

Finite action principle revisited

John D. Barrow 

*DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road,
Cambridge CB3 0WA United Kingdom*



(Received 30 December 2019; published 28 January 2020)

We reconsider and extend the cosmological predictions that can be made under the assumption that the total action of the Universe is finite. When initial and final singularities in curvature invariants are avoided, it leads to singularities in the gravitational action of the Universe. The following properties are required of a universe with finite action: Compact spatial sections (i.e., a closed universe) giving a finite total lifetime for the Universe. Compactification of flat and open universes is excluded. The Universe can contain perfect fluids with $-1 < p/\rho < 2$ on approach to singularities. The Universe cannot display a bounce or indefinite cyclic behavior to the past or the future. Here, we establish new consequences of imposing finite action: the Universe cannot be dominated by massless scalar fields or the kinetic energy of self-interacting scalar fields or a $p = \rho$ perfect fluid on approach to the initial or final singularity. The ekpyrotic scenario with an effective fluid obeying $p/\rho > 2$ in a closed, flat or open universe is excluded. Any bouncing loop quantum gravity model with indefinite past or future evolution is ruled out. The Einstein static and steady-state universes are ruled out along with past or future eternal inflating universes anisotropies of Kasner or Mixmaster type cannot dominate the dynamics on approach to singularities. This excludes density inhomogeneity spectra versus mass, of the form $\delta\rho/\rho \propto M^{-q}$, with $q > 2/3$. Higher-order Lagrangian theories of gravity are significantly constrained. Quadratic Lagrangians are excluded with fluids satisfying $p/\rho > -1/3$. Lagrangians with $L_g = R^{1+\delta}$ have infinite actions on approach to a singularity when $2\delta(1 - 3\gamma) + 2 - 3\gamma < 0$, where $p = (\gamma - 1)\rho$ for the fluid. As shown by Barrow and Tipler, the Gauss-Bonnet quadratic combination causes a cosmological action singularity even though it does not contribute terms to the field equations. Scalar-tensor theories like Brans-Dicke dominated by the scalar field on approach to singularities have action singularities. Dark energy cannot be a simple cosmological constant, as it would create an action singularity to the future: the Universe cannot be asymptotically de Sitter as $t \rightarrow \infty$. The dark energy must be an evolving energy density in a closed universe that produces collapse to a future singularity and cannot be dominated by the kinetic energy of the scalar field.

DOI: [10.1103/PhysRevD.101.023527](https://doi.org/10.1103/PhysRevD.101.023527)

I. INTRODUCTION

Earlier, Barrow and Tipler (BT) [1] explored the cosmological consequences of requiring the gravitational and matter actions of the Universe to be finite. One feature of focusing attention upon the action is that there have been many attempts to avoid the appearance of physical infinities (singularities) in cosmological models by means, for example, of dynamical bounces at finite scale factor values or past and future asymptotes that are nonsingular. Yet, typically, these attempts to avoid singularities give rise to infinities in the action. In this paper we briefly summarize our original finite action conjecture of Ref. [1] and update its consequences and predictions in the light of subsequent developments in observational and theoretical cosmology. There has also been a recent rediscovery of this focus on the action, with a specific emphasis on quadratic gravity and a computation of the action between an initial time and the present rather than for the entire spacetime, by Lehnert and Stelle [2].

In what follows we define the finite action proposal in Sec. II, followed by a series of applications to constrain the fluid content of the Universe, the requirement of no massless scalar fields in the Universe, the ruling out of anisotropy domination at singularities and constraints on the statistics of inhomogeneities, the exclusion of sudden finite-time singularities, and the conclusion that the dark energy cannot be contributed by a simple cosmological constant: it must be a variable energy source in a closed universe that ultimately recollapses to a future singularity. In Sec. III we consider the implications of finite action for modified gravity theories that generalize the Einstein-Hilbert Lagrangian to higher order in the curvature, the Gauss-Bonnet combination, and scalar tensor theories. Our conclusions are listed in Sec. IV.

II. THE FINITE ACTION PROPOSAL

As discussed in BT [1], there are a variety of motivations for the fundamental importance of the action, S . Planck

appears to have been the first 20th century physicist to argue for the primacy of the action as the basis for physical theories [3]. In modern physics, the action is always the starting point because it is invariant under gauge transformations of Yang-Mills and supersymmetric fields and its importance in the path-integral method of quantization [4,5]. We know that the finiteness of the action places important constraints on some theories in Minkowski and Euclidean spaces. For example, on solutions to the Yang-Mills-Higgs equations and the classical solutions with finite action lead to a semiclassical approximation to the Euclidean path integral that can be analytically continued back to Minkowski space to get the physical path integral.

In Einstein's gravitational theory the general action of Einstein-Hilbert-York consists of three pieces (with units such that $8\pi G = c = 1$): the gravitational, S_g , matter, S_m , and boundary action terms, respectively. Therefore, we have for the universal action

$$S = \frac{1}{2} \int_M (R + 2\Lambda) \sqrt{-g} d^4x + \int_M L_m \sqrt{-g} d^4x + \int_{\partial M} (\text{tr}K)(\sqrt{\pm h}) d^3x, \quad (1)$$

where R is the four-dimensional Ricci scalar, Λ is the cosmological constant, and L_m is the matter Lagrangian. The boundary of M is ∂M where ∂M has induced metric $h_{\mu\nu}$ and extrinsic curvature K_{ab} . The plus (minus) sign is chosen in $(\pm h)^{1/2}$ if the boundary is spacelike (timelike), respectively. Unless we state otherwise, we drop the boundary term since we want the integrations to be performed over the whole spacetime, so there will be no boundary.

The universal action will be finite if each term in Eq. (1) is finite. Finite action requires the Universe to have compact space sections (i.e., be closed) or else the integrals over 3-space in Eq. (1) will diverge to infinity, so for example the steady-state universe has infinite action. Whereas a finite spatial volume if necessary for finite action is not sufficient. The Einstein static universe is closed but has infinite action when the time integration is carried out over the time-independent quantities $R\sqrt{-g}$ and $\Lambda\sqrt{-g}$ in Eq. (1). An oscillating closed universe that is either nonsingular to the past or the future (or both) also has infinite action, as does a closed universe that undergoes a single bounce at finite radius. These examples illustrate how the avoidance of a curvature singularity generally leads to a singularity in the action.

The imposition of compact topologies on flat or open universes, for example, the flat and open Friedmann universes or the Bianchi-type universes, [6–8], does not produce models with finite action because although the space integral is finite in Eq. (1), these models will expand for ever and create a divergence in the time integration to the future. For an explicit example of the compact, T^3 , the

Bianchi type I model displaying the expected indefinite future expansion which approaches isotropy in the presence of a perfect fluid, see [9].

A. Fluid cosmologies

For homogeneous and isotropic closed universes, we need to consider the behavior of the matter action contribution in Eq. (1). For perfect fluid models with equation of state $p = (\gamma - 1)\rho$ in a Friedmann universe with scale factor $a(t) = t^{2/3\gamma}$ near the initial singularity (the final singularity behavior near t_f is just a linear time translation of this, via $t \rightarrow t_f - t$), where t is the comoving proper time, we have

$$\int R\sqrt{-g}d^4x \propto (\gamma - 4/3)t^{(2-\gamma)/\gamma}, \quad \gamma \neq 0, 2, \quad (2)$$

$$\int L_m\sqrt{-g}d^4x \propto \frac{1}{2} \int (3p - \rho)\sqrt{-g}d^4x \propto t^{(2-\gamma)/\gamma}, \quad \gamma \neq 0, 2. \quad (3)$$

Importantly, we note that for $\gamma = 2$ the $t^{(2-\gamma)/\gamma}$ factor is replaced by $\ln(t)$ which diverges as $t \rightarrow 0$. The $\gamma = 0$ case is de Sitter or anti-de Sitter spacetime and the action also diverges because the range of t integration is infinite, as in nonclosed universes. When $\gamma > 2$ the action also diverges as $t \rightarrow 0$ and so finite universal action also excludes the so-called ekpyrotic models [10] which behave like $\gamma > 2$ perfect fluid cosmologies as $t \rightarrow 0$ discussed in Ref. [11]. Also excluded are oscillating cosmologies [12], and loop quantum gravity cosmologies that experience a bounce and past eternal inflationary scenarios like those in Ref. [13], whose past geodesic incompleteness was well known from the properties of the steady-state universe [14,15].

B. Scalar-field cosmologies

If we have a scalar field, ϕ , with self-interaction potential $V(\phi) \geq 0$, in a closed Friedmann universe, then we see that if the kinetic part of the scalar field action $L_m \propto (d\phi dt)^2$ dominates on approach to a singularity (or $V = 0$), then we will have $a(t) \propto t^{1/3}$ and $\phi \propto \ln(t)$ and the action diverges as $\int t^{-2} \times t dt \propto \ln(t)$ as $t \rightarrow 0$ just like in the $\gamma = 2$ fluid model. In the borderline case where the kinetic and potential energy densities are proportional, $\dot{\phi}^2 \propto V \propto \exp[-\lambda\phi]$, we have the asymptotic solution (which is the $k = 0$ exact solution) [16],

$$\phi \propto \frac{2}{\lambda} \ln(t), \quad a(t) \propto t^{2/\lambda^2}. \quad (4)$$

Hence the matter action term is proportional to $\int \dot{\phi}^2 a^3 dt \propto \int t^{-2+6/\lambda^2} dt \propto t^{6/\lambda^2-1}$ and this diverges as $t \rightarrow 0$ only when $\lambda^2 > 6$. Scalar-field cosmologies that create a bounce as

$t \rightarrow 0$, as discussed Refs [17–20] for $V \propto \phi^2$, are also ruled out by the finite action principle because they produce a $t \rightarrow \infty$ divergence in S .

C. Anisotropic and inhomogeneous cosmologies

If we extend our consideration to homogeneous and anisotropic universes, then similar results hold. The Bianchi IX models and their axisymmetric Taub counterparts with fluids and S^3 spatial topology have $\sqrt{-g} \propto a^3 \propto t$ for the geometric-mean scale factor and $\rho \propto a^{-3\gamma} \propto t^{-\gamma}$, on approach to singularities; therefore,

$$S_m \propto (\gamma - 4/3) \int \rho t dt \propto t^{2-\gamma}, \quad \gamma \neq 2, \quad (5)$$

$$S_m \propto \ln(t), \quad \gamma = 2. \quad (6)$$

However, if the cosmology is dominated by anisotropy on approach to either singularity, then the gravitational action diverges because, with a Kasner-like vacuum-dominated asymptote, $a(t) \propto t^{1/3}$, and

$$S_g \propto \int R \sqrt{-g} d^4x \propto \int t^{-2} dt \propto \ln(t), \quad (7)$$

which diverges on approach to the singularity, the anisotropy mimicking the behavior of a $\gamma = 2$ anisotropy fluid, [21,22]. This is the case for the most general Bianchi type IX universes: S_g diverges as $t \rightarrow 0$ for all $\gamma < 2$ fluids and S_m and S_g diverge when $\gamma = 2$. The $\gamma > 2$ cases reduce to the isotropic universe situation studied above and also all have divergent actions.

The other permitted compact topology for a closed universe which permits a maximal hypersurface (i.e., an expansion maximum) [15,23,24] is $S^2 \times S^1$, which characterizes the Kantowski-Sachs universes [25,26]. The asymptotes are again Kasner-like and the same results hold for fluid and scalar field sources as for the S^3 topologies. Inhomogeneity does not alter these results and the S^3 Tolman-Bondi models [27] and the S^3 or $S^2 \times S^1$ Szekeres dust models [27,28] all have finite action, whereas the closed $p = \rho$ inhomogeneous closed universes with massless scalar field found by Belinskii [27,29] have infinite actions like other $\gamma = 2$ fluid cosmologies (although these solitonic solutions are slightly peculiar because the scalar field depends only on time).

In all anisotropic and inhomogeneous cosmologies containing black body radiation, magnetic and electric fields, and Yang-Mills fields, we find that the matter action, S_m , is finite. The latter two types of source require anisotropy to be present. These are acceptable finite action matter sources in the Universe, as we would expect.

If inhomogeneities are added, then a density inhomogeneity spectrum of the power-law form $\delta\rho/\rho \propto M^{-q}$ with mass scale M produces divergent metric fluctuations

$\delta g/g \propto M^{2/3-q}$ of a non-Friedmann type on small scales, as $M \rightarrow 0$ for $q > 2/3$. This would lead to large anisotropies and divergent action and so is excluded. On large scales we cannot be so conclusive because we require a closed universe and the inhomogeneity spectrum does not extend to infinite mass values. The initial singularity for $q \leq 2/3$ is quasi-isotropic with no divergence of the action because of dominant anisotropies.

D. Sudden singularities

Finite-time singularities of the sudden sort were first introduced into relativistic cosmology in Ref. [30] in order to sharpen the conditions needed to ensure the recollapse of closed universes with S^3 or $S^2 \times S^1$ topologies, and then defined and explored in detail by the author in Refs. [31–36]. Other finite-time singularities where the Hubble parameter evolves as $H(t) = h_1(t) + h_2(t)(t - t_s)^\lambda$ were subsequently defined and investigated; see Refs. [37,38] for a classification. The finite-time singularities occurring are $\lambda < -1$ for “big rip,” $-1 < \lambda < 0$ for sudden (also known as type III), $0 < \lambda < 1$ for “type II,” and $\lambda > 1$ for “type IV.”

Sudden singularities create no geodesic incompleteness: they are soft singularities [39–42] and have been shown to be part of the general nine-function solution of the Einstein equations in the absence of an equation of state [42]. So, in an ever-expanding universe, we might integrate the action to future infinity to obtain a singularity. However, suppose we assume that the past singularity displayed no action singularity because it was like a perfect fluid Friedmann model and the evolution did not proceed beyond the finite-time singularity at t_s . Then, in the simplest example, at a sudden singularity the quantities $a(t)$, $H(t)$ and $\rho(t)$ are finite as $t \rightarrow t_s$, but there are infinities in the pressure, p , and the acceleration, \ddot{a} , [31]. On approach to the sudden singularity as $t \rightarrow t_s$ we have

$$a(t) = \left(\frac{t}{t_s}\right)^q (a_s - 1) + 1 - \left(1 - \frac{t}{t_s}\right)^n \rightarrow a_s + q(1 - a_s) \left(1 - \frac{t}{t_s}\right), \quad (8)$$

with $1 < n < 2$, where at early times we can have a standard Friedmann fluid evolution with

$$a(t) \approx \left(\frac{t}{t_s}\right)^q, \quad (9)$$

as $t \rightarrow 0$, with $q = 1/2$ for radiation domination. Both $a(t_s)$ and $\dot{a}(t_s)$ are finite but $\ddot{a}(t_s) \rightarrow -\infty$ when $1 < n < 2$, as

$$\ddot{a} \rightarrow q(q-1)Bt^{q-2} - \frac{n(n-1)}{t_s^2(1-\frac{t}{t_s})^{2-n}} \rightarrow -\infty. \quad (10)$$

Thus assuming finiteness of the space integration in Eq. (1) and the behavior as $t \rightarrow 0$, we have on approach to the sudden singularity,

$$S_g \propto \int_0^{t_s} \frac{\ddot{a}}{a} \sqrt{-g} dt \propto - \int_0^{t_s} \frac{n(n-1)a_s^2}{t_s^2(1-\frac{t}{t_s})^{2-n}} dt. \quad (11)$$

Hence, we have

$$S_g \propto \frac{na_s^2}{t_s} \left(1 - \frac{t}{t_s}\right)^{n-1}, \quad (12)$$

and this converges to a finite value as $t \rightarrow t_s$ in the sudden singularity regime with $1 < n < 2$. The action is infinite for finite-time singularities in the $n < 1$ domain.

E. Dark energy and Λ

If we return to the scalar field models with exponential potential in a closed Friedmann universe together with a perfect fluid with equation of state parameter $2/3 < \gamma < 2$, then when $\lambda^2 > 2$ all solutions start and end with a curvature singularity and have finite action. When $\lambda^2 < 2$ solutions can either recollapse to a singularity or expand forever, approaching the exact power-law inflationary solutions given above, [43]. In order to have a description of dark energy that is consistent with the finite action requirement, we see that we cannot have an explicit Λ term because this leads to indefinite future expansion towards de Sitter spacetime [44–47] and a divergence of the time integral for the action in Eq. (1). Therefore, for finite action, we require the dark energy to be an evolving scalar field, either explicitly in general relativity as in the action (1), or via an effective scalar in a modified theory of gravity. This allows the closed cosmology eventually to cease to be dominated by the scalar field and collapse to a future singularity, yielding finite total action. An unusual example of this sort is the late scenario with the Albrecht-Skordis potential [48] that was investigated by Barrow *et al.* [49]. A potential of the form

$$V(\phi) = e^{-\mu\phi} P(\phi), \quad (13)$$

where $P(\phi)$ is a polynomial in the scalar field ϕ with several minima [for example $P(\phi) = A + (\phi - B)^n$ in the simplest case], with μ being constant, allows the scalar field to get caught in a succession of local minima as it rolls down the steep exponential “cliff.” If the field comes to rest in a local minimum then $\dot{\phi} = 0$ there, and the expansion dynamics inflate [48]. The effect of the crenellations

created by the polynomial $P(\phi)$ is a sequence of accelerating expansion episodes that can end in nonaccelerating expansion or collapse to a future curvature singularity. However, the ϕ field can overshoot or tunnel through the barrier at a minimum leading to different versions of this inflation [49]. In a closed universe it is possible for this sequence of inflations to end in collapse to a future singularity and the Universe then has finite total action. A variety of other scenarios are possible for closed universes to accelerate transiently before recollapsing to a future curvature singularity. We require the dark energy to display this evolutionary behavior culminating in a future curvature singularity and the Universe to be closed.

III. MODIFIED GRAVITY

When higher-order terms, for example, of quadratic and higher orders in the scalar curvature, [50], or higher-order matter terms [51] on the right-hand side of the field equations, are added to the Einstein-Hilbert action we expect it to be easier to create an action singularity at an initial or final singularity of a closed universe. The finite action principle is at its most powerful when confronting action singularities in higher-order Lagrangian theories. One class of examples derives from the choice of gravitational Lagrangian [50],

$$L_g = f(R), \quad (14)$$

whose variation along with the matter action gives the field equations that generalize Einstein’s,

$$f_R R_{ab} - \frac{1}{2} f g_{ab} + f_R^{;cd} (g_{ab} g_{cd} - g_{ac} g_{bd}) = T_{ab}, \quad (15)$$

where $f_R = df/dR$.

A. Power-law Lagrangians: $f(R) = R^{1+\delta}$

Barrow and Clifton have examined the case of power-law Lagrangians that reduce to general relativity as the constant $\delta \rightarrow 0$ in some detail [52,53]. It is possible to find exact isotropic and anisotropic solutions [54,55] of the field equations in vacuum and with perfect fluids. For equation of state $p = (\gamma - 1)\rho$, the zero-curvature Friedmann metric has the exact solution for the scale factor

$$a(t) = t^{2(1+\delta)/3\gamma}, \quad \gamma \neq 0, \quad (16)$$

where the constraint on δ and the value of the scalar curvature are given by

$$(1 - 2\delta)[2 - 3\delta\gamma - 2\delta^2(1 + 3\gamma)] = \frac{1}{4}(1 - \delta)\gamma^2 \rho_c, \quad (17)$$

$$R = 3\delta(1 + \delta)t^{-2}, \quad (18)$$

where ρ_c is the Friedmann critical density.¹ This zero-curvature solution is the behavior of the closed models as they approach their initial and final singularities [52,53]. As in general relativity ($\delta = 0$), there are simple deductions from the finite action principle. The Universe must have compact space sections and have finite total lifetime, from initial to final singularity. Integrating out the space part of the action in (1), we have for the $\gamma \neq 0$ behavior on approach to a singularity (the $\gamma = 0$ solution has infinite action and is excluded here),

$$S_g \propto \int R^{1+\delta} \sqrt{-g} dt \propto \int t^{-2(1+\delta)} t^{2(1+\delta)/3\gamma} dt \propto t^{[2\delta(1-3\gamma)+2-3\gamma]/3\gamma}, \quad (19)$$

which diverges as $t \rightarrow 0$ if $\gamma > 0$ and

$$2\delta(1-3\gamma) + 2 - 3\gamma < 0. \quad (20)$$

Hence, in the radiation case ($\gamma = 4/3$) we exclude $\delta > -1/3$ and for dust ($\gamma = 1$) we exclude $\delta > -1/4$.

In the anisotropic case of the generalized Kasner vacuum solution in this theory found in Refs. [54,55], $\sqrt{-g} \propto t^{1+2\delta}$, and so

$$S_g \propto \int R^{1+\delta} \sqrt{-g} dt \propto \int t^{-2(1+\delta)} t^{(1+2\delta)} dt \propto \int t^{-1} dt \propto \ln(t), \quad (21)$$

and there is a logarithmic singularity in the action as $t \rightarrow 0$ just as in general relativity.

Another two exact Friedmann exact solutions of this theory for zero-curvature Friedmann universes are

$$a(t) = t^{\frac{\delta(1+2\delta)}{1-\delta}}, \quad (22)$$

$$a(t) = t^{1/2}, \quad (23)$$

independent of the matter content γ . Note that the second solution does not require radiation to be present, as was first found in Refs. [56,57] and exists in vacuum, as do the radiation solutions with nonzero curvature $a(t) \propto (t - kt^2)^{1/2}$.

B. Quadratic gravity

If we choose a Lagrangian of quadratic form,

$$L_g = R + AR^2, \quad A \text{ constant}, \quad (24)$$

then if the dynamics on approach to a singularity are close to Friedmann in the case of a closed universe containing

¹When $\gamma = 0$, the solution becomes the exact de Sitter metric with $a = \exp(nt)$ with the constraint

$$3(1-2\delta)n^2 = (1-\delta)\rho_c.$$

perfect fluid, then the contribution of the R^2 Lagrangian term to the universal action is

$$S \propto \int R^2 \sqrt{-g} dt \propto \int t^{-4} t^{2/\gamma} dt \propto t^{2/\gamma-3}. \quad (25)$$

This diverges on approach to the initial or final singularity for $\gamma > 2/3$, which includes the physically interesting cases of dust, radiation, and stiff fluid and the divergence in the latter case is stronger than in the general relativity case, as expected from the R^2 term. Likewise any R^n addition to the Einstein-Hilbert Lagrangian creates a stronger divergence in the action on approach to a singularity.

C. Gauss-Bonnet Lagrangian

Consider the general quadratic Lagrangian without the Λ term,

$$L_g = R + \alpha R^2 + \beta R_{ab} R^{ab} + \mu R_{abcd} R^{abcd}, \quad (26)$$

where R_{ab} is the Ricci tensor and R_{abcd} is the Riemann tensor, and α, β and μ are arbitrary constants. If there is an isotropic and homogeneous cosmological model with scale factor $a = t^n$ on approach to an initial singularity at $t = 0$, then the total gravitational action is [1]

$$S_g = \frac{2n(1-2n)t^{3n-1}}{3n-1} + \frac{4n^2 t^{3(n-1)}}{n-1} \times \{3\alpha + \beta + \mu + n(n-1)(12\alpha + 3\beta + 2\mu)\}. \quad (27)$$

We recognize the first term on the right-hand side as the general relativity contribution from the variation of R , which diverges as $t \rightarrow 0$ for $n \leq 1/2$, as shown above. When $\alpha:\beta:\mu = 1:1:-4$ the quadratic terms in (27) create a complete divergence that leads to no contributions to the field equations in four spacetime dimensions, so the field equations are the same as for general relativity. However, the quadratic terms still contribute to the gravitational action and can create a divergence as $L_{\text{quad}} \propto n^3 t^{3(n-1)}$. Therefore, there is an action singularity as $t \rightarrow 0$ when $n \leq 1$, and hence for all perfect fluid models with $\gamma \geq 2/3$ (since $n = 2/3\gamma$ for perfect fluid Friedmann solutions). The special Gauss-Bonnet combination $\alpha:\beta:\mu = 1:1:-4$ is therefore excluded in Friedmann models even though it does not affect the field equations. We expect anisotropic models to be excluded also. The finite action principle only allows (27) to be finite on approach to a singularity for the special (radiationlike) case,

$$n = 1/2, \quad \beta = -2\mu/5. \quad (28)$$

The higher-order terms in the Lovelock Lagrangian [58] in more than four spacetime dimensions could also be examined for action singularities. We expect them to be singular in an analogous way since they include higher powers of the curvature invariants.

D. Brans-Dicke

Brans-Dicke (BD) cosmological models of Friedmann type fall into two classes depending on the boundary condition imposed on the BD scalar field [59]. As discussed in detail in Ref. [1], the “Machian” models which are matter dominated near the initial and final singularity in closed universes have finite action except when $\gamma = 2$. By contrast, in the general case when the cosmologies are dominated by the BD scalar field on approach to the singularities, there are infinite actions. This is confirmed by analysis of the general vacuum solutions which are approached at the singularity by the scalar field (vacuum)-dominated solutions: all have infinite action and behave as if they are $\gamma = 2$ general relativistic cosmologies, displaying a logarithmic singularity $S_g \propto \ln(t)$ as $t \rightarrow 0$. Horndeski Lagrangians [60] can also have their cosmological conclusions tested against the finite action requirements and again will be challenged by the presence of higher-order curvature and scalar-field terms and possibly singularities at finite times for particular choices of coupling that are linked to the well-posedness of the initial value problem [61,62].

IV. CONCLUSIONS

We have reconsidered and extended the cosmological predictions that can be made under the assumption that the total action of the Universe is finite. This is an interesting cosmological constraint because attempts to avoid cosmological singularities in curvature invariants appear to lead generally to singularities in the gravitational action. Specifically, we have shown that the simple ansatz that the total action of the Universe be finite has a large number of powerful consequences. We require the following properties to be possessed by a universe with finite action:

- (a) There must be compact spatial sections (i.e., a closed universe), but compact spatial sections in flat and open universes created by topological identifications lead to infinite actions and are excluded.
- (b) There must be initial and final singularities (i.e., a finite total lifetime for the Universe).
- (c) The Universe cannot be dominated by massless scalar fields or the kinetic energy of self-interacting scalar fields or a $p = \rho$ perfect fluid on approach to an initial or final singularity.
- (d) The Universe can contain perfect fluids with $-1 < p/\rho < 2$ on approach to initial and final singularities.
- (e) The Universe cannot display a bounce or indefinite cyclic behavior to the past or the future.
- (f) An ekpyrotic scenario with an effective fluid obeying $p/\rho > 2$ in a closed, flat or open universe is ruled out.
- (g) Any loop quantum gravity model experiencing a bounce and indefinite past or future evolution is ruled out.

- (h) The Einstein static and steady-state universes are ruled out along with past eternal inflating universes and future ever-expanding eternally inflating universes.
- (i) Anisotropies (notably those of Kasner or Mixmaster type) cannot dominate the dynamics of the Universe on approach to initial and final singularities. Inhomogeneities cannot dominate if they induce dominant anisotropies of Kasner or Mixmaster type. For example, this excludes density inhomogeneity spectra versus mass scale with $\delta\rho/\rho \propto M^{-q}$, for $q > 2/3$.
- (j) Higher-order Lagrangian theories of gravity are significantly constrained because the action diverges faster than in general relativity when powers of R exceeding unity dominate on approach to a singularity. For example, quadratic Lagrangians are excluded with fluids satisfying $p/\rho > -1/3$. Gravitational Lagrangians with $L_g = R^{1+\delta}$ and $p = (\gamma - 1)\rho$ fluids have infinite actions on approach to a singularity when $2\delta(1 - 3\gamma) + 2 - 3\gamma < 0$. We also find that the Gauss-Bonnet quadratic Lagrangian combination causes an action singularity even though it does not contribute terms to the field equations.
- (k) Dark energy cannot be provided by a simple cosmological constant, which would create an action singularity to the future. The Universe cannot be asymptotically de Sitter as $t \rightarrow \infty$. The dark energy therefore needs to be an evolving energy density (or effective energy density) in a closed universe that produces collapse to a future singularity after a de Sitter-like accelerated expansion phase of finite duration. An Albrecht-Skordis potential for a scalar field in a closed universe that recollapses to the future is an admissible example with finite action [48,49]. There are many others.
- (l) Scalar-tensor theories like Brans-Dicke and its generalizations cannot have cosmological solutions that are dominated by the scalar field on approach to singularities. They must track the special matter-dominated (Machian) solutions [59,63–65] and must have $p < \rho$.

In conclusion, we have shown that the requirement that the total gravitational and matter actions of the Universe be finite produces a number of powerful predictions about the geometrical and topological structure of the Universe, its early expansion dynamics, the equation of state of its material content, the presence of scalar fields, the nature of the dark energy, and the allowed form of modifications to general relativity. We have confirmed the constraints found in [1] with some more recent applications added in points (a), (b), and (e). We have established new consequences of finite action in points (c), (d), and (f)–(k), and indicated further theoretical developments in modified gravity that will reveal new conclusions in the context of Lovelock and Horndeski actions.

ACKNOWLEDGMENTS

I thank Frank Tipler for earlier collaboration and thank J. Magueijo, J.-L. Lehnert and G. Gibbons for helpful

discussions and communications. Support by the Science and Technology Facilities Council of the UK (STFC) is acknowledged.

-
- [1] J. D. Barrow and F. J. Tipler, *Nature (London)* **331**, 31 (1988).
- [2] J.-L. Lehnert and K. S. Stelle, *Phys. Rev. D* **100**, 083540 (2019).
- [3] M. Planck, *Scientific Autobiography and Other Papers* (Greenwood, New York, 1968), pp. 48–180.
- [4] P. Ramond, *Field Theory: A Modern Primer* (Benjamin, New York, 1981).
- [5] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Vol. 1 (Cambridge University Press, Cambridge, England, 1987).
- [6] A. Ashtekar and J. Samuel, *Classical Quantum Gravity* **8**, 2191 (1991).
- [7] J. D. Barrow and H. Kodama, *Classical Quantum Gravity* **18**, 1753 (2001).
- [8] J. D. Barrow, *Phys. Rev. D* **89**, 064022 (2014).
- [9] H. Barzegar, D. Fajman, and G. Heißel, arXiv:1904.13290.
- [10] P. J. Steinhardt and N. Turok, *Science* **296**, 1436 (2002).
- [11] A. A. Coley and W. C. Lim, *Classical Quantum Gravity* **22**, 3073 (2005).
- [12] J. D. Barrow and M. Dąbrowski, *Mon. Not. R. Astron. Soc.* **275**, 850 (1995).
- [13] A. Borde, A. H. Guth, and A. Vilenkin, *Phys. Rev. Lett.* **90**, 151301 (2003).
- [14] S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1973).
- [15] J. D. Barrow and F. J. Tipler, *The Anthropic Cosmological Principle* (Oxford University Press, Oxford, 1986), p. 107.
- [16] J. D. Barrow, *Phys. Lett. B* **235**, 40 (1990).
- [17] A. A. Starobinski, *Sov. Astron. Lett. [Pisma. v Astron. J.]* **4**, 155 (1978).
- [18] J. D. Barrow and R. A. Matzner, *Phys. Rev. D* **21**, 336 (1980).
- [19] D. N. Page, *Classical Quantum Gravity* **1**, 417 (1984).
- [20] H.-J. Schmidt, *Astron. Nachr.* **311**, 99 (1990).
- [21] C. W. Misner, *Astrophys. J.* **151**, 431 (1968).
- [22] J. D. Barrow, *Nature (London)* **272**, 211 (1978).
- [23] J. D. Barrow, G. J. Galloway, and F. J. Tipler, *Mon. Not. R. Astron. Soc.* **223**, 835 (1986).
- [24] J. D. Barrow, *Nucl. Phys.* **B296**, 697 (1988).
- [25] R. Kantowski and R. K. Sachs, *J. Math. Phys. (N.Y.)* **7**, 443 (1966).
- [26] A. S. Kompanyets and A. S. Chernov, *Zh. Eksp. Teor. Fiz.* **47**, 1939 (1964) [*Sov. Phys. JETP* **20**, 1303 (1965)].
- [27] A. Krasinski, *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge, England, 2006).
- [28] P. Szekeres, *Commun. Math. Phys.* **41**, 55 (1975).
- [29] V. Belinski and E. Verdaguer, *Gravitational Solitons* (Cambridge University Press, Cambridge, England, 2001).
- [30] J. D. Barrow, G. Galloway, and F. J. Tipler, *Mon. Not. R. Astron. Soc.* **223**, 835 (1986).
- [31] J. D. Barrow, *Classical Quantum Gravity* **21**, L79 (2004).
- [32] J. D. Barrow, *Classical Quantum Gravity* **21**, 5619 (2004).
- [33] J. D. Barrow and C. G. Tsagas, *Classical Quantum Gravity* **22**, 1563 (2005).
- [34] J. D. Barrow and S. Z. W. Lip, *Phys. Rev. D* **22**, 1563 (2009).
- [35] J. D. Barrow, S. T. C. Cotsakis, and A. Tsokaros, *Classical Quantum Gravity* **27**, 165017 (2010).
- [36] J. D. Barrow and A. A. H. Graham, *Phys. Rev. D* **91**, 083513 (2015).
- [37] S. Nojiri, S. D. Odintsov, and S. Tsujikawa, *Phys. Rev. D* **71**, 063004 (2005).
- [38] P. Singh, *Classical Quantum Gravity* **26**, 125005 (2009).
- [39] F. J. Tipler, *Phys. Lett. A* **64**, 8 (1977).
- [40] A. Królak, *Classical Quantum Gravity* **3**, 267 (1986).
- [41] L. Fernandez-Jambrina and R. Lazkoz, *Phys. Rev. D* **70**, 121503 (2004).
- [42] J. D. Barrow and S. T. C. Cotsakis, *Phys. Rev. D* **88**, 067301 (2013).
- [43] A. Coley and M. Goliath, *Phys. Rev. D* **62**, 043526 (2000).
- [44] R. Wald, *Phys. Rev. D* **28**, 2118 (1983).
- [45] J. D. Barrow, *The Very Early Universe*, edited by G. W. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge University Press, Cambridge, England, 1983), pp. 267–272.
- [46] W. Boucher and G. W. Gibbons, *The Very Early Universe*, edited by G. W. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge University Press, Cambridge, England, 1983).
- [47] A. A. Starobinski, *Sov. Phys. JETP Lett.* **37**, 55 (1983).
- [48] A. Albrecht and C. Skordis, *Phys. Rev. Lett.* **84**, 2076 (2000).
- [49] J. D. Barrow, R. Bean, and J. Magueijo, *Mon. Not. R. Astron. Soc.* **316**, L41 (2000).
- [50] J. D. Barrow and A. C. Ottewill, *J. Phys. A* **16**, 2757 (1983).
- [51] J. D. Barrow and C. Board, *Phys. Rev. D* **96**, 123517 (2017).
- [52] J. D. Barrow and T. Clifton, *Phys. Rev. D* **72**, 103005 (2005).
- [53] T. Clifton, *Phys. Rev. D* **78**, 083501 (2008).
- [54] J. D. Barrow and T. Clifton, *Classical Quantum Gravity* **23**, L1 (2005).
- [55] J. D. Barrow and T. Clifton, *Classical Quantum Gravity* **23**, 2951 (2006).
- [56] J. D. Barrow and S. Hervik, *Phys. Rev. D* **73**, 023007 (2006).
- [57] J. D. Barrow and S. Hervik, *Phys. Rev. D* **74**, 124017 (2006).
- [58] D. Lovelock, *J. Math. Phys. (N.Y.)* **12**, 498 (1971).

- [59] L. E. Gurevich, A. M. Finkelstein, and V. A. Ruban, *Astrophys. Space Sci.* **22**, 231 (1973).
- [60] G. W. Horndeski, *Int. J. Theor. Phys.* **10**, 363 (1974).
- [61] J. D. Barrow, M. Thorsrud, and K. Yamamoto, *J. High Energy Phys.* **02** (2013) 146.
- [62] G. Papallo, Causality and the initial value problem in Modified Gravity (Doctoral thesis), <https://doi.org/10.17863/CAM.24726> (2019).
- [63] C. Brans and R. H. Dicke, *Phys. Rev.* **124**, 925 (1961).
- [64] P. Jordan, *Schwerkraft und Weltal* (Vieweg und Sohn, Braunschweig, 1955).
- [65] C. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1993).