Electromagnetic solitons in quantum vacuum

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In the limit of extremely intense electromagnetic fields the Maxwell equations are modified due to the photon-photon scattering that makes the vacuum refraction index depend on the field amplitude. In the presence of electromagnetic waves with small but finite wave numbers the vacuum behaves as a dispersive medium. We show that the interplay between the vacuum polarization and the nonlinear effects in the interaction of counter-propagating electromagnetic waves can result in the formation of Kadomtsev-Petviashvily solitons and, in one-dimension configuration, of Korteveg-de-Vries type solitons that can propagate over a large distance without changing their shape.

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I. INTRODUCTION

Fast progress in the laser and free electron laser technology aimed at developing sources of extremely high power electromagnetic radiation has called into being a vast area of nonlinear physics related to the behavior of matter and vacuum irradiated by ultraintense electromagnetic fields [1].

Among the rich variety of nonlinear effects induced by a relativistically strong light, we will choose as the topic of the present paper the formation and evolution of electromagnetic solitary waves in the quantum vacuum.

Relativistic electromagnetic solitons propagating in collisionless plasma have been extensively studied theoretically [2–13], with computer simulations [6,14–30], and in the experiments on the laser-plasma interaction [31–37]. Typically solitons in relativistic plasmas can be regarded as pulses of electromagnetic radiation trapped inside the cavities formed in the plasma electron density by the pulse ponderomotive pressure. In the limit of small but finite soliton amplitude they are described by the nonlinear Schroedinger equation (for the properties of the NSE solitons see Refs. [38–41]). Relativistic electromagnetic solitons can provide one of the ways of anomalous absorption of the laser energy by transforming it into energy of fast particles and into energy of high and low frequency electromagnetic radiation.

The properties of solitons that are formally similar to the NSE solitons were analyzed theoretically in Refs. [42,43]. We note that NSE solitons are predicted to be formed in the quantum vacuum. They correspond to electromagnetic pulses trapped in the local modulations of the refraction index of the vacuum. In classical electrodynamics the vacuum refraction index equals unity, i.e., electromagnetic waves do not interact with each other. On the contrary, in quantum electrodynamics (QED) electromagnetic waves interact in vacuum via virtual electron-positron pair excitation which is related to vacuum polarization [44–48]. In the other words, the electromagnetic field can excite a virtual electron-positron plasma. The experiments on the detection of the photon-photon scattering using high power laser facilities [49–56] is one of the most attracting goals in fundamental science.

The electromagnetic field intensity required for the observation of the vacuum polarization is characterized by the QED critical electric field. It is also known as the Schwinger field [44] $E_S = m_e^2 c^3/e\hbar$, where e and m_e are the electron charge and mass, c the speed of light in vacuum, and $\hbar$ is the Planck constant. The corresponding normalized wave amplitude $a_S = eE_S/m_e\omega c = m_e c^2/\hbar\omega$ and light intensity are 5.1×10^5 and 10^{29} $\mathrm{W/cm^2}$, respectively. By virtue of the Lorentz invariance, a plane electromagnetic wave does not induce the vacuum polarization. In other words, there is no self-action of a single plane wave. The situation becomes different for counterpropagating electromagnetic pulses, when they mutually change the vacuum refraction index seen by the other wave. The refraction index depends nonlinearly on the colliding electromagnetic wave amplitude [44,45,57], and the resulting wave self-action can lead to wave steepening and wave breaking [58,59]. In the long-wavelength limit the QED vacuum is a dispersionless medium, i.e., the phase and group velocity of the electromagnetic wave are equal. The vacuum dispersion effects seen at small but finite photon momentum have been analyzed in Refs. [60,61]. These effects can also be found by using an approach developed in Refs. [62,63]. In general, the nonlinearity and dispersion balance provides the condition for the formation of solitary waves [38–40], which can propagate over large distance without changing their shape. Along with the solitons described by the NSE equation, were solitons described by the Korteveg-de-Vries (KdV) equation [64] (a generalization of the KdV equation to the multidimensional case is known as the Kadomtsev-Petviashvili (KP) equation [65]) [38–40].

Below we show that the vacuum polarization and the nonlinear effects in the interaction of counterpropagating electromagnetic waves can result in the formation of the relativistic electromagnetic solitons and nonlinear waves described by the KP, KdV, and dispersionless Kadomtsev-Petviashvili (dKP) equations. Realizing conditions for the soliton formation in the superstrong laser beam collisions we might be able to understand better vacuum behavior testing the appearance of excitation of the electron positron Dirac sea.

The paper is organized as follows. In Sec. II we discuss the EM wave dispersion in the QED vacuum as well as the long wavelength limit. The nonlinear EM waves in vacuum are discussed in Sec. III. First, equations of nonlinear electrodynamcis are written down, then the case of the counterpropagating EM waves is investigated. In Sec. IV we discuss EM waves in the QED vacuum described by Kadomtsev-Petviashvili, dispersionless Kadomtsev-Petviashvili, and Korteveg-de-Vries equations. We conclude in Sec. V.

II. ELECTROMAGNETIC WAVE DISPERSION IN THE QED VACUUM

A. Dispersion equation

The dispersion equation giving the relationship between the frequency ω and the wave vector $\mathbf{k}$ of a relatively high frequency small amplitude electromagnetic wave colliding in the QED vacuum with a low frequency wave can be written in the form

$$\omega^2 - k^2c^2 - \frac{\mu_{\parallel,\perp}^2 c^4}{\hbar^2} = 0. \tag{1}$$

In this case, the low frequency wave is approximated by the crossed field wave with electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields orthogonal to each other of equal amplitude, $E_0 = B_0$. Here $\mu_{\parallel,\perp}$ is the "invariant photon mass" [63] (on the effects of the field inhomegeity and the limits of applicability of the crossed field approximation see Refs. [66,67]). The subscripts $\parallel, \perp$ of $\mu_{\parallel,\perp}$ correspond to the parallel and perpendicular polarizations of the colliding electromagnetic waves in the reference frame where they are counterpropagating to each other. For the sake of brevity we assume below that the wave polarizations are parallel and denote by μ the invariant photon mass.

The invariant mass depends on the photon frequency (it is the photon energy expressed in terms of the quantum parameter χ_γ). The invariant

$$\chi_\gamma = \frac{\hbar\sqrt{-k_\rho F^{\rho\sigma}F_{\sigma\tau}k^\tau}}{m_e c E_S}, \tag{2}$$

characterizes the QED processes of photons interacting with an electromagnetic field. Here k^ρ is the 4-moment of the photon, $F_{\rho\sigma}$ is the electromagnetic field tensor given by

$$F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho, \tag{3}$$

with A_ρ being the 4-vector potential of the electromagnetic field, $\rho = 0,\ 1,\ 2,\ 3$, ∂_ρ denotes partial derivative with respect to the 4-coordinate x_ρ. Here and below summation over repeating indices is assumed.

For a photon counterpropagating to the crossed E_0—B_0 fields, the invariant equals

$$\chi_\gamma = \frac{E_0}{E_S}\frac{\hbar(\omega + k_x c)}{m_e c^2}, \tag{4}$$

where k_x is the x component of the wave vector.

According to Ref. [63] the square of the invariant photon mass μ is given by

$$\mu^2 = \frac{\alpha m_e^2}{6\pi}\int_1^\infty du \frac{8u-2}{\zeta u\sqrt{u(u-1)}}\frac{df}{d\zeta} \tag{5}$$

with

$$\zeta = \left(\frac{4u}{\chi_\gamma}\right)^{2/3} \tag{6}$$

and

$$f(\zeta) = i\int_0^\infty dt\, \exp\left[-i\left(\zeta t + \frac{t^3}{3}\right)\right]. \tag{7}$$

In the right-hand side (r.h.s.) of Eq. (5) $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant.

The function $f(\zeta)$ can be written as a linear combination of the Airy function $Ai(\zeta)$ and the inhomogeneous Airy function $Gi(\zeta)$, as

$$f(\zeta) = \pi[iAi(\zeta) + Gi(\zeta)]. \tag{8}$$

Using the analytical properties of the Airy functions $Ai(\zeta)$ and $Gi(\zeta)$ (see Refs. [68,69] and Appendix A; $Gi(\zeta)$ is also known as the Scorer function [70]) we can present the dependence of $f(\zeta)$ on the variable ζ in the limit $\zeta \to +\infty$ (i.e., in the limit $\chi_\gamma \ll 1$) as

$$f(\zeta) = \frac{1}{\zeta} + \frac{2}{\zeta^4} + \cdots + i\frac{\pi}{2\zeta^{1/4}}\exp\left(-\frac{2}{3}\zeta^{3/2}\right). \tag{9}$$

and its derivative is then

$$f'(\zeta) = -\frac{1}{\zeta^2} - \frac{8}{\zeta^5} - \cdots - i\frac{\pi}{8\zeta^{5/4}}\exp\left(-\frac{2}{3}\zeta^{3/2}\right) - i\frac{\pi}{2}\zeta^{1/4}\exp\left(-\frac{2}{3}\zeta^{3/2}\right). \tag{10}$$

We can neglect the last term in the derivative in the limit $\chi_\gamma \ll 1$. Substituting this expression into the integrand in the r.h.s. of Eq. (5) and calculating the integral we find expansions of the real and the imaginary parts of the square of the photon mass at $\chi_\gamma \ll 1$,

$$\Re[\mu^2] = -\alpha m_e^2 \frac{4}{45\pi}\left[\chi_\gamma^2 + \frac{1}{3}\chi_\gamma^4 + \mathcal{O}(\chi_\gamma^6)\right], \tag{11}$$

$$\Im[\mu^2] = -\alpha m_e^2 \frac{1}{8}\sqrt{\frac{3}{2}}\chi_\gamma \exp\left(-\frac{8}{3\chi_\gamma}\right) + \ldots. \tag{12}$$

Furthermore we neglect the effects of the exponentially small imaginary part (12) which describes the electron-positron pair creation via the Breit-Wheeler process [71,72].

We assume here that the Poynting vector of the strong low frequency wave $c\mathbf{E}\times\mathbf{B}/4\pi$ is directed in the negative direction along the x-axis. For definiteness we set $\mathbf{E} = \mathbf{e}_z E_0$, $\mathbf{B} = \mathbf{e}_y E_0$. The high frequency electromagnetic wave propagates in the negative direction along the x-axis.

Substituting the real part of the photon mass given by the first two terms in Eq. (11) into the dispersion equation (1) we obtain for the relationship between the electromagnetic wave frequency and the wave number

$$\omega^2 - k^2 + \frac{4\alpha W_0^2}{45\pi}(\omega + k_x)^2 + \frac{4\alpha W_0^4}{135\pi}(\omega + k_x)^4 = 0, \tag{13}$$

where $W_0 = E_0/E_S$ is the electric field of the cross field electromagnetic wave normalized on E_S, $k^2 = k_x^2 + k_y^2$, the wave number and frequency are normalized on $\lambda\!\!\!^-_C{}^{-1} = m_c/\hbar$ and $c\lambda\!\!\!^-_C{}^{-1} = m_c^2/\hbar$.

We can rewrite the dispersion equation (13) as

$$\omega^2 - k_x^2 + \kappa_1(\omega + k_x)^2 + \kappa_2(\omega + k_x)^4 = k_y^2. \tag{14}$$

Here parameters κ_1 and κ_2 are given by

$$\kappa_1 = 4\alpha W_0^2/45\pi \quad \text{and} \quad \kappa_2 = 4\alpha W_0^4/135\pi. \tag{15}$$

The parameters κ_1 and κ_2 give a measure of the vacuum polarization and vacuum dispersion effects, respectively. The term k_y^2 in the r.h.s. of Eq. (14) describes the electromagnetic wave diffraction.

We note that the expression for the electromagnetic wave dispersion in the QED vacuum given by last terms in the r.h.s. of Eqs. (13) and (14) being the consequence of the 6-photon mixing is different from the dependence found in [60] and used in Ref. [42] for describing the envelope solitons. Due to the symmetry of dispersion equation (1) with the invariant photon mass given by relationships (5)–(7) the first nonvanishing dispersion term corresponds to the 6-photon mixing.

Using the relationships between the frequency ω and wave-number k and the partial derivatives with respect to time and spatial coordinates,

$$\omega \leftrightarrow -i\partial_t, \qquad k_x \leftrightarrow i\partial_x, \quad \text{and} \quad k_y \leftrightarrow i\partial_y \tag{16}$$

we obtain from Eq. (14)

$$\partial_-(\partial_+ a - \kappa_1\partial_- a - 2\kappa_2\partial_{---}a) = -\frac{1}{2}\partial_{yy}a \tag{17}$$

with $\partial_- = \partial_{x^-}$, $\partial_+ = \partial_{x^+}$, and $\partial_{---} = \partial_-^3$. $\partial_\pm = \partial_{x^\pm} = (\partial/\partial x^\pm)$.

In Eq. (17) $a(x^-, x^+, y)$ is the z component of the 4 vector potential. Here and below we use the so-called Dirac's light cone coordinates x^- and x^+ defined as (see e.g., Ref. [73])

$$x^+ = \frac{x + ct}{\sqrt{2}}, \qquad x^- = \frac{x - ct}{\sqrt{2}}, \tag{18}$$

As is well known, the coordinates (x, t) in the laboratory frame of reference are related to the coordinates (x', t') in the frame of reference moving with the normalized velocity β as

$$x' = x\cosh\eta - ct\sinh\eta, \\ t' = t\cosh\eta - (x/c)\sinh\eta, \tag{19}$$

where

$$\eta = \ln\sqrt{\frac{1+\beta}{1-\beta}}. \tag{20}$$

The Lorentz transform of the light-cone variables, x'^{+}, x'^{-}, defined in Eq. (18) is

$$x'^{+} = \frac{x' + ct'}{\sqrt{2}} = e^{-\eta}\frac{x + ct}{\sqrt{2}} = e^{-\eta}x^{+},$$
$$x'^{-} = \frac{x' - ct'}{\sqrt{2}} = e^{+\eta}\frac{x - ct}{\sqrt{2}} = e^{+\eta}x^{-}. \qquad (21)$$

As a result

$$(\partial_-)' = e^{-\eta}\partial_- \quad \text{and} \quad (\partial_+)' = e^{+\eta}\partial_+. \qquad (22)$$

Now we introduce the field variables u and w defined as

$$u = \partial_- a \quad \text{and} \quad w = \partial_+ a. \qquad (23)$$

They are related to the electric, $e_z = -\partial_t a$ (along z), and magnetic, $b_y = -\partial_x a$ (along y), fields by the following relations

$$u = \frac{e_z - b_y}{\sqrt{2}}, \qquad w = -\frac{e_z + b_y}{\sqrt{2}}. \qquad (24)$$

The Lorentz transform of the fields u and w is

$$u' = \frac{e'_z - b'_y}{\sqrt{2}} = e^{-\eta}\frac{e_z - b_y}{\sqrt{2}} = e^{-\eta}u,$$
$$w' = -\frac{e'_z + b'_y}{\sqrt{2}} = -e^{+\eta}\frac{e_z + b_y}{\sqrt{2}} = e^{+\eta}w. \qquad (25)$$

The field product $uw = (b_y^2 - e_z^2)/2$,

$$u'w' = uw, \qquad (26)$$

is Lorentz invariant in the (t, x)-plane. It is proportional to the first Poincaré invariant $\mathfrak{F}$ of the Maxwell equations, which will be introduced below. We note that W_0 transforms like w:

$$W'_0 = e^{\eta}W_0. \qquad (27)$$

B. Dispersionless vacuum in the long wavelength limit

1. Counterpropagating electromagnetic waves

Equation (13) is obtained within the framework of the approximation, which assumes that the parameter χ_γ is small. Neglecting the dispersion and diffraction effects we can write Eq. (14) as

$$(\omega + k_x c)[\omega(1 + \kappa_1) - k_x c(1 - \kappa_1)] = 0. \qquad (28)$$

Taking into accounts the relations given by Eqs. (16) and (18) this equation leads to the wave equation

$$\partial_-(\partial_+ a - \kappa_1\partial_- a) = 0, \qquad (29)$$

with the solution

$$a(x^-, x^+) = f(x^+) + g(x^- + \kappa_1 x^+). \qquad (30)$$

Functions $f(x)$ and $g(x)$ are determined by the initial conditions $a_0(x)$ and $\dot{a}_0(x)$. A dot denotes a differentiation with respect to time.

Using the fact that $\kappa_1 \ll 1$, the solution (30) can be written in the following form

$$a(x, t) = f(x + ct) + g(x - vt), \qquad (31)$$

where $v = c(1 - \kappa_1)/(1 + \kappa_1)$.

The Cauchy problem is determined by the initial conditions

$$a_0(x) = f(x) + g(x),$$
$$\dot{a}_0(x) = cf'(x) - vg'(x). \qquad (32)$$

A prime here and below denotes a differentiation with respect to the function argument.

Since $a'_0(x) = f'(x) + g'(x)$ we can find that

$$f(x) = \frac{v}{c + v}a_0(x) + \frac{1}{c + v}\int^x \dot{a}_0(s)ds,$$
$$g(x) = \frac{c}{c + v}a_0(x) - \frac{1}{c + v}\int^x \dot{a}_0(s)ds. \qquad (33)$$

Substituting these expressions into Eq. (31) we obtain the solution to the wave equation (29)

$$a(x, t) = \frac{va_0(x + ct) + ca_0(x - vt)}{c + v} - \frac{1}{c + v}\int_{x-vt}^{x+t} \dot{a}_0(s)ds. \qquad (34)$$

In the case $v = c$ it becomes a standard d'Alembert formula.

Figure 1 shows the electromagnetic waves in the (x, t) plane, which are determined by the initial conditions $\dot{a}_0 = 0$ and $a_0 = \exp(-x^2/l^2)$ with $l = 0.125$ and $\kappa_1 = 1/3$, i.e., $v = 0.5c$. There are two waves. One of them propagates from the right to the left with the speed of light in vacuum. The other wave propagates from the left to the right with speed equal to v. The ratio of their amplitudes is equal to v/c.

The vacuum polarization in the field of interacting electromagnetic waves changes the electromagnetic wave propagation velocity making Cherenkov radiation possible in vacuum [74–76].

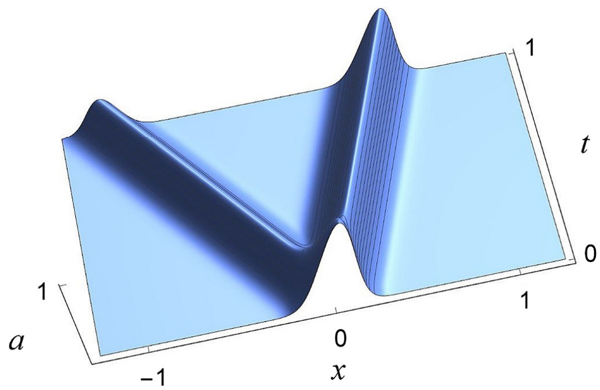

FIG. 1. Electromagnetic waves in the (x, t) plane for $\dot{a}_0 = 0$ and $a_0 = \exp(-x^2/l^2)$ with $l = 0.125$ and $\kappa_1 = 1/3$.

2. Frederick's diagrams

In the long-wavelength limit, when $k_x \to 0$ one can neglect the last term in the l.h.s. of Eq. (13), i.e., neglect the dispersion but retaining the diffraction effects. Then the dispersion equation can be written as

$$\omega^2 - k^2c^2 + \kappa_1(\omega + kc\cos\theta)^2 = 0, \qquad (35)$$

where $k = |\mathbf{k}| = \sqrt{k_x^2 + k_\perp^2}$. Here we introduce the angle between the wave vector direction and the x-axis equal to $\theta = \arccos(k_x/k)$ in the polar coordinate system.

The solution of Eq. (35) gives the wave frequency

$$\omega = -kc\frac{\kappa_1\cos\theta + \sqrt{1 + \kappa_1\sin^2\theta}}{1 + \kappa_1}. \qquad (36)$$

This relationship yields the phase diagram representing the dependence of normalized phase velocity $\beta_{\mathrm{ph}} = \omega/kc$ on the angle θ,

$$\beta_{\mathrm{ph}} = -\frac{\kappa_1\cos\theta + \sqrt{1 + \kappa_1\sin^2\theta}}{1 + \kappa_1}. \qquad (37)$$

Frederick's diagram (it is the polar diagram for group velocity of the wave, for details, e.g., see [77]) can be obtained by calculating the group velocity $\mathbf{v}_g = \partial\omega/\partial\mathbf{k}$. Taking into account that $\omega = kv_{\mathrm{ph}}$, where the phase velocity v_{ph} depend on the direction of $\mathbf{k}$ only we obtain

$$\mathbf{v}_g = v_{\mathrm{ph}}\frac{\mathbf{k}}{k} + v_{\perp g}\frac{\mathbf{k}_\perp}{k}. \qquad (38)$$

Here the perpendicular to the wave vector component equals $\mathbf{v}_{\perp g} = k\partial v_{\mathrm{ph}}/\partial\mathbf{k}$, i.e., $|\mathbf{v}_{\perp g}| = v_{\perp g} = (\partial\omega/\partial\theta)/k$

For the normalized value of the perpendicular component $\beta_{g\perp} = v_{g\perp}/c$ we have

$$\beta_{\mathrm{g}\perp} = \frac{\kappa_1\sin\theta(\cos\theta + \sqrt{1 + \kappa_1\sin^2\theta})}{(1 + \kappa_1)\sqrt{1 + \kappa_1\sin^2\theta}}. \qquad (39)$$

Fig. 2 presents the polar phase diagrams for the phase β_{ph} velocity and the group velocity $\beta_g = \sqrt{\beta_{\mathrm{ph}}^2 + \beta_{\perp g}^2}$: a) $\kappa_1 = 0.3$ and b) $\kappa_1 = 0.9$. As it is clearly seen the phase and group velocities are equal to each other for waves propagating along the x-axis being equal to speed of light in vacuum for copropagating waves, i.e., for $\theta = 0$ when $\beta_g = \beta_{ph} < 1$ and $\beta_g = \beta_{ph} = -1$ for $\theta = \pi$. In the cases $\theta \neq 0$ and $\theta \neq \pi$, the phase velocity is smaller than the group velocity.

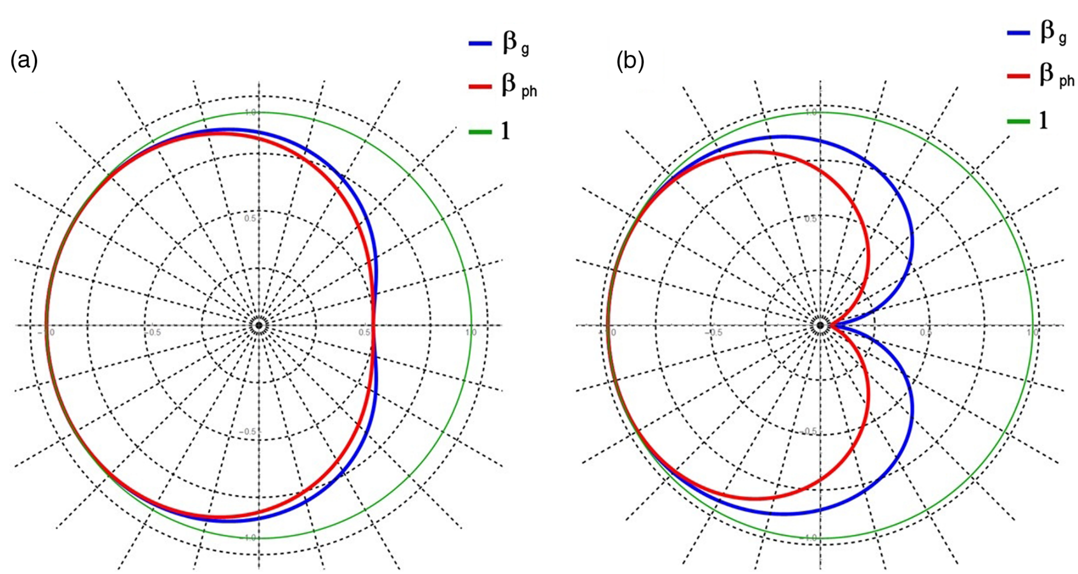

FIG. 2. Polar phase diagrams for group velocity β_g (blue), phase velocity β_{ph} (red) and speed of light in vacuum $\beta = 1$ (green): (a) $\kappa_1 = 0.3$; (b) $\kappa_1 = 0.9$.

III. NONLINEAR ELECTROMAGNETIC WAVES IN VACUUM

A. Equations of nonlinear electrodynamics

Our consideration here is based on using the Euler–Heisenberg Lagrangian describing the electromagnetic field in the long-wavelength limit. It is given by [44,78]

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}', \tag{40}$$

where

$$\mathcal{L}_0 = -\frac{m_e^4}{16\pi\alpha} F_{\mu\nu}F^{\mu\nu} \tag{41}$$

is the Lagrangian in classical electrodynamics, $F_{\mu\nu}$ is the electromagnetic field tensor determined by Eq. (3).

In the Euler–Heisenberg theory, the QED radiation corrections are described by $\mathcal{L}'$ on the right-hand side of Eq. (40), which can be written as [44]

$$\mathcal{L}' = \frac{m_e^4}{8\pi^2}\mathcal{M}(\mathfrak{e},\mathfrak{b}) = \frac{m_e^4}{8\pi^2}\int_0^\infty \frac{\exp(-\eta)}{\eta^3} \times \left[-(\eta\mathfrak{e}\cot\eta\mathfrak{e})(\eta\mathfrak{b}\coth\eta\mathfrak{b}) + 1 - \frac{\eta^2}{3}(\mathfrak{e}^2 - \mathfrak{b}^2)\right] d\eta. \tag{42}$$

Here the invariant fields $\mathfrak{e}$ and $\mathfrak{b}$ are expressed in terms the Poincaré invariants

$$\mathfrak{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad \mathfrak{G} = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{43}$$

as

$$\mathfrak{e} = \sqrt{\sqrt{\mathfrak{F}^2 + \mathfrak{G}^2} - \mathfrak{F}} \quad \text{and} \quad \mathfrak{b} = \sqrt{\sqrt{\mathfrak{F}^2 + \mathfrak{G}^2} + \mathfrak{F}}, \tag{44}$$

respectively. Here $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol in four dimensions.

Here and in the following text, we use the units $c = \hbar = 1$, and the electromagnetic field is normalized on the QED critical field E_S.

In the 3D notations the Poincaré invariants are

$$\mathfrak{F} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2), \qquad \mathfrak{G} = \mathbf{B}\cdot\mathbf{E}. \tag{45}$$

As explained in Ref. [44] the Euler–Heisenberg Lagrangian in the form given by Eq. (42) should be used for obtaining an asymptotic series over the invariant electric field $\mathfrak{e}$ assuming its smallness. The resulting expression is

$$\mathcal{L}' = \kappa\left[\left(\mathfrak{F}^2 + \frac{7}{4}\mathfrak{G}^2\right) + \frac{8}{7}\mathfrak{F}\left(\mathfrak{F}^2 + \frac{13}{16}\mathfrak{G}^2\right)\right] + \ldots \tag{46}$$

with $\kappa = e^4/90\pi^2 m_e^4$.

In the Lagrangian (46) the first two terms on the right-hand side and the last two correspond respectively to four and to six photon mixing.

B. Counterpropagating electromagnetic waves

In what follows we consider the interaction of counter-propagating electromagnetic waves with the same linear polarization. In this case the invariant $\mathfrak{G}$ vanishes identically. This field configuration can be described in a transverse gauge by a vector potential having a single component, $\mathbf{A} = A\mathbf{e}_z$, with $\mathbf{e}_z$ the unit vector along the z axis. In terms of the light cone coordinates [see Eq. (18)] the vector potential $\mathbf{A}$ is given by

$$A = a(x^+, x^-). \tag{47}$$

In these variables the Lagrangian (40) takes the form

$$\mathcal{L} = -\frac{m^4}{4\pi\alpha}[wu - \epsilon_2(wu)^2 - \epsilon_3(wu)^3], \tag{48}$$

where the field variables u and w are defined by Eq. (23). The dimensionless parameters ϵ_2 and ϵ_3 in Eq. (48) are given by

$$\epsilon_2 = \frac{2e^2}{45\pi} = \frac{2}{45\pi}\alpha \quad \text{and} \quad \epsilon_3 = \frac{32e^2}{315\pi} = \frac{32}{315\pi}\alpha, \tag{49}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant, i.e., $\epsilon_2 = 7\epsilon_3/8 \approx 10^{-4}$.

The field equations can be found by varying the electromagnetic action

$$\mathcal{S}(a) = \int dx^+ \int dx^- \mathcal{L}(a), \tag{50}$$

with respect to the vector potential $a(x^+, x^-)$ which gives

$$\partial_-(\partial_u\mathcal{L}) + \partial_+(\partial_w\mathcal{L}) = 0. \tag{51}$$

As a result, we obtain the system of equations

$$\partial_- w = \partial_+ u, \tag{52}$$

$$[1 - uw(4\epsilon_2 + 9\epsilon_3 uw)]\partial_+ u = w^2(\epsilon_2 + 3\epsilon_3 uw)\partial_- u + u^2(\epsilon_2 + 3\epsilon_3 uw)\partial_+ w, \tag{53}$$

Equation (52), is a consequence of the symmetry of the second derivatives, $\partial_{-,+}a = \partial_{+,-}a$ and it expresses the vanishing of the 4-divergence of the dual electromagnetic field tensor $\tilde{F}^{\mu\nu}$.

The solution to Eq. (52) can be found to be

$$w(x^+, x^-) = \int^{x^-} \partial_+ u \, dx^- + w_0(x^+), \tag{54}$$

where $w_0(x^+)$ corresponds to the electromagnetic wave propagating from the right to the left along the x-axis with a speed equal to the light speed in vacuum.

C. The Hopf equation

The system of equations (52) and (53) is a system of quasilinear equations. It admits a rich variety of solutions including those solutions that describe the formation of singularities during the electromagnetic field evolution (e.g., see Ref. [79,80]). This system also admits solutions in the form of simple waves [58] in which w is a function of u, i.e., $w = w(u)$. In this case, Eqs. (52) and (53) take the form

$$J\partial_- u = \partial_+ u, \tag{55}$$

$$\partial_- u = \frac{1 - uw(4\epsilon_2 + 9\epsilon_3 uw) - Ju^2(\epsilon_2 + 3\epsilon_3 uw)}{w^2(\epsilon_2 + 3\epsilon_3 uw)} \partial_+ u \tag{56}$$

where $J = dw/du$ is the Jacobian. Consistency of these equations implies that

$$u^2 J^2 - \frac{1 - uw(4\epsilon_2 + 9\epsilon_3 uw)}{\epsilon_2 + 3\epsilon_3 uw} J + w^2 = 0. \tag{57}$$

Introducing the new variables

$$r = uw \quad \text{and} \quad l = \ln u, \tag{58}$$

for which

$$J = \frac{1}{u^2}\left(\frac{dr}{dl} - r\right), \tag{59}$$

we can write the solution to Eq. (57) as

$$\int^{uw} \frac{2(\epsilon_2 + 3\epsilon_3 r)dr}{\mathcal{F}(r)} = l \tag{60}$$

where

$$\mathcal{F}(r) = 1 - 2\epsilon_2 r - 3\epsilon_3 r^2 \pm \sqrt{1 - 8\epsilon_2 r + 6(2\epsilon_2^2 - 3\epsilon_3)r^2 + 48\epsilon_2\epsilon_3 r^3 + 45\epsilon_3^2 r^4}, \tag{61}$$

Expanding this solution up to linear terms in ϵ_2 and ϵ_3 we obtain for the Jacobian J

$$J = w^2(\epsilon_2 + 3\epsilon_3 uw) + \ldots. \tag{62}$$

We assume that the electromagnetic wave corresponding to the variable u in Eq. (62) counterpropagates with respect to the unperturbed wave $w_0 = w_0(x^+)$, taken to depend on x^+ only. Using Eqs. (29) and (54) and the smallness of the parameter ϵ_2 we can obtain that

$$w(x^+, x^-) \approx w_0(x^+) + \epsilon_2 w_0(x^+) u(x^+, x^-). \tag{63}$$

Further we consider the electromagnetic wave $w_0(x^+)$ to have a constant amplitude $w_0 = -\sqrt{2} W_0 = \text{const}$, i.e., to correspond to crossed fields. In this case, the Jacobian (62) is

$$J = 2\epsilon_2 W_0^2 - 6\sqrt{2}\epsilon_3 W_0^3 u + \ldots. \tag{64}$$

Substitution of J given by Eq. (64) into Eq. (55) yields

$$\partial_+ u - (2\epsilon_2 W_0^2 - 6\sqrt{2}\epsilon_3 W_0^3 u)\partial_- u = 0. \tag{65}$$

In the third term describing nonlinear effects we retain the 6-photon mixing effects because as it is shown in Refs. [58,59] the 4-photon mixing photon effects is of higher order of a small parameter α. We note that the envelope solitons considered in Refs. [42,43] have been discussed within the framework of the 4-photon mixing approximation.

Introducing the new dependent variable

$$\bar{u} = -(2\epsilon_2 W_0^2 - 6\sqrt{2}\epsilon_3 W_0^3 u) \tag{66}$$

we can rewrite Eq. (65) as the Hopf equation

$$\partial_+ \bar{u} + \bar{u}\partial_- \bar{u} = 0. \tag{67}$$

Equation (67) has two groups of symmetries. It means that it remains the same if we make two sets of substitution:

$$\bar{u} \to u_0 + \bar{u} \qquad x^- \to x^- + u_0 x^+; \tag{68}$$

and

$$x^- \to \frac{x^-}{X^-}, \qquad x^+ \to \frac{x^+}{X^+}, \qquad \bar{u} \to \frac{X^+}{X^-}\bar{u}; \tag{69}$$

where u_0, X^- and X^+ are arbitrary real constants. Under the transformations (68), (69) solutions of Eq. (67) go into solutions.

As is well known, the Hopf equation describes the steepening of nonlinear waves (see Ref. [38]). In the case of a finite amplitude electromagnetic wave in the QED vacuum this equation has been obtained and analysed in Refs. [58,59].

The solutions to the Hopf equation (67) can be obtained as follows [38,77]. The l.h.s. of the Hopf equation is a full derivative of the function $\bar{u}(x^+, x^-)$ along the characteristics of Eq. (67) determined by the equation

$$\frac{dx^-}{dx^+} = \bar{u}. \tag{70}$$

The function $\bar{u}$ in Eq. (70) is constant, defined by initial conditions at $x^+ = 0$, and this equation can be rewritten as

$$\frac{dx^-}{dx^+} = \bar{u}(0, x_0^-) \equiv \bar{u}_0(x^-). \tag{71}$$

Relationship between variables x^- and x^+ on the characteristics can be represented as

$$x^- = x_0^- + \bar{u}_0(x_0^-)x^+, \tag{72}$$

whereas the general solution of the Hopf equation, defined by the initial condition $\bar{u}_0(x^-)$, can be written implicitly as

$$\bar{u}(x^-, x^+) = \bar{u}_0(x^- - \bar{u}(x^-, x^+)x^+). \tag{73}$$

The value x_0^-, appearing in Eqs. (72)–(74), is the x^--coordinate on the characteristic at $x^+ = 0$. In other words the variables (x^+, x_0^-) are the Lagrange coordinates. Equation (72) gives relationship between the Euler (x^+, x^-) and Lagrange coordinates (x^+, x_0^-).

Evaluating the gradient of the function $\bar{u}(x^-, x^+)$ we obtain

$$\partial_- \bar{u} = (\partial_- x_0^-)(\partial \bar{u}_0 / \partial x_0^-), \tag{74}$$

where

$$\partial_- x_0^- = \frac{1}{1 + x^+(\partial \bar{u}_0 / \partial x_0^-)} \tag{75}$$

is the Jacobian of the transformation from the Lagrange to the Euler variables. In the region where $\partial \bar{u}_0 / \partial x_0^-$ is negative the gradient (74) grows. The growing of the Jacobian corresponds to the steepening of the wave and to the generation of high order harmonics (e.g., see discussion in Refs. [58,77]). At the coordinate

$$x^+{}_{br} = 1/|\partial \bar{u}_0 / \partial x_0^-| \tag{76}$$

the wave gradient tends to infinity: i.e., the wave breaks. This corresponds to the so called gradient catastrophe. Using the relationship (18) between the light cone coordinates (x^-, x^+) and variables (x, t) we can find that the breaking time equals $t_{br} = 1/c|\partial \bar{u}_0 / \partial x_0|$. Here x_0 is the Lagrange coordinate if the Euler coordinates are (x, t).

In a dispersive medium the nonlinear wave steepening can be balanced by the dispersion effects resulting in formation of quasistationary nonlinear waves such as collisionless shock waves and solitons.

IV. ELECTROMAGNETIC WAVES IN THE QED VACUUM DESCRIBED BY THE KADOMTSEV-PETVIASHVILI, THE DISPERSIONLESS KADOMTSEV-PETVIASHVILI, AND THE KORTEVEG-DE VRIES EQUATIONS

Below we show that combining the effects of difraction, dispersion, and nonlinearity we obtain the nonlinear KP, dKP, and the KdV wave equations.

The Cauchy problem for these equations can be solved exactly [81] (see also Refs. [38–40] and the literature cited therein). The solution includes in particular breaking nonlinear waves and interacting solitons.

From the theory of nonlinear waves we know that in the limit of small but finite nonlinearity, the terms that describe the dispersion and the diffraction appear in the nonlinear wave equations additively. As far as it concerns the diffraction, it can be implemented into the wave equation by considering the case when the electromagnetic potential (47) depends not only on the variables x^+ and x^- but also on the transverse coordinate y (i.e., the beam size the z direction is substantially larger than along the y coordinate). The nonlinear effects resulting in the finite amplitude wave breaking are described by Eq. (67).

A. Kadomtsev-Petviashvili equation

Combining Eqs. (17) and (67) we obtain the Kadomtsev-Petviashvili equation

$$\partial_-(\partial_+ \bar{u} + \bar{u}\partial_- \bar{u} - \partial_{---}\bar{u}) = -\partial_{yy}\bar{u}. \tag{77}$$

In Eq. (77) the variables are normalized as $x^- \to x^-/L$, $x^+ \to x^+/L$, $y \to \sqrt{2}y/L$ with $L = (2\kappa_2)^{1/2}$. If one uses the units with $\hbar = c = 1$, and fields are measured in E_S, then the coefficient κ_2 defined in Eq. (15) equals $\kappa_2 = (4\alpha/135\pi)W_0^4 m_e^{-2}$.

Equation (77) remains unchanged under the transform (68). It is invariant also against the transforms:

$$\begin{aligned} &x^+ \to x^+/X, \qquad x^- \to x^-/X^{1/3}, \\ &y \to y/X^{2/3}, \qquad \bar{u} \to \bar{u}/X^{2/3}, \end{aligned} \tag{78}$$

where X is an arbitrary positive number.

In non-normalized variables Eq. (77) takes the form

$$\begin{aligned} \partial_- \bigg[&\partial_+ u - \left(\frac{4e^2}{45\pi} W_0^2 - \frac{32\sqrt{2}e^2}{105\pi} W_0^3 u \right) \partial_- u \\ &- \frac{8e^2}{135\pi m_e^2} W_0^4 \partial_{---} u \bigg] = -\frac{1}{2}\partial_{yy} u. \end{aligned} \tag{79}$$

Typical examples of the Kadomtsev-Petviashvili equation solitons are discussed in Appendix B.

To demonstrate the Lorentz-invariance of the problem under consideration it is convenient to rewrite Eq. (79) in the equivalent form

$$\partial_+ u - \left(\frac{4e^2}{45\pi} W_0^2 - \frac{32\sqrt{2}e^2}{105\pi} W_0^3 u\right)\partial_- u - \frac{8e^2}{135\pi m_e^2} W_0^4 \partial_{---} u = -\frac{1}{2}\partial_{yy} a, \tag{80}$$

$$\partial_- a = u. \tag{81}$$

As it can be seen from Eqs. (21)–(25) each of the terms in the Eq. (80) is Lorentz-invariant in the (t, x)-plane.

Moreover, the way of derivation of this equation shows that, if we skip the nonlinear term, then remaining equation is completely Lorentz-invariant in the (t, x, y)-hyperplane. Returning back to the whole equation, we may say, that it remains the same under action of any weak rotation in (x, y, z)-hyperplane and any weak Lorentz-transformation in (t, y, z)-hyperplane. The "weak rotation" means neglecting the quadratic terms relative to the rotation angles and to the angle η in Eqs. (19). As far it concerns "weak Lorentz-transformation," according to Ref. [82] it is sufficient to satisfy the requirement that the components of one vector be small compared to those of another in just one frame of reference; by virtue of relativistic invariance, the four-dimensional formulas obtained on the basis of such an assumption will be valid in any other reference frame.

In regard with a relationship between the KdV and KP solitons discussed in the present paper and the solitons which can be obtained with NSE, here we briefly discuss the evolution of a packet of quasimonochromatic waves described by the 2D KP equation Eq. (B1). We assume that the carrier wave wavelength is short enough: $k_0\ell \gg 1$. In this case, a multiscale expansion technique can be applied [38] for finding the solution describing the wave packet evolution. It can be easily shown, the approach developed in Refs. [83,84] in the present case results in obtaining the 3D version of the nonlinear Schroedinger equation (NSE). From the NSE analysis, in this case, it follows that the wave packets are neither subject to self-focusing nor to bunching and, hence, during their evolution the NSE solitons are not formed.

B. Dispersionless KP equation

Neglecting the dispersion effects in Eq. (77) we obtain the so-called dispersionless KP equation

$$\partial_-(\partial_+ \bar{u} + \bar{u}\partial_- \bar{u}) = -\partial_{yy}\bar{u}, \tag{82}$$

The dispersionless KP equation describes the nonlinear wave breaking in a non-one-dimensional configuration [85–89].

As noted above, formally the nonlinear wave steepening and breaking correspond to the growth of the field gradient and to the appearance of the gradient catastrophe. This process can be demonstrated by analysing the self-similar solution of Eq. (82) of the form

$$\bar{u}(x^+, x^-, \mathbf{x}_\perp) = g(x^+)x^- - \frac{\sigma}{2} k_y^2 y^2, \tag{83}$$

where g and k_y are the longitudinal and the transverse inverse scale-lengths of the field $\bar{u}$, and $\sigma = \pm 1$. Substituting (83) into Eq. (82) we obtain an ordinary differential equation for the function $g(x^+)$,

$$g' + g^2 = \sigma k_y^2, \tag{84}$$

where a prime denotes a differentiation with respect to the variable x^+. Its solution reads

$$g = -k_y \tan\left[k_y x^+ - \arctan\left(\frac{g_0}{k_y}\right)\right] \tag{85}$$

if $\sigma = -1$ and

$$g = k_y \tanh\left[k_y x^+ - \operatorname{arctanh}\left(\frac{g_0}{k_y}\right)\right] \tag{86}$$

for $\sigma = +1$. Here g_0 is equal to $g|_{x^+=0}$.

In the case $\sigma = -1$, as

$$x^+ \to \frac{1}{2k_y}\left[\pi + 2\arctan\left(\frac{g_0}{k_y}\right)\right] \tag{87}$$

the gradient of g tends to minus infinity. This corresponds to the wave breaking and to the formation of the shock wavelike structure.

C. Korteveg-de Vries equation

If one neglects the effects of the transverse inhomogeneity by assuming $\partial_{yy}\bar{u} = 0$, then Eq. (77) reduces to the Korteveg-de Vries equation [64]

$$\partial_+ \bar{u} + \bar{u}\partial_- \bar{u} - \partial_{---}\bar{u} = 0. \tag{88}$$

It has the same symmetries as Eq. (77) in Sec. IV A.

We may not distinguish solutions of this equations related with each other by these symmetries. Equation (88) rewritten in physical variables looks as:

$$\partial_+ u - \left(\frac{4e^2}{45\pi} W_0^2 - \frac{32\sqrt{2}e^2}{105\pi} W_0^3 u\right)\partial_- u = \frac{8e^2}{135\pi m_e^2} W_0^4 \partial_{---} u. \tag{89}$$

Equation (88) has the well known single soliton solutions [38–40,64]. They can be presented in terms of Eq. (89), when $u \to 0$ at $|x^-| \to \infty$, and $W_0 u_m > 0$, as:

$$u = \frac{u_m}{\cosh^2[q(x^- + vx^+)]}, \tag{90}$$

This is a constant shape localized nonlinear wave propagating with constant velocity $(1-v)/(1+v)$. Its amplitude, u_m, and v, are related to each other, as well as the soliton width, q^{-1}, as

$$v = \frac{4e^2}{45\pi} W_0^2 \left(1 + \frac{8\sqrt{2}}{7} W_0 \bar{u}_m\right),$$
$$q^2 = \frac{3\sqrt{2}}{7} m_e^2 \frac{\bar{u}_m}{W_0}. \tag{91}$$

Evaluation of the soliton characteristic width, $\ell_s = 1/q$, yields

$$\ell_s \approx 2\frac{1}{m_e}\sqrt{\frac{W_0}{\bar{u}_m}}\left(= 2\lambda\!\!\!^{-}_C\sqrt{\frac{E_0}{E_m}}\right), \tag{92}$$

whereas the soliton formation length is approximately equal to

$$\ell_f \approx \frac{100}{e^2}\frac{W_0^{-4}}{m_e}\left(\frac{W_0}{\bar{u}_m}\right)^{3/2}\left(= \frac{100}{\alpha}\lambda\!\!\!^{-}_C\left(\frac{E_S}{E_0}\right)^4\left(\frac{E_0}{E_m}\right)^{3/2}\right). \tag{93}$$

Here E_0 and E_m are the amplitudes of the counterpropagating waves $\lambda\!\!\!^{-}_C = \hbar/m_e c$ is the Compton wavelength. Assuming $E_0/E_m = 100$ and $E_0/E_S \approx 1$ we obtain for the soliton width $\ell_s = 1.5 \times 10^{-2}$ nm and for the soliton formation length $\ell_f = 4$ μm.

We note that the field invariant $\mathfrak{F}$ for this soliton, as determined by Eq. (43), is negative, i.e., it can be considered as an electromagnetic object where the electron-positron pair creation can occur via the Schwinger mechanism [44,78,90,91].

V. CONCLUSIONS

We have obtained an analytical description of relativistic electromagnetic solitons that can be formed in a configuration consisting of two countercrossing electromagnetic waves propagating in the QED vacuum. These extreme high intensity electromagnetic waves in the QED vacuum are described by partial differential equations that belong to the family of the canonical equations in the theory of nonlinear waves such as the Hopf, the Korteweg-de Vries, the dispersionless Kadomtsev-Petviashvili, and the Kadomtsev-Petviashvili equations.

In the case of the soliton solution of the KdV and KP equations the nonlinearity effects are balanced by the wave dispersion. The description of the nonlinearity leading to the wave steepening requires to take into account the 6-photon mixing process (for details see Refs. [58,59]) within the framework of the theoretical model based on the Heisenberg-Euler Lagrangian (42), which being principally dispersionless corresponds to the long wavelength limit. An adequate approach for calculating the QED vacuum dispersion is based on the perturbation theory developed in Refs. [62,63] where the expression for the invariant photon mass (5), which is a pole of the photon Green's function in a crossed field, has been obtained. The approach elaborated in Refs. [62,63] is valid as long as $\alpha\chi_\gamma^{2/3} \ll 1$. As known, e.g., see [48] in the small amplitude long wavelength limit, when one can neglect the effects dispersion and the nonlinearity is weak the approaches based on the perturbation theory and on the Heisenberg-Euler paradigm are equivalent. Analysis of analytical properties of the $f(\zeta)$ function (see Appendix A allowed us to derive the dispersion term in Eq. (11) which leads to the wave equation in the form (17). The dispersion term combination with the nonlinear term results in the KdV and KdP equations.

These equations have a wide range of applications in mathematics and physics that spans from fluid mechanics to solid state physics and to plasma physics. The soliton theory is also used in quantum field theory [92–95]. In the present paper we extend the field of applications of the KdV, KP and dKP equations to the QED vacuum.

The QED vacuum polarization effects are planned to be studied with the next generation lasers (see for details [47,48,52,53,55,96,97]).

In particular these effects can be revealed by measuring the phase difference between the phase of the electromagnetic pulse colliding with the counterpropagating wave and the phase of the pulse which does not interact with high intensity wave, as well as by analyzing the wave frequency spectrum with specific features corresponding to the soliton formation.

Revealing the change in the parameters of colliding extremely intense laser beams will shed a light on the space-time properties and vacuum texture.

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APPENDIX A: ANALYTICAL PROPERTIES OF THE $f(\zeta)$ FUNCTION

According to Eqs. (7) and (8) the function $f(\zeta)$ can be presented in terms of the Airy functions $Ai(\zeta)$ and $Gi(\zeta)$ (see also Ref. [98]). Integral representations of the standard Airy function $Ai(\zeta)$ and of the inhomogeneous Airy function $Gi(\zeta)$ (it is also known as the Scorer function) are [68,69]

$$Ai(\zeta) = \frac{1}{\pi}\int_0^\infty dt \cos\left(\zeta t + \frac{t^3}{3}\right) \quad \text{(A1)}$$

and

$$Gi(\zeta) = \frac{1}{\pi}\int_0^\infty dt \sin\left(\zeta t + \frac{t^3}{3}\right), \quad \text{(A2)}$$

respectively. They obey the differential equations

$$Ai'' - \zeta Ai = 0 \quad \text{(A3)}$$

and

$$Gi'' - \zeta Gi = -\frac{1}{\pi}. \quad \text{(A4)}$$

Here a prime denotes a differentiation with respect to the variable ζ. The equations should be solved with the initial conditions corresponding to expressions (A1) and (A2) and to (A5) and (A6) below.

The functions $Ai(\zeta)$ and $Gi(\zeta)$ can be expanded into the Maclaurin series as follows.

$$Ai(\zeta) = \frac{3^{-2/3}}{\pi}\sum_{n=0}^\infty \Gamma\left(\frac{n+1}{3}\right)\sin\left(\frac{3n-1}{3}\pi\right)\frac{(3^{1/3}\zeta)^n}{n!}, \quad \text{(A5)}$$

and

$$Gi(\zeta) = \frac{3^{-2/3}}{\pi}\sum_{n=0}^\infty \Gamma\left(\frac{n+1}{3}\right)\cos\left(\frac{3n-1}{3}\pi\right)\frac{(3^{1/3}\zeta)^n}{n!}. \quad \text{(A6)}$$

In the limit $\zeta \to 0$, i.e., for $\chi_\gamma \to \infty$ with the relationship between ζ and χ_γ given by Eq. (6) expressions (A5), (A6) give

$$f(\zeta) = i\frac{3^{-2/3}}{2}\left[\Gamma\left(\frac{1}{3}\right)(\sqrt{3}+i) + \Gamma\left(\frac{2}{3}\right)(-\sqrt{3}+i)3^{1/3}\zeta\right] + \ldots. \quad \text{(A7)}$$

For large ζ, when $\zeta \to \infty$ and $|\arg\zeta| < \pi$, asymptotic expansions of $Ai(\zeta)$ and $Gi(\zeta)$ yield

$$Ai(\zeta) = \frac{\zeta^{-1/4}}{2\pi}\exp\left(-\frac{2}{3}\zeta^{3/2}\right) \times \sum_{n=0}^\infty (-1)^n\Gamma\left(3n+\frac{1}{2}\right)\frac{(9\zeta^{3/2})^{-n}}{(2n)!}, \quad \text{(A8)}$$

$$Gi(\zeta) = \frac{1}{\pi\zeta}\sum_{n=0}^\infty \frac{(3n)!}{n!}(3\zeta^3)^{-n}. \quad \text{(A9)}$$

As a result we obtain for the asymptotic expansion of the function $f(\zeta)$ in the limit $\zeta \gg 1$

$$f(\zeta) = \frac{1}{\zeta} + \frac{2}{\zeta^4} + \frac{120}{\zeta^7} + \cdots + \frac{i\pi}{2\zeta^{1/4}}\exp\left(-\frac{2\zeta^{3/2}}{3}\right). \quad \text{(A10)}$$

APPENDIX B: SOLITONS OF KADOMTSEV-PETVIASHVILI EQUATION

Here several typical examples of the solitons of Kadomtsev-Petviashvili equation [99–101] are presented. By rescaling independent and depended variables the KP equation can be reduced to the normalized form

$$\partial_-(\partial_+ u + 6u\partial_- + \partial_{---}u) = 3\partial_{yy}u. \quad \text{(B1)}$$

A rich variety of the soliton solutions of the KP equation can be found with using the Hirota method [102], Backlund transformation or the Wronskian technique [103].

The localized solution of the KP equation (B1) is known as the "lump" and has the form [101,104–106]

$$u(x^+, x^-, y) = 24v\frac{3 - v[(x^- + vx^+)^2 - vy^2]}{\{3 + v[(x^- + vx^+)^2 + vy^2]\}^2}. \quad \text{(B2)}$$

It is shown in Fig. 3(a). The propagation velocity in the $x^\pm$ variables v of the lump soliton and its maximum amplitude are related as $\bar{u}_m = 8v$. The lump width is $\sqrt{3/v}$ and is inversely proportional to the square root of its amplitude as in the case described by Eq. (92). At $x^- = vx^+$, along the y axis the function $\bar{u}$ monotonically decreases as $\bar{u} \sim y^{-2}$. In the plane (x^-, y) it changes sign on the hyperbola given by equation

$$(x^- + vx^+)^2 - vy^2 = 3/v. \quad \text{(B3)}$$

This hyperbola is clearly seen in Fig. 3(b), where the isocontours of $\bar{u}(0, x^-, y)$ are plotted.

Multi-solitons for the KP equation can be found from equation $u = 2\partial_{--}(\ln f)$ (see Ref. [102] and literature cited therein). To single-soliton solution to the KP equation the function $f(x^-, x^+, y)$ equals

$$f = 1 + \exp(\theta_1), \quad \text{(B4)}$$

where $\theta_1 = k_1(x^- + \omega_1 x^+ + p_1 y) + \xi_1$ with $\omega_1 = k_1^3 + 3p_1^2$. The corresponding KP soliton is given by

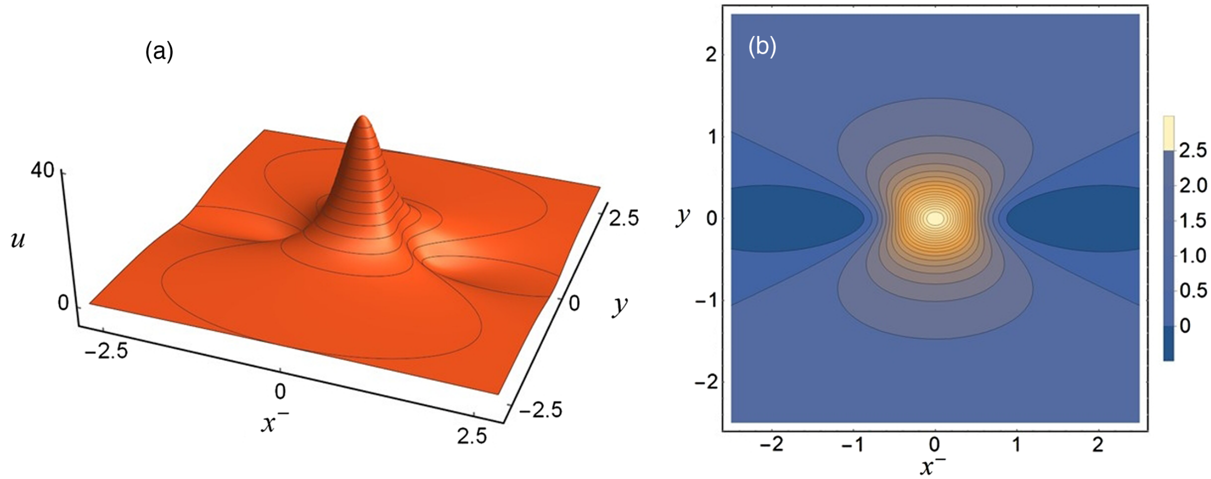

FIG. 3. Lump soliton for $v = 5$: a) $\bar{u}(x, y, 0)$; b) contours of equal value of $\bar{u}(x, y, 0)$.

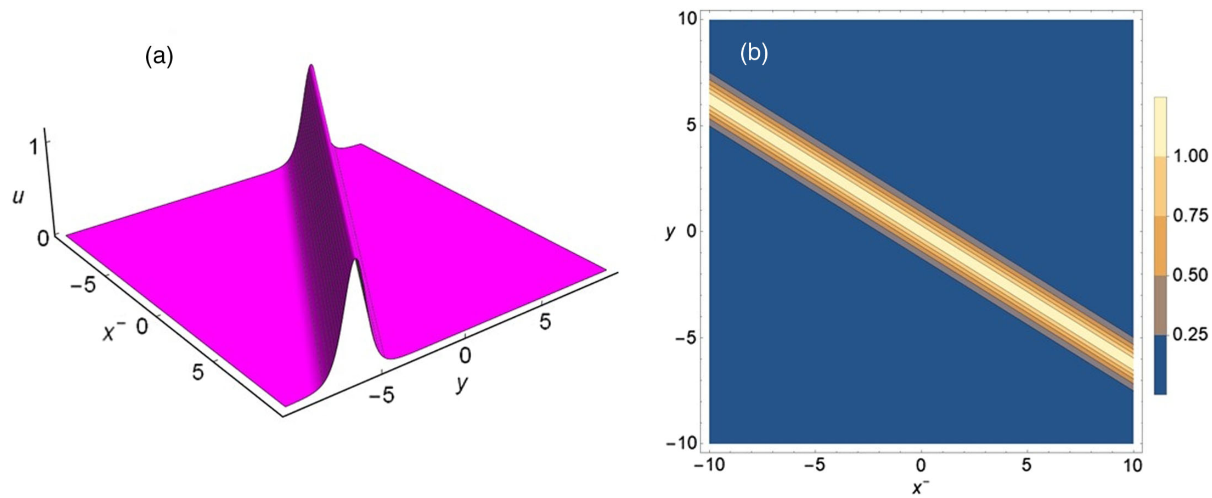

FIG. 4. Single soliton for KP equation.

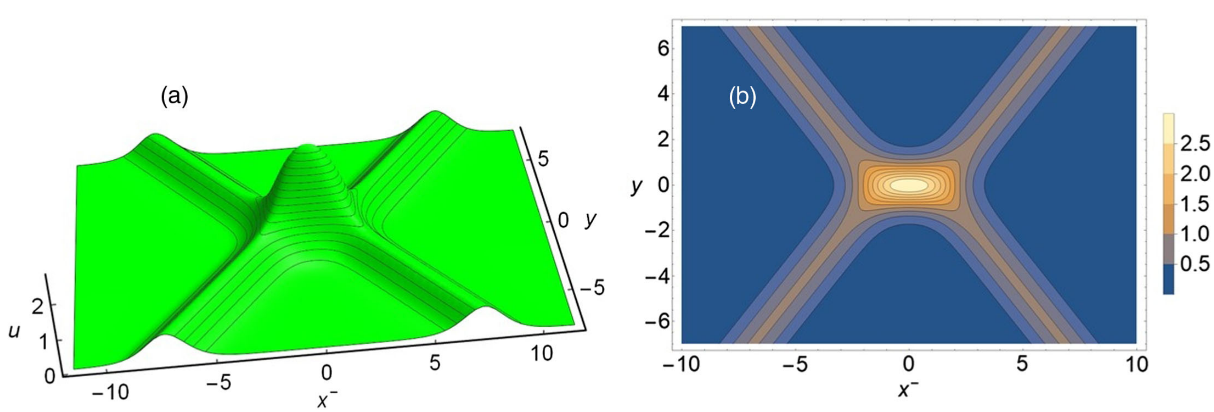

FIG. 5. Double soliton for KP equation.

$$u = \frac{k_1^2}{2\cosh^2(\theta_1/2)}. \quad \text{(B5)}$$

It describes "oblique KdV" soliton whose maximum is localized on the line $\theta_1 = 0$. It is shown in Fig. 4.

Double soliton solution for the KP equation is given by the function f equal to

$$f = 1 + b_1 \exp(\theta_1) + b_2 \exp(\theta_2) + b_{12} \exp(\theta_1 + \theta_2), \quad \text{(B6)}$$

where $\theta_i = k_i(x^- + \omega_i x^+ + p_i y) + \xi_i$ with $\omega_i = k_i^3 + 3p_i^2$ $(i = 1, 2)$ and

$$b_1 = -\frac{k_1 + k_2 + p_1 - p_2}{k_1 - k_2 - p_1 + p_2}, \qquad b_2 = \frac{k_1 + k_2 - p_1 + p_2}{k_1 - k_2 - p_1 + p_2},$$
$$b_3 = -\frac{k_1 - k_2 + p_1 - p_2}{k_1 - k_2 - p_1 + p_2}. \quad \text{(B7)}$$

Double soliton for the KP equation is shown in Fig. 5.

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