Trimaximal TM_1 mixing with two modular S_4 groups

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We discuss a minimal flavor model with twin modular symmetries, leading to trimaximal TM_1 lepton mixing in which the first column of the tri-bimaximal lepton mixing matrix is preserved. The model involves two modular S_4 groups, one acting in the neutrino sector, associated with a modulus field value τ_{SU} with residual Z_2^{SU} symmetry, and one acting in the charged lepton sector, associated with a modulus field value τ_T with residual Z_3^T symmetry. Apart from the predictions of TM_1 mixing, the model leads to a new neutrino mass sum rule which implies lower bounds on neutrino masses close to current limits from neutrinoless double beta decay experiments and cosmology.

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I. INTRODUCTION

The discovery of neutrino masses and lepton mixing opened up a new direction in physics beyond Standard Model (SM) focused on understanding their theoretical origin. An elegant possibility remains the classical type-Ia seesaw mechanism [1–7] involving right-handed neutrinos, which, after being integrated out, yield the Weinberg operators $H_u H_u L_i L_j$ with $H_u = H$ being the SM Higgs doublet and L_i a lepton doublet of the ith flavour.[1] To explain the observed approximate tri-bimaximal (TBM) lepton mixing, one has to go beyond the seesaw mechanism and consider to impose a non-Abelian discrete flavor symmetry [9,10]. For example, S_4 can be used to account for trimaximal TM_1 mixing [11,12], which is imposed by a residual Z_2^{SU} symmetry in the neutrino sector and a residual Z_3^T symmetry in the charged lepton sector.[2] However, all existing realistic models typically involve several flavon fields with nontrivial vacuum alignments.

Non-Abelian discrete flavor symmetries have been widely used in models of lepton flavor mixing for decades, but the nature of non-Abelian discrete flavor symmetry is still unclear. It might be an effective remnant symmetry after a continuous non-Abelian symmetry breaking [13–20], or a fundamental symmetry of spacetime in extra dimensions [21–32]. In the latter case, a non-Abelian discrete symmetry could either arise as an accidental symmetry of orbifolding (see [29,33–35] for recent discussion with two extra dimensions) or as a subgroup of the so-called modular symmetry. The modular symmetry [36] is an infinite symmetry of the extradimensional lattice arising from superstring theory [37,38].[3] Indeed, it has been suggested that a finite subgroup of the modular group, when interpreted as a flavor symmetry, might be helpful for an explanation for lepton mixing [41–43].

Recently, such a finite modular symmetry has been proposed as the direct origin of flavor mixing. In this approach, Yukawa and mass textures arise not from flavon fields, but modular forms with even modular weights which are holomorphic functions of a modulus field [44].[4] The complex modulus field τ acquires a vacuum expectation value (VEV) and eventually determines the flavor structure. The finite modular groups $\Gamma_2 \simeq S_3$ [46,47], $\Gamma_3 \simeq A_4$ [44,47–52], $\Gamma_4 \simeq S_4$ [53,54], and $\Gamma_5 \simeq A_5$ [55,56] have been considered, in which special Yukawa textures are consequences of the modular forms. Compared with the framework of traditional flavor model constructions, only a minimal set of flavons (or no flavons at all) need to be introduced in this framework,[5] making such an approach very attractive.

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[1]An alternative type-Ib seesaw mechanism, yielding the new Weinberg operators $H_u \tilde{H}_d L_i L_j$ with $\tilde{H}_d$ being a charge conjugated second Higgs doublet with opposite hypercharge, was proposed in [8] recently.

[2]We apply the standard convention of the S_4 generators S, T, and U where $S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = I$ [9] hold.

[3]Recently, the geometric connection between the origin of the flavor symmetry due to modular symmetry and that due to orbifolding with two extra dimensions has been discussed, e.g., in [39,40].

[4]Very recently, this approach has been extended to include odd weight modular forms [45].

[5]Extensions to flavor mixing in the quark sector are given in [47,50,57,58].

For flavor models with finite modular symmetry outlined above, only one single modulus field τ is usually included, corresponding to a single finite modular group symmetry Γ_N. It has been pointed out that particular modular forms at some special values of the modulus VEV preserve a residual subgroup of the finite modular symmetry. Such an idea was discussed in [51] where residual symmetries are considered as subgroups of the modular A_4 symmetry. Making use of two moduli fields with VEVs preserving different residual symmetries, i.e., Z_3 in the charged lepton sector and Z_2 in the neutrino sector, it was shown how trimaximal TM_2 mixing might be realized [51]. A brief discussion of residual symmetry after the breaking of modular S_4 symmetry has also been given in [54].

In a recent paper [59], two of us extended the formalism of finite modular symmetry to the case of multiple moduli fields τ_J $(J = 1, \ldots M)$ associated with the finite modular symmetry $\Gamma^1_{N_1} \times \Gamma^2_{N_2} \times \cdots \times \Gamma^M_{N_M}$. This is motivated by superstring theory which involves six compact extra dimensions, suggesting the introduction of three modular symmetries associated with three different factorized tori in the simplest compactifications. As an example, we presented the first consistent example of a flavor model of leptons with multiple modular S_4 symmetries interpreted as a flavor symmetry. The considered model involved three finite modular symmetries S_4^A, S_4^B, and S_4^C, associated with two right-handed neutrinos and the charged lepton sector, respectively, broken by two bi-triplet scalars to their diagonal subgroup. The low energy effective theory consisted of a single S_4 modular symmetry with three independent modular fields τ_A, τ_B, and τ_C, which preserve the residual modular subgroups Z_3^A, Z_2^B, and Z_3^C, in their respective sectors leading to trimaximal TM_1 lepton mixing, in which the first column of the tri-bimaximal mixing matrix is achieved, in excellent agreement with current data, without requiring any flavons.

In the present paper, we discuss a simpler model of TM_1 lepton mixing via two modular S_4 groups, one S_4^ν acting in the neutrino sector, associated with a modulus field value τ_{SU} with residual Z_2^{SU} symmetry, and one S_4^l acting in the charged lepton sector, associated with a modulus field value τ_T with residual Z_3^T symmetry. The two moduli fields are assumed to be "stabilized" at these symmetric points, and there are no other flavons, making the model very economical and predictive. In particular, it leads to a new neutrino mass sum rule which implies sizeable neutrino masses sensitive to neutrinoless double beta decay and cosmological probes. The main difference between the present model and the one in [59], is that here we assume that there are three right-handed neutrinos in a triplet of an S_4, whereas the previous model assumed two right-handed neutrinos which were S_4 singlets. The resulting model here is very similar to the "semidirect" models of traditional flavor symmetry. However, the predictions are different due to the smaller number of parameters, leading to a new and testable neutrino mass sum rule.

The rest of the paper is organized as follows. In Sec. II, we first focus on the case of the single finite modular S_4 symmetry, with residual symmetry arising from the moduli stabilizers. We then generalize the results to the case of two modular S_4 groups. In Sec. III, we propose a model based on $S_4^\nu \times S_4^l$ with two moduli fields, which is broken to a single diagonal S_4 with two independent moduli fields at low energies, whose stabilizers lead to different remnant symmetry in the different sectors, which may be used to enforce trimaximal TM_1 mixing with the new neutrino mass sum rule. Section IV concludes the paper.

II. S_4 MODULAR SYMMETRIES

A. A single S_4 modular group

The modular group $\overline{\Gamma}$ acting on the modulus field τ as linear fractional transformations

$$\gamma\colon\ \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{1}$$

where the modulus field τ is defined on the upper complex plane $\mathrm{Im}(\tau) > 0$, a, b, c, and d are integers and satisfy $ad - bc = 1$. It is convenient to represent each element of $\overline{\Gamma}$ by a two by two matrix.[6] In this way, $\overline{\Gamma}$ is expressed as

$$\overline{\Gamma} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / (\pm \mathbf{1}),\ a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}. \tag{2}$$

The modular group is isomorphic to the projective spatial linear group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$. It has two generators, S_τ and T_τ, satisfying $S_\tau^2 = (S_\tau T_\tau)^3 = \mathbf{1}$. These generators act on the modulus τ in the following way:

$$S_\tau\colon\ \tau \to -\frac{1}{\tau}, \qquad T_\tau\colon\ \tau \to \tau + 1, \tag{3}$$

respectively. Representing them by two by two matrices, we obtain

$$S_\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T_\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

$\overline{\Gamma}$ is a discrete but infinite group. By requiring a, $d = 1$ (mod 4) and b, $c = 0$ (mod 4), i.e.,

$$a = 4k_a + 1, \qquad d = 4k_d + 1, \qquad b = 4k_b, \qquad c = 4k_c, \tag{5}$$

where k_a, k_b, k_c, and k_d are all integers; we obtain a subset of $\overline{\Gamma}$ labeled as

[6] Note that it may not be a unitary matrix.

$$\bar{\Gamma}(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (\mathrm{mod}\ 4) \right\}. \quad (6)$$

It is also an infinite group. The quotient group $\Gamma_4 = \overline{\Gamma}/\overline{\Gamma}(4)$ is a finite modular group. It is equivalently obtained by imposing $T_\tau^4 = \mathbf{1}$. As Γ_4 is a subgroup of $\overline{\Gamma}$, its elements can also be represented as two by two matrices, but the representation matrices are not unique. Since S_4 is the quotient group $\overline{\Gamma}/\overline{\Gamma}(4)$, with the help of Eq. (5), we know that any element γ of S_4 which can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (7)$$

is identical to be represented in the form

$$\eta \begin{pmatrix} 4k_a + a & 4k_b + b \\ 4k_c + c & 4k_d + d \end{pmatrix}, \quad (8)$$

where the integers k_a, k_b, k_c, and k_d satisfy $4k_ak_d + ak_d + dk_a = 4k_bk_c + bk_c + ck_b$ and $\eta = \pm 1$. This is just a mathematical redundancy. Selecting a different two by two representation matrix gives no physical difference.

The finite modular group Γ_4 is isomorphic to S_4, the permutation group of four objects. In other word, S_τ and T_τ, which satisfy $S_\tau^2 = (S_\tau T_\tau)^3 = T_\tau^4 = 1$, can be used as generators of S_4. In the literature of flavor symmetry studies, it is more popular to use a different set of generators, S, T, and U, which satisfy $S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = 1$, to generate S_4. These generators can be represented by S_τ and T_τ as

$$T = S_\tau T_\tau, \qquad S = T_\tau^2, \qquad U = T_\tau S_\tau T_\tau^2 S_\tau. \quad (9)$$

With the requirement $\tau = \tau + 4$, S, T, and U can be represented by two by two matrices such as

$$T = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}. \quad (10)$$

Again, we mention that representation matrices of these elements are not unique. Different representation matrices are obtained by considering the correlation between Eqs. (7) and (8). We also list a two by two matrix for $SU = S_\tau T_\tau S_\tau T_\tau^{-1} S_\tau$,[7]

$$SU = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}. \quad (11)$$

This generator is important for the trimaximal TM_1 mixing in the classical flavor model building (see, e.g., [12]) and will also be used for our model construction in the next section.

In the framework of $\mathcal{N} = 1$ supersymmetry with the S_4 modular symmetry, the superpotential $W(\phi_i;\tau)$ is in general a function of the modulus field τ and superfields ϕ_i. Under the modular transformation, the superpotential should be invariant [37]. Expanding the superpotential $W(\phi_i;\tau)$ in powers of the superfields ϕ_i, we obtain

$$W(\phi_i;\tau) = \sum_n \sum_{\{i_1,\ldots,i_n\}} \sum_{I_Y} (Y_{I_Y}\phi_{i_1}\cdots\phi_{i_n})_{\mathbf{1}}, \quad (12)$$

where Y_{I_Y} represents a collection of coefficients of the couplings. The chiral superfield ϕ_i, as a function of τ (but does not need to be a modular form), transforms as [37]

$$\phi_i(\tau) \to \phi_i(\gamma\tau) = (c\tau + d)^{-2k_i} \rho_{I_i}(\gamma)\phi_i(\tau), \quad (13)$$

where $-2k_i$ (with k_i being an integer) is the modular weight of ϕ_i, I_i denotes the representation of ϕ_i, and $\rho_{I_i}(\gamma)$ is a unitary representation matrix of γ with $\gamma \in S_4$. The coefficients Y_{I_Y} transform as a multiplet modular form of weight $2k_Y$ and with the representation I_Y,

$$Y_{I_Y}(\tau) \to Y_{I_Y}(\gamma\tau) = (c\tau + d)^{2k_Y} \rho_{I_Y}(\gamma) Y_{I_Y}(\tau), \quad (14)$$

where $k_Y = k_{i_1} + \cdots + k_{i_n}$ is required to be a non-negative integer. The representation and weight of Y_{I_Y} are constrained due to the invariance of the operator under the S_4 modular transformation. For $k_Y = 1$, there are five modular forms $Y_i(\tau)$ for $i = 1, 2, 3, 4, 5$, which form a doublet $\mathbf{2}$ and a triplet $\mathbf{3}'$ of S_4,

$$Y_{\mathbf{2}}^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \qquad Y_{\mathbf{3}'}^{(2)}(\tau) = \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}. \quad (15)$$

Specifically, an algebra between Y_3, Y_4, and Y_5,

$$(Y_3^2 + 2Y_4Y_5)^2 = (Y_4^2 + 2Y_3Y_5)(Y_5^2 + 2Y_3Y_4) \quad (16)$$

is satisfied [53]. This constraint is independent of the value of τ and essential to cover the modular space of Γ_4. Contracting these modular forms gives rise to modular forms with weights $2k_Y = 4$,

[7]The product SU gives

$$\begin{pmatrix} 5 & -3 \\ 2 & -1 \end{pmatrix} = (-1)\begin{pmatrix} 4\times(-1) - 1 & 4\times 1 - 1 \\ 4\times(-1) + 2 & 4\times 0 + 1 \end{pmatrix}.$$

Applying Eq. (8), we arrive at Eq. (11).

$$Y_{\mathbf{1}}^{(4)}(\tau) = Y_1 Y_2,$$
$$Y_{\mathbf{2}}^{(4)}(\tau) = \begin{pmatrix} Y_2^2 \\ Y_1^2 \end{pmatrix},$$
$$Y_{\mathbf{3}}^{(4)}(\tau) = \begin{pmatrix} Y_1Y_4 - Y_2Y_5 \\ Y_1Y_5 - Y_2Y_4 \\ Y_1Y_3 - Y_2Y_4 \end{pmatrix},$$
$$Y_{\mathbf{3'}}^{(4)}(\tau) = \begin{pmatrix} Y_1Y_4 + Y_2Y_5 \\ Y_1Y_5 + Y_2Y_4 \\ Y_1Y_3 + Y_2Y_4 \end{pmatrix}. \tag{17}$$

Modular forms with higher weights can all be constructed from Y_i. We refer to [53] for detailed discussions.

It is helpful to summarize the special properties of stabilizers and their relations with residual modular symmetries. We gave a thorough discussion on this issue in [59]. Here we will mention four stabilizers which are relevant to the current work,

$$\tau_T = \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad \tau_S = i\infty,$$
$$\tau_U = \frac{1}{2} + \frac{i}{2}, \qquad \tau_{SU} = -\frac{1}{2} + \frac{i}{2}. \tag{18}$$

Given any element γ in a modular group, a stabilizer of γ is a special value of the modulus field, denoted as τ_γ, which satisfies $\gamma\tau_\gamma = \tau_\gamma$. If the modulus τ gains a VEV at the stabilizer, $\langle\tau\rangle = \tau_\gamma$, an Abelian residual modular symmetry generated by γ is preserved. Specifically, for $\langle\tau\rangle = \tau_T, \tau_S, \tau_U, \tau_{SU}$, residual symmetries Z_3^T, Z_2^S, Z_2^U, and Z_2^{SU} are preserved, respectively.[8]

A modular form at a stabilizer takes an interesting weight-dependent direction. Starting from $Y_I(\gamma\tau_\gamma) = Y_I(\tau_\gamma)$ and following the standard transformation property in Eq. (14), one arrives at

$$\rho_I(\gamma) Y_I(\tau_\gamma) = (c\tau_\gamma + d)^{-2k} Y_I(\tau_\gamma). \tag{19}$$

Therefore, a modular form at a stabilizer $Y_I(\tau_\gamma)$ is an eigenvector of the representation matrix $\rho_I(\gamma)$ with respective eigenvalue $(c\tau_\gamma + d)^{-2k}$. If $(c\tau_\gamma + d)^{-2k} = 1$ is satisfied, $\rho_I(\gamma)Y_I(\tau_\gamma) = Y_I(\tau_\gamma)$, the residual modular symmetry is reduced to the residual flavor symmetry. Otherwise, the residual modular symmetry is different from a residual flavor symmetry.

We consider triplet modular forms $Y_{\mathbf{3}^{(\prime)}}^{(2k)}(\tau)$ at $\tau_\gamma = \tau_S$, τ_T, τ_U, and τ_{SU}. The eigenvalue $(c\tau_\gamma + d)^{-2k}$ at these stabilizers is, respectively, given by

$$(c\tau_\gamma + d)^{-2k} = \begin{cases} (0\tau_S + 1)^{-2k} \equiv 1 & \text{for } \gamma = S \text{ and } \tau_\gamma = \tau_S \\ (-\tau_T - 1)^{-2k} = \omega^{2k} & \text{for } \gamma = T \text{ and } \tau_\gamma = \tau_T \\ (2\tau_U - 1)^{-2k} = (-1)^k & \text{for } \gamma = U \text{ and } \tau_\gamma = \tau_U \\ (2\tau_{SU} + 1)^{-2k} = (-1)^k & \text{for } \gamma = SU \text{ and } \tau_\gamma = \tau_{SU} \end{cases}, \tag{20}$$

where values of c and d for $\gamma = S, T, U, SU$ are obtained from Eq. (10). Given triplet ($\mathbf{3}, \mathbf{3'}$) representation matrices for S, T, and U in Table II, it is straightforward to obtain

$$Y_{\mathbf{3}^{(\prime)}}^{(6j+2)}(\tau_T) \propto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad Y_{\mathbf{3}^{(\prime)}}^{(6j+4))}(\tau_T) \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad Y_{\mathbf{3}^{(\prime)}}^{(6j+6))}(\tau_T) \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$
$$Y_{\mathbf{3}^{(\prime)}}^{(2k)}(\tau_S) \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$
$$Y_{\mathbf{3}}^{(4j+2)}(\tau_U) \propto Y_{\mathbf{3'}}^{(4j+4)}(\tau_U) \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$
$$Y_{\mathbf{3}}^{(4j+2)}(\tau_{SU}) \propto Y_{\mathbf{3'}}^{(4j+4)}(\tau_{SU}) \propto \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \tag{21}$$

[8]The stabilizer of an element γ may not be unique. We will not discuss other stabilizers that preserve Z_3^T, Z_2^S, Z_2^U, or Z_2^{SU}.

where j is a non-negative integer.[9] These results are obtained without knowing explicit expressions of modular forms. However, there are some exceptions of modular forms whose directions cannot be directly determined by the above argument, $Y_{\mathbf{3}}^{(4j+4)}(\tau_U)$, $Y_{\mathbf{3'}}^{(4j+2)}(\tau_U)$ and $Y_{\mathbf{3}}^{(4j+4)}(\tau_{SU})$, $Y_{\mathbf{3'}}^{(4j+2)}(\tau_{SU})$. These modular forms correspond to eigenvectors of degenerate eigenvalues. For instance, $Y_{\mathbf{3'}}^{(2)}(\tau_{SU})$ is the eigenvalue of $\rho_{\mathbf{3'}}(SU)$ with respect to the degenerate eigenvalue 1. To fully determine the direction of these modular forms, we can apply the algebra in Eq. (16). Take $Y_{\mathbf{3'}}^{(2)}(\tau_{SU})$ again as an example. It corresponds to the eigenvalue 1 of $\rho_{\mathbf{3'}}(SU)$. The latter has two linearly independent eigenvectors $(1,1,1)^T$ and $(0,1,-1)^T$, and $Y_{\mathbf{3'}}^{(2)}(\tau_{SU})$ should be a linear combination of them,

$$Y_{\mathbf{3'}}^{(2)}(\tau_{SU}) = a\begin{pmatrix}1\\1\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\-1\end{pmatrix}. \tag{22}$$

Taking it to Eq. (16),[10] we obtain the identity, $[a^2+2(a+b)(a-b)]^2 = [(a+b)^2+2a(a-b)][(a-b)^2+2a(a+b)]$, which leads to the ratio $b=\pm\sqrt{6}a$. The sign difference, which cannot be determined by the above algebra, is determined by calculating the exact modular functions. Taking the value $\tau_{SU}=-1/2+i/2$ into the formula of modular forms, we obtain numerically $Y_3(\tau_{SU}) = -1.09422i$, $Y_4(\tau_{SU}) = -3.7745i$, and $Y_5(\tau_{SU}) = 1.58606i$, i.e., $a=-1.09422i$ and $b=2.68028i$. Therefore, we arrive at $b=-\sqrt{6}a$. Here, together with $Y_{\mathbf{3'}}^{(2)}(\tau_{SU})$, we list some interesting modular forms respecting to degenerate eigenvalues with modular weights ≤ 4,

$$Y_{\mathbf{3'}}^{(2)}(\tau_{SU}) \propto \begin{pmatrix}1\\1-\sqrt{6}\\1+\sqrt{6}\end{pmatrix}, \qquad Y_{\mathbf{3}}^{(4)}(\tau_U) \propto \begin{pmatrix}\sqrt{2}+2i\\\sqrt{2}-i\\\sqrt{2}-i\end{pmatrix},$$
$$Y_{\mathbf{3}}^{(4)}(\tau_{SU}) \propto \begin{pmatrix}\sqrt{2}\\\sqrt{2}-\sqrt{3}\\\sqrt{2}+\sqrt{3}\end{pmatrix}. \tag{23}$$

In addition, we list double modular forms $Y_{\mathbf{2}}^{(2k)}(\tau_S)$, $Y_{\mathbf{2}}^{(2k)}(\tau_U)$, and $Y_{\mathbf{2}}^{(2k)}(\tau_{SU})$,

$$Y_{\mathbf{2}}^{(4j+2)}(\tau_U) \propto Y_{\mathbf{2}}^{(4j+2)}(\tau_{SU}) \propto \begin{pmatrix}1\\-1\end{pmatrix},$$
$$Y_{\mathbf{2}}^{(2k)}(\tau_S) \propto Y_{\mathbf{2}}^{(4j+4)}(\tau_U) \propto Y_{\mathbf{2}}^{(4j+4)}(\tau_{SU}) \propto \begin{pmatrix}1\\1\end{pmatrix}. \tag{24}$$

[9] Note that for $j=0$, $Y_{\mathbf{3}}^{(2)}$ should be considered since it does not exist.

[10] Although the modular symmetry is broken by VEV of the modular field, this identity, which is independent of the value of the modular field, is always satisfied.

B. Two S_4 modular groups

In our recent paper [59], we discussed how to generalize the discussion from a single S_4 to multiple S_4 modular symmetries. Here we will give a brief review, limiting the discussion to the case of two S_4 modular groups relevant to the model discussed later.

Given two infinite modular groups $\bar{\Gamma}^l$ and $\bar{\Gamma}^\nu$, where the moduli fields are denoted as τ_l and τ_ν, respectively. Following Eq. (1), any two modular transformations $\gamma_l\times\gamma_\nu$ in $\overline{\Gamma}^l\times\overline{\Gamma}^\nu$ take forms as

$$\gamma_l\times\gamma_\nu\colon\ (\tau_l,\tau_\nu)\to(\gamma_l\tau_l,\gamma_\nu\tau_\nu) = \left(\frac{a_l\tau_l+b_l}{c_l\tau_l+d_l},\frac{a_\nu\tau_\nu+b_\nu}{c_\nu\tau_\nu+d_\nu}\right). \tag{25}$$

Two finite modular groups S_4^l and S_4^ν can be obtained by imposing $T_{\tau_l}^4 = T_{\tau_\nu}^4 = \mathbf{1}$ following the discussion in the former section. Their generators $(S,\ T,\ U)$ are denoted by $(S_l,\ T_l,\ U_l)$ and $(S_\nu,\ T_\nu,\ U_\nu)$, respectively, where the subscripts are only used to distinguish groups.

The superpotential $W(\phi_i;\tau_l,\tau_\nu)$, which is invariant under any modular transformations, is in general a holomorphic function of the moduli fields τ_l, τ_ν and superfields ϕ_i. It is expressed in powers of ϕ_i as

$$W(\phi_i;\tau_l,\tau_\nu) = \sum_n \sum_{\{i_1,\ldots,i_n\}} (Y_{(I_{Y,l},I_{Y,\nu})}\phi_{i_1}\cdots\phi_{i_n})_{(\mathbf{1},\mathbf{1})}, \tag{26}$$

the weights of $Y_{(I_{Y,l},I_{Y,\nu})}$ are given by $k_{Y,l}=k_{1,l}+\cdots k_{n,l}$ and $k_{Y,\nu}=k_{1,\nu}+\cdots k_{n,\nu}$. The chiral field ϕ_i and the modular form $Y_{(I_{Y,l},I_{Y,\nu})}$, respectively, transform as

$$\begin{aligned}\phi_i(\tau_l,\tau_\nu) &\to \phi_i(\gamma_l\tau_l,\gamma_\nu\tau_\nu)\\ &= (c_l\tau_l+d_l)^{-2k_{i,l}}(c_\nu\tau_\nu+d_\nu)^{-2k_{i,\nu}}\rho_{I_{i,l}}(\gamma_l)\\ &\quad\times\phi_i(\tau_l,\tau_\nu)\rho^T_{I_{i,\nu}}(\gamma_\nu),\\ Y_{(I_{Y,l},I_{Y,\nu})}(\tau_l,\tau_\nu) &\to Y_{(I_{Y,l},I_{Y,\nu})}(\gamma_l\tau_l,\gamma_\nu\tau_\nu)\\ &= (c_l\tau_l+d_l)^{2k_{Y,l}}(c_\nu\tau_\nu+d_\nu)^{2k_{Y,\nu}}\\ &\quad\times\rho_{I_{Y,l}}(\gamma_l)Y_{(I_{Y,l},I_{Y,\nu})}(\tau_l,\tau_\nu)\rho^T_{I_{Y,\nu}}(\gamma_\nu).\end{aligned} \tag{27}$$

Here, we have arranged ϕ_i and $Y_{(I_{Y,l},I_{Y,\nu})}$ as matrices, and let γ_l act on them vertically and γ_ν act on them horizontally.

Including two modular symmetries allows us to break modular symmetries into different subgroups in charged lepton sector and neutrino sector, respectively. For example, $\langle\tau_l\rangle = \tau_T$ and $\langle\tau_\nu\rangle = \tau_S$ or $\langle\tau_\nu\rangle = \tau_U$. We will discuss phenomenological consequences of these different breaking chains in the next section in the model building.

III. A MINIMAL MODEL WITH $S_4^l \times S_4^\nu$ MODULAR SYMMETRIES

The extension from one single modulus field to multiple moduli fields [59] opens the door to new directions in modular model building. Following the approach of multiple modular symmetries, we will construct a flavor model with two modular symmetries, S_4^l and S_4^ν, with moduli fields labeled by τ_l and τ_ν, respectively. After moduli fields gain different VEVs, different textures of mass matrices are realized in charged lepton and neutrino sectors.

The transformation properties of leptons are given in Table I. Leptons, including right-handed neutrinos ν^c are arranged in the following way: (1) the right-handed leptons e^c, μ^c, and τ^c are singlets $\mathbf{1}'$ of S_4^l and trivial singlets 1 of S_4^ν, and have different weights $2k_l = -6, -4, -2$, respectively, and the same weight $2k_\nu = -2$; (2) the lepton doublets L form a triplet of S_4^l with zero weight, but a singlet of S_4^ν with weight $2k_\nu = +2$; (3) we introduce three right-handed neutrinos ν^c which form a triplet of S_4^ν with weight $2k_\nu = -2$.

Superpotential terms for generating charged lepton and neutrino mass matrices are, respectively, given by

$$\begin{aligned} w = {} & [LY_e(\tau_l)e^c + LY_\mu(\tau_l)\mu^c + LY_\tau(\tau_l)\tau^c]H_d \\ & + \frac{y_\nu}{\Lambda}L\Phi\nu^c H_u + \frac{1}{2}M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} \\ & + \frac{1}{2}M_{\mathbf{2}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{2}} + \frac{1}{2}M_{\mathbf{3}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{3}}. \end{aligned} \quad (28)$$

TABLE I. Transformation properties of leptons, Yukawa couplings, and right-handed neutrino masses in $S_4^l \times S_4^\nu$.

Fields	S_4^l	S_4^ν	$2k_l$	$2k_\nu$
e^c	$\mathbf{1}'$	$\mathbf{1}$	−6	−2
μ^c	$\mathbf{1}'$	$\mathbf{1}$	−4	−2
τ^c	$\mathbf{1}'$	$\mathbf{1}$	−2	−2
L	$\mathbf{3}$	$\mathbf{1}$	0	+2
ν^c	$\mathbf{1}$	$\mathbf{3}$	0	−2
Φ	$\mathbf{3}$	$\mathbf{3}$	0	0
$H_{u,d}$	$\mathbf{1}$	$\mathbf{1}$	0	0

Yukawas/masses	S_4^l	S_4^ν	$2k_l$	$2k_\nu$
$Y_e(\tau_l)$	$\mathbf{3}'$	$\mathbf{1}$	+6	0
$Y_\mu(\tau_l)$	$\mathbf{3}'$	$\mathbf{1}$	+4	0
$Y_\tau(\tau_l)$	$\mathbf{3}'$	$\mathbf{1}$	+2	0
$M_{\mathbf{1}}(\tau_\nu)$	$\mathbf{1}$	$\mathbf{1}$	0	+4
$M_{\mathbf{2}}(\tau_\nu)$	$\mathbf{1}$	$\mathbf{2}$	0	+4
$M_{\mathbf{3}}(\tau_\nu)$	$\mathbf{1}$	$\mathbf{3}$	0	+4

To be invariant under the modular transformation, $Y_{e,\mu,\tau}$ are $\mathbf{3}'$-plet modular forms of S_4^l with weights $2k_l = 6, 4, 2$, respectively, y_ν can only be a modulus-independent coefficient in this model instead of a modular form. Masses for right-handed neutrinos all take the same modular weight $2k_\nu = +4$. $M_{\mathbf{1}}(\tau_\nu)$, $M_{\mathbf{2}}(\tau_\nu)$, and $M_{\mathbf{3}}(\tau_\nu)$ represent **1**-, **2**-, and **3**-plets modular forms appearing in right-handed neutrino mass terms. The dimension-five operator $L\Phi\nu^c H_u$ is understood as an effective operator after integrating out heavy particles. A typical example is including a pair of electroweak-neutral superfields, F, F^c, with couplings $LF^cH_u + M_F FF^c + F\Phi\nu^c$, where $F, F^c \sim (\mathbf{3}, \mathbf{1})$ of (S_4^l, S_4^ν) and $(2k_l, 2k_\nu) = (0, \pm 2)$. Decoupling of these fields introduces no additional relevant dimension-five operator but the one in Eq. (28).

A. $S_4^l \times S_4^\nu \to S_4$

In order to achieve this breaking, we have introduced a scalar Φ, which is arranged as a bi-triplet, i.e., $\Phi \sim (\mathbf{3}, \mathbf{3})$ of $S_4^l \times S_4^\nu$, and its modular weights $2k_l$ and $2k_\nu$ are arranged at zero. This scalar is not supposed to generate special Yukawa textures for leptons. Instead, it is used for the connection between two S_4's and its VEV is the key to break two S_4's to a single S_4. This idea and relevant technique for how to obtain the required the VEV was introduced and developed in [59]. We will not repeat them in this article. Without loss of generality, we can fix the VEV of Φ at $\langle\Phi\rangle_{\alpha i} = v_\Phi (P_{23})_{\alpha i}$ with

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (29)$$

Here, $\alpha = 1, 2, 3$ corresponds the entries of the triplet of S_4^l, while $i = 1, 2, 3$ corresponds to those of S_4^ν. With this VEV, we can realize the breaking $S_4^l \times S_4^\nu \to S_4$.

As mentioned, $S_4^l \times S_4^\nu$ is broken after Φ gains the above VEV. The scalar Φ connects S_4^l with S_4^ν via the effective dimension-five operator $\frac{y_\nu}{\Lambda}L\Phi\nu^c H_u$, responsible for Dirac neutrino Yukawa couplings. This operator is explicitly expanded as

$$\frac{y_\nu}{\Lambda}(L_1, L_2, L_3)P_{23}\begin{pmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{pmatrix} P_{23}\begin{pmatrix} \nu_1^c \\ \nu_2^c \\ \nu_3^c \end{pmatrix} H_u. \quad (30)$$

Given the VEV $\langle\Phi\rangle = P_{23}v_\Phi$, this term is not invariant under transformations γ_l and γ_ν of S_4^l and S_4^ν and thus the modular symmetry $S_4^l \times S_4^\nu$ is broken. However, given any γ_l of S_4^l, we can perform the same transformation $\gamma_\nu = \gamma_l$ of S_4^ν, such that the VEV of Φ keeps invariant, namely,

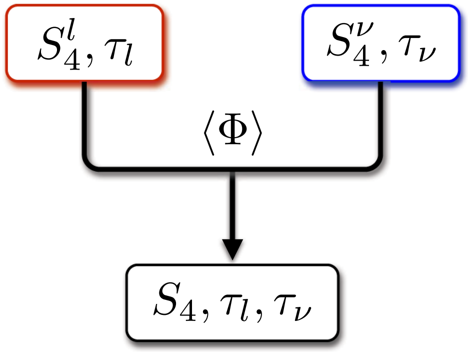

FIG. 1. Diagram of the breaking of $S_4^l \times S_4^\nu \to S_4$, their diagonal subgroup, through the VEV of Φ.

$$\langle\Phi\rangle \to \rho_{\mathbf{3}}(\gamma_l)\langle\Phi\rangle\rho_{\mathbf{3}}^T(\gamma_\nu) = \langle\Phi\rangle \tag{31}$$

for $\gamma_l = \gamma_\nu$. This equation is simply proven after we write it in the following matrix form:

$$\begin{aligned}\rho_{\mathbf{3}}(\gamma_l)\langle\Phi\rangle\rho_{\mathbf{3}}^T(\gamma_\nu) &= \rho_{\mathbf{3}}(\gamma_l)P_{23}\rho_{\mathbf{3}}^T(\gamma_\nu)v_\Phi = \rho_{\mathbf{3}}(\gamma_l\gamma_\nu^{-1})P_{23}v_\Phi \\ &= \rho_{\mathbf{3}}(\gamma_l\gamma_\nu^{-1})\langle\Phi\rangle,\end{aligned} \tag{32}$$

where $P_{23}\rho_{\mathbf{3}}^T(\gamma) = \rho_{\mathbf{3}}(\gamma^{-1})P_{23}$ has been used. It is obvious that $\langle\Phi\rangle$ is invariant if $\gamma_l = \gamma_\nu$. Therefore, the diagonal part of $S_4^l \times S_4^\nu$ is preserved in the vacuum. $\frac{y_\nu}{\Lambda}L\Phi\nu^c H_u$ is the only term which breaks $S_4^l \times S_4^\nu$ to a single S_4. Fix Φ at its VEV, this term is left with $y_D(L_1\nu_1^c + L_2\nu_3^c + L_3\nu_2^c)H_u$, where we have denoted $y_D = y_\nu v_\Phi/\Lambda$. It appears as a renormalizable Dirac neutrino Yukawa interaction at low energy, which is proportional to P_{23}. Therefore, all neutrino mixing arises from the heavy Majorana neutrino mass matrix.

To summarize, after Φ gains the VEV, superpotential w is effectively given by

$$\begin{aligned}w_{\text{eff}} &= [LY_e(\tau_l)e^c + LY_\mu(\tau_l)\mu^c + LY_\tau(\tau_l)\tau^c]H_d \\ &\quad + y_D L\nu^c H_u + \frac{1}{2}M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} + \frac{1}{2}M_{\mathbf{2}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{2}} \\ &\quad + \frac{1}{2}M_{\mathbf{3}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{3}}.\end{aligned} \tag{33}$$

The full effective superpotential involves two moduli fields. It is not invariant in $S_4^l \times S_4^\nu$ but their diagonal subgroup S_4.

Under this symmetry, a modular transformation appears to be

$$\gamma\colon\ (\tau_l, \tau_\nu) \to (\gamma\tau_l, \gamma\tau_\nu) = \left(\frac{a\tau_l + b}{c\tau_l + d}, \frac{a\tau_\nu + b}{c\tau_\nu + d}\right) \tag{34}$$

for any $\gamma \in S_4$. We also write out transformation properties of leptons

$$\begin{aligned}L(\tau_\nu) &\to L(\gamma\tau_\nu) = (c\tau_\nu + d)^2\rho_{\mathbf{3}}(\gamma)L(\tau_\nu), \\ \alpha^c(\tau_l, \tau_\nu) &\to \alpha^c(\gamma\tau_l, \gamma\tau_\nu) \\ &\quad = (c\tau_l + d)^{-2k_\alpha}(c\tau_\nu + d)^{-2}\alpha^c(\tau_l, \tau_\nu), \\ \nu^c(\tau_\nu) &\to \nu^c(\gamma\tau_\nu) = (c\tau_\nu + d)^{-2}\rho_{\mathbf{3}}(\gamma)\nu^c(\tau_\nu)\end{aligned} \tag{35}$$

and those for modular forms

$$\begin{aligned}Y_\alpha(\tau_l) &\to Y_\alpha(\gamma\tau_l) = (c\tau_l + d)^{2k_\alpha}\rho_{\mathbf{3}}(\gamma)Y_\alpha(\tau_l), \\ M_{\mathbf{r}}(\tau_\nu) &\to M_{\mathbf{r}}(\gamma\tau_\nu) = (c\tau_\nu + d)^4\rho_{\mathbf{r}}(\gamma)M_{\mathbf{r}}(\tau_\nu),\end{aligned} \tag{36}$$

where $\alpha = e, \mu, \tau$, $k_{e,\mu,\tau} = 3, 2, 1$ and $\mathbf{r} = \mathbf{1}, \mathbf{2}, \mathbf{3}$. Note that in the residual S_4 symmetry, we have not induced any correlation between the moduli fields τ_l and τ_ν. Namely, τ_l and τ_ν can gain independent VEVs. Furthermore, there is no flavon fields involved in the effective superpotential.

Geometrically, we represent the idea of $S_4^l \times S_4^\nu \to S_4$ in the sketch shown in Fig. 1.

B. Flavor structure after S_4 breaking

In the charged lepton sector, we assume the VEV of τ_l fixed at $\langle\tau_l\rangle = \tau_T = \omega$, which is a stabilizer of T. At this stabilizer, a residual modular Z_3^T symmetry is preserved in the charged lepton sector. It has been proven in [59] that $Y_{e,\mu,\tau}(\tau_T)$ are eigenvectors of the 3×3 representation matrix of T for eigenvalues 1, ω, and ω^2, respectively. Namely, the Yukawa coupling vectors are

$$Y_e(\tau_T) \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y_\mu(\tau_T) \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad Y_\tau(\tau_T) \propto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{37}$$

for weights $2k_l = 6, 4, 2$, respectively. These modular forms will lead to diagonal Yukawa couplings for the charged leptons. We have also seen that the Dirac neutrino Yukawa matrix is proportional to P_{23}. Therefore, all lepton mixing arises from the heavy Majorana neutrino mass matrix, to which we now turn.

In the neutrino sector, the right-handed neutrino mass matrix is explicitly written to be

$$\begin{aligned}M_R &= \begin{pmatrix} M_{\mathbf{1}} & 0 & 0 \\ 0 & 0 & M_{\mathbf{1}} \\ 0 & M_{\mathbf{1}} & 0 \end{pmatrix} + \begin{pmatrix} 0 & M_{\mathbf{2},1} & M_{\mathbf{2},2} \\ M_{\mathbf{2},1} & M_{\mathbf{2},2} & 0 \\ M_{\mathbf{2},2} & 0 & M_{\mathbf{2},1} \end{pmatrix} \\ &\quad + \begin{pmatrix} 2M_{\mathbf{3},1} & -M_{\mathbf{3},3} & -M_{\mathbf{3},2} \\ -M_{\mathbf{3},3} & 2M_{\mathbf{3},2} & -M_{\mathbf{3},1} \\ -M_{\mathbf{3},2} & -M_{\mathbf{3},1} & 2M_{\mathbf{3},3} \end{pmatrix},\end{aligned} \tag{38}$$

where $M_{\mathbf{r},i}$ is the ith component of $M_{\mathbf{r}}(\tau)$, $i = 1, 2$ for $\mathbf{r} = \mathbf{2}$ and $i = 1, 2, 3$ for $\mathbf{r} = \mathbf{3}$. The Dirac mass matrix is trivially given by

$$M_D = y_D P_{23} v_u. \tag{39}$$

The active neutrino mass matrix is obtained by applying the seesaw formula

$$M_\nu = -M_D M_R^{-1} M_D^T = -y_D^2 v_u^2 P_{23} M_R^{-1} P_{23}. \tag{40}$$

Specifically, the mass eigenvalues of M_ν, m_i for $i = 1, 2, 3$ are given by $m_i = y_D^2 v_u^2 / M_i$. The (1,1) entry of M_ν gives rise to the effective mass parameter in neutrinoless double beta decay $m_{ee} \equiv |(M_\nu)_{(1,1)}| = y_D^2 v_u^2 |(M_R^{-1})_{(1,1)}|$.

Since the charged lepton mass matrix is diagonal, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix is determined by the structure of neutrino mass matrix which is governed by the VEV of τ_ν. We assume the stabilizer in the neutrino sector,[11] $\langle \tau_\nu \rangle = \tau_{SU} = -\frac{1}{2} + \frac{i}{2}$. At this stabilizer, we are left with a residual Z_2^{SU} symmetry. In former discussion in the framework of flavor symmetry, the Z_2^{SU} residual symmetry is crucial to realize the TM_1 mixing [59]. $M_\mathbf{2}$ and $M_\mathbf{3}$ take directions $M_\mathbf{2} \propto (1,1)^T$ and $M_\mathbf{3} \propto (\sqrt{2}, \sqrt{2}-\sqrt{3}, \sqrt{2}+\sqrt{3})^T$, respectively. Together with $M_\mathbf{1}$, we write them in the following way:

$$M_\mathbf{1}(\tau_{SU}) = a, \qquad M_\mathbf{2}(\tau_{SU}) = b\begin{pmatrix}1\\1\end{pmatrix},$$
$$M_\mathbf{3}(\tau_{SU}) = c\begin{pmatrix}\sqrt{2}\\ \sqrt{2}-\sqrt{3}\\ \sqrt{2}+\sqrt{3}\end{pmatrix}. \tag{41}$$

Thus, the Majorana mass matrix for right-handed neutrinos is written in the form

$$M_R = a\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} + b\begin{pmatrix}0&1&1\\1&1&0\\1&0&1\end{pmatrix} + c\sqrt{2}\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix} - c\sqrt{3}\begin{pmatrix}0&1&-1\\1&2&0\\-1&0&-2\end{pmatrix}, \tag{42}$$

where a, b, and c are complex parameters. As discussed in the next subsection, the above heavy Majorana neutrino mass matrix, together with a Dirac neutrino Yukawa matrix proportional to P_{23}, and a diagonal charged lepton mass matrix, will lead to trimaximal TM_1 lepton mixing which preserves the first column on the tri-bimaximal mixing matrix,

$$U_{\mathrm{TM}_1} = \begin{pmatrix}\frac{2}{\sqrt{6}} & - & -\\ -\frac{1}{\sqrt{6}} & - & -\\ -\frac{1}{\sqrt{6}} & - & -\end{pmatrix}. \tag{43}$$

It is worth mentioning that in classical flavor models without modular symmetry, such as [12], coefficients for the third and fourth terms on the right-hand side of Eq. (42) are fully arbitrary, but here they are constrained by a fixed ratio $-\sqrt{2/3}$. Thus, in the modular symmetry model here, M_R depends on three complex parameters, while in the classical (nonmodular symmetry) model in [12] M_R depends on four complex parameters. We will show that having fewer parameters leads to a new neutrino mass sum rule, not present in the previous flavon models of TM_1 mixing which do not rely on modular symmetry.

C. Results for neutrino mass and mixing

The heavy Majorana mass matrix M_R in Eq. (42) can be put into block diagonal form by applying the TBM mixing matrix,

$$U_{\mathrm{TBM}}^T M_R U_{\mathrm{TBM}} = \begin{pmatrix}-\beta-2\gamma & 0 & 0\\ 0 & \alpha & \gamma\\ 0 & \gamma & \beta\end{pmatrix}, \tag{44}$$

where $\alpha = a + 2b$, $\beta = b - a + 3\sqrt{2}c$, and $\gamma = -3\sqrt{2}c$. Since the remaining (2,3) rotations required to diagonalize M_R leave the first column of the TBM matrix unchanged, this implies that M_R is diagonalized by the TM_1 matrix in Eq. (43). Then, since the Dirac neutrino Yukawa matrix proportional to P_{23}, the seesaw mass matrix M_ν in Eq. (40) will also be diagonalized by U_{TM_1}. Hence, as claimed, we have trimaximal TM_1 lepton mixing, given that the charged lepton mass matrix is diagonal.

Returning to Eq. (44), the reparametrized mass parameters α, β, and γ are independent complex parameters. Namely, the bottom right 2×2 submatrix in Eq. (44) is an arbitrary complex symmetric matrix. Thus, it can be diagonalized by a 2×2 unitary matrix

$$V = e^{i\alpha_3}\begin{pmatrix}\cos\theta_R e^{-i\alpha_1} & \sin\theta_R e^{i\alpha_2}\\ \sin\theta_R e^{-i\alpha_2} & -\cos\theta_R e^{i\alpha_1}\end{pmatrix} \tag{45}$$

with two real eigenvalues M_2 and M_3. Here, M_2, M_3, and V are arbitrary. However, the first eigenvalue of M_R, i.e., M_1, is not arbitrary, but determined by M_2, M_3, and the mixing parameters in V via

[11] Note that if we had selected $\langle \tau_\nu \rangle = \tau_S = i\infty$, we would have obtained $M_\mathbf{2}(\tau_S) \propto (1,1)^T$ and $M_\mathbf{3}(\tau_S) \propto (1,1,1)^T$, with a residual Z_2^S *flavor symmetry* preserved in the neutrino sector [59], leading to tri-bimaximal mixing. Alternatively, the choice $\langle \tau_\nu \rangle$ by $\tau_U = 1/2 + i/2$ would preserve a residual Z_2^U flavor symmetry corresponding to a mu-tau permutation symmetry in the neutrino sector. Since both patterns are excluded, due to the prediction of vanishing θ_{13}, we will not discuss them any further here.

$$M_1 = |\beta + 2\gamma| = |M_2(\sin^2\theta_R e^{i2\alpha_2} + \sin 2\theta_R e^{i(\alpha_1+\alpha_2)}) + M_3(\cos^2\theta_R e^{-i2\alpha_1} - e^{-i(\alpha_1+\alpha_2)})|. \quad (46)$$

According to the above discussion, the model predicts lepton mixing to be of the TM_1 form, $U_{PMNS} = U_{TM_1}$, with the general form of TM_1 mixing in Eq. (43) parametrized as

$$U_{TM_1} = U_{TBM}\begin{pmatrix} e^{\alpha'_3} & 0 & 0 \\ 0 & \cos\theta_R e^{i\alpha_1} & \sin\theta_R e^{-i\alpha_2} \\ 0 & -\sin\theta_R e^{i\alpha_2} & \cos\theta_R e^{-i\alpha_1}\end{pmatrix}, \quad (47)$$

where $\alpha'_3 = \frac{1}{2}\arg(-\beta - 2\gamma)$. The mixing angles and Dirac-type CP-violating phase are determined to be [12]

$$\sin\theta_{13} = \frac{\sin\theta_R}{\sqrt{3}},$$
$$\tan\theta_{12} = \frac{\cos\theta_R}{\sqrt{2}},$$
$$\tan\theta_{23} = \left|\frac{\cos\theta_R + \sqrt{\frac{2}{3}}e^{i(\alpha_1-\alpha_2)}\sin\theta_R}{\cos\theta_R - \sqrt{\frac{2}{3}}e^{i(\alpha_1-\alpha_2)}\sin\theta_R}\right|,$$
$$\delta = \arg[(5\cos 2\theta_R + 1)\cos(\alpha_1 - \alpha_2) - i(\cos 2\theta_R + 5)\sin(\alpha_1 - \alpha_2)]. \quad (48)$$

The above TM_1 mixing implies three equivalent relations,

$$\tan\theta_{12} = \frac{1}{\sqrt{2}}\sqrt{1 - 3s_{13}^2} \quad \text{or} \quad \sin\theta_{12} = \frac{1}{\sqrt{3}}\frac{\sqrt{1 - 3s_{13}^2}}{c_{13}} \quad \text{or} \quad \cos\theta_{12} = \sqrt{\frac{2}{3}}\frac{1}{c_{13}}, \quad (49)$$

leading to a prediction $\theta_{12} \approx 34°$, in excellent agreement with current global fits, assuming $\theta_{13} \approx 8.5°$. By contrast, the corresponding TM_2 relations imply $\theta_{12} \approx 36°$ [60], which is on the edge of the three sigma region, and hence disfavored by current data. TM_1 mixing also leads to an exact sum rule relation for $\cos\delta$ in terms of the other lepton mixing angles [60],

$$\cos\delta = -\frac{\cot 2\theta_{23}(1 - 5s_{13}^2)}{2\sqrt{2}s_{13}\sqrt{1 - 3s_{13}^2}}, \quad (50)$$

which, for approximately maximal atmospheric mixing, predicts $\cos\delta \approx 0$, $\delta \approx \pm 90°$. Such atmospheric mixing sum rules may be tested in future experiments [61].

Apart from predicting TM_1 lepton mixing, the model also predicts a neutrino mass sum rule [62] between the light physical effective Majorana neutrino mass eigenvalues m_i (i.e., the active neutrino masses relevant for low energy experiments). Using the correlation of M_1 and $M_{2,3}$ in Eq. (46) and $M_i = -y_D^2 v_u^2/m_i$ for $i = 1, 2, 3$, we obtain a new neutrino mass sum rule for the active neutrino masses (beyond those reported in [62]),

$$\frac{1}{m_1} = \left|\frac{1}{m_2}(\sin^2\theta_R e^{-i2\alpha_2} + \sin 2\theta_R e^{-i(\alpha_1+\alpha_2)}) + \frac{1}{m_3}(\cos^2\theta_R e^{i2\alpha_1} - \sin 2\theta_R e^{i(\alpha_1+\alpha_2)})\right|. \quad (51)$$

Furthermore, we can predict the effective neutrino mass parameter m_{ee} in neutrinoless double beta decay experiments. It is effectively represented as

$$m_{ee} = y_D^2 v_u^2 |(M_R^{-1})_{(1,1)}| = y_D^2 v_u^2 \left|\frac{2}{3(\beta + 2\gamma)} - \frac{\beta}{3(\alpha\beta - \gamma^2)}\right|$$
$$= \left|\frac{2m_2 m_3}{3(m_2(\cos^2\theta_R e^{i2\alpha_1} - \sin 2\theta_R e^{i(\alpha_1+\alpha_2)}) + m_3(\sin^2\theta_R e^{-i2\alpha_2} + \sin 2\theta_R e^{-i(\alpha_1+\alpha_2)}))} + \frac{1}{3}(m_2\cos^2\theta_R e^{2i\alpha_1} + m_3\sin^2\theta_R e^{-2i\alpha_2})\right|. \quad (52)$$

In Fig. 2, we display the prediction of m_{lightest} vs m_{ee}, where $m_{\text{lightest}} = m_1$ for neutrino masses with normal ordering (NO) and $m_{\text{lightest}} = m_3$ for inverted ordering (IO). 1σ and 3σ ranges of oscillation parameters from [63,64] have been taken as inputs in the left and right panels, respectively. In this plot, we also show the upper limit from KamLAND-Zen experiment, $(m_{ee})_{\text{upper limit}} = 0.061$–$0.165$ eV, which is the current best experimental constraint for m_{ee}, and cosmological constraints from Planck 2018 [65], for comparison. The latter set limits on $\sum_i m_i$. Depending on data inputs, different limits are obtained. In the figure, we consider two limits $\sum_i m_i < 0.12$ eV (95%, Planck TT, TE, EE + lowE + lensing + BAO + θ_{MC}) and $\sum_i m_i < 0.60$ eV (95%, Planck lensing + BAO + θ_{MC}), which we refer to "disfavored" and "very disfavored" regimes, respectively. The first limit was obtained earlier in [66]. In the 1σ range, the model has no points compatible with data in the NO case.

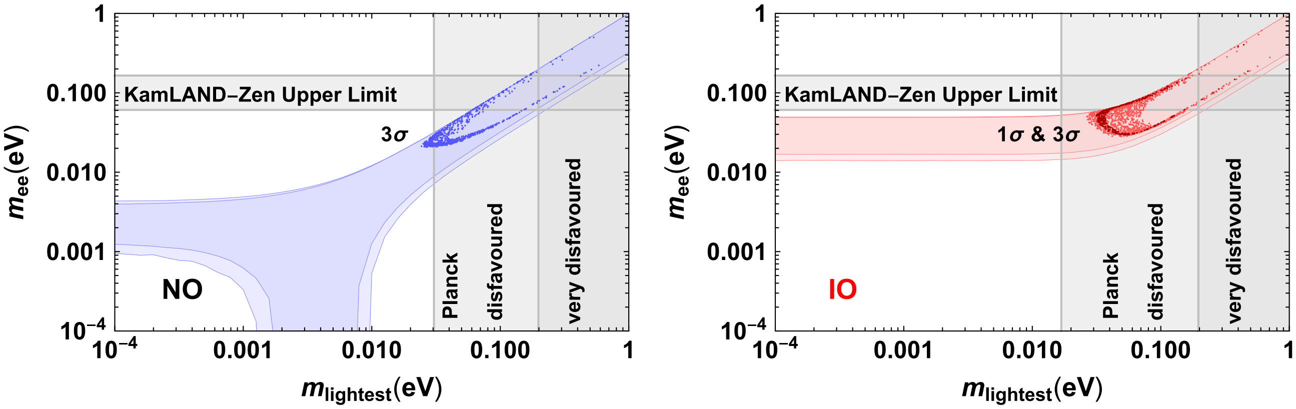

FIG. 2. Predictions of $m_{\rm lightest}$ vs m_{ee} for both normal ordering (NO, left panel) and inverted ordering (IO, right panel) of neutrino masses, allowed by the model, where $m_{\rm lightest} = m_1$ for NO and $m_{\rm lightest} = m_3$ for IO. 1σ and 3σ range data of oscillation parameters from [63,64] are taken as inputs. The general parameter space of m_{ee} allowed by oscillation data and current upper limit from KamLAND-Zen and cosmological constraints from PLANCK 2018 [65] (disfavored region 0.12 eV $< \sum m_i <$ 0.60 eV and very disfavored region $\sum m_i >$ 0.60 eV) are shown for comparison.

In the IO case, the minimum values of both $m_{\rm lightest}$ and m_{ee} are around 0.03 eV. Given the 3σ ranges, both mass orderings are compatible with data. The minimum values of $m_{\rm lightest}$ and m_{ee} compatible with data are given by

$$(m_{\rm lightest})^{\rm NO}_{\rm min} \approx 0.025~{\rm eV}, \qquad (m_{ee})^{\rm NO}_{\rm min} \approx 0.021~{\rm eV},$$
$$(m_{\rm lightest})^{\rm IO}_{\rm min} \approx 0.026~{\rm eV}, \qquad (m_{ee})^{\rm IO}_{\rm min} \approx 0.029~{\rm eV}, \quad (53)$$

respectively. Making use of the best cosmological constraint, $\sum m_i < 0.12$ eV, we arrive at $m_{\rm lightest} < 0.31$ eV for NO and < 0.17 eV for IO. Most points in NO and all points in IO lie in this “disfavored” region. On the other hand, few points lie in the “very disfavored” region.

IV. CONCLUSION

In this paper, we have discussed a minimal model of trimaximal mixing in which the first column of the tri-bimaximal lepton mixing matrix is achieved via two modular S_4 groups, namely $S_4^l \times S_4^\nu$. The associated moduli fields are assumed to be “stabilized” at these two different symmetric points, where the misalignment leads to the lepton mixing. To be precise, one of these factors, S_4^ν, acts in the heavy Majorana neutrino sector, under which the right-handed neutrinos transform as triplets, and is associated with a modulus field value τ_{SU} with residual Z_2^{SU} symmetry. The other factor S_4^l acts in the Dirac charged lepton sector and is associated with a modulus field value τ_T with residual Z_3^T symmetry.

In addition, there is a Higgs scalar Φ introduced to break the $S_4^l \times S_4^\nu$ down to a diagonal S_4 subgroup, yielding a Dirac neutrino Yukawa matrix proportional to P_{23} at low energy (but above the seesaw scale). The model here represents a simpler example of multiple modular symmetries than a previous model based on three modular symmetries, in which two Higgs scalars were required to break the three modular symmetries down to their diagonal subgroup.

In our chosen basis, the model leads to a diagonal charged lepton mass matrix, together with a heavy Majorana neutrino mass matrix which depends on three complex parameters, one fewer than previous flavon models of TM_1 mixing which do not use modular symmetry at all. Together with the Dirac neutrino Yukawa matrix proportional to P_{23}, this implies that the light effective left-handed neutrino Majorana mass matrix and the heavy Majorana mass matrix are diagonalized by the same unitary matrix, namely the TM_1 lepton mixing matrix. The model therefore is subject to the usual TM_1 lepton mixing sum rules.

Apart from the usual predictions of TM_1 lepton mixing, the model also leads to a new neutrino mass sum rule, which implies sizeable, quite degenerate, neutrino masses, with a marked preference for IO over NO. Much of the parameter space for the IO region falls well inside the cosmologically disfavored region. By contrast, some of the parameter space for the NO case falls outside the cosmologically disfavored region, with most points being not very disfavored at the moment, although this conclusion could change with modest improvements in the cosmological limits. In both IO and NO cases, the entire parameter space of the model can be probed by the planned neutrinoless double beta decay experiments.

In conclusion, we have proposed a minimal model of TM_1 lepton mixing based on having an independent modular S_4 symmetry acting in each of the charged lepton and neutrino sectors, respectively, where the two associated moduli respect different residual symmetries. The model, at the intermediate scale where only a single S_4 symmetry is conserved, does not involve any flavons but it does reply on a Higgs field breaking the two S_4 symmetries down to their diagonal subgroup. The combination of the TM_1 lepton mixing sum rules and the new neutrino mass sum rule makes the proposed model highly testable in the near future.

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APPENDIX: GROUP THEORY OF S_4

S_4 is the permutation group of four objects; see, e.g., [67,68]. The Kronecker products between different irreducible representations can be easily obtained,

TABLE II. The representation matrices for the S_4 generators T, S, and U used in the main text, where ω is the cube root of unit $\omega = e^{2\pi i/3}$.

	$\rho(T)$	$\rho(S)$	$\rho(U)$
1	1	1	1
1′	1	1	−1
2	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
3′	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\mathbf{1'} \otimes \mathbf{1'} = \mathbf{1}, \quad \mathbf{1'} \otimes \mathbf{2} = \mathbf{2}, \quad \mathbf{1'} \otimes \mathbf{3} = \mathbf{3'}, \quad \mathbf{1'} \otimes \mathbf{3'} = \mathbf{3}, \quad \mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{2},$$
$$\mathbf{2} \otimes \mathbf{3} = \mathbf{2} \otimes \mathbf{3'} = \mathbf{3} \oplus \mathbf{3'}, \quad \mathbf{3} \otimes \mathbf{3} = \mathbf{3'} \otimes \mathbf{3'} = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3'}, \quad \mathbf{3} \otimes \mathbf{3'} = \mathbf{1'} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3'}. \tag{A1}$$

The generators of S_4 on the basis we used in the main text in different irreducible representations are listed in Table II. This basis is widely used in the literature since the charged lepton mass matrix invariant under T is diagonal on this basis. The products of $a \sim b \sim \mathbf{3}$ (or $a \sim b \sim \mathbf{3'}$) are expressed as

$$\begin{aligned}
(ab)_{\mathbf{1}} &= a_1 b_1 + a_2 b_3 + a_3 b_2, \\
(ab)_{\mathbf{2}} &= (a_2 b_2 + a_1 b_3 + a_3 b_1, a_3 b_3 + a_1 b_2 + a_2 b_1)^T, \\
(ab)_{\mathbf{3}} &= (2a_1 b_1 - a_2 b_3 - a_3 b_2, 2a_3 b_3 - a_1 b_2 - a_2 b_1, 2a_2 b_2 - a_3 b_1 - a_1 b_3)^T, \\
(ab)_{\mathbf{3'}} &= (a_2 b_3 - a_3 b_2, a_1 b_2 - a_2 b_1, a_3 b_1 - a_1 b_3)^T.
\end{aligned} \tag{A2}$$

Here, **3** and **3′** represent the symmetric and antisymmetric triplet contractions, respectively.[12] For $a \sim \mathbf{3}$ and $b \sim \mathbf{3'}$, the contractions are given by

$$\begin{aligned}
(ab)_{\mathbf{1'}} &= a_1 b_1 + a_2 b_3 + a_3 b_2, \\
(ab)_{\mathbf{2}} &= (a_2 b_2 + a_1 b_3 + a_3 b_1, -(a_3 b_3 + a_1 b_2 + a_2 b_1))^T, \\
(ab)_{\mathbf{3'}} &= (2a_1 b_1 - a_2 b_3 - a_3 b_2, 2a_3 b_3 - a_1 b_2 - a_2 b_1, 2a_2 b_2 - a_3 b_1 - a_1 b_3)^T, \\
(ab)_{\mathbf{3}} &= (a_2 b_3 - a_3 b_2, a_1 b_2 - a_2 b_1, a_3 b_1 - a_1 b_3)^T.
\end{aligned} \tag{A3}$$

The products of two doublets $a = (a_1, a_2)^T$ and $b = (b_1, b_2)^T$ are divided into

$$(ab)_{\mathbf{1}} = a_1 b_2 + a_2 b_1, \qquad (ab)_{\mathbf{1'}} = a_1 b_2 - a_2 b_1, \qquad (ab)_{\mathbf{2}} = (a_2 b_2, a_1 b_1)^T. \tag{A4}$$

[12] Note that the difference of conventions for **3** and **3′** in this paper from those in e.g., [69,12], where **3** represents the antisymmetric triplet contraction for two **3** (or two **3′**) and **3′** represents the symmetric triplet contraction.

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