

Kinematical gravitational charge algebra

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When formulated in terms of connection and coframes, and in the time gauge, the phase space of general relativity consists of a pair of conjugate fields: the flux 2-form and the Ashtekar connection. On this phase space, one has to impose the Gauss constraints, the vector, and scalar Hamiltonian constraints. These are respectively generating local SU(2) gauge transformations, spatial diffeomorphisms, and time diffeomorphisms. We write the Gauss and space diffeomorphism constraints as conservation laws for a set of boundary charges, representing spin and momenta, respectively. We prove that these kinematical charges generate a local Poincaré ISU(2) symmetry algebra. This gives strong support to the recent proposal of Poincaré charge networks as a new realm for discretized general relativity [*Classical Quantum Gravity* **36**, 195014 (2019)].

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I. INTRODUCTION

General relativity is a fully constrained theory of spacetime geometry, which means that its whole physical content is encoded in its gauge transformations and the resulting symmetries and conserved charges induced on the boundaries of space-time. The goal of the present work is to revisit the Hamiltonian analysis of general relativity in its first order formulation *à la* Cartan in terms of coframe and connection, with a special focus on boundary terms and symmetries.

This analysis is based on a new perspective in the way one handles differentiability of gauge generators in the presence of boundaries. In this approach one allows for the presence of boundary degrees of freedom (d.o.f.), the edge modes, which are acting upon by the boundary symmetry charges. There is growing understanding that edge modes must play a key role in quantum gravity. They are central in the quantization of 2d Yang-Mills and 2d gravity [1–3]. They play a key role in 3d quantum gravity [4–7]. They are essential to the understanding of boundary dynamics and to the construction of defects operators in Chern-Simons theory [8–11]. In four and higher space-time dimensions, edge modes have been argued to be essential to understanding black hole entropy [12–15]. Finally they are now understood to be a key ingredient in the holographic nature of gravity [16–19] and they provide a new understanding in the quantization of geometric observables [20–22].

In [20], relying on the assumption that curvature on the boundary is distributional (i.e., it vanishes on the boundary, except at the location of a given set of points or punctures), we exploited the local holographic nature of gravity to put forward an extension of the kinematical quantum d.o.f. given by the spin networks in loop quantum gravity. The analysis of [20], motivated by a realization of quantum gravity as dynamical networks of quantum edge modes, led to a novel construction of tubular networks dressed by representations of the Euclidean $\mathfrak{iso}(3)$ algebra generated by the fluxes and momentum operators (plus the additional higher mode quantum numbers). This is done by gluing together 3D regions bounded by 2D surfaces with punctures where the edge modes live and which are presumed to carry all the dynamical information of the 3D bulk geometry. The most natural reduction or coarse-graining which traces out the higher modes of these tubular networks leads to these generalized spin networks, dubbed *Poincaré networks*, which then carry a representation of the SU(2) charges from the Gauss constraint and also of the momentum charges associated with the diffeomorphism constraint.

In the present paper, we review the classical setup and provide further details, both at the conceptual and the technical level, of the novel framework developed in [20]. The key idea we develop is twofold. First we show that the constraints on the phase space of general relativity can be written as conservation laws of local charges. These charges can be integrated along a 2-dimensional surface after smearing with local symmetry parameters. Then we also show that the conserved charges, properly supplemented with boundary terms, do generate the correct gauge transformations. These boundary terms involve edge modes

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propagating on the boundary and carrying a boundary symplectic form. The edge modes are would-be-gauge d.o.f. induced on the boundary, shadows of the bulk gauge transformations, which ensure that the conserved charges are properly differentiable with respect to field variations, both in the bulk and on the boundary.

In our analysis, we assume the time gauge, i.e., we fix the time direction in the internal space to the 4-vector $n^I = (1, 0, 0, 0)$, which partially gauge-fixes the local Lorentz transformations down to local $SU(2)$ transformations. The phase space of general relativity then consists of the canonical pair of conjugate fields, the Ashtekar-Barbero $SU(2)$ connection A and the cotriad or flux 2-form $\Sigma = e \wedge e$, supplemented with first class constraints generating local $SU(2)$ gauge transformations and space-time diffeomorphisms. We focus on the kinematics of the theory, analyzing the $SU(2)$ gauge transformations and space diffeomorphisms while postponing studying the fate of time diffeomorphisms and the time evolution of the geometry to future work. In the first part of this work, we show that the constraints generating the $SU(2)$ gauge transformations and space diffeomorphisms can be recast in terms of conservation of boundary charges satisfying a Poincaré algebra $ISU(2)$. While the $SU(2)$ sector of Poincaré obviously corresponds to the local gauge transformations, the space diffeomorphisms are now written as field dependent translations.

More precisely, we consider a bounded 3D region \mathcal{B} within the canonical space-like Cauchy hypersurface and call its boundary S . We establish the validity of the Poincaré algebra for the boundary charges G_α and P_φ which are respectively the local generators of local internal rotations and translations:

$$\begin{aligned} \{G_\alpha, G_\beta\} &= G_{(\alpha \times \beta)}, & \{G_\alpha, P_\varphi\} &= P_{(\alpha \times \varphi)}, \\ \{P_\varphi, P_\xi\} &= 0, \end{aligned} \quad (1)$$

where $\alpha, \beta, \varphi, \xi$ are $\mathfrak{su}(2)$ -valued scalar fields on the hypersurface and $\alpha \times \beta$ denotes the cross product. Now, coming back to the diffeomorphisms, we can map an arbitrary vector field $\hat{\varphi} = \hat{\varphi}^a \partial_a$ to a $\mathfrak{su}(2)$ -valued scalar field $\varphi^i = \iota_{\hat{\varphi}} e^i = \hat{\varphi}^a e_a^i$ and identify the charge corresponding to the 3D diffeomorphism as a field-valued translational charge:

$$D_{\hat{\varphi}} = P_{\iota_{\hat{\varphi}} e}. \quad (2)$$

This allows us to show that, while the translational charges commute with each other, the covariant diffeomorphism algebra admits an extension proportional to the boundary curvature on S :

$$\{D_{\hat{\xi}}, D_{\hat{\varphi}}\} = D_{[\hat{\xi}, \hat{\varphi}]} + G_{\iota_{\hat{\xi}} \iota_{\hat{\varphi}} F}. \quad (3)$$

This realization of the algebra of conserved charges as a Poincaré algebra comes in support of the recent proposal of

discretizing general relativity in terms of Poincaré charge networks [20].

In the Sec. II below, we review the bulk phase space for general relativity with the constraints written as conservation laws for electric, magnetic and translation charges. In the following Sec. III, we introduce the edge modes and correct the conserved charges with the appropriate boundary terms. We show that they generate as expected the $SU(2)$ gauge transformations and 3D diffeomorphisms. In the final Sec. IV, we compute the algebra of charges and show that they are effectively repackaged as a Poincaré algebra.

II. BULK PHASE SPACE AND CONSTRAINTS

Given a spacelike Cauchy hypersurface M in a $3+1$ decomposition of spacetime, we consider a bulk 3D region $\mathcal{B} \subset M$ with a 2D boundary S . The bulk phase space of first order gravity in connection formulation is parametrized by the Ashtekar–Barbero $SU(2)$ connection and the $\mathfrak{su}(2)$ -valued flux 2-form, namely

$$A^i := \Gamma^i + \gamma K^i, \quad \Sigma_i = \frac{1}{2} (e \times e)_i, \quad (4)$$

where we denote the $SU(2)$ bracket $(f \times g)^i = \epsilon^{ijk} f^j \wedge g^k$ for arbitrary $\mathfrak{su}(2)$ -valued forms f and g , with the indices $i, j, k \in \{1, 2, 3\}$ labeling the Pauli matrices as a basis of the $\mathfrak{su}(2)$ Lie algebra. Above, Γ^i is the 3d spin connection, e^i the normalized¹ 3d-frame field, and K^i the extrinsic curvature one-form:

$$d_{\Gamma} e^i = 0, \quad K^i := d_{\omega} n^i, \quad (5)$$

with n^i the hypersurface internal normal and ω the Lorentz connection. In the time gauge we have $e^0 = n$.

The Ashtekar-Barbero connection and the flux 2-form form a pair of conjugate fields, so that the bulk presymplectic 2-form reads (see [23] for a review of the canonical analysis of general relativity):

$$\Omega_{\mathcal{B}} = \int_{\mathcal{B}} (\delta A^i \wedge \delta \Sigma_i). \quad (6)$$

We focus our attention on the kinematical theory, which means that the phase space is restricted by the action of two sets of kinematical constraints: the Gauss law and the space diffeomorphism constraints. We postpone to future work the investigation of the full dynamical theory that takes into account the time diffeomorphism constraint. These kinematical constraints are usually [23] written in terms of the canonical variables A_a^i and the densitized triad $\tilde{E}_i^a := \epsilon^{abc} \Sigma_{bc}^i$ as

¹This means that we have rescaled the coframe field by $e^i \rightarrow \frac{e^i}{\sqrt{\kappa \gamma}}$.

$$G_i = \nabla_a \tilde{E}_i^a = 0, \quad D_a = F_{ab}^i \tilde{E}_i^b = 0. \quad (7)$$

It is nevertheless convenient to keep working with differential forms, in which case the Gauss law is written as a conservation law $G_i := d_A \Sigma_i = 0$ for the Lie algebra valued two-form Σ^i identified as the electric charge aspect. A central point of our approach [20] is to show that the diffeomorphism constraint can similarly be written as a conservation law for a momentum aspect defined as:

$$P_i := d_A e_i. \quad (8)$$

To do so, we use an isomorphism between vector fields, and Lie-algebra valued functions

$$\begin{aligned} \Gamma(T\mathcal{B}) &\rightarrow C(\mathfrak{su}(2)) \\ \hat{\varphi} &\mapsto \varphi^i := \iota_{\hat{\varphi}} e^i. \end{aligned} \quad (9)$$

In the following, we will denote vector fields with a hat $\hat{\varphi} = \hat{\varphi}^a \partial_a$, which should be distinguished from the corresponding scalar functions $\varphi^i := \iota_{\hat{\varphi}} e^i = \hat{\varphi}^a e_a^i$. The diffeomorphism constraint associated with the vector field $\hat{\varphi}$ is denoted $D_{\hat{\varphi}}$. We can now witness the ‘‘caterpillar to butterfly’’ transformation of the 3D diffeomorphism generator:

$$\begin{aligned} D_{\hat{\varphi}} &= (\hat{\varphi}^a F_{ab}^i \tilde{E}_i^b) \\ &= \iota_{\hat{\varphi}} F^i \wedge \Sigma_i \\ &= -F^i \wedge \iota_{\hat{\varphi}} \Sigma_i \\ &= F^i \wedge (e \times \iota_{\hat{\varphi}} e)_i \\ &= (F \times e)_i \varphi^i \\ &= (d_A^2 e_i) \varphi^i \\ &= (d_A P_i) \varphi^i, \end{aligned} \quad (10)$$

where we have used the useful relation $(F \times \eta)^i = d_A^2 \eta^i$ valid for any $\mathfrak{su}(2)$ -valued n -form η^i . This shows that the diffeomorphism constraint follows from a conservation law, namely the conservation of momenta $d_A P_i = 0$.

This momentum aspect $P_i = d_A e_i$ actually measures the torsion of the Ashtekar-Barbero connection A . Let us remember that the Ashtekar-Barbero connection $A = \Gamma + \gamma K$ combines the 3D spin-connection Γ and the extrinsic curvature K . Since the spin-connection is by definition torsionless, $d_{\Gamma} e = 0$, the Ashtekar-Barbero torsion $d_A e_i$ simply measures the extrinsic curvature up to a factor given by the Immirzi parameter, $P = d_A e = \gamma(K \times e)$. This provides a direct geometrical interpretation of the momentum aspect 2-form P in terms of the extrinsic curvature of the hypersurface.

Putting the Gauss law $d_A \Sigma^i = 0$ together with the 3D diffeomorphism constraint $d_A P^i = 0$, we get two conservation laws, to which it is natural to add the Bianchi identity $d_A F^i = 0$ satisfied automatically by the Ashtekar-Barbero

connection. This means that the theory at the kinematical level is defined by the following three conservation laws in the bulk:

$$d_A \Sigma^i \simeq 0, \quad \text{Electric Gauss law}, \quad (11)$$

$$d_A P^i \simeq 0, \quad \text{Translation constraint}, \quad (12)$$

$$d_A F^i \simeq 0, \quad \text{Magnetic Gauss law}. \quad (13)$$

The first two are first class constraints to be imposed on the phase space variables, while the last one is a topological constraint directly implied by the definition of the curvature tensor $F(A)$ and the exterior derivative d for the differential calculus. An interesting relationship exists between these constraints:

$$d_A \Sigma = P \times e, \quad d_A P = F \times e, \quad d_A F = 0, \quad (14)$$

which hints at an intriguing hierarchy and order between these conservation laws. The first identity says that any source to the Gauss constraint is a source of angular momenta since $P \times e$ can be understood as the angular momenta density associated with the momenta P . The second equality is more surprising as it suggests that the momenta density, which appears on the right-hand side (RHS) of the momentum constraints, is itself an ‘‘angular momenta’’ associated with the monopole charge aspect, or curvature F . Note that the last equations stay sourceless since any source would be a gravitational monopole, which is excluded. The full meaning of these equations remains to be unraveled.

The associated charges that are covariantly conserved are the electric, the translational and the magnetic charges²

$$Q_{\alpha}^e = \int_S \alpha^i \Sigma_i, \quad Q_{\varphi}^t = \int_S \varphi^i P_i, \quad Q_{\beta}^m = \int_S \beta^i F_i, \quad (15)$$

where α^i and β^i respectively denote the SU(2) electric and magnetic gauge parameters. Using the isomorphism

²The conservation law in the bulk $d_A \Sigma = 0$ implies by an integration by parts that the boundary charge is given by the bulk integral of the covariant variation of the electric gauge parameter:

$$d_A \Sigma = 0 \Rightarrow Q_{\alpha}^e = \int_S \alpha^i \Sigma_i = \int_B d_A \alpha^i \wedge \Sigma_i.$$

If the electric gauge parameter is held covariantly constant in the bulk, then the boundary charge vanishes $Q_{\alpha}^e = 0$. In particular, if the boundary consists in two disjoint parts, e.g., if the 3D bulk is an open cylinder between an initial surface and a final surface, then the initial surface charge and final surface charge are equal. The same holds for the translational and magnetic boundary charges.

between vector fields $\hat{\varphi} = \hat{\varphi}^a \partial_a$ and $\mathfrak{su}(2)$ -valued scalar functions φ^i , the diffeomorphism charge can then be written as a field dependent translation associated with the vector $\hat{\varphi}$

$$Q_{\hat{\varphi}}^d := Q_{i_{\hat{\varphi}} e}^t = \int_S i_{\hat{\varphi}} e^i d_A e_i = \frac{1}{2} \int_S e^i L_{\hat{\varphi}} e_i, \quad (16)$$

where $L_{\hat{\varphi}} := d_A i_{\hat{\varphi}} + i_{\hat{\varphi}} d_A$ is the covariant Lie derivative along the vector field $\hat{\varphi}$.

As it is well known, any first class constraint also plays the role of canonical generators for an associated gauge transformation. This dual role of the constraints, often phrased by pointing out that each first class constraint kills two d.o.f., reflects the fact that initial data differing by an infinitesimal gauge transformation again solve the constraints. In other words, (A, Σ) and an infinitesimal variation $(A + \delta A, \Sigma + \delta \Sigma)$ represent the same data but in different gauges, as long as

$$\delta_{\alpha} A = \{A, H_{\alpha}\}, \quad \delta_{\alpha} \Sigma = \{\Sigma, H_{\alpha}\}, \quad (17)$$

where H_{α} is the Hamiltonian generating the given transformation δ_{α} , namely

$$\delta H_{\alpha} = \Omega_B(\delta_{\alpha}, \delta) = I_{\delta_{\alpha}} \Omega_B, \quad (18)$$

where we denote by I contraction in field space, namely $I_{\delta} \Omega$ represents the contraction between the (field space) vector field δ and the (field space) 2-form Ω .

In order for δ_{α} to be a gauge transformation one needs its Hamiltonian generator H_{α} to vanish identically on solutions. On the other hand, the Hamiltonian generator of a symmetry transformation that does not vanish on-shell generates truly physical canonical transformations that change the system. In the presence of a boundary, it is well known that some of the would-be gauge transformations are in fact symmetries. This becomes especially important when analyzing the edge modes propagating on the boundary, which are the modes conjugated to the boundary symmetry generators. The Poisson bracket of two Hamiltonian generators is given by:

$$\{H_{\alpha}, H_{\beta}\} = \delta_{\beta} H_{\alpha} = \Omega_B(\delta_{\alpha}, \delta_{\beta}), \quad (19)$$

where $\delta_{\alpha}, \delta_{\beta}$ are the Hamiltonian vector fields generated by the two Hamiltonians.

As such, the covariant phase space formalism provides efficient tools to identify the gravitational gauge charges and to study their algebra. An important feature of this formalism is the requirement that the Hamiltonian generator be differentiable with respect to (all) field variations, as demanded by the definition of the Poisson bracket. This requirement has far reaching implications in the presence of boundaries [16]. To understand this crucial aspect clearly,

let us first notice that differentiability of the Hamiltonians implies that some of the constraints need to be integrated by parts. For instance, when the bulk region \mathcal{B} is bounded by a surface S , the generator of electric gauge transformations requires an appropriate boundary term in order to take a differentiable form:

$$G_{\alpha} = \int_B \alpha^i \wedge d_A \Sigma_i - \int_S \alpha^i \wedge \Sigma_i = - \int_B d_A \alpha^i \wedge \Sigma_i. \quad (20)$$

We see that in the presence of a boundary and in the absence of edge modes,³ the gauge transformations are only the ones associated with a gauge parameter vanishing at the boundary, while symmetries correspond to transformations with nonvanishing boundary parameter. We could then follow the same strategy also for the translation constraint (12) and the magnetic Gauss law (13) and introduce the corresponding bulk generators

$$P_{\varphi} = - \int_B d_A \varphi^i \wedge d_A e_i, \quad (21)$$

$$F_{\beta} = - \int_B d_A \beta^i \wedge F_i(A). \quad (22)$$

However, even after this integration by parts, the generators (21), (22) are still not differentiable, as their variation still yields a boundary contribution. At this point, to define the generators of translations and magnetic transformations as proper Hamiltonians, one can follow two strategies. The first option is the most standard: one imposes boundary conditions by demanding that the phase space field variations vanish on the boundary. In particular, $\delta e^i \stackrel{S}{=} 0$ ensures the differentiability of the translation generators. While $\delta A^i \stackrel{S}{=} 0$ ensures the differentiability of the magnetic gauss generator (here $\stackrel{S}{=}$ denotes an equality for forms pulled-back to S). However, such substantial restrictions can kill boundary d.o.f., which may play an important physical role (see, e.g., [24] for the role of edge modes in the description of the quantum Hall effect). Physically, this means that we consider physical processes restricted to a charge superselection sector. That is processes that do not change the value of the boundary charges: $\delta Q_{\alpha} = 0$. The second strategy aims to allow such processes and more flexible boundary conditions by extending the bulk phase space with boundary edge modes. This allows for a physical interpretation of the boundary charges, in particular for momentum charges associated with the diffeomorphism constraint, and a symmetric treatment of the Gauss and diffeomorphism constraints. We describe in detail this alternative approach in the next section.

³That is, in the absence of the introduction of explicit new boundary d.o.f.

III. EXTENDED PHASE SPACE

In this section we perform a phase space extension by introducing the electric edge modes. This allows us to define differentiable Hamiltonian generators of both electric and translational gauge transformations, which we compute explicitly. We postpone to future investigation the study of magnetic gauge transformations associated to the Bianchi identity.

A. Boundary presymplectic 2-form

In order to allow for boundary field variations, we can extend the bulk phase space by adding a boundary term to the presymplectic 2-form (6). In the present work, we focus on electric gauge transformations and translations, and we postpone the detailed study of magnetic gauge transformations to future investigation. This means that we will allow arbitrary boundary field variations for the triad field e while keeping the Ashtekar-Barbero connection fixed on the boundary. In this setting, we see from the expression (21) that, in order for the translation generator to become differentiable, we need to parametrize the boundary presymplectic 2-form in terms of a boundary coframe field e^i , which we distinguish from the bulk frame e^i . Therefore, if we allow for arbitrary boundary variations δe^i , while still keeping the boundary connection fixed $\delta A^i \stackrel{S}{=} 0$, and thus the boundary curvature fixed, $\delta F^i(A) \stackrel{S}{=} 0$, we introduce the extended presymplectic 2-form, as shown in the earlier work [21,22]:

$$\Omega = \Omega_B + \Omega_S = \int_B (\delta A^i \wedge \delta \Sigma_i) + \frac{1}{2} \int_S (\delta e_i \wedge \delta e^i). \quad (23)$$

Within this extended phase space, the translation generator (21) becomes differentiable. However, this spoils the Hamiltonian nature of the electric generator (20). In order to remedy this, it is necessary to add a boundary term to (20) in terms of the boundary coframe field. The requirement that the Hamiltonian generator vanishes on-shell dictates the form of such boundary term. More precisely, if we demand the boundary simplicity constraint

$$S^i := \Sigma_i - \frac{1}{2} (e \times e)_i \stackrel{S}{=} 0 \quad (24)$$

to hold,⁴ then the canonical generators for the gauge transformations associated with the electric and translational constraints (11), (12) are given by:

⁴This can be equivalently written as the constraint that the boundary coframe equals the pull-back of the bulk frame field $e^i \stackrel{S}{=} e^i$.

$$G_\alpha = - \int_B d_A \alpha^i \wedge \Sigma_i + \frac{1}{2} \int_S \alpha^i (e \times e)_i, \quad \text{Electric gauge} \quad (25)$$

$$P_\varphi = - \int_B d_A \varphi^i \wedge d_A e_i + \int_S \varphi^i d_A (e_i - e_i), \quad \text{Spatial translation.} \quad (26)$$

We see that with the introduction of the boundary electric edge modes e^i , G_α vanishes on-shell even for parameters α_i that are nonvanishing on S ; explicitly,

$$G_\alpha = \int_B \alpha^i d_A \Sigma_i - \int_S \alpha^i S_i \stackrel{S}{=} 0, \quad (27)$$

where $\stackrel{S}{=}$ denotes on-shell of (24), or equivalently $S^i := e^i - e^i \stackrel{S}{=} 0$. This is in agreement with the general philosophy of [18] where it is shown that the restoration of boundary gauge symmetry goes hand-in-hand with the introduction of edge modes.

On the other hand, the translation Hamiltonian generator (26) does not vanish on-shell in general for translations that do not vanish on the boundary; explicitly,

$$P_\varphi = \int_B \varphi^i d_A P_i - \int_S \varphi^i d_A e_i \stackrel{S}{=} - \int_S \varphi^i P_i, \quad (28)$$

where we have used the translation constraint $d_A P_i \simeq 0$. In [20] the boundary condition $P^i \stackrel{S}{=} 0$ was assumed, where S_P is a punctured boundary sphere. This means that nonzero symmetry charges were associated only to the punctures, where the source of momenta were located. While we are not going to introduce distributional curvature and momentum at the punctures here, as we are simply interested in the classical algebra of the gravitational kinematical charges, one can think of (26) as the generator of gauge translation in the bulk and on the boundary at the locus of vanishing P^i , and as a generator of boundary symmetry otherwise.

Before looking in more detail at the symmetry algebra, let us conclude this part with an interesting observation concerning the momentum P_i (26) which plays a key role in the symplectic structure. P_i is simply the canonical momentum conjugate to the coframe e^i . In fact, assuming the simplicity constraint in the form $e = e$ on the boundary, and denoting ω the integrand of the symplectic structure, we have:

$$\begin{aligned} \omega &= \delta A^i \wedge \delta \Sigma_i + \frac{1}{2} d(\delta e_i \wedge \delta e^i) \\ &= \delta A^i \wedge (e \times \delta e)_i + \frac{1}{2} d_A \delta e_i \wedge \delta e^i - \frac{1}{2} \delta e^i \wedge d_A \delta e_i \\ &= (\delta A \times e)_i \wedge \delta e^i + d_A \delta e_i \wedge \delta e^i \\ &= \delta P_i \wedge \delta e^i. \end{aligned} \quad (29)$$

This shows that the bulk plus boundary phase space (A^i, Σ_i, e^i) , supplemented by the matching condition $e = \mathbf{e}$ on the boundary, can be expressed equivalently in terms of the canonical pair (P_i, e^i) .

In Sec. III C, we establish explicitly that the generators (25), (26) give respectively the electric and the translational gauge transformations. As a first step, we introduce a useful duality map between $\mathfrak{su}(2)$ -valued 1-forms and $\mathfrak{su}(2)$ -valued 2-forms below in Sec. III B.

B. A duality map

In order to write the phase space field transformations in a symplectic manner, we need to introduce the following map from Lie algebra valued 2-forms to Lie algebra valued 1-forms:

$$\begin{aligned} \rho: \Omega_2(\mathfrak{su}(2)) &\rightarrow \Omega_1(\mathfrak{su}(2)) & \text{such that } (\tilde{B} \times e)_i &= B_i. \\ B^i &\mapsto \rho(B)_i := \tilde{B}_i \end{aligned} \quad (30)$$

This ‘‘tilde’’ map is the inverse of $A^i \rightarrow (A \times e)^i$ from $\Omega_1(\mathfrak{su}(2)) \rightarrow \Omega_2(\mathfrak{su}(2))$. It exists as long as e is invertible. An explicit formula can be given. In order to do so, we expand the forms in the corresponding basis

$$B_i = B_i^j \Sigma_j, \quad \tilde{B}^i = \tilde{B}^j e^j. \quad (31)$$

The components of the forms are related using shifts by their traces $B = B_i^i$ and $\tilde{B} = \tilde{B}^i_i$:

$$B_i^j = \tilde{B} \delta_i^j - \tilde{B}^j_i, \quad \tilde{B}^j_i = \frac{B}{2} \delta_i^j - B_i^j. \quad (32)$$

The trace $B = B_i^i$ appears naturally when evaluating the 3-form

$$B_i \wedge e^i = 2\tilde{B}^i \wedge \Sigma_i = B \det(e), \quad (33)$$

where $\det(e) = e^1 \wedge e^2 \wedge e^3$ is the volume form.⁵ This duality map $B^i \rightarrow \tilde{B}_i$ defines a symmetric product on $\Omega_2(\mathfrak{su}(2))$ since

$$\tilde{A}^i \wedge B_i = A_i \wedge \tilde{B}^i = (\tilde{A} \tilde{B} - \tilde{A}^i_j \tilde{B}^j_i) \det(e). \quad (35)$$

Examples of the value of the map on some relevant $\mathfrak{su}(2)$ -valued 2-forms include

⁵One uses that

$$e^i \wedge \Sigma_j = \delta_j^i \det(e). \quad (34)$$

$$\tilde{\Sigma}^i = \frac{1}{2} e^i, \quad (\widetilde{\alpha \times \Sigma})^i = (\alpha \times e)^i, \quad \tilde{P}^i = \widetilde{d_A e^i} = \gamma K^i. \quad (36)$$

Another useful identity that follows from $(\varphi \times B)^i = (\iota_{\hat{\varphi}} B \times e)^i + \iota_{\hat{\varphi}}(e \times B)^i$, where $\hat{\varphi}^i = \iota_{\hat{\varphi}} e^i$, is

$$(\widetilde{\varphi \times B})^i = \iota_{\hat{\varphi}} B^i + \iota_{\hat{\varphi}}(\widetilde{e \times B})^i. \quad (37)$$

C. Gauge transformations

1. Electric gauge transformations

The electric gauge transformations generated by the constraint G_α given in (25) act on the bulk and boundary fields in the following way:

$$\delta_\alpha^e \Sigma_i = (\alpha \times \Sigma)_i, \quad \delta_\alpha^e A^i = -d_A \alpha^i, \quad \delta_\alpha^e e^i = (\alpha \times e)^i. \quad (38)$$

This in turn implies that

$$\delta_\alpha^e e^i = (\alpha \times e)^i, \quad \delta_\alpha^e P_i = (\alpha \times P)_i, \quad \delta_\alpha^e F_i = (\alpha \times F)_i. \quad (39)$$

In particular, e and \mathbf{e} transform in the same way, so that the simplicity constraint is preserved under electric gauge transformations. We also see that all the charge aspects (Σ, P, F) transform in the same manner.

In order to derive these gauge transformations, one evaluates the extended presymplectic form Ω given in (23) on the Hamiltonian vector field generated by the constraint G_α :

$$\begin{aligned} I_{\delta_\alpha} \Omega &= - \int_B d_A \alpha^i \wedge \delta \Sigma_i - \int_B \delta A^i \wedge (\alpha \times \Sigma)_i \\ &\quad + \int_S (\alpha \times \mathbf{e})_i \wedge \delta e^i \\ &= - \int_B (\delta A \times \alpha)^i \wedge \Sigma_i - \int_B d_A \alpha^i \wedge \delta \Sigma_i \\ &\quad + \int_S \alpha^i (\mathbf{e} \times \delta \mathbf{e})_i = \delta G_\alpha. \end{aligned} \quad (40)$$

This establishes (38).

2. Translational gauge transformations

The translational transformations generated by the constraint P_φ given in (26) act on the bulk and boundary fields in the following way:

$$\delta_\varphi^t \Sigma_i = d_A(\varphi \times e)_i, \quad \delta_\varphi^t A^i = (\widetilde{\varphi \times F})^i, \quad \delta_\varphi^t e^i \stackrel{S}{=} d_A \varphi^i. \quad (41)$$

These relations imply that the action of the translations takes the same form on all the electric, momentum and curvature aspects:

$$\begin{aligned}\delta_\varphi^t \Sigma_i &= d_A(\widetilde{\varphi \times \Sigma})_i, & \delta_\varphi^t P_i &= d_A(\widetilde{\varphi \times P})_i, \\ \delta_\varphi^t F_i &= d_A(\widetilde{\varphi \times F})_i.\end{aligned}\quad (42)$$

In order to show that these expressions are gauge transformations canonically generated by P_φ , we evaluate the variation of the momentum constraint:

$$\begin{aligned}\delta P_\varphi &= - \int_B (\delta A \times \varphi)^i \wedge d_A e_i - \int_B d_A \varphi^i \wedge (\delta A \times e)_i \\ &\quad - \int_B d_A \varphi^i \wedge d_A \delta e_i + \int_S d_A \varphi^i \wedge \delta(e_i - e_i) \\ &\quad - \int_S \delta A^i \wedge (\varphi \times s)_i \\ &= - \int_B \delta A_i \wedge (\varphi \times d_A e)_i - \int_B \delta A_i \wedge (d_A \varphi \times e)^i \\ &\quad - \int_B (F \times \varphi)_i \wedge \delta e^i + \int_S d_A \varphi_i \wedge \delta e^i \\ &\quad - \int_S \delta A^i \wedge (\varphi \times s)_i,\end{aligned}\quad (43)$$

where we have denoted the boundary simplicity constraint by $s^i := e^i - e^i$. Using the duality map, we can now write the variation of P_φ as

$$\begin{aligned}\delta P_\varphi &= - \int_B \delta A_i \wedge d_A(\varphi \times e)^i - \int_B (\widetilde{F \times \varphi})_i \wedge (e \times \delta e)^i \\ &\quad + \int_S d_A \varphi^i \wedge \delta e_i - \int_S \delta A^i \wedge (\varphi \times s)_i \\ &= - \int_B \delta A_i \wedge d_A(\varphi \times e)^i + \int_B (\widetilde{\varphi \times F})_i \wedge \delta \Sigma^i \\ &\quad + \int_S d_A \varphi^i \wedge \delta e_i - \int_S \delta A^i \wedge (\varphi \times s)_i \\ &= I_{\delta_\varphi^t} \Omega - \int_S \delta A^i \wedge (\varphi \times s)_i,\end{aligned}\quad (44)$$

where in the last equality we have used the action of the translations given in (41). This shows that P_φ is the generator of translation (41) if the boundary Gauss law $s_i = 0$ is satisfied. From the transformation of the flux 2-form Σ we deduce the transformation of the frame field e :

$$\delta_\varphi^t e^i = d_A \varphi^i + (\widetilde{\varphi \times P})^i. \quad (45)$$

This shows that the translation acts on the momenta as follows:

$$\begin{aligned}\delta_\varphi^t P_i &= \delta_\varphi^t (d_A e)_i = (\delta_\varphi^t A \times e)_i + d_A \delta_\varphi^t e_i \\ &= (\varphi \times F)_i + d_A^2 \varphi^i + d_A (\widetilde{\varphi \times P})_i \\ &= d_A (\widetilde{\varphi \times P})_i,\end{aligned}\quad (46)$$

which allows us to recover the transformation law for the momentum P anticipated in (42).

As one can see from above by comparing the action of the translation given above in (41) and (45), the bulk frame field e does not transform in the same way as the edge mode e on the boundary. This means that boundary translations do not generally preserve the boundary simplicity constraint. More precisely, we find that

$$\delta_\varphi^t s_i = (d_A \varphi \times s)_i + (\varphi \times P)_i, \quad \delta_\varphi^t s^i = (\widetilde{\varphi \times P})^i. \quad (47)$$

As we will see in the next section, this puzzling feature does not affect the computation of the charge algebra, but it simply means that the charge algebra only closes on the support of the boundary constraint $s^i = 0$.

D. Translation versus diffeomorphism

Now that we have derived the expressions of the translational gauge transformations, we have to compare them to the diffeomorphisms. First, using the expressions (41), and the identity (37), we can relate the action of translations to the action of the covariant diffeomorphisms given by the covariant Lie derivative $L_{\hat{\varphi}} := \iota_{\hat{\varphi}} d_A + d_A \iota_{\hat{\varphi}}$:

$$\begin{aligned}\delta_\varphi^t e^i &= L_{\hat{\varphi}} e^i - \iota_{\hat{\varphi}} P^i, & \delta_\varphi^t \Sigma_i &= L_{\hat{\varphi}} \Sigma_i - \iota_{\hat{\varphi}} (d_A \Sigma)_i, \\ \delta_\varphi^t A^i &= \iota_{\hat{\varphi}} F^i - \iota_{\hat{\varphi}} (d_A P^i).\end{aligned}\quad (48)$$

We see that translations induce an extra-term besides the Lie derivative, which vanishes on-shell for both the flux 2-form Σ and the Ashtekar-Barbero connection A .

Diffeomorphism can actually be understood as field dependent translations. In order to show this, let us start with the covariant diffeomorphism generator defined by

$$D_{\hat{\varphi}} = - \int_B \iota_{\hat{\varphi}} \Sigma^i \wedge F_i - \frac{1}{2} \int_S e^i \wedge L_{\hat{\varphi}} e_i. \quad (49)$$

This operator generates the covariant diffeomorphism symmetry $\delta_{\hat{\varphi}}^d$ ⁶

⁶ $\delta_{\hat{\varphi}}^d$ is equivalent on-shell to the action of the covariant Lie derivative. More precisely we have that

$$(L_{\hat{\varphi}} - \delta_{\hat{\varphi}}^d) \Sigma_i = \iota_{\hat{\varphi}} d_A \Sigma_i, \quad (L_{\hat{\varphi}} - \delta_{\hat{\varphi}}^d) A^i = 0, \quad (L_{\hat{\varphi}} - \delta_{\hat{\varphi}}^d) e^i = 0. \quad (50)$$

$$\delta_{\hat{\varphi}}^d \Sigma_i = d_A \iota_{\hat{\varphi}} \Sigma_i, \quad \delta_{\hat{\varphi}}^d A^i = \iota_{\hat{\varphi}} F^i(A), \quad \delta_{\hat{\varphi}}^d e^i = L_{\hat{\varphi}} e^i. \quad (51)$$

We derive these diffeomorphism transformation properties (51) by evaluating the extended presymplectic form Ω on the constraint $D_{\hat{\varphi}}$ and using that the boundary value of $\hat{\varphi}$ is tangent to S :

$$\begin{aligned} \delta D_{\hat{\varphi}} &= - \int_B \iota_{\hat{\varphi}} \delta \Sigma_i \wedge F_i - \int_B \iota_{\hat{\varphi}} \Sigma^i \wedge d_A \delta A_i \\ &\quad - \int_S (\delta A^i \times \iota_{\hat{\varphi}} e_i) \wedge e^i - \int_S \delta e^i \wedge L_{\hat{\varphi}} e_i, \\ &= \int_B \iota_{\hat{\varphi}} F_i \wedge \delta \Sigma_i - \int_B \delta A^i \wedge d_A \iota_{\hat{\varphi}} \Sigma_i \\ &\quad - \int_S \delta A^i \wedge \iota_{\hat{\varphi}} S_i + \int_S L_{\hat{\varphi}} e_i \wedge \delta e^i \\ &= I_{\delta_{\hat{\varphi}}} \Omega - \int_S \delta A^i \wedge \iota_{\hat{\varphi}} S_i, \end{aligned} \quad (52)$$

from which we can read the transformations (51). We have used that $S_i = \Sigma_i - \frac{1}{2}(e \times e)_i$.

We can then compute the difference between translations and diffeomorphisms:

$$\begin{aligned} (\delta_{\hat{\varphi}}^d - \delta_{\hat{\varphi}}^t) \Sigma_i &= 0, \quad (\delta_{\hat{\varphi}}^d - \delta_{\hat{\varphi}}^t) A^i = \iota_{\hat{\varphi}} (\widetilde{d_A P^i}), \\ (\delta_{\hat{\varphi}}^d - \delta_{\hat{\varphi}}^t) e^i &= \iota_{\hat{\varphi}} P_i. \end{aligned} \quad (53)$$

As expected, the difference between transformations vanishes on-shell (when $d_A \Sigma = d_A P = 0$) for the bulk variables. The key difference between the translation and diffeomorphism shows up in their action on the boundary variable e . This in turn implies that the diffeomorphism generator always preserves the boundary simplicity constraints, namely $\delta_{\hat{\varphi}}^d s^i = 0$, even if the translations do not.

Finally, it is straightforward to show that the difference between a field dependent translation and the diffeomorphism generator vanishes on-shell of the boundary simplicity constraints.

$$D_{\hat{\varphi}} \hat{=} P_{\iota_{\hat{\varphi}} e}. \quad (54)$$

IV. CHARGE ALGEBRA

Now that we have established the Hamiltonians (25), (26) as the canonical generators of electric and translational gauge symmetry, we can study the algebra of the Hamiltonian charges associated to these gauge symmetries. We rely on the expression of the Poisson bracket between two Hamiltonian generators in terms of the generators variations, namely

$$\{H_{\alpha}, H_{\beta}\} = \delta_{\beta} H_{\alpha}. \quad (55)$$

Moreover, since the generators of gauge transformations $\delta_{\alpha} = \{H_{\alpha}, \cdot\}$ form a closed algebra $[\delta_{\alpha}, \delta_{\beta}] = -\delta_{[\alpha, \beta]}$, the Jacobi identity implies the consistency condition $\{H_{\alpha}, H_{\beta}\} = H_{[\alpha, \beta]} + c(\alpha, \beta)$ where $c(\alpha, \beta)$ is central.

We can thus analyze the various sectors of the algebra formed by the generators G_{α} and P_{φ} by means of the transformation properties under electric gauge transformations (38) and under translations (41) derived in the previous section. This will lead us to show that the constraints G_{α} and P_{φ} form an $\mathfrak{isu}(2)$ Poincaré Lie algebra.

A. Electric-electric sector

The electric-electric sector consists of the Poisson brackets of the electric gauge generators G with each other. This is the simplest case, as it is immediate to check that using the electric gauge transformations (38), we obtain the following brackets:

$$\begin{aligned} \{G_{\alpha}, G_{\beta}\} &= \delta_{\beta}^e G_{\alpha} = - \int_B (\delta_{\beta} A \times \alpha)^i \wedge \Sigma_i - \int_B d_A \alpha^i \wedge \delta_{\beta} \Sigma_i + \int_S \alpha^i (e \times \delta_{\beta} e)_i \\ &= \int_B (d_A \beta \times \alpha)^i \wedge \Sigma_i - \int_B d_A \alpha^i \wedge (\beta \times \Sigma)_i + \int_S \alpha^i (e \times (\beta \times e))_i \\ &= - \int_B d_A (\alpha \times \beta)^i \wedge \Sigma_i + \frac{1}{2} \int_S (\alpha \times \beta)^i (e \times e)_i \\ &= G_{(\alpha \times \beta)}. \end{aligned} \quad (56)$$

The electric-electric sector thus closes and we recover the expected $\mathfrak{su}(2)$ Lie algebra, as for the flux observables in loop quantum gravity.

B. Electric-translation sector

The electric-translation sector consists of the Poisson brackets of the electric gauge generators G_{α} with the translation generators P_{φ} . For this mixed sector we start with the definition of the electric constraint (25) and compute its variation under translations using (41). After a sequence of elementary operations involving the duality map, the cross product and integrations by part, we obtain:

$$\begin{aligned}
\{G_\alpha, P_\varphi\} &= \delta_\varphi^t G_\alpha = - \int_B (\delta_\varphi^t A \times \alpha)_i \wedge \Sigma^i - \int_B d_A \alpha^i \wedge \delta_\varphi^t \Sigma_i + \int_S \alpha^i (\delta_\varphi^t \mathbf{e} \times \mathbf{e})_i \\
&= - \int_B (\widetilde{\varphi \times F})^i \wedge (\alpha \times \Sigma)_i - \int_B d_A \alpha^i \wedge d_A (\varphi \times e)^i + \int_S \alpha^i (d_A \varphi \times \mathbf{e})_i \\
&= - \int_B (\varphi \times F)_i \wedge (\widetilde{\alpha \times \Sigma})^i - \int_B d_A^2 \alpha_i \wedge (\varphi \times e)^i + \int_S d_A \alpha_i \wedge (\varphi \times e)^i + \int_S (\alpha \times d_A \varphi)_i \wedge \mathbf{e}^i \\
&= \int_B (F \times \varphi) \wedge (\alpha \times e)_i - \int_B ((F \times \alpha) \times \varphi)_i \wedge e^i + \int_S d_A (\alpha \times \varphi)_i \wedge e^i + \int_S (\alpha \times d_A \varphi)_i \wedge (e^i - e^i) \\
&= \int_B ((F \times (\varphi \times \alpha))_i \wedge e^i + \int_S d_A (\alpha \times \varphi)_i \wedge e^i + \int_S (\alpha \times d_A \varphi)_i \wedge (e^i - e^i) \\
&= - \int_B d_A (\alpha \times \varphi)^i \wedge d_A e_i - \int_S d_A (\alpha \times \varphi)_i \wedge (e^i - e^i) + \int_S (d_A \alpha \times \varphi)_i \wedge (e^i - e^i), \\
&= P_{(\alpha \times \varphi)} + \int_S (d_A \alpha \times \varphi)_i \wedge s^i.
\end{aligned} \tag{57}$$

The extra-term vanishes on-shell of the boundary Gauss Law $s^i = e^i - e^i \stackrel{S}{=} 0$. Thus, assuming the boundary simplicity $e^i \stackrel{S}{=} e^i$, we recover the expected Poisson bracket, namely

$$\{G_\alpha, P_\varphi\} \stackrel{\text{on-shell}}{=} P_{(\alpha \times \varphi)}. \tag{58}$$

Instead of starting with the generator G_α and computing its variation under translations, we can do the reverse and start with the generator P_φ as defined in (26) and compute

its variation under electric gauge transformations using (38). This allows to check that the gauge transformations (38), (41) are indeed consistent with the antisymmetry of the Poisson bracket, namely that

$$\delta_\varphi^t G_\alpha = -\delta_\alpha^e P_\varphi. \tag{59}$$

So, in order to check this, we plug in the electric variations (38) in the general expression (44) for the variation of the translation generator δP_φ and compute:

$$\begin{aligned}
\{P_\varphi, G_\alpha\} &= \delta_\alpha^e P_\varphi = - \int_B \delta_\alpha^e A_i \wedge d_A (\varphi \times e)^i + \int_B (\widetilde{\varphi \times F})_i \wedge \delta_\alpha^e \Sigma^i + \int_S d_A \varphi^i \wedge \delta_\alpha^e \mathbf{e}_i - \int_S \delta_\alpha^e A^i \wedge (\varphi \times s)_i \\
&= \int_B d_A \alpha^i \wedge d_A (\varphi \times e)_i + \int_B (\varphi \times F)_i \wedge (\alpha \times e)^i + \int_S d_A \varphi^i \wedge (\alpha \times \mathbf{e})_i + \int_S d_A \alpha^i \wedge (\varphi \times s)_i \\
&= \int_B (F \times \alpha)_i \wedge (\varphi \times e)^i - \int_B (F \times \varphi)_i \wedge (\alpha \times e)^i - \int_S d_A \alpha^i \wedge (\varphi \times e)_i + \int_S (d_A \varphi \times \alpha)_i \wedge \mathbf{e}^i \\
&= \int_B (\varphi \times \alpha)_i \wedge (F \times e)^i + \int_S d_A (\varphi \times \alpha)_i \wedge e^i \\
&= - \int_B d_A (\varphi \times \alpha)_i \wedge d_A e^i + \int_S (\varphi \times \alpha)_i \wedge d_A (e - e)^i \\
&= P_{(\varphi \times \alpha)}.
\end{aligned} \tag{60}$$

This shows that the electric and translation transformations are indeed consistent when $s^i = 0$. And we have established that the electric-translation sector closes when the boundary simplicity constraint is satisfied, in which case the bracket (58) holds.

C. Translation-translation sector

Finally, we turn to the translation-translation sector. We now want to prove that the translation generators commute on-shell, that is we want to establish that

$$\{P_\xi, P_\varphi\} \simeq 0, \tag{61}$$

where the equality \simeq means we are on-shell of the three conservation laws (11), (12), (13). Although this is the simplest commutation relation in the end, it turns out that this is the most involved evaluation. We start by plugging the transformations (41) into (44) which, together with the identity (37) for the duality map, yields when $s^i = e^i - e^i \stackrel{S}{=} 0$:

$$\begin{aligned}
\{P_\xi, P_\varphi\} &= \delta_\varphi P_\xi \hat{=} - \int_B (\widetilde{\varphi \times F})_i \wedge d_A(\xi \times e)^i + \int_B (\widetilde{\xi \times F})_i \wedge d_A(\varphi \times e)^i + \int_S d_A \xi^i \wedge d_A \varphi_i \\
&= - \int_B \iota_{\hat{\varphi}} F_i \wedge d_A(\xi \times e)^i + \int_B \iota_{\hat{\xi}} F_i \wedge d_A(\varphi \times e)^i - \int_S (\varphi \times \xi)^i F_i \\
&\quad - \int_B \iota_{\hat{\varphi}} (\widetilde{e \times F})^i \wedge d_A(\xi \times e)^i + \int_B \iota_{\hat{\xi}} (\widetilde{e \times F})^i \wedge d_A(\varphi \times e)^i,
\end{aligned} \tag{62}$$

where we have integrated by parts on the sphere and introduced vectors $(\hat{\varphi}, \hat{\xi})$ related to the translation gauge parameters (φ^i, ξ^i) via $\iota_{\hat{\varphi}} e^i = \varphi^i$.

Let us focus on the term

$$\iota_{\hat{\varphi}} F_i \wedge d_A(\xi \times e)^i = -F_i \wedge \iota_{\hat{\varphi}} d_A \iota_{\hat{\xi}} \Sigma^i. \tag{63}$$

The key commutation of Cartan calculus, $[L_{\hat{\xi}}, \iota_{\hat{\varphi}}] = \iota_{[\hat{\xi}, \hat{\varphi}]}$ between Lie derivative and interior product, implies the identity

$$\iota_{\hat{\xi}} d_A \iota_{\hat{\varphi}} - \iota_{\hat{\varphi}} d_A \iota_{\hat{\xi}} = \iota_{[\hat{\xi}, \hat{\varphi}]} - d_A(\iota_{\hat{\xi}} \iota_{\hat{\varphi}}) - \iota_{\hat{\xi}} \iota_{\hat{\varphi}} d_A. \tag{64}$$

This allows us to compute:

$$\begin{aligned}
\iota_{\hat{\varphi}} F_i \wedge d_A(\xi \times e)^i - \iota_{\hat{\xi}} F_i \wedge d_A(\varphi \times e)^i &= F_i \wedge (\iota_{\hat{\xi}} d_A \iota_{\hat{\varphi}} - \iota_{\hat{\varphi}} d_A \iota_{\hat{\xi}}) \Sigma^i, \\
&= F_i \wedge (\iota_{[\hat{\xi}, \hat{\varphi}]} + \iota_{\hat{\varphi}} \iota_{\hat{\xi}} d_A - d_A \iota_{\hat{\xi}} \iota_{\hat{\varphi}}) \Sigma^i.
\end{aligned} \tag{65}$$

We thus obtain

$$\begin{aligned}
\{P_\xi, P_\varphi\} \hat{=} & \int_B \iota_{[\hat{\xi}, \hat{\varphi}]} F_i \wedge \Sigma^i - \int_B F^i \wedge d_A(\iota_{\hat{\varphi}} \iota_{\hat{\xi}} \Sigma_i) - \int_B F^i \wedge \iota_{\hat{\varphi}} \iota_{\hat{\xi}} d_A \Sigma_i - \int_S (\varphi \times \xi)^i F_i \\
& - \int_B \iota_{\hat{\varphi}} (\widetilde{e \times F})^i \wedge d_A(\xi \times e)^i + \int_B \iota_{\hat{\xi}} (\widetilde{e \times F})^i \wedge d_A(\varphi \times e)^i \\
& = \int_B F_i \wedge \iota_{[\hat{\varphi}, \hat{\xi}]} \Sigma^i + \int_B d_A F^i (\iota_{\hat{\varphi}} \iota_{\hat{\xi}} \Sigma_i) - \int_B \iota_{\hat{\varphi}} \iota_{\hat{\xi}} F^i d_A \Sigma_i \\
& - \int_B \iota_{\hat{\varphi}} (\widetilde{e \times F})^i \wedge d_A(\xi \times e)^i + \int_B \iota_{\hat{\xi}} (\widetilde{e \times F})^i \wedge d_A(\varphi \times e)^i \\
& = - \int_B (\iota_{[\hat{\varphi}, \hat{\xi}]} e^i) d_A P_i + \int_B (\iota_{\hat{\varphi}} \iota_{\hat{\xi}} \Sigma_i) d_A F^i - \int_B (\iota_{\hat{\varphi}} \iota_{\hat{\xi}} F^i) d_A \Sigma_i \\
& + \int_B \iota_{\hat{\varphi}} \widetilde{d_A P^i} \wedge d_A(\xi \times e)^i - \int_B \iota_{\hat{\xi}} \widetilde{d_A P^i} \wedge d_A(\varphi \times e)^i \\
& \simeq 0.
\end{aligned} \tag{66}$$

This expression vanishes on-shell, i.e., when assuming the constraints and Bianchi identity, $d_A \Sigma = d_A P = d_A F = 0$. This concludes our proof that the electric and translational charges form indeed a closed Poincaré algebra.

D. Diffeomorphisms algebra

As we have checked that the electric and momentum constraints, with the appropriate boundary terms, form a

Poincaré Lie algebra on-shell, it is interesting to turn back to the diffeomorphisms. We come back to the expression (49) of the diffeomorphism constraints as field dependent translations, and we would like to verify that we recover the correct diffeomorphisms algebra.

Plugging the diffeomorphism transformations (51) in the general variation of the diffeomorphism generators (52), we can compute the bracket

$$\begin{aligned}
\{D_{\hat{\xi}}, D_{\hat{\varphi}}\} &= \delta_{\hat{\varphi}}^{\hat{\xi}} D_{\hat{\xi}} = \int_B d_A l_{\hat{\varphi}} \Sigma_i \wedge l_{\hat{\xi}} F_i - \int_B l_{\hat{\varphi}} F^i \wedge d_A l_{\hat{\xi}} \Sigma_i - \int_S l_{\hat{\varphi}} F^i \wedge (l_{\hat{\xi}} \mathbf{e} \times \mathbf{e})_i - \int_S L_{\hat{\varphi}} \mathbf{e}^i \wedge L_{\hat{\xi}} \mathbf{e}_i \\
&= \int_B (l_{\hat{\varphi}} d_A l_{\hat{\xi}} \Sigma_i - l_{\hat{\xi}} d_A l_{\hat{\varphi}} \Sigma_i) \wedge F^i - \int_S (d_A l_{\hat{\varphi}} \mathbf{e}_i \wedge d_A l_{\hat{\xi}} \mathbf{e}_i + l_{\hat{\xi}} l_{\hat{\varphi}} d_A \mathbf{e}_i \wedge d_A \mathbf{e}_i) \\
&\quad - \int_S (l_{\hat{\xi}} d_A l_{\hat{\varphi}} \mathbf{e}^i - l_{\hat{\varphi}} d_A l_{\hat{\xi}} \mathbf{e}^i) \wedge d_A \mathbf{e}_i - \frac{1}{2} \int_S l_{\hat{\xi}} l_{\hat{\varphi}} F^i \wedge (\mathbf{e} \times \mathbf{e})_i.
\end{aligned} \tag{67}$$

We now use again the Cartan calculus identity (64) to obtain

$$\begin{aligned}
\{D_{\hat{\xi}}, D_{\hat{\varphi}}\} &= - \int_B l_{[\hat{\xi}, \hat{\varphi}]} \Sigma_i \wedge F^i + \int_B l_{\hat{\xi}} l_{\hat{\varphi}} d_A \Sigma_i \wedge F^i + \int_B d_A (l_{\hat{\xi}} l_{\hat{\varphi}} \Sigma_i) \wedge F^i \\
&\quad - \int_S l_{[\hat{\xi}, \hat{\varphi}]} \mathbf{e}_i d_A \mathbf{e}_i - \int_S d_A \varphi^i \wedge d_A \xi_i - \frac{1}{2} \int_S l_{\hat{\xi}} l_{\hat{\varphi}} F^i \wedge (\mathbf{e} \times \mathbf{e})_i \\
&= D_{[\hat{\xi}, \hat{\varphi}]} + \int_B d_A \Sigma_i l_{\hat{\xi}} l_{\hat{\varphi}} F^i - \int_B l_{\hat{\xi}} l_{\hat{\varphi}} \Sigma_i d_A F^i + \int_S l_{\hat{\xi}} l_{\hat{\varphi}} \Sigma_i F^i + \int_S (\xi \times \varphi)^i F_i - \frac{1}{2} \int_S l_{\hat{\xi}} l_{\hat{\varphi}} F^i \wedge (\mathbf{e} \times \mathbf{e})_i \\
&= D_{[\hat{\xi}, \hat{\varphi}]} + \int_B l_{\hat{\xi}} l_{\hat{\varphi}} F^i d_A \Sigma_i - \int_B l_{\hat{\xi}} l_{\hat{\varphi}} \Sigma_i d_A F^i - \frac{1}{2} \int_S l_{\hat{\xi}} l_{\hat{\varphi}} F^i \wedge (\mathbf{e} \times \mathbf{e})_i \\
&\simeq D_{[\hat{\xi}, \hat{\varphi}]} + G_{l_{\hat{\xi}} l_{\hat{\varphi}} F}.
\end{aligned} \tag{68}$$

In the last line we have used the Bianchi identity—our magnetic Gauss law— $d_A F = 0$ and the definition (27) of the electric gauge generator in terms of the bulk electric Gauss law and boundary simplicity constraint, leading to the gauge generator $G_{l_{\hat{\xi}} l_{\hat{\varphi}} F}$ which generates gauge transformation with gauge parameter $l_{\hat{\xi}} l_{\hat{\varphi}} F$. This is consistent with the action of covariant diffeomorphism since the covariant Cartan calculus implies that

$$[L_{\hat{\xi}}, L_{\hat{\varphi}}] = L_{[\hat{\xi}, \hat{\varphi}]} + l_{\hat{\xi}} l_{\hat{\varphi}} F \times, \tag{69}$$

showing how the connection curvature F deforms the commutator of the covariant Lie derivatives.

V. CONCLUSION

We have revisited general relativity in its first order formulation in terms of frame field and connection in the presence of boundaries. We have seen that the gravity bulk phase space needs to be appropriately extended by a set of boundary edge modes to allow for arbitrary frame field variations on the boundary. We have shown that, quite remarkably, the kinematical constraints can be understood as conservation of boundary charges. We have demonstrated that these charges are the Hamiltonian generators of electric gauge transformations and translations. We have proven that these charges form a closed Poincaré algebra, as anticipated in [20].

One has to wonder whether we can extend these results to the dynamical constraints. It has already been established in [25], that Einstein equations projected along null surfaces can be understood, in the metric formulation, as conservation equations for charges aspects associated with energy and momenta. The canonical structure of these

charges has not been revealed yet. A relevant analysis in the first order as recently been performed in [26].

At this point, a fascinating question would be whether magnetic gauge transformations can also be included or not as phase space transformation. The physical implications of including the dual magnetic sector in the boundary phase space have been explored in [27] for electromagnetism. One may expect the magnetic charges to play an equally important role also in the case of gravity in order to unravel the full boundary symmetry group. Indeed, although the Bianchi law $d_A F = 0$ is a purely geometric identity, it is tempting to interpret it as a magnetic Gauss law. In this scenario, the integrated generator $F_\beta = \int d_A \beta^i \wedge F_i[A]$ would play the role of the generator of magnetic gauge transformations. One expects, following [27], that it is necessary to introduce magnetic edge modes to render this generator differentiable. It seems natural to expect these magnetic edge modes to be encoded in a boundary connection field. Either by providing with a Chern-Simons-like boundary symplectic term [28–30] or with a mixed term coupling the boundary connection to the boundary frame field [31] as was derived in the case of isolated horizon boundary conditions.

Similarly if one wants to ensure that the translations respect the boundary Gauss law, one would also have, according to the general philosophy of boundary charges and symmetries [18], to allow for nontrivial translational edge modes. This is left to future investigation.

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