other than the massless transverse spin-2 graviton propagate around the Minkowski background.

Since we have chosen $f = g$, the metric potentials of Eqs. (29) and (30) coincide

$$\phi(r) = \psi(r) = -\frac{Gm}{r}\text{Erf}\left(\frac{M_s r}{2}\right), \qquad (93)$$

where

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad (94)$$

is the error function [48].

Note that, since the metric potentials are equal, we automatically obtain $\gamma = 1$ as in GR, which means that, at least from our study, there is no additional contribution to the neutrino oscillation phase that can be directly related to SEP violation. Indeed, because of our assumptions, for this theory, it is straightforward to check that the reduced Nordtvedt parameter in Eq. (47) identically vanishes.

By introducing the complementary error function [48],

$$\text{Erfc}(x) = 1 - \text{Erf}(x), \qquad (95)$$

one can prove that

$$\phi_Q(r) = \frac{Gm}{r}\text{Erfc}\left(\frac{M_s r}{2}\right). \qquad (96)$$

Therefore, the phase associated to this quadratic model is equal to

$$\varphi_Q = \frac{\Delta m^2 L_p}{2E_\ell}\left\{-\frac{Gm}{r_B}\text{Erfc}\left[\frac{M_s r_B}{2}\right] + \frac{Gm}{L_p}\ln\left(\frac{r_B}{r_A}\right)\right.$$
$$\left. -\frac{Gm}{L_p}\left[\frac{M_s r}{\sqrt{\pi}}{}_2F_2\left(\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};-\frac{M_s^2 r^2}{4}\right)\right]_{r_A}^{r_B}\right\}, \qquad (97)$$

where we have employed the generalized hypergeometric function [48]

$${}_pF_q(a_1,...,a_p;b_1,...,b_q;z) = \sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{z^n}{n!}, \qquad (98)$$

with $(x)_n$ being the Pochhammer symbol [48]

$$(x)_0 = 1, \quad (x)_n = x(x+1)(x+2)...(x+n-1). \qquad (99)$$

E. Nonlocal gravity with nonanalytic form factors

For the last case, we consider two models of a nonlocal infrared extension of Einstein's GR, in which form factors are nonanalytic functions of $\Box$. These theories are inspired by quantum corrections to the effective action of quantum gravity [63–69].

1. First model

The first model is described by the following choice of the form factors:

$$\mathcal{F}_1 = \frac{\alpha}{\Box}, \qquad \mathcal{F}_2 = 0 \Rightarrow f = 1, \qquad g = 1 - 2\alpha. \quad (100)$$

The two metric potentials are infrared modifications of the Newtonian one:

$$\phi(r) = -\frac{Gm}{r}\left(\frac{4\alpha - 1}{3\alpha - 1}\right), \qquad (101)$$

$$\psi(r) = -\frac{Gm}{r}\left(\frac{2\alpha - 1}{3\alpha - 1}\right). \qquad (102)$$

Since we expect α to be small, we can deduce that the Eddington-Robertson-Schiff parameter for this model is represented by

$$\gamma = \frac{2\alpha - 1}{4\alpha - 1} \simeq 1 + 2\alpha. \qquad (103)$$

Starting from (101) and (102), we obtain

$$\phi_Q(r) = -\frac{\alpha}{3\alpha - 1}\frac{Gm}{r}, \qquad (104)$$

and consequently

$$\varphi_Q = \frac{\alpha}{3\alpha - 1}\varphi_{\text{GR}}. \qquad (105)$$

The SEP violating phase takes the form

$$\varphi_{\text{SEP}} = -\frac{\Delta m^2 G m}{E_\ell} \frac{\alpha}{3\alpha - 1} \ln\left(\frac{r_B}{r_A}\right). \tag{106}$$

2. Second model

The nonlocal form factors for the second model are

$$\mathcal{F}_1 = \frac{\beta}{\Box^2}, \quad \mathcal{F}_2 = 0 \Rightarrow f = 1, \quad g = 1 - \frac{2\beta}{\Box}. \tag{107}$$

In this framework, the infrared modification is not a constant, but the metric potentials show a Yukawa-like behavior,

$$\phi(r) = -\frac{4}{3}\frac{Gm}{r}\left(1 - \frac{1}{4}e^{-\sqrt{3\beta}r}\right), \tag{108}$$

$$\psi(r) = -\frac{2}{3}\frac{Gm}{r}\left(1 + \frac{1}{2}e^{-\sqrt{3\beta}r}\right). \tag{109}$$

Also, for the current nonlocal model, GR is recovered in the limit $\beta \to 0$. Therefore, an expansion around this parameter allows us to cast γ of Eq. (44) in the following form:

$$\gamma = \frac{1 + \frac{1}{2}e^{-\sqrt{3\beta}r}}{2 - \frac{1}{2}e^{-\sqrt{3\beta}r}} \simeq 1 - \frac{2}{3}\sqrt{3\beta}r. \tag{110}$$

The gravitational potential associated to the purely quadratic part of this model reads

$$\phi_Q(r) = -\frac{1}{3}\frac{Gm}{r}(1 - e^{-\sqrt{3\beta}r}). \tag{111}$$

The phase related to the previous potential is given by

$$\varphi_Q = \frac{\Delta m^2 L_p}{2E_\ell}\left\{\frac{Gm}{3r_B}(1 - e^{-\sqrt{3\beta}r_B})\right.$$
$$\left. -\frac{Gm}{3L_p}\ln\left(\frac{r_B}{r_A}\right) + \frac{Gm}{3L_p}[\text{Ei}(-\sqrt{3\beta}r)]_{r_A}^{r_B}\right\}. \tag{112}$$

The SEP violating phase now reads

$$\varphi_{\text{SEP}} = \frac{\Delta m^2 G m}{3E_\ell}\left\{[\text{Ei}(-\sqrt{3\beta}r)]_{r_A}^{r_B} - \ln\left(\frac{r_B}{r_A}\right)\right\}. \tag{113}$$

VI. CONCLUSIONS

In this work, we have studied neutrino oscillation within the framework of quadratic theories of gravity. Specifically, we have shown to what extent the quadratic part of the action (17) contributes to the covariant phase φ_{12} via the emergence of extra terms into the flavor oscillation probability. In light of this, we have stressed that it is always possible to split φ_{12} into different terms, among which we have recognized the analog of the flat phase φ_0, the GR-induced phase φ_{GR}, and the corrections pertaining to the quadratic sector φ_Q. Calculations have been performed for neutrino oscillations both in vacuum and matter, noticing that formulas in the latter case can be obtained from the corresponding equations in vacuum by accounting for the redefinitions (65) and (66).

Apart from their intrinsic theoretical relevance, it would be interesting to analyze our results in connection with possible experimental applications. For instance, it has been shown that nontrivial gravitational contributions to the neutrino oscillation phase might have significant effects in supernova explosions [40], for which also matter effects play an important role. In light of this and by exploiting the existing data on neutrino oscillations, the present study may provide an important step toward a deeper understanding of gravity, since it could help us to shed some light in the current zoo of theories, both validating or ruling them out at a fundamental level. This aspect, however, will be investigated in more detail in a future work.

Another crucial result we have pointed out is the possibility of identifying a contribution associated to the violation of the strong equivalence principle in the expression of the oscillation phase. Indeed, for different gravitational potentials, $\phi \neq \psi$, we have observed that the Nordtvedt parameter η does not vanish, which in turn implies SEP violation. This occurrence has been achieved by requiring all post-Newtonian terms of the examined models to be equivalent to the GR ones, except for the Eddington-Robertson-Schiff parameter γ. A more rigorous treatment which includes the whole set of post-Newtonian expansion coefficients would require a full-fledged analysis that goes beyond the linearized approximation. However, the generality of the aforesaid outcome is not affected by the regime in which we have investigated such an intriguing issue.

Finally, we have implemented the above reasoning in several quadratic theories of gravity. The purpose of this application is to draw attention on the expressions for the neutrino oscillation phase φ_Q related to the quadratic part of the action. Furthermore, we have explicitly written the contribution arising from the presence of SEP violation φ_{SEP}, which enters in φ_Q and not in φ_{GR}, as expected.

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