BTZ black hole with Korteweg–de Vries-type boundary conditions: Thermodynamics revisited

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The thermodynamic properties of the Bañados-Teitelboim-Zanelli black hole endowed with Korteweg–de Vries (KdV)-type boundary conditions are reconsidered. This family of boundary conditions is labeled by a non-negative integer n, and gives rise to a dual theory which possesses anisotropic Lifshitz scaling invariance with dynamical exponent $z = 2n + 1$. We show that from the scale invariance of the action for stationary and circularly symmetric spacetimes, an anisotropic version of the Smarr relation arises. Finally, we analyze the global and local thermal stability of the system and we found that the specific heat of the black hole is positive for all possible values of z, and at the self-dual temperature $T_s = \frac{1}{2\pi}\left(\frac{1}{z}\right)^{\frac{z}{z+1}}$, there is a Hawking-Page phase transition between the BTZ black hole and thermal AdS_3 spacetime.

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I. INTRODUCTION

In the pursuit of a better understanding of quantum gravity, in the past two decades, a lot of interest has been put into the so called gauge/gravity correspondence, whose most celebrated example is the AdS/CFT duality [1,2]. In this context, AdS_3/CFT_2 correspondence has played an important role. One of the first main results was the renowned article from Brown and Henneaux [3], where they showed that the asymptotic symmetries of General Relativity in three dimensions with negative cosmological constant correspond to the conformal algebra in two dimensions with a classical central charge given by $c = 3\ell/2G$, where ℓ is the AdS radius and G the Newton constant. This result naturally suggest that a quantum theory of gravity in three dimensions could be described by a CFT at the boundary. Based on this result, Strominger [4] proved that the entropy of the Bañados-Teitelboim-Zanelli (BTZ) black hole [5,6] can be recovered by a microscopic counting of states by means of the Cardy formula [7]. This simple example gave rise to an active field of research regarding the thermodynamic properties of lower dimensional black holes and how they could be holographically related to a dual field theory that describes the much sought after quantum theory of gravity.

Several efforts have been made to generalize the gauge/gravity proposal for non-AdS asymptotics (see, e.g., [8–11]). In this scenario, a lot of attention has been placed on 2D gravity dual theories with anisotropic scaling properties [12–15], which are found in the context of nonrelativistic condense matter physics [16]. The main work on this subject has been done along the lines of nonrelativistic holography [17–19] and more specifically, Lifshitz holography, where the gravity counterparts are given by asymptotically Lifshitz geometries (see, e.g., [20] and references therein). However, this class of spacetimes are not free of controversies. In particular, the Lifshitz spacetime (which would play the role of ground state in the thermodynamic description) suffer from divergent tidal forces. Additionally, asymptotically Lifshitz black holes are not vacuum solutions to general relativity, and it is mandatory to include extra matter fields as in the case of

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Proca fields [20], p-form gauge fields [16,21], to name a couple of examples.

Following [22], we will adopt an unconventional approach to this holographic realization for field theories possessing anisotropic scaling properties, where the spacetime anisotropy at the boundary emerge from a very special choice of boundary conditions for general relativity on AdS_3, instead of Lifshitz asymptotics. This new set of boundary conditions is labeled by a nonnegative integer n, and is related with the Korteweg–de Vries (KdV) hierarchy of integrable systems.[1]

The paper is organized as follows. In Sec. II we provide a brief review of the main results found in [22]. In Sec. III it is shown that an anisotropic Smarr formula emerges from the radially conserved charge associated with the scale invariance of the reduced Einstein-Hilbert action endowed with KdV-type boundary conditions. The local and global thermal stability of BTZ black hole with KdV-type boundary conditions is deeply analyzed in Sec. IV. Finally, we conclude with some comments in Sec. V.

II. REVIEW OF KDV-TYPE BOUNDARY CONDITIONS AND THE ANISOTROPIC CARDY FORMULA

By allowing the Lagrange multipliers depend on the global charges in the asymptotic region, it has been shown [22] that the Einstein field equations reduce to

$$\partial_t \mathcal{L}_\pm = \pm \mathcal{D}^\pm \mu_\pm \tag{2.1}$$

where

$$\mathcal{D}^\pm := (\partial_\phi \mathcal{L}_\pm) + 2\mathcal{L}_\pm \partial_\phi - 2\partial_\phi^3. \tag{2.2}$$

Here $\mathcal{L}_\pm(t,\phi)$ stand for the state dependent dynamical fields and $\mu_\pm(t,\phi)$ correspond to the values of the Lagrange multipliers at infinity. In this context, the Lagrange multipliers are chosen to be given by the nth Gelfand-Dikii polynomial evaluated on $\mathcal{L}_\pm$ and can be obtained by the functional derivative with respect to $\mathcal{L}_\pm$ of the nth Hamiltonian of the KdV hierarchy, i.e.,

$$\mu_\pm^{(n)}[\mathcal{L}_\pm] = \frac{\delta H_\pm^{(n)}}{\delta \mathcal{L}_\pm}, \tag{2.3}$$

where the following recursion relation is satisfied

$$\partial_\phi \mu_\pm^{(n+1)} = \frac{n+1}{2n+1} \mathcal{D}^\pm \mu_\pm^{(n)}. \tag{2.4}$$

This unconventional choice of the Lagrange multipliers at infinity does not spoil a well-defined action principle, since by allowing them to be functionals of the $\mathcal{L}_\pm$, the integrability of the boundary term is guaranteed. For the case $n=0$, one recovers the Brown-Henneaux boundary conditions ($\mu_\pm^{(0)} = 1$), and in consequence, according to (2.1), the dynamical fields are chiral. In the case $n=1$, the Lagrange multipliers are given by $\mu_\pm^{(1)} = \mathcal{L}_\pm$, and then the field equations reduce to two copies of the KdV equation, while for the remaining cases ($n>1$) the field equations are given by the corresponding nth member of the KdV hierarchy.

It is worth to emphasize that, although these boundary conditions describe asymptotically locally AdS_3 spacetimes, the associated dual field theory at the boundary[2] possesses an anisotropic scaling of Lifshitz type,

$$t \to \lambda^z t, \qquad \phi \to \lambda \phi, \tag{2.5}$$

provided that $\mathcal{L}_\pm \to \lambda^{-2}\mathcal{L}_\pm$, where the dynamical exponent z is related to the KdV label n by $z = 2n+1$ (for further details see [22]). This feature, give rise to a different alternative for the study of holography of nonrelativistic field theories possessing anisotropic scaling invariance, where the gravitational counterpart now take place in general relativity on AdS_3 and its associated infinite-dimensional conformal symmetry, being preserved by KdV-type boundary conditions for all n.

A. The BTZ black hole with KdV-type boundary conditions

For each allowed choice of n (or equivalently z), the spectrum of solutions is quite different. Nonetheless, BTZ black hole and AdS_3 spacetime fits within every choice of boundary conditions of the KdV-type. Indeed, this class of configurations are described by constant dynamical fields, $\ell^2 \mathcal{L}_\pm^{\mathrm{BTZ}} = (r_+ \pm r_-)^2$ and $\mathcal{L}_\pm^{\mathrm{AdS}} = -1$, respectively, which both trivially solves (2.1) for all possible values of n. In these cases, according to the normalization choice in (2.4), it is possible to show that the Lagrange multipliers generically acquire a remarkably simple form, $\mu_\pm^{(n)} = \mathcal{L}_\pm^n N_\pm$, where $N_\pm$ is assumed to be fixed without variation at the boundary ($\delta N_\pm = 0$). Note that $\mu_\pm^{(0)} = N_\pm$, so in that special case, the Lagrange multipliers at infinity are held constants but given by arbitrary values, and the standard Brown-Henneaux analysis is recovered by setting $N_\pm = 1$.

[1] Other examples of this relationship between 2D integrable systems and gravity in $2+1$, have been also made for the cases of "flat" and "soft hairy" boundary conditions in [23,24], respectively.

[2] As shown in [25], by performing the Hamiltonian reduction of KdV-type boundary conditions, the equations (2.1) actually corresponds to the conservation law of the energy-momentum tensor of the corresponding theory at the boundary. For the particular case $n=0$, the field equations are equivalent to the aforementioned conservation law. See also [26], for a recent related result, in the case of "near horizon" boundary conditions.

In this scenario, along the lines of [27] (see also [28]), the Lagrange multipliers are allowed to depend on the dynamical fields, which amounts to a different fixing of the "chemical potentials" at the boundary, implying that we are dealing with the same configuration but in a different thermodynamic ensemble. In what follows, we will use the dynamical exponent z, instead of the KdV-label n, so by using $z = 2n + 1$, we can rewrite the KdV-type Lagrange multipliers as,[3]

$$\mu_\pm = \mathcal{L}_\pm^{\frac{z-1}{2}} N_\pm. \tag{2.6}$$

According to the canonical approach [33], the energies of the left and right movers also takes a simple form for a generic choice of n, namely $E_\pm = \frac{\ell}{32\pi G} H_\pm^{(n)} = \frac{\ell}{16G}\frac{1}{n+1}\mathcal{L}_\pm^{n+1}$. Therefore, in terms of the dynamical exponent we can rewrite them as

$$E_\pm = \frac{\ell}{8G}\frac{1}{z+1}\mathcal{L}_\pm^{\frac{z+1}{2}}. \tag{2.7}$$

From the gravitational perspective, the energy of the BTZ black hole and AdS_3 spacetime are determined by $E = E_+ + E_-$.

B. The anisotropic Cardy formula

As was discussed above, it is well known that the simplest version of the Cardy formula, namely for free bosons and fermions, successfully reproduce the entropy of the BTZ black hole. This formula is just the Hardy-Ramanujan count for partitions of an integer [34]. The key in the derivation of the Cardy formula lies in the S-modular invariance of the partition function of the CFT_2, $Z[\beta] = Z[4\pi^2\beta^{-1}]$, where β corresponds to the inverse temperature of the system. As argued in [22,35], using a generalization of the usual S-modular transformation, given by

$$Z[\beta_\pm; z] = Z[(2\pi)^{1+\frac{1}{z}}\beta_\pm^{-\frac{1}{z}}; z^{-1}], \tag{2.8}$$

and assuming a gap in the spectrum of the theory, one can perform the saddle-point approach on the asymptotic growth of the number of states and derive a z-dependent version of the Cardy formula[4] [22],

$$S = 2\pi(z+1)\left[\left(\frac{|E_+^0[z^{-1}]|}{z}\right)^z E_+\right]^{\frac{1}{z+1}} + 2\pi(z+1)\left[\left(\frac{|E_-^0[z^{-1}]|}{z}\right)^z E_-\right]^{\frac{1}{z+1}}. \tag{2.9}$$

For the case $z = 1$, this *anisotropic Cardy formula* reduces to the standard one written in terms of the left and right ground state energies, $E_\pm^0$, rather than the central charges. The above formula exactly agree with the leading term of the entropy of the free boson with Lifshitz scaling [36], or equivalently, two copies of the anisotropic chiral boson [37]. Further, each copy of (2.9) is nothing else than the Hardy-Ramanujan formula for the counting of partitions of an integer into zth powers [38]. Remarkably, plugging into (2.9) the left and right energies of the BTZ black hole and using AdS_3 as the ground state energy, both with KdV-type boundary conditions (2.7), the Bekenstein-Hawking formula for the entropy is exactly recovered,

$$S = \frac{\pi r_+}{2G} = \frac{A}{4G}. \tag{2.10}$$

It is worth highlighting that the S-duality relation (2.8), is proposed as a generic property of the dual field theory possessing anisotropic scaling and defined on a torus, so it should be considered as a property that defines the theories in which we are interested. However, as argued in [38], there is a good geometric reason that supports this relationship. First, the lattice that defines a generic torus is invariant under the swap of thermal and angular cycles, which is expressed through the standard S-duality relation [39]. Secondly, performing a swap between the generators of Euclidean time and space translations, it can be show that 2D Lifshitz algebras with dynamical exponents z and z^{-1} are isomorphic [35]. From the aforementioned, it is clear that the partition function of a theory with Lifshitz scaling and defined on a torus, should be invariant under the anisotropic S-duality relation (2.8).

III. THE ANISOTROPIC SMARR FORMULA

In [40], the authors showed that the reduced Einstein-Hilbert action coupled to a scalar field on AdS_3 is invariant under a set of scale transformations which leads to a radial conservation law by means of the Noether theorem. When this conserved quantity is evaluated in a black hole solution of the theory, one obtains a Smarr relation [41]. This method has been successfully applied to several cases in the literature [42–49] for different theories. By following this procedure, we will show that a new anisotropic version of Smarr formula for the BTZ black hole naturally emerges as a consequence of the scale invariance of the reduced Einstein-Hilbert action, as long as we consider KdV-type boundary conditions.

By considering stationary and circularly symmetric spacetimes described by the following line element

[3]In the context of AdS/CFT holography, the relationship between the chemical potentials and conserved charges is known as "multitrace deformations" of the dual theory [29–32].

[4]As explained in [22], for odd values of $n = (z-1)/2$, Euclidean BTZ with KdV-type boundary conditions is diffeomorphic to thermal AdS_3, but with reversed orientation, and in consequence, there is a opposite sign between Euclidean and Lorentzian energies of the ground state. As it will be shown in Sec. IV, this leads to a local thermodynamic instability of the system for odd values of n. So, it is mandatory to adopt $E_\pm^0[z^{-1}] \to -|E_\pm^0[z^{-1}]|$, in the Lorentzian ground state energies of the anisotropic Cardy formula.

$$ds^2 = -\mathcal{N}(r)^2\mathcal{F}(r)^2dt^2 + \frac{dr^2}{\mathcal{F}(r)^2} + r^2(d\phi + \mathcal{N}^\phi(r)dt)^2, \tag{3.1}$$

the reduced action principle of general relativity in the canonical form, is given by

$$I = -2\pi(t_2 - t_1)\int dr(\mathcal{N}\mathcal{H} + \mathcal{N}^\phi\mathcal{H}_\phi) + B, \tag{3.2}$$

where $\mathcal{N}$, $\mathcal{N}^\phi$ stand for their corresponding Lagrange multipliers associated to the surface deformation generators,

$$\mathcal{H} = -\frac{r}{8\pi G\ell^2} + 32\pi G r(\pi^{r\phi})^2 + \frac{(\mathcal{F}^2)'}{16\pi G}, \tag{3.3}$$

$$\mathcal{H}_\phi = -2(r^2\pi^{r\phi})', \tag{3.4}$$

where prime denotes derivative with respect to r. The only nonvanishing component of the momenta π^{ij} is explicitly given by

$$\pi^{r\phi} = -\frac{(\mathcal{N}^\phi)' r}{32\pi G\mathcal{N}}, \tag{3.5}$$

The boundary term B must be added in order to have a well-defined variational principle. Straightforwardly, we can see that the above action turns out to be invariant under the following set of scale transformations

$$\bar{r} = \xi r, \quad \bar{\mathcal{N}} = \xi^{-2}\mathcal{N}, \quad \bar{\mathcal{N}^\phi} = \xi^{-2}\mathcal{N}^\phi, \quad \bar{\mathcal{F}^2} = \xi^2\mathcal{F}^2, \tag{3.6}$$

where ξ is a positive constant. By applying the Noether theorem, we obtain a radially conserved charge associated with the aforementioned symmetries,

$$C(r) = \frac{1}{4G}\left[-\mathcal{N}\mathcal{F}^2 + \frac{r\mathcal{N}(\mathcal{F}^2)'}{2} - \frac{r^3(\mathcal{N}^\phi)'\mathcal{N}^\phi}{\mathcal{N}}\right], \tag{3.7}$$

which means that $C' = 0$ on-shell. We will find a Smarr formula by exploiting the fact that this radial conserved charge must satisfy $C(r_+) = C(\infty)$, where left and right hand sides denotes evaluations on the event horizon of the black hole and the asymptotic region, respectively.

At the horizon, $\mathcal{F}^2(r_+) = 0$, and in order to the Euclidean configuration being smooth around this point, the metric functions must to fulfill the regularity conditions [50],

$$\mathcal{N}(r_+)\mathcal{F}^2(r_+)' = 4\pi, \qquad \mathcal{N}^\phi(r_+) = 0. \tag{3.8}$$

In consequence, the value of the radial charge at the event horizon is

$$C(r_+) = \frac{\pi r_+}{2G} = S, \tag{3.9}$$

which corresponds to the entropy of the BTZ black hole.

In order to obtain the radial charge in the asymptotic region, we use the explicit form of the BTZ metric functions in (3.1), namely

$$\mathcal{N}(r) = \frac{\ell}{2}(\mu_+ + \mu_-),$$
$$\mathcal{N}^\phi(r) = \frac{1}{2}(\mu_+ - \mu_-) + \frac{\ell^2}{8r^2}(\mathcal{L}_+ - \mathcal{L}_-)(\mu_+ + \mu_-),$$
$$\mathcal{F}^2(r) = \frac{r^2}{\ell^2} - \frac{1}{2}(\mathcal{L}_+ + \mathcal{L}_-) + \frac{\ell^2}{16r^2}(\mathcal{L}_+ - \mathcal{L}_-)^2, \tag{3.10}$$

and then we replace the Lagrange multipliers at infinity according to the ones fixed by KdV-type boundary conditions (2.6),

$$C(\infty) = \frac{\ell}{8G}(N_+\mathcal{L}_+^{\frac{z+1}{2}} + N_-\mathcal{L}_-^{\frac{z+1}{2}}), \tag{3.11}$$

which in terms of the left and right energies (2.7), reads

$$C(\infty) = (z+1)N_+E_+ + (z+1)N_-E_-. \tag{3.12}$$

Therefore, equalizing both expressions we finally obtain an anisotropic version of the Smarr formula,[5]

$$S = (z+1)N_+E_+ + (z+1)N_-E_-, \tag{3.13}$$

which corresponds to a generalization of the one previously found in the literature.

Identifying the Lagrange multipliers as the inverse of left and right temperatures $T_\pm = N_\pm^{-1}$, and then turning off the angular momentum, the above expression reduces to[6]

$$E = \frac{1}{(z+1)}TS, \tag{3.14}$$

which fits with the Smarr formula for Lifshitz black holes in 3D (see, e.g., [57,58]).

It is worth to point out that despite of the BTZ black hole corresponds to an asymptotically AdS_3 spacetime, the anisotropic nature of (3.13), also present in asymptotically

[5]Resembling expressions for the entropy as a bilinear combination of the global charges times the chemical potentials have been previously found for three dimensional black holes and cosmological configurations in the context of higher spin gravity [51–53], hypergravity [54,55] and extended supergravity [56]. The factor in front of each term corresponds to the conformal weight (spin) of the corresponding generator.

[6]The left and right temperatures are related with the Hawking temperature through $T = \frac{2T_+T_-}{(T_+ + T_-)}$, so in the absence of rotation $T_+ = T_- = T$.

Lifshitz black holes, in this case is induced by the scaling properties of KdV-type boundary conditions. This common anisotropic feature leads to an interesting consequence. Since (3.13) was derived from a stationary metric (BTZ with $\mathcal{L}_+ \neq \mathcal{L}_-$), the contribution due to the rotation naturally appears, even though the fact that, as far as the knowledge of the present authors, there is no a rotating Lifshitz black hole in three dimensions.

It is also worth mentioning that in the limit $z \to 0$, (3.13) fits with the corresponding Smarr relation of soft hairy horizons in three spacetimes dimension [59], where the infinite-dimensional symmetries at the horizon span soft hair excitations in the sense of Hawking, Perry and Strominger [60–62].

A. Relationship with the anisotropic Cardy formula

As was mentioned in Sec. II B, one can derive the anisotropic Cardy formula (2.9) by making use of the saddle-point approximation on the asymptotic growth of the number of states, provided the anisotropic S-duality relation between high and low temperatures regime of the partition function (2.8). This procedure throws a critical point $\beta^* = \{\beta_+, \beta_-\}$ in terms of $E^0_\pm$ and $E_\pm$. If we solve that relation for the ground state energies,

$$|E^0_\pm[z^{-1}]| = zE_\pm[z]\left(\frac{\beta_\pm}{2\pi}\right)^{1+\frac{1}{z}}, \tag{3.15}$$

and then we replace the above in (2.9), we found that identifying $\beta_\pm \to N_\pm$, the entropy exactly matches with the anisotropic Smarr formula (3.13). Here we show explicitly that the relationship between Cardy and Smarr formulas[7] is given by the critical point (3.15), which remarkably, is nothing else than the anisotropic version of the Stefan-Boltzmann law

$$E_\pm[z] = \frac{1}{z}|E^0_\pm[z^{-1}]|(2\pi)^{1+\frac{1}{z}}T_\pm^{1+\frac{1}{z}}. \tag{3.16}$$

Interestingly enough, from equation (2.9), it is clear that the entropy written as a function of left and right energies scales as $(z+1)^{-1}$, hence it is reassuring to prove that the anisotropic Smarr formula (3.13) can be recovered by simply applying the Euler theorem for homogeneous functions.

The following section is devoted to the local and global thermal stability of the BTZ black hole endowed with KdV-type boundary conditions.

IV. THERMODYNAMNIC STABILITY AND PHASE TRANSITIONS

In concordance with the spirit of the previous sections, we will analyze the thermodynamic stability of the BTZ black hole in the KdV-type ensemble from a holographic perspective. That is, we will first study the behavior of the presumed dual field theory, described by two independent left and right movers, whose energies and entropy are given by the anisotropic Stefan-Boltzmann law and the anisotropic Smarr formula, respectively. Then we will compute the thermodynamic quantities associated with the gravitational system endowed with KdV-type boundary conditions and show that both descriptions match.

We analyze the thermodynamic stability at fixed chemical potentials. Local stability condition can be determined by demanding a negative defined Hessian matrix of the free energy of the system (see, e.g., [64]). Nonetheless, in the current ensemble it can equivalently be performed by the analysis of the left and right specific heats with fixed chemical potential. From the anisotropic Stefan-Boltzmann law (3.16) one finds that left and right specific heats are given by

$$C_\pm[z] = \frac{\partial E_\pm}{\partial T_\pm} = \frac{z+1}{z^2}|E^0_\pm[z^{-1}]|(2\pi)^{1+\frac{1}{z}}T_\pm^{\frac{1}{z}}. \tag{4.1}$$

We see that, for all possible values of z, the specific heats are continuous monotonically increasing functions of $T_\pm$, and always positive,[8] which means that the system is at least locally stable. Identifying the ground state energies with the energy of AdS_3 spacetime, i.e., $E^0_\pm \to \frac{1}{2}E_{\mathrm{AdS}}[z] = -\frac{\ell}{8G}\frac{1}{z+1}$, and the left and right temperatures with the ones associated to the inner and outer horizon of the BTZ metric in the KdV-type ensemble, namely, $r_\pm = \ell(2\pi T_\pm)^{\frac{1}{z}}$, we found that the specific heat of the black hole is proportional to its entropy,

$$C_{\mathrm{BTZ}}[z] = C_+[z] + C_-[z] = \frac{1}{z}\frac{\pi r_+}{2G}, \tag{4.2}$$

and tends to zero for large values of the dynamical exponent z. This means that dual field theories with a high level of anisotropy between their temporal and spatial scales, will be holographically related to a black hole for which a small change in its energy would significantly increase its temperature. Since the specific heats are finite and positive regardless of the value of z, the BTZ black hole with generic KdV-type boundary conditions can always reach local thermal equilibrium with the heat bath at any temperature. It is important to remark that, as mentioned at footnote 4, if we had not warned on the correct sign of the ground state energies for odd n, the sign of the specific heats would have depended on z, and in consequence, for

[7]This link between both expressions has been previously suggested in the literature [63].

[8]Strictly speaking, specific heats $C_\pm$ are always positive provided that $T_\pm > 0$. In terms of the temperature and angular velocity of the black hole, the above is equivalent to the nonextremality condition; $0 < T$, $-1 < \Omega < 1$. Since in the present paper we are not dealing with the extremal case, we will consider that this condition is always fulfilled.

odd values of n the black hole would be thermodynamically unstable.

Once local stability is assured, it makes sense to ask about the global stability of the system. Following the seminal paper of Hawking and Page [65], we use the free energies at fixed values of the chemical potentials in order to realize which of the phases present in the spectrum is thermodynamically preferred.

In the semiclassical approximation, the on-shell Euclidean action is proportional to the free energy of the system. Taking into account the contributions of the left and right movers, the action acquires the following form

$$I = N_+ E_+ + N_- E_- - S. \tag{4.3}$$

Assuming a nondegenerate ground state with zero entropy and whose left and right energies are equal and negative defined, $E_\pm \to -|E^0[z]|$, we see that the value of the action of the ground state is given by

$$I_0 = -|E^0[z]| \left(\frac{1}{T_+} + \frac{1}{T_-} \right). \tag{4.4}$$

On the other hand, considering in (4.3) a system whose entropy is given by the formula (3.13), we obtain that

$$I = -z(N_+ E_+ + N_- E_-), \tag{4.5}$$

hence, using the formulas for the energies in (3.16), the action then reads

$$I = -|E^0[z^{-1}]| (2\pi)^{\frac{z+1}{z}} \left(T_+^{\frac{1}{z}} + T_-^{\frac{1}{z}} \right). \tag{4.6}$$

Therefore, it is straightforward to see that, regardless of the value of z, the partition function $Z = e^{I+I_0}$, will be dominated by (4.4) at low temperatures, and by (4.6) at the high temperatures regime. Consistently, it can also be shown that the same ground state action can be found by making use of the anisotropic S-duality transformation (2.8) on (4.6).

In what follows, we will focus on the simplest case where the whole system is in equilibrium at a fixed temperature $T_\pm = T$. Then, the free energy of the system, $F = TI$, at high and low temperatures will respectively given by

$$F = -2|E^0[z^{-1}]| (2\pi)^{1+\frac{1}{z}} T^{1+\frac{1}{z}}, \quad F_0 = -2|E^0[z]|. \tag{4.7}$$

Finally, comparing them, we can obtain the self-dual temperature, at where both free energies coincide,

$$T_s[z] = \frac{1}{2\pi} \left| \frac{E^0[z]}{E^0[z^{-1}]} \right|^{\frac{z}{z+1}}, \tag{4.8}$$

which manifestly depend on the dynamical exponent. At this point, a highly nontrivial detail is worth to be mentioned. The fact that the self-dual temperature T_s depends on the specific choice of z, is because the S-duality transformation involves an inversion of the dynamical exponent between the high and low temperature regimes, namely, $z \to z^{-1}$. This is a defining property of the partition function of the theories that we are dealing with. If one does not take this detail into account, the self-dual temperature would be the same for all values of z.

From the gravitational hand, according to the formulas written in the Sec. II A and the Smarr relation (3.14), the free energy, $F = E - TS$, of the static BTZ black hole and thermal AdS_3 spacetime with KdV-type boundary conditions are given by

$$F_{\mathrm{BTZ}} = -\frac{\ell}{4G} \frac{z}{z+1} (2\pi T)^{\frac{z+1}{z}}, \qquad F_{\mathrm{AdS}} = -\frac{\ell}{4G} \frac{1}{z+1}, \tag{4.9}$$

and then, the self-dual temperature for which the two phases are equally likely is

$$T_s[z] = \frac{1}{2\pi} \left(\frac{1}{z} \right)^{\frac{z}{z+1}}. \tag{4.10}$$

Therefore, it is reassuring to verify that if one identifies the ground state energy of the field theory with the one of the AdS_3 spacetime with KdV-type boundary conditions in (4.8), it exactly matches with the above gravitational self-dual temperature.

As it is shown in Fig. 1, for an arbitrary temperature below the self-dual temperature ($T < T_s$), the thermal AdS_3 phase has less free energy than the BTZ, and therefore the former one is the most probable configuration, while if $T > T_s$, the black hole phase dominates the partition function and hence is the preferred one. Note that for higher values of z, the self-dual temperature becomes lower. The latter point entails to a remarkable result. In the case of Brown-Henneaux boundary conditions

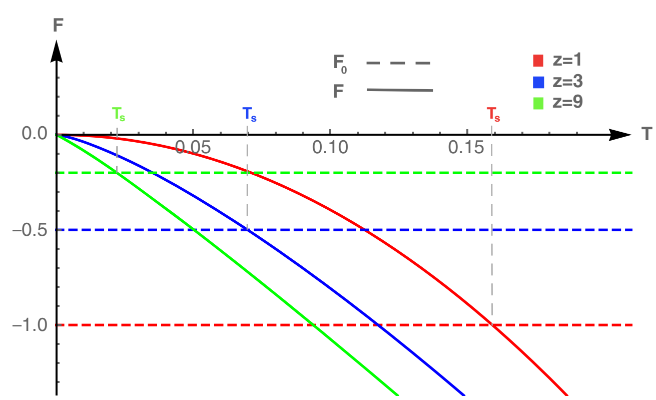

FIG. 1. Free energies of the BTZ black hole and thermal AdS_3 spacetime with KdV-type boundary conditions associated to the dynamical exponents $z = 1; 3; 9$.

($z=1$), one can deduce that in order for the black hole reach the equilibrium with a thermal bath at the self-dual point T_s, the event horizon must be of the size of the AdS_3 radius, i.e., $r_+ = \ell$. Nonetheless, for a generic choice of z, the horizon size has to be

$$r_+^s[z] = \ell\left(\frac{1}{z}\right)^{\frac{1}{z+1}}. \tag{4.11}$$

This means that the size of the black hole at the self-dual temperature decreases for higher values of z. In the same way, at T_s, the energy of the BTZ, $E_{\rm BTZ} = E_+ + E_-$, endowed with generic KdV-type boundary conditions, acquires the following form

$$E_s[z] = \frac{\ell}{4G}\frac{1}{z(z+1)}, \tag{4.12}$$

and when compared to the AdS_3 spacetime energy,

$$\Delta E = E_{\rm BTZ} - E_{\rm AdS} = \frac{\ell}{4G}\frac{1}{z}, \tag{4.13}$$

we can observe that at the self-dual temperature there is an endothermic process, where the system absorbs energy from the surround thermal bath at a lower rate for higher values of z.

From these last points we can conclude that the global stability of the system is certainly sensitive to which KdV-type boundary condition is chosen, since the free energy of the possible phases of the system are explicitly z-dependent. Moreover, the temperature at which both phases have equal free energy, the size of the black hole horizon and the internal energy of the system at that temperature, decrease for higher values of z, giving rise to a qualitatively different behavior of the thermodynamic stability of the system, compared to the standard analysis defined by $z=1$.

V. OUTLOOK AND ENDING REMARKS

The purpose of this work was to analyze in depth the thermodynamic properties of the BTZ black hole in the KdV-type ensemble. In the first place, we have shown that an anisotropic version of the Smarr relation can be obtained by means of the Noether theorem. We obtained a radial conserved quantity, which once evaluated in the BTZ solution naturally leads to an expression for the entropy as a z-dependent bilinear combination of the conserved charges times the chemical potentials at infinity. Then we prove that our generalized Smarr relation exactly matches with the microscopic count of states given by the previously reported anisotropic Cardy formula, ones the chemical potentials are expressed in terms of the ground state energies. Second, we were devoted to the study of the thermodynamical stability of the system. We have shown that, as it is expected for a black hole solution in a Chern-Simons theory, the specific heat of the BTZ black hole is a positive, monotonically increasing function of the temperature [66–71], independently of the choice of KdV-type boundary condition. In contrast, it was shown that the global stability of the system is sensitive to the specific choice of boundary conditions. There is Hawking-Page phase transition at an specific z-dependent self-dual temperature T_s, for which, at temperatures below this point, the preferred phase is the AdS_3 spacetime, and for higher temperatures, the BTZ black hole is the more stable phase. This self-dual temperature decreases for higher values of z, as well as the size of the black hole horizon and the energy that the system absorbs from the environment in order for the transition occurs.

It is worth to remark that the scaling (3.6) is equivalent to the scaling of the Lifshitz type (2.5) introduced in [22]. By redefining the scale factor as $\lambda \to \xi^{-1}$ in (3.6), we obtain

$$t \to \xi^{-z} t, \quad \phi \to \xi^{-1}\phi, \quad r \to \xi r, \quad \mathcal{L}_\pm \to \xi^2 \mathcal{L}_\pm. \tag{5.1}$$

Then, in the case of KdV-type boundary conditions (2.6), we can see that the Lagrange multipliers at infinity scales as $\bar{\mu}_\pm = \xi^{z-1}\mu_\pm$, therefore from (3.10), we can deduce that $\mathcal{N}$ and $\mathcal{N}^\phi$ scales accordingly. Now, since the reduced Hamiltonian action does not depend on t and ϕ, before integrate them, we can absorb its scalings on the Lagrange multipliers, and in consequence, they must scale as $\bar{\mathcal{N}} = \xi^{-2}\mathcal{N}$ and $\bar{\mathcal{N}}^\phi = \xi^{-2}N^\phi$, which is in full agreement with (3.6).

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