Quantum and classical correlations inside the entanglement wedge

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We show that the entanglement wedge cross section (EWCS) can become larger than the quantum entanglement measures such as the entanglement of formation in the AdS/CFT correspondence. We then discuss a series of holographic duals to the optimized correlation measures, finding a novel geometrical measure of correlation, the entanglement wedge mutual information (EWMI), as the dual of the Q-correlation. We prove that the EWMI satisfies the properties of the Q-correlation as well as the strong superadditivity, and that it can become larger than the entanglement measures. These results imply that both of the EWCS and the EWMI capture more than quantum entanglement in the entanglement wedge, which enlightens a potential role of classical correlations in holography.

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I. INTRODUCTION

Quantum entanglement has provided a key tool to study various aspects of modern physics from condensed matter theory to the black hole evaporation. In the AdS/CFT correspondence [1–3], quantum entanglement also plays a central role in the investigation of how the bulk geometrical data are encoded in the boundary field theory [4–8]. The Ryu-Takayanagi (RT) formula [9,10] (or the Hubeny-Rangamani-Takayanagi (HRT) formula [11,12] for covariant cases) tells us that the von Neumann entropy associated with a spacial subregion A in CFTs $S_A \equiv S(\rho_A) = -\mathrm{Tr}\rho_A \log \rho_A$ is equivalent to the area of codimension-2 minimal surface γ_A that is anchored on the entangling surface ∂A and homologous to A,

$$S_A = \min_{\gamma_A} \frac{\mathrm{Area}(\gamma_A)}{4G_N}, \tag{1}$$

at the leading order of the large N limit. The von Neumann entropy S_A is commonly called the entanglement entropy (EE) because this quantifies an amount of quantum entanglement between A and its complement A^c when the total state is pure [13]. For mixed states, however, the von Neumann entropy no longer deserves to be a measure of correlation, and thus we need to find another geometrical way to measure correlations.

A generalization of the Ryu-Takayanagi surface, the entanglement wedge cross section (EWCS), was introduced in [14,15] as the minimal cross section of the entanglement wedge [16–18]. This is a geometrical measure of correlations between the boundary subsystems connected by the entanglement wedge that are usually in mixed states. Thus the EWCS in boundary theories is expected to be dual to some correlation measure that is a generalization of EE for mixed states.

The EWCS was originally conjectured to be the dual of the entanglement of purification (EOP) [19], based on agreements of their various information-theoretic properties [14,15] as well as compatibility with the tensor network description of AdS/CFT [20,21]. The proposal has passed further consistency checks in the multipartite generalization [22] and in the conditional generalization [23,24]. Refer to [25–40] for recent progress.

Surprisingly, several correlation measures other than EOP have been shown to be essentially equal to the EWCS with appropriate coefficients, including the logarithmic negativity [41–44], the odd entropy [45], and the reflected entropy [46]. With the monogamy of holographic mutual information [47] in mind, which strongly suggests that quantum entanglement dominates holographic correlations, we may speculate that some axiomatic measure of quantum entanglement (see, e.g., [48]) would also be equivalent to the EWCS in holographic CFTs.

In this paper, however, we present a no-go theorem in this direction: the EWCS is *not* dual of various entanglement measures. Furthermore, we show that the EWCS can be strictly larger than various entanglement measures at the leading order $O(N^2)$. It is particularly shown in a holographic configuration near to the saturation of the Araki-Lieb inequality [49,50]. We also point out that the EWCS is also larger than another type of quantum correlation, the quantum discord [51,52]. It implies that the EWCS captures more than quantum entanglement in the entanglement wedge, and it must be sensitive to classical correlations as well.

Next, we introduce a series of holographic duals for the optimized correlation measures, which are akin to the EOP. This class includes two entanglement measures, the squashed entanglement [53] and the conditional entanglement of mutual information (CEMI) [54], and three total correlation measures, the EOP, the Q-correlation, and the R-correlation [55]. We show that the CEMI reduces to half of the holographic mutual information as the R-correlation does to the EWCS, when they are optimized over the geometrical extensions. These two duals thus do not lead to new geometrical object in the bulk.

However, we find that the holographic dual of the Q-correlation provides us with a new bulk measure of correlation inside the entanglement wedge, which we call the entanglement wedge mutual information (EWMI). This quantity appropriately satisfies all of the properties of the Q-correlation, as well as the strong superadditivity like the EWCS. Furthermore, we show that the EWMI can also strictly become larger than the various quantum correlation measures in the same holographic configurations. It again implies that classical correlations are included in holographic correlations and they are geometrically encoded in the entanglement wedge.

This paper is organized as follows: In Sec. II, we review the basic notion of the EWCS and information-theoretic correlation measures. In Sec. III, we show that the EWCS is strictly larger than various measures of quantum correlation in a holographic configuration near to the saturation of the Araki-Lieb inequality. In Sec. IV, we argue holographic duals of the optimized correlation measures, introduce the EWMI, and discuss the aspects of the EWMI. In Sec. V, we discuss some future problems. In the Appendix, we prove new inequalities of the multipartite EOP and the multipartite EWCS, complementing the work of [22].

II. PRELIMINARIES

A. Entanglement wedge cross section

In the present paper we deal with static spacetime for simplicity (a generalization to nonstatic spacetime is straightforward using the HRT formula [11,12] instead of the RT formula). The boundary subsystems are denoted by A and B and the entanglement wedge of $AB \equiv A \cup B$ (on a canonical time slice) is denoted by $\mathcal{M}_{AB}$ [16–18]. Given an entanglement wedge $\mathcal{M}_{AB}$, we may define the minimal cross section as follows [14,15].

Suppose the boundary of $\mathcal{M}_{AB}$ is divided into two "subsystems" $\mathcal{A}$ and $\mathcal{B}$, i.e., $\partial\mathcal{M}_{AB} = \mathcal{A} \cup \mathcal{B}$ under the condition $\mathcal{A} = A \cup A'$, $\mathcal{B} = B \cup B'$. We include the asymptotic AdS boundary and (if it exists) black hole horizon in the boundary of $\mathcal{M}_{AB}$. The EWCS of $\mathcal{M}_{AB}$, $E_W(A:B)$, is defined as the minimum of the holographic entanglement entropy $S_\mathcal{A}$ optimized over all possible partitions (Fig. 1)

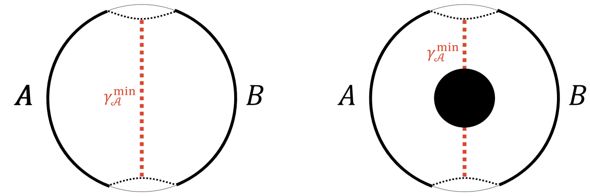

FIG. 1. The EWCS (red dashed lines) on a time slice of the entanglement wedge.

$$E_W(A:B) := \min_{\mathcal{A}:\partial\mathcal{M}_{AB}=\mathcal{A}\cup\mathcal{B}, A\subset\mathcal{A}, B\subset\mathcal{B}} S_\mathcal{A} \tag{2}$$

$$= \min_{\gamma_\mathcal{A}} \frac{\text{Area}(\gamma_\mathcal{A})}{4G_N}, \tag{3}$$

where $\gamma_\mathcal{A}$ is the RT surface of $\mathcal{A}$. It gives a generalization of (1) for mixed states in the sense that $\gamma_\mathcal{A}$ reduces to the usual RT surface when ρ_{AB} is a pure state. The EWCS always satisfies the inequalities $\frac{1}{2}I(A:B) \le E_W(A:B) \le \min\{S_A, S_B\}$, where $I(A:B) := S_A + S_B - S_{AB}$ is the mutual information. The above definition can be generalized to n-partite subsystems [22]. Remarkably, the EWCS can be regarded as a generalization of the area of a wormhole horizon in the canonical purification [46].

B. Information-theoretic correlation measures

The EWCS was originally conjectured to be dual to the EOP at the leading order $O(N^2)$. The EOP is defined for a bipartite state ρ_{AB} by [19]

$$E_P(A:B) := \min_{|\psi\rangle_{AA'BB'}} S_{AA'} = \frac{1}{2}\min_{|\psi\rangle_{AA'BB'}} I(AA':BB'), \tag{4}$$

where the minimization is performed over all possible purifications. The information-theoretic properties of EOP [19,56] are proven for the EWCS geometrically, including the multipartite cases [22]. Moreover, the surface/state correspondence of the tensor network description [21] allows us to find a heuristic derivation of $E_W = E_P$ [14].

The EOP, mutual information, Q-correlation, and R-correlation [55] (which are defined in Sec. IV) are monotonically nonincreasing under local operations (LO), but may increase by classical communication (CC). We call such non-negative quantities on ρ_{AB} (bipartite) total correlation measures. On the other hand, entanglement measures are defined by monotonicity under local operations and classical communication (LOCC). There is a class of entanglement measures that satisfies additional axioms such as asymptotic continuity, which we collectively call (bipartite) axiomatic entanglement measures (see, e.g., [48]). There are various choices of additional

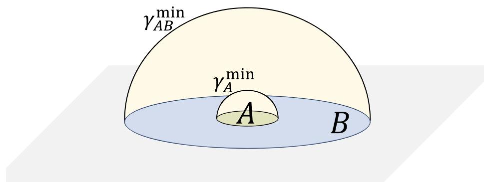

FIG. 2. A holographic configuration for which the Araki-Lieb inequality is saturated, $S_A + S_{AB} = S_B$.

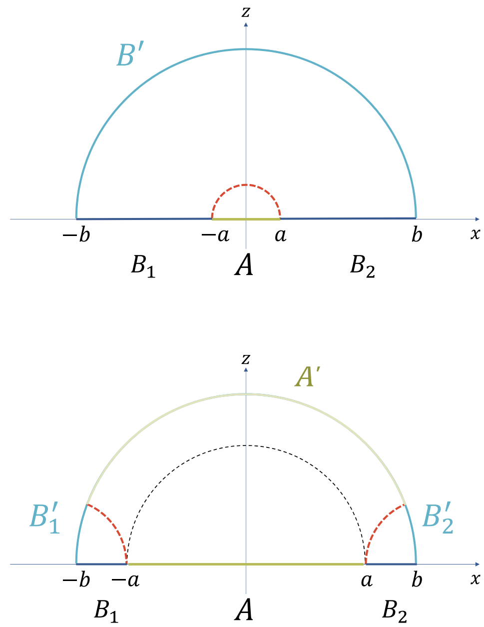

FIG. 3. The two configurations of the EWCS $E_W(A:B) = E_W(A:B_1B_2)$, denoted by the orange dashed line, for the symmetric setup in the Poincaré AdS_3. The left (right) configuration is preferred when the relative size $p < p^*_{\mathrm{EW}}$ $(p > p^*_{\mathrm{EW}})$. The primed symbols allocated on the upper semicircle denote the partition in (3) with $\mathcal{A} = A \cup A'$ and $\mathcal{B} = B_1 \cup B_2 \cup B_1' \cup B_2'$.

axioms one can impose. In what follows we make a somewhat minimal requirement motivated by the uniqueness theorem [13]: They coincide with EE for pure states. This may be regarded as a normalization condition for different measures. Such a class includes, for instance, the distillable entanglement E_D [57,58], the squashed entanglement E_{sq} [53,59], the conditional entanglement of mutual information E_I [54], the relative entropy of entanglement E_{RE} [60], the entanglement cost E_C [57,61], and the entanglement of formation E_F [57]. There is another measure of quantum correlation, called the quantum discord D [51,52]. It captures wider types of quantum correlation than quantum entanglement, and coincides with EE for pure states.

III. EWCS IS NOT DUAL OF AXIOMATIC ENTANGLEMENT MEASURES

First of all, we can use the generic upper bounds $E_D, E_{sq}, E_I, E_{RE}, D \le I$ to exclude E_D, E_{sq}, E_I, E_{RE}, and D as a dual candidate of E_W, since $E_W(A:B) > I(A:B)$ can be observed near to the $O(1)$ phase transition of $I(A:B)$ [14]. It already gives us intuition that the entanglement (or quantum correlation) measures are usually less than the EWCS in holographic CFTs. In this way, however, we cannot exclude E_C and E_F since they may exceed $I(A:B)$ [they can be greater than $I(A:B)/2$ [62]]. In order to do that, we consider another particular holographic setup as follows.

A. The EWCS in the Araki-Lieb transition

One of outstanding characteristics of holographic CFTs is the fact that the Araki-Lieb inequality,

$$S_A + S_{AB} \ge S_B, \tag{5}$$

can be saturated at the leading order $O(N^2)$ in some particular configurations [49,50]. It is typically realized by a subsystem A completely surrounded by sufficiently large B (Fig. 2). Though the following discussion is valid for the more generic setups, we focus on a configuration in Poincaré AdS_3 with the metric

$$ds^2 = \frac{dz^2 - dt^2 + dx^2}{z^2}. \tag{6}$$

Suppose the subsystems A and B are given by $A = [-a, a], B = [-b, -a] \cup [a, b] \equiv B_1 \cup B_2$ for $0 < a < b$ w.l.o.g. We also define the relative size of subsystems by $p \equiv \frac{a}{b}$ for $p \in (0, 1)$. The mutual information $I(A:B)$ exhibits a phase transition due to that of $S_B = S_{B_1B_2}$ depending on the relative size p. The connected phase $I(B_1:B_2) > 0$ is preferred if p is small, and the disconnected phase $I(B_1:B_2) = 0$ is if it is large. Thus $I(A:B)$ can be computed as

$$I(A:B) = S_A + S_B - S_{AB}$$
$$= \min\left\{\frac{2c}{3}\log\frac{2a}{\epsilon}, \frac{2c}{3}\log\frac{\sqrt{a/b}(b-a)}{\epsilon}\right\}, \tag{7}$$

where c is the central charge of holographic two-dimensional CFTs and ϵ is the UV cutoff. It is divergent since we are taking the adjacent limit. The phase transition point of $I(A:B)$ can be read off as

$$p^*_{\mathrm{MI}} \equiv \frac{a^*_{\mathrm{MI}}(b)}{b} = 3 - 2\sqrt{2}. \tag{8}$$

The Araki-Lieb inequality is saturated for $0 < p < p^*_{\mathrm{MI}}$ but not for $p^*_{\mathrm{MI}} < p < 1$.

The EWCS also exhibits a phase transition depending on the relative size of A (Fig. 3). The formula for the EWCS in Poincaré AdS_3 is given in [14] by

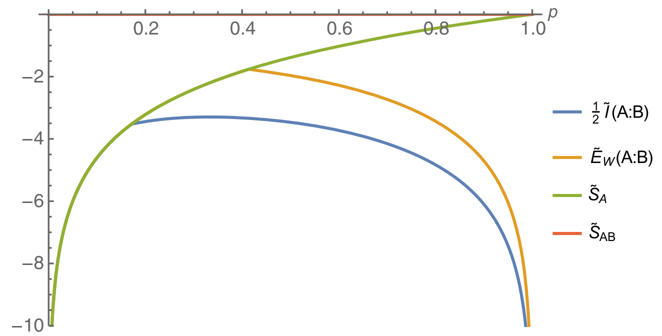

FIG. 4. Half of the mutual information and the EWCS for the Araki-Lieb transition (normalized by subtracting S_{AB}).

$$E_W = \min\{S_A, 2E_W(AB_1 : B_2)\}$$
$$= \min\left\{\frac{c}{3}\log\frac{2a}{\epsilon}, \frac{c}{3}\log\left[\frac{b^2-a^2}{b\epsilon}\right]\right\}. \quad (9)$$

The phase transition of EWCS therefore happens at

$$p^*_{\mathrm{EW}} \equiv \frac{a^*_{\mathrm{EW}}(b)}{b} = \sqrt{2} - 1. \quad (10)$$

The phase transition points of the mutual information and the EWCS do not match, and the strict inequality $p^*_{\mathrm{MI}} < p^*_{\mathrm{EW}}$ holds (Fig. 4). This means that the EWCS saturates its upper bound $E_W(A:B) = S_A$ while the Araki-Lieb inequality is not saturated for $p \in (p^*_{\mathrm{MI}}, p^*_{\mathrm{EW}})$. This observation provides us a crucial benchmark: A correlation measure E cannot be dual to the EWCS if $E(A:B) = S_A$ automatically implies the saturation of the Araki-Lieb inequality $S_A + S_{AB} = S_B$.

The Araki-Lieb inequality is also holographically saturated in the global Banados-Teitelboim-Zanelli (BTZ) black hole

$$ds^2 = \frac{f^{-1}(z)dz^2 - f(z)dt^2 + dx^2}{z^2}, \quad (11)$$

$$f(z) = 1 - \frac{z^2}{z_H^2}, \quad (12)$$

with the inverse temperature $\beta = 2\pi z_H$ and the periodic boundary condition $x \simeq x + 2\pi$. We choose $A = [-l/2, l/2]$ for $l \in (0, \pi)$ and B as the remainder. It exhibits the Araki-Lieb saturation when the size of A is small enough. We find the phase transition points (see, e.g., [14,15,49])

$$l^*_{\mathrm{MI}}(z_H) = \pi - z_H \log\cosh\left(\frac{\pi}{z_H}\right), \quad (13)$$

$$l^*_{\mathrm{EW}}(z_H) = 2z_H \log(1 + \sqrt{2}). \quad (14)$$

This leads to $l^*_{\mathrm{MI}}(z_H) < l^*_{\mathrm{EW}}(z_H)$ for any $z_H > 0$, which confirms the above conclusion.

B. The axiomatic entanglement measures and the Araki-Lieb saturation

We can now cite the following fact: the entanglement of formation saturates its upper bound $E_F(A:B) \le S_A$ if and only if the Araki-Lieb inequality is saturated [63]. This immediately means that E_F is not dual of E_W from the above observation. Furthermore, this statement can be generalized to another measure E, which satisfies (i) the monotonicity $E(A:B_1B_2) \ge E(A:B_1)$, (ii) $E(A:B) = S_A = S_B$ for pure states, and (iii) $E \le E_F$. Indeed, the saturation $E(A:B) = S_A$ leads to $E_F(A:B) = S_A$ from (iii), which is equivalent to the Araki-Lieb saturation. The opposite is shown by using the unique structure (up to isometries) of states that saturate the Araki-Lieb inequality [64]

$$\rho_{AB} = |\psi\rangle\langle\psi|_{AB_L} \otimes \rho_{B_R}, \quad (15)$$

where the Hilbert space of B is decomposed into $\mathcal{H}_B = \mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}$. This form with (i) and (ii) leads to the saturation by $S_A \ge E(A:B_LB_R) \ge E(A:B_L) = S_A$. This class of correlation measures especially includes the entanglement cost E_C, and the other entanglement measures mentioned at the beginning of this section as well. Thus we can also exclude E_C and the other measures as a dual candidate of E_W.

In addition, it also means that this class of entanglement measures must be strictly less than E_W for the states of $p \in (p^*_{\mathrm{MI}}, p^*_{\mathrm{EW}})$. Since there seems to be no reason to believe that these states are singular among the holographic states, we argue that the EWCS is generically larger than these entanglement measures in holographic CFTs (unless the Araki-Lieb inequality is saturated).

C. Interpretation from the holographic entanglement of purification

By contrast, the EOP evades the above criteria since (iii) does not hold, and still deserves consideration as a possible dual of the EWCS. In addition, there exists a class of states for which $E_P(A:B) = S_A$, but the Araki-Lieb inequality is not saturated [65]. Similarly, the logarithmic negativity, the odd entropy, and the reflected entropy do not satisfy (iii), and we expect that they also coincide with S_A with an appropriate coefficient for some mixed states without the Araki-Lieb saturation.

Furthermore, we can understand the behavior of the EWCS through the Araki-Lieb transition based on the surface/state correspondence. First, we note a remarkable equality that holds after the phase transition $p > p^*_{\mathrm{EW}}$,

$$E_W(A:B_1B_2) = E_W(AB_1:B_2) + E_W(AB_2:B_1). \quad (16)$$

This equation can be explained as follows: the two configurations $p > p^*_{\mathrm{EW}}$ and $p < p^*_{\mathrm{EW}}$ are equivalent to whether the correlation $I(\mathcal{B}_1:\mathcal{B}_2)$ vanishes or not. For $p > p^*_{\mathrm{EW}}$, we see $I(\mathcal{B}_1:\mathcal{B}_2)=0$, and it immediately leads to the unique form (up to isometries on $\mathcal{H}_{A}$) of any purification [66],

$$|\psi\rangle_{\mathcal{A}\mathcal{B}_1\mathcal{B}_2}=|\phi^1\rangle_{\mathcal{A}\mathcal{B}_1}\otimes|\phi^2\rangle_{\mathcal{A}\mathcal{B}_2}. \tag{17}$$

This form of optimal purification, common to each of the three EWCSs, clearly establishes the equality (16).

On the other hand, if $p<p^*_{\mathrm{EW}}$, remaining correlation $I(\mathcal{B}_1:\mathcal{B}_2)>0$ drastically changes the structure of purifications from (17). In this case, the optimal purification would be simply given by the standard purification [19], i.e., setting A' as empty. In this sense, the phase transition point p^*_{EW} is thus understood as a point at which the standard purification switches with the decoupled purification (17) as the optimal purification.

IV. HOLOGRAPHIC DUALS OF THE OPTIMIZED CORRELATION MEASURES

We observed that the EWCS cannot be the dual of any axiomatic entanglement measures. Then a natural question is as follows: Is there any axiomatic entanglement measure that deserves a geometrical dual?

A. Holographic dual of the optimized entanglement measures

Here we discuss two possible candidates: the squashed entanglement E_{sq} [53] and the conditional entanglement of mutual information E_I [54]. Their definitions are reminiscent of the EOP (4). The squashed entanglement is defined as

$$E_{sq}(A:B)\coloneqq\frac{1}{2}\min_{\rho_{ABE}}I(A:B|E) \tag{18}$$

$$=\frac{1}{2}I(A:B)-\frac{1}{2}\max_{\rho_{ABE}}I_3(A,B,E), \tag{19}$$

where ρ_{ABE} is an extension such that $\mathrm{Tr}_E\rho_{ABE}=\rho_{AB}$, and $I_3(A,B,C)=S_A+S_B+S_C-S_{AB}-S_{BC}-S_{CA}+S_{ABC}$ is the tripartite information.

We now impose a crucial assumption to find a possible geometrical dual of E_{sq}: Performing the minimization over a class of extensions that have classical geometrical duals is sufficient to achieve the minimum. It implies that the monogamy of mutual information $I_3(A,B,E)\leq 0$ [47] must hold for the extensions ρ_{ABE}. A holographic dual of the squashed entanglement is then given by half of the holographic mutual information [47],

$$E_{sq}(A:B)=\frac{1}{2}I(A:B). \tag{20}$$

This is achieved by a trivial extension $E=\emptyset$. This relation implies that the holographic mutual information should satisfy the properties of the squashed entanglement, such as the monogamy relation $E_{sq}(A:BC)\geq E_{sq}(A:B)+E_{sq}(A:C)$ [67], which is generically considered as a characteristic of quantum entanglement. The holographic mutual information indeed satisfies the monogamy relation as mentioned above. It is worth noting that the saturation of $E_{sq}\leq\frac{1}{2}I$ occurs if ρ_{AB} saturates Araki-Lieb inequality, but this is not the only possibility [65].

The relation (20), or the monogamy property of the mutual information, suggests a striking conclusion: the mutual information captures only quantum entanglement in holography, even though it is usually a total correlation measure [47].

We give support for this argument by elaborating on the conditional entanglement of mutual information E_I [54]. It is defined by

$$E_I(A:B)\coloneqq\frac{1}{2}\min_{\rho_{ABA'B'}}\left(I(AA':BB')-I(A':B')\right) \tag{21}$$

$$=\frac{1}{2}I(A:B)+\frac{1}{2}\min_{\rho_{ABA'B'}}\left(I(AA':BB')-I(A:B)-I(A':B')\right), \tag{22}$$

where $\rho_{ABA'B'}$ is again any extension of ρ_{AB}. It is an additive measure of quantum entanglement [54]. Suppose the monogamy of mutual information for some geometric extensions $\rho_{AA'BB'}$ is enough to find the minimum. Then we find $I(AA':BB')-I(A:B)-I(A':B')\geq I(A:B')+I(B:A')\geq 0$, which leads to the holographic dual of the CEMI as half of the holographic mutual information (with a trivial extension $A'B'=\emptyset$),

$$E_I(A:B)=\frac{1}{2}I(A:B). \tag{23}$$

It again implies that the holographic mutual information only captures quantum entanglement. This is in contrast to the EWCS, which still captures classical correlations in holography. Indeed, it was pointed out in [32] that the EOP could be more sensitive to classical correlations than the mutual information.

These proposals about E_{sq} and E_I are obviously consistent with the Araki-Lieb transition discussed above, since the holographic dual of E_{sq} and E_I would be the holographic mutual information itself.

We emphasize the fact that two differently defined measures of entanglement reduce to the same quantity $\frac{1}{2}I$ in holography. To our knowledge, there seems to be no obstruction to speculate that the other entanglement measures such as E_C and E_F also coincide with $\frac{1}{2}I$. We leave investigating their holographic duals as an interesting future work.

B. Holographic duals of the optimized total correlation measures

All of the correlation measures E_P, E_{sq}, E_I are defined as the minimum of a linear combination of von Neumann entropies over all possible purifications or extensions. This class of correlation measures is called the optimized correlation measures [55]. There are two other such measures, the Q-correlation and the R-correlation, introduced in [55]

$$E_Q(A:B) \coloneqq \frac{1}{2}\min_{\rho_{ABE}}(S_A + S_B + S_{AE} - S_{BE}) \quad (24)$$

$$\equiv \min_{\rho_{ABE}} f^Q(A, B, E). \quad (25)$$

$$E_R(A:B) \coloneqq \frac{1}{2}\min_{\rho_{ABE}}(S_{AB} + 2S_{AE} - S_{ABE} - S_E) \quad (26)$$

$$\equiv \min_{\rho_{ABE}} f^R(A, B, E). \quad (27)$$

The symmetry between A and B becomes obvious in the equivalent expression in terms of purifications ($E \equiv A'$),

$$E_Q(A:B) = \frac{1}{2}\min_{|\psi\rangle_{AA'BB'}}\left(S_A + S_B + \frac{S_{AA'} + S_{BB'} - S_{BA'} - S_{AB'}}{2}\right) \quad (28)$$

$$\equiv \min_{|\psi\rangle_{AA'BB'}} f^Q(A, A', B, B'). \quad (29)$$

$$E_R(A:B) = \frac{1}{2}\min_{|\psi\rangle_{AA'BB'}}(S_{AB} + S_{AA'} + S_{BB'} - S_{A'} - S_{B'}) \quad (30)$$

$$\equiv \min_{|\psi\rangle_{AA'BB'}} f^R(A, A', B, B'). \quad (31)$$

The Q-correlation and the R-correlation are nonincreasing under local operations, but not necessarily under LOCC. They satisfy the inequality [55]

$$\frac{1}{2}I \leq E_Q, \qquad E_R \leq E_P. \quad (32)$$

We note a close relationship between the R-correlation and the CEMI, which is clear from the following expression of E_R,

$$E_R(A:B) = \frac{1}{2}\min_{|\psi\rangle_{AA'BB'}}(I(AA':BB') - I(A':B')). \quad (33)$$

It is similar to the CEMI (21), though the minimization of the CEMI is performed over all possible extensions.

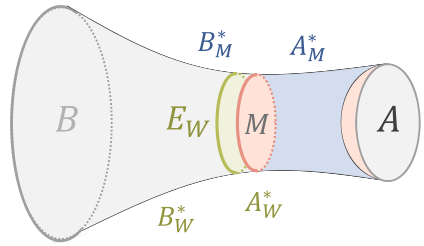

FIG. 5. The entanglement wedge mutual information E_M in the entanglement wedge. In the above picture, E_M is given by the area of red codimension-2 surfaces subtracted by the area of blue codimension-2 surface (divided by $2 \cdot 4G_N$), which may be understood as the mutual information $\frac{1}{2}I(A:M)$. The symmetry $E_M(A:B) = E_M(B:A)$ stems from the fact that the RT surface of $S_{BA'}$ and $S_{AB'}$ has the same configurations. The optimal partition A^*_M and B^*_M of the EWMI located on the RT surface of S_{AB} is not necessarily equivalent to these A^*_W and B^*_W of the EWCS.

1. The holographic counterparts

Here we investigate holographic duals of the Q-correlation and the R-correlation. The definition of the holographic dual candidate of E_Q is stated as follows (we focus on static geometries):

Given an entanglement wedge $\mathcal{M}_{AB}$, divide its boundary into $\partial\mathcal{M}_{AB} = \mathcal{A} \cup \mathcal{B}$ so that $\mathcal{A} = A \cup A'$ and $\mathcal{B} = B \cup B'$. Then minimize the combination of holographic entanglement entropy $f^Q(A, A', B, B')$ over all possible partitions. We define the minimum as the EWMI, denoted by E_M,

$$E_M(A:B) \coloneqq \min_{A' \cup B'} f^Q(A, A', B, B'). \quad (34)$$

An example of the EWMI is depicted in Fig. 5. It may be regarded as the half of the mutual information between A (or B) and the subsystem M assigned to the codimention-2 cross section of $S_{AA'}(= S_{BB'})$.

Generically, the EWMI requires us to consider many complicated configurations of A' and B' in order to minimize f^Q. For some simple cases, however, such as the two disjoint intervals in AdS_3/CFT_2 or the (symmetric) Araki-Lieb saturating configurations, there is an intuitive way to compute E_M owing to the symmetry of setup: Minimize (half of) the mutual information $\max\{I(A:M), I(B:M)\}$ over all possible choices of the cross sections,

$$E_M(A:B) = \frac{1}{2}\min_M(\max\{I(A:M), I(B:M)\}), \quad (35)$$

where M corresponds to the cross section of some partition $A' \cup B'$. This form also clarifies a useful relation

$$I(A:M^*) = I(B:M^*), \quad (36)$$

for at least one of the optimal cross sections M^*. Note that the optimal purification for E_M is not necessarily unique; nor does it necessarily agree with that of E_W (Fig. 5). We see both concrete examples in the below discussion of the Araki-Lieb transition of E_M. There is another suggestive form of E_M for these cases,

$$E_M(A:B) = \frac{1}{2}\left[\frac{1}{2}I(A:B) + \min_{A'\cup B'}\left(S_{AA'} + \frac{I(A:B') + I(B:A')}{2}\right)\right], \quad (37)$$

where we have used $I(A':B') = 0$, which holds for the ancillary subsystems on the RT surface. At least one of $I(A:B')$ and $I(B:A')$ must vanish at this point because the whole system is homologically trivial. Moreover, the balancing condition (36) is equivalent to the condition $I(A:B^*) = I(B:A^*)$. Thus we can conclude that both $I(A:B')$ and $I(B:A')$ should vanish for the balanced optimal partition. As a result, we reach a formula

$$E_M(A:B) = \frac{1}{2}\left[\frac{1}{2}I(A:B) + S_{AA_b^*}\right], \quad (38)$$

where A_b^* is the balanced optimal partition. We may define the deviation from the EWCS due to the balancing term $S_{BA'}$ as

$$D_b(A:B) \coloneqq S_{AA_b^*} - E_W(A:B) \geq 0. \quad (39)$$

We check this formula (38) by direct computation in the Araki-Lieb transition.

A caveat is that neither formula (35) nor (38) is necessarily valid for any configurations, and there possibly exists other types of optimal configurations of A' and B' for more complicated subsystems. Indeed, for example, if we set $|B_1| > |B_2|$ in the Araki-Lieb saturating configuration, E_M can be realized by an optimal configuration neither of $I(A:M^*)$ nor $I(B:M^*)$, but of a combination $I(A:M_A^*) + I(B:M_B^*)$ where $M_A^* \cup M_B^* = M$. Such a configuration is not preferred for the disjoint two intervals [68] or for the symmetric Araki-Lieb configuration. This example indicates that we need to replace $\max\{I(A:M), I(B:M)\}$ in (35) with $\max_{M_A\cup M_B=M}\{I(A:M_A) + I(B:M_B)\}$ in general. We leave proving or disproving it for generic configurations as an important future work.

The EWMI satisfies the properties of E_Q. For example, it cannot be greater than the EWCS,

$$E_M \leq E_W, \quad (40)$$

which must hold to be consistent with $E_W = E_P$ from (32). One can prove this inequality by drawing a picture, but an easier way is to use the von Neumann entropy to represent the corresponding geometrical areas. Suppose the optimal partition of E_W is given by A_W^* and B_W^*. Then we can show $E_W = S_{AA_W^*} \geq \frac{1}{2}(S_A + S_B + S_{AA_W^*} - S_{BA_W^*}) \geq E_M$, where we have used strong subadditivity.

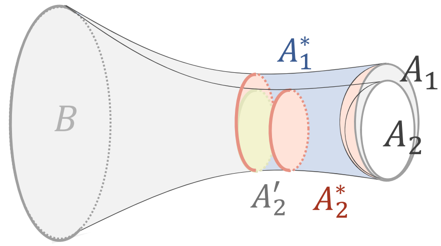

FIG. 6. The extensivity $E_M(A_1:B) \geq E_M(A_2:B)$ for $A_1 \supset A_2$. We abbreviate the B' labels. From the optimal partition A_1^* for A_1, one can induce a partition A_2' on $\partial\mathcal{M}_{A_2B}$ so that $A_1^* \cap \gamma_{A_2B} = A_2' \cap \gamma_{A_2B}$. Then $E_M(A_1:B) \geq f^Q(A_2, B, A_2')$ holds due to the minimality of the RT surface, and $f^Q(A_2, B, A_2') \geq E_M(A_2:B)$ is clear by definition.

Similarly, E_M cannot be less than half of the holographic mutual information,

$$\frac{1}{2}I \leq E_M. \quad (41)$$

It is clear from (38) as $S_{AA_b^*} \geq E_W(A:B) \geq \frac{1}{2}I(A:B)$. These properties also guarantee that $E_M(A:B) = S_A = S_B$ for pure states, and that E_M vanishes if and only if $I(A:B) = 0$ (with $A^* = \gamma_A$ and $B^* = \gamma_B$). It also shows the extensivity $E_M(A_1:B) \geq E_M(A_2:B)$ when $A_1 \supset A_2$ (Fig. 6). The additivity $E_M(\rho_{A_1B_1} \otimes \sigma_{A_2B_2}) = E_M(\rho_{A_1B_1}) + E_M(\sigma_{A_2B_2})$ is also clear because the decoupled state corresponds to disjoint geometries. All of these consistent properties tempt us to propose the relation [at the leading order $O(N^2)$]

$$E_Q = E_M. \quad (42)$$

In pure AdS_3, the E_M for two disjoint intervals has a simple expression (Fig. 7). In such cases, the optimal partition coincides with that of E_W, as it is obvious from the conformal symmetry. Thus E_M becomes just the average of $\frac{1}{2}I$ and E_W by (38),

$$E_M(A:B) = \frac{1}{2}\left[\frac{1}{2}I(A:B) + E_W(A:B)\right]. \quad (43)$$

From this expression, we can easily confirm all of the properties of E_M mentioned above. This expression is not

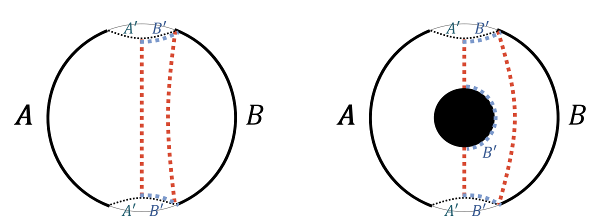

FIG. 7. The E_M in the pure global AdS_3 (left) and in the global BTZ (right) for the symmetric two disjoint intervals, in which $E_M = \frac{1}{2}(\frac{1}{2}I + E_W)$. In the vacuum, one can map the two disjoint subsystems into this setup by the conformal symmetry.

necessarily true in generic setups such as three or more multipartite intervals or black hole geometry.

Surprisingly, the EWMI also satisfies the strong superadditivity

$$E_M(\rho_{A_1A_2B_1B_2}) \geq E_M(\rho_{A_1B_1}) + E_M(\rho_{A_2B_2}), \qquad (44)$$

which can be proven geometrically (Fig. 8). It is similar to the proof of the strong superadditivity of E_W [14]. The relation (44) is not a generic property of E_Q. Thus we may regard it as a characteristic of holographic correlations, as with the holographic entropy cone [47,69–72].

The dual of E_R is defined in the same manner, replacing f^Q with f^R in the above procedure. However, it turns out that this definition is equivalent to that of the EWCS. It stems from the fact we implicitly used in the definition of E_M (and E_W) that it is sufficient to consider the ancillary systems A' and B' located only on the RT surface S_{AB} for minimization. For such subsystems we find $I(A':B') = S_{A'} + S_{B'} - S_{AB} = 0$, resulting in $E_R = E_P = E_W$ from (33). We also state it as a holographic proposal

$$E_R = E_W. \qquad (45)$$

The additivity of the EWCS is consistent with that of the R-correlation [55]. The relation between E_R and E_I then gives an interesting perspective on the geometrical extensions: if only pure geometries are available, the correlations can reduce to E_W at most. If mixed geometries are also allowed, then the inaction gives a further reduction to $\frac{1}{2}I$.

C. The EWMI in the Araki-Lieb transition

Let us study E_M in the Araki-Lieb transition discussed in Sec. III A in detail. First, we remark that $E_M = S_A$ should hold for $p < p^*_{\mathrm{MI}}$ from the inequality $\frac{1}{2}I \leq E_M \leq E_W$, while it also can be checked by direct computation. For $p > p^*_{\mathrm{MI}}$, the situation is more complicated than E_W due to the four configurations of $S_{AA'} - S_{BA'}$. For simplicity, we fix the size b to unit size in the setup and always deal with the relative size p as the parameter.

The two phases of $S_{AA'}$ and the two phases of $S_{BA'}$ are depicted in Fig. 9. The minimal configuration depends not only on the parameter p but also on the size of A', parametrized by $q \in (0,1)$. We can easily find out the minimal configurations in the extremal cases: in the small A' limit ($q \to 0$), the phase (A1) for $S_{AA'}$ and the phase (B1) for $S_{BA'}$ are preferred [recall $I(B_1:B_2) = 0$ for $p > p^*_{\mathrm{MI}}$]. Similarly, we have the phase (A2) for $S_{AA'}$ and the phase (B2) for $S_{BA'}$ in the large A' limit ($q \to 1$). Therefore, as we increase q from 0 to 1, we see phase transitions of $S_{AA'} - S_{BA'}$ for the fixed $p > p^*_{\mathrm{MI}}$ in either path,

$$\text{(I)}\ (\text{A1},\text{B1}) \to (\text{A1},\text{B2}) \to (\text{A2},\text{B2}), \qquad (46)$$

$$\text{(II)}\ (\text{A1},\text{B1}) \to (\text{A2},\text{B1}) \to (\text{A2},\text{B2}). \qquad (47)$$

Note that $S_{BA'}$ is always in (B2) regardless of q for $p < p^*_{\mathrm{MI}}$.

It is not hard to show that increasing q may decrease $S_{AA'} - S_{BA'}$ only in the phase (A2, B1). In the phase (A1, B1), changing q has no effect at all, and in the phase (A1, B2) and (A2, B2), increasing q does increase $S_{AA'} - S_{BA'}$. Therefore, a nontrivial optimal partition for E_M is observed only when the phase transition follows the path (II).

With this in mind, we find the phase transition points q^* of $S_{AA'}$ and $S_{BA'}$ as a function of p,

$$q^*_{AA'}(p) = \frac{(1-p)^2}{4p}, \qquad (48)$$

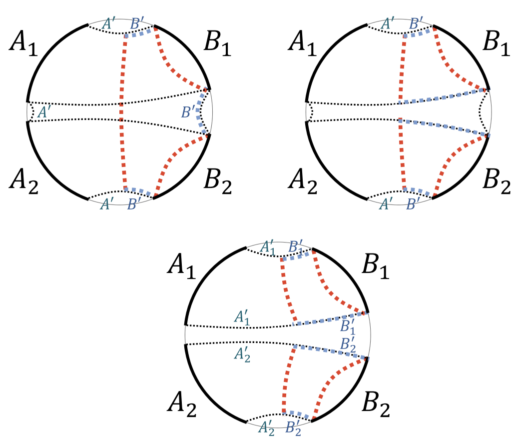

FIG. 8. A proof of the strong superadditivity for E_M. The left figure corresponds to $E_M(\rho_{A_1A_2B_1B_2})$, and the right one corresponds to $E_M(\rho_{A_1B_1}) + E_M(\rho_{A_2B_2})$. As the total of the areas, (left) $\geq$ (middle) $\geq$ (right) is obvious (plus sign for red and minus sign for blue).

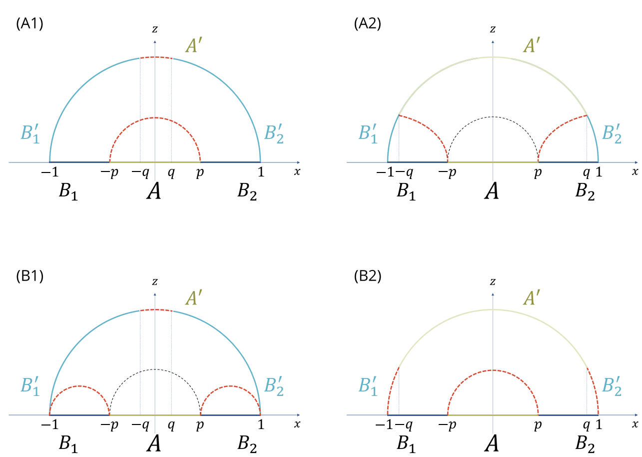

FIG. 9. The two phases of $S_{AA'}$ (top panels) and these of $S_{BA'}$ (bottom panels). The RT surfaces of $S_{AA'}$ and $S_{BA'}$ are denoted as red dashed lines.

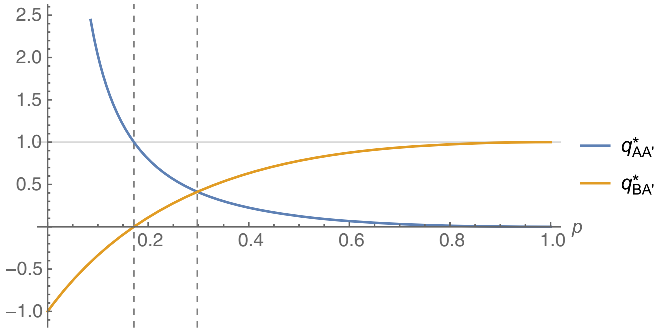

FIG. 10. The phase transition points of $S_{AA'}$ (blue) and $S_{BA'}$ (yellow) with respect to the size q. The vertical dashed lines denote $p^*_{\mathrm{MI}} = 3 - 2\sqrt{2} \simeq 0.17$ and $p^*_{\mathrm{EM}} = -1 + 2\sqrt{2} - 2\sqrt{2-\sqrt{2}} \simeq 0.30$.

$$q^*_{BA'}(p) = -\frac{1-6p+p^2}{(1+p)^2}. \tag{49}$$

These are plotted in Fig. 10. The region $q < q^*_{AA'}$ corresponds to the phase (A1) for $S_{AA'}$, and the region $q < q^*_{BA'}$ to the phase (B1) for $S_{BA'}$. The phase transition of E_M happens at the crossing point p^*_{EM} at which $q^*_{AA'}(p^*_{\mathrm{EM}}) = q^*_{BA'}(p^*_{\mathrm{EM}})$ holds,

$$p^*_{\mathrm{EM}} = -1 + 2\sqrt{2} - 2\sqrt{2-\sqrt{2}} \simeq 0.30. \tag{50}$$

The ancillary system A' of any size $q \le q^*_{BA'}$ achieves the minimum $E_M = S_A$ for $p < p^*_{\mathrm{EM}}$. For $p > p^*_{\mathrm{EM}}$, the minimum of $S_{AA'} - S_{BA'}$ is obtained at $q = q^*_{BA'}(p)$.

Therefore, we have found $E_M = S_A$ for $p \le p^*_{\mathrm{EM}}$, and $E_M = \frac{1}{2}(S_A + S_B + S_{AA^*} - S_{BA^*}) = \frac{1}{2}(S_A - S_{A^*} + S_{AA^*})$ for $p > p^*_{\mathrm{EM}}$. In the latter case, the size of A^* is given by $q^*_{BA'}(p)$, and S_{AA^*} is in the phase (A2) and S_{BA^*} is at the phase transition point (B1) = (B2). We may compute S_{A^*} as $S_{A^*} = \frac{1}{2}(S_A + S_{AB} - S_B)$ from the equality condition (B1) = (B2).

After all, we obtain E_M in the Araki-Lieb transition as

$$E_M(A\!:\!B) = \begin{cases} S_A & (p < p^*_{\mathrm{EM}}) \\ \frac{1}{2}\left[\frac{1}{2}I(A\!:\!B) + S_{AA^*}(p, q^*_{BA'}(p))\right] & (p > p^*_{\mathrm{EM}}) \end{cases}, \tag{51}$$

where $S_{AA^*} \equiv S_{AA^*}(p, q^*_{BA'}(p))$ denotes a contribution from the geodesics between ∂A and ∂A^*,

$$S_{AA^*}(p, q^*_{BA'}(p)) = \frac{c}{3}\log\left(\frac{(1-p)(1+6p+p^2)}{4\sqrt{p}\epsilon}\right). \tag{52}$$

This result (51) confirms the shortcut formula (38). Note that $\frac{1}{2}I(A\!:\!B) = S_A$ for $p < p^*_{\mathrm{MI}}$ and that the balanced optimal partition for $p \in (p^*_{\mathrm{MI}}, p^*_{\mathrm{EM}})$ is given by A' of the size $q = q^*_{BA'}(p)$, not the trivial partition (though it is also optimal). The balancing condition $I(A\!:\!M^*) = I(B\!:\!M^*)$ generically corresponds to the condition (B1) = (B2). The deviation (39) is given by

$$D_b(p) = S_{AA^*}(p, q^*_{BA'}) - E_W(p) = \frac{c}{3}\log\frac{1+6p+p^2}{4\sqrt{p}(1+p)}. \tag{53}$$

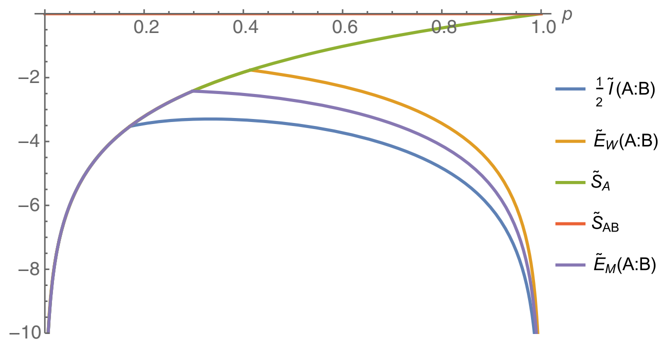

FIG. 11. The E_M for the Araki-Lieb transition (normalized by subtracting S_{AB}).

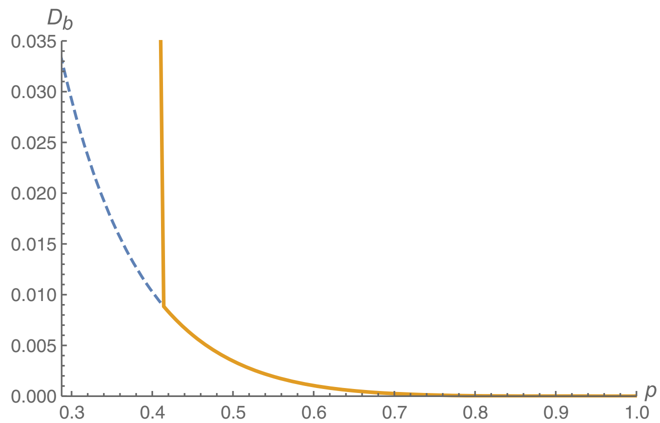

FIG. 12. The deviation D_b for the Araki-Lieb transition (yellow solid line), and the D_b replaced E_W with the nonoptimal configuration in the Fig. 3 (blue dashed line).

The plots of E_M and D_b are given in Figs. 11 and 12.

In particular, the strict inequality $p^*_{\mathrm{MI}} < p^*_{\mathrm{EM}}$ indicates that E_M must be strictly greater than the axiomatic entanglement measures for $p \in (p^*_{\mathrm{MI}}, p^*_{\mathrm{EM}})$, based on the same logic as the EWCS. One can also confirm that E_M exhibits the same kind of phase transition in the global BTZ black hole.

V. DISCUSSION

We have introduced a series of possible holographic duals to the optimized correlation measures. The crucial assumption for the equivalence was that the geometrical extensions are enough to achieve their minimum in holographic CFTs. They demonstrate many properties that are completely consistent with the original information-theoretic measures.

We showed that the EWCS and the EWMI can be larger than the wide class of entanglement measures, while the holographic mutual information is not necessarily. This implies that the EWCS and the EWMI should be more sensitive to classical correlations than the holographic mutual information. Note that both the EWCS and the EWMI satisfy the strong superaddivity, which is a weaker property of quantum entanglement than the monogamy relation (since the latter induces the former). In addition, the EOP or the reflected entropy is supposed to be more sensitive to classical correlations than the mutual information [32,40]. It will be interesting future work to investigate a role of classical correlation in holographic CFTs.

There is a caveat that all of our discussions are restricted to the leading order $O(N^2)$. In particular, the Araki-Lieb transition relies on the property of the holographic entanglement entropy at this order. If one includes quantum corrections from bulk entanglement entropy at $O(N^0)$ [73,74], the rigorous relation will be violated. For instance, the structure of state (15) is not robust against small correction to the exact saturation [75]. The saturation of the EWCS or the EWMI should also be found only in the large N limit. We expect, however, that our conclusion itself still survives: the EWCS and the EWMI with some appropriate quantum corrections will still capture classical correlations.

The bit thread formalism [76] has been cooperative with these holographic optimized correlation measures. The bit threads for the bipartite EWCS were discussed in [77–80] and generalized to the multipartite EWCS [80]. It is interesting to seek a bit thread formalism for E_M as well. Also, a multipartite generalization of E_M would provide us a new tool to probe a specific aspect of the holographic correlations.

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Note.—We recently became aware that an independent work [68] that partially overlaps with the present paper will appear soon.

APPENDIX: THE MULTIPARTITE GENERALIZATION

In this Appendix, we complement some missing pieces in the previous study [22] of the multipartite generalization of the mutual information, the EOP, and the squashed entanglement as well as their holographic duals. The mutual information $I(A:B)$ has various multipartite generalizations. One of them is called the total correlation defined by

$$T_n(A_1:\cdots:A_n) \coloneqq S(\rho_A||\rho_{A_1}\otimes\cdots\otimes\rho_{A_n}) \tag{A1}$$

$$= \sum_{i=1}^{n} S_{A_i} - S_A \tag{A2}$$

$$= I(A_1:A_2) + I(A_1A_2:A_3) + \cdots + I(A_1\cdots A_{n-1}:A_n), \tag{A3}$$

where $S(\rho||\sigma) = \mathrm{Tr}\rho(\log\rho - \log\sigma)$ is the relative entropy. There is another generalization called the dual total correlation,

$$D_n(A_1:\cdots:A_n) \coloneqq S_{A_1\cdots A_n} - \sum_{i=1}^{n} S(A_i|A_1\overset{i}{\check{\cdots}}A_n) \tag{A4}$$

$$= I(A_1:A_2\cdots A_n) + I(A_2:A_3\cdots A_n|A_1) + \cdots + I(A_{n-1}:A_n|A_1\cdots A_{n-2}), \tag{A5}$$

where $S(A|B) = S_{AB} - S_B$ is the conditional entropy and $\overset{i}{\check{\cdots}}$ denotes the exclusion of A_i. The T_n and D_n are monotonically nonincreasing under strict local operations, vanish if and only if the state is totally decoupled, and $T_n = D_n = \sum_{i=1}^{n} S_{A_i}$ if the state is pure.

A multipartite generalization of the EOP [22,24] and the squashed entanglement [81,82] are given as follows:

$$E_P(A_1:\cdots:A_n) = \frac{1}{2}\min_{|\psi\rangle_{A_1A_1'\cdots A_nA_n'}} T_n(A_1A_1':\cdots:A_nA_n'). \tag{A6}$$

$$E_{sq}(A_1:\cdots:A_n) = \frac{1}{2}\min_{\rho_{A_1\cdots A_nE}} T_n(A_1:\cdots:A_n|E), \tag{A7}$$

where $T_n(A_1:\cdots:A_n|E) = I(A_1:A_2|E) + I(A_1A_2:A_3|E) + \cdots + I(A_1\cdots A_{n-1}:A_n|E)$. The multipartite EOP is monotonically nonincreasing under strict local operations. The holographic dual of the multipartite EOP was proposed as the multipartite EWCS [22].

We can generalize the discussion of the holographic dual of the bipartite squashed entanglement as follows. The multipartite squashed entanglement can be written as

$$E_{sq} = \frac{1}{2} T_n(A_1 : \cdots : A_n) + \frac{1}{2} \min_{\rho_{A_1 \cdots A_n E}} Q_n(A : E), \tag{A8}$$

where we define $Q_n(A;E) := I(A_1 \cdots A_n : E) - \sum_{i=1}^n I(A_i : E)$. This can be both positive and negative in the generic quantum system. In holography, however, the monogamy of mutual information implies $Q_n \geq 0$. For $n=2$, it reproduces the nonpositivity of tripartite information $Q_2(AB;E) = I(AB:E) - I(A:E) - I(B:E) = -I_3(A:B:E) \geq 0$. It again results in a conjecture that holographic multipartite squashed entanglement is equivalent to half of the total correlation,

$$E_{sq}(A_1 : \cdots : A_n) = \frac{1}{2} T_n(A_1 : \cdots : A_n). \tag{A9}$$

For the latter convenience, we introduce two nonnegative quantities for $n \geq 3$,

$$X_n := \frac{(n-1)T_n - D_n}{n-2}, \quad Y_n := \frac{(n-1)D_n - T_n}{n-2}. \tag{A10}$$

They are normalized so that $X_n = Y_n = \sum_{i=1}^n S_{A_i}$ holds for pure states. They are positive semidefinite as it is clear from the following expressions,

$$X_n(A_1 : \cdots : A_n) = \frac{1}{n-2} \sum_{i=1}^n T_{n-1}(A_1 : \overset{i}{\check{\cdots}} : A_n), \tag{A11}$$

$$Y_n(A_1 : \cdots : A_n) = \frac{1}{n-2} \sum_{i=1}^n D_{n-1}(A_1 : \overset{i}{\check{\cdots}} : A_n | A_i), \tag{A12}$$

where $D_n(A_1 : \cdots : A_n | E) = I(A_1 : A_2 \cdots A_n | E) + I(A_2 : A_3 \cdots A_n | A_1 E) + \cdots + I(A_{n-1} : A_n | A_1 \cdots A_{n-2} E)$. X_n is monotonically nonincreasing under strict local operations, while the Y_n is not necessarily. Both X_n and Y_n are not faithful; i.e., there exists a state that is not decoupled $\rho_A \neq \rho_{A_1} \otimes \cdots \otimes \rho_{A_n}$ but $X_n = 0$ or $Y_n = 0$. Thus we do not consider each of them as a good correlation measure. Note a balance equation,

$$T_n + D_n = X_n + Y_n = \sum_{i=1}^n I(A_1 : \overset{i}{\check{\cdots}} : A_n). \tag{A13}$$

For holographic states, the monogamy of mutual information leads to a generic ordering,

$$X_n \leq T_n \leq D_n \leq Y_n. \tag{A14}$$

Indeed, $T_n \leq D_n$ follows from the monogamy of mutual information,

$$\begin{aligned} D(A_1 : \cdots : A_n) &= I(A_1 : A_2 \cdots A_n) + I(A_2 : A_3 \cdots A_n | A_1) \\ &\quad + \cdots + I(A_{n-1} : A_n | A_1 \cdots A_{n-2}) \\ &\geq I(A_1 : A_2 \cdots A_n) + I(A_2 : A_3 \cdots A_n) \\ &\quad + \cdots + I(A_{n-1} : A_n) \\ &= T(A_1 : \cdots : A_n). \end{aligned} \tag{A15}$$

Then $X_n \leq T_n$ and $D_n \leq Y_n$ are obvious by their definition.

Now we present some lower bounds on the multipartite EOP, which generalizes and complements the inequalities proven in [22]. The multipartite EOP is bounded from below by half of any multipartite correlation measure Θ, which satisfies (i) $\Theta = \sum_{i=1}^n S_{A_i}$ for pure n-partite states, and (ii) is nonincreasing under strict local operations,

$$E_P \geq \frac{1}{2} \Theta. \tag{A16}$$

It is obvious from the definition (A6) following the same logic as in [22]. Here T_n, D_n, and X_n satisfy both conditions, but Y_n does not satisfy (ii). Thus, we get three inequalities for generic multipartite states

$$E_P \geq \frac{1}{2} \max\{X_n, T_n, D_n\}. \tag{A17}$$

The lower bounds by T_n and X_3 were proven in [22], and the above inequality gives n-partite generalization for X_n. On the other hand, the bound by D_n is totally new. Interestingly, D_n gives a stricter lower bound on E_P than T_n in holography by the ordering (A14). One can check that $E_W \geq \frac{1}{2} D_n$ always holds, while $E_W \geq \frac{1}{2} Y_n$ is not true in general.

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