Universality of the area product: Solutions with conical singularity

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It has been observed that the area product of horizons for many black hole solutions is mass independent and satisfies the universality relation $A_+A_- = (8\pi)^2 N$, where N is related to the quantized charges of the solution as angular momentum and electric charge. In this work we study the area product for black hole and black ring solutions with conical singularity. We find that the area product is still mass independent and regardless of the horizon topology, the conical characteristic (κ) of the solutions, appears in the universality relation as $\kappa A_+A_- = (8\pi)^2 N$. We also check that the first law of black hole inner mechanics is satisfied for these solutions.

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I. INTRODUCTION

Within the context of string theory, it has been suggested [1–5] that the area of black hole event horizon might be quantized as

$$A = 8\pi\ell_p^2(\sqrt{N_1} + \sqrt{N_2}), \qquad N_1, N_2 \in \mathbb{N}. \tag{1}$$

In the case of black hole solutions that possess both inner and outer horizons, there is also a string theory-inspired conjecture for the area product [6–9]

$$A_+A_- = (8\pi\ell_P^2)^2 N, \qquad N \in \mathbb{N}. \tag{2}$$

In recent years the area product of multihorizon black holes has received much attention [8–22]. It has been observed that the product of horizon areas in many solutions is independent of the mass and it depends only on the quantized charges as angular momentum and electric charge. This property is called "universality" of area product [8,10,11]. It is also shown [8,9] that this universality can be generalized to the higher dimensional black hole and black ring solutions. Black rings [23,24] are solutions with the horizon topology of $S^1 \times S^{d-3}$, where $d \geq 5$ is the space-time dimension.

Horizons of a solution can be found from $g^{rr} = 0$ which may have negative or complex roots. The inner and outer horizons are the smallest and the largest physical (positive real) roots of g^{rr}, respectively. It has been shown in [14] that by considering just the inner and outer horizons, the entropy product S_+S_- is mass independent when $T_+S_+ = T_-S_-$, where $T_\pm$ are Hawking temperature of the inner and outer horizons. It is observed [14] that for some solutions such as the Myers-Perry black holes in more than five dimensions, $T_+S_+ \neq T_-S_-$ so the entropy product S_+S_- is mass dependent. However it is shown in [8] that by considering all the horizons, the entropy product becomes mass independent. Thermodynamics of black hole (ring) horizons can also be related to the conformal field theory (CFT) duals of black hole (ring) [25–29], this relation is studied in [14,15].

The universality of area product is also investigated for some other solutions: in the case of black holes in higher curvature gravity, it is observed [16] that the universality of area product fails in general. For the black hole solutions containing Newman-Unti-Tamburino (NUT) charge, it is also shown in [17,18] that the area product is not universal. However the mass independence of the area product is observed for the regular black holes [21] and for the acoustic black holes [22].

This paper is organized as follows. In Sec. II we review, very briefly, some examples of the universality of the area product for both black hole and black ring solutions. In Sec. III the conical characteristic of black hole/ring solutions is introduced. In Sec. IV we concentrate on the area product for some black hole and ring solutions that contain conical singularity. We also verify the first law of thermodynamics and the Smarr relation for the inner horizon of these solutions. Finally, Sec. V is devoted to the concluding remarks.

II. UNIVERSALITY OF THE AREA PRODUCT

In this section we review some examples to reveal that the area product is mass independent. Hereafter, throughout the paper, we set the Newton's gravitational constant to unity, i.e., $G = 1$.

(i) *The Kerr-Newman black hole.*—This four-dimensional solution [30] is characterized by mass (M), spin (J) and electric charge (Q). It has been shown [10] that the area product of inner and outer horizons for this solution is universal. Specifically, the universality relation is

$$A_+A_- = 16\pi^2(4J^2 + Q^4). \tag{3}$$

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In four and five dimensions, asymptotically flat charged and rotating black hole solutions generally can be characterized by six parameters [31,32]. In four dimensions, these parameters are M, J and four charges Q_i, $i = 1, 2, 3, 4$ and in five dimensions the black hole possesses M, two angular momentum J_ϕ, J_ψ and three charges Q_1, Q_2, Q_3. It is shown [8,14] that the area product in four and five dimensions respectively goes to

$$A_+A_- = 64\pi^2\left[J^2 + \prod_{i=1}^{4} Q_i\right],$$
$$A_+A_- = 64\pi^2\left[J_\phi J_\psi + \prod_{i=1}^{3} Q_i\right], \tag{4}$$

which are universal explicitly.

(ii) *Double rotating black ring*.—In the case of asymptotically flat black solutions, the area product is universal regardless of the topology of the horizons. For a five-dimensional double rotating neutral black ring (the horizon topology of $S^1 \times S^2$) [33], it has been checked [9] that the area product is universal as

$$A_+A_- = 64\pi^2 J_\phi^2, \tag{5}$$

where the S^1 circle of the black ring lies at the ψ direction and J_ϕ is the angular momentum perpendicular to it.

(iii) *Single rotating dipole black ring*.—The universality of area product is also observed for the single spin dipole black ring [34]. This solution possesses dipole charge q in addition to its angular momentum J_ψ along the S^1 of the ring. It has been shown for this solution that [9,15]

$$A_+A_- = 64\pi^2 J_\psi q^3, \tag{6}$$

which means the universality for the dipole black ring.

III. THE CONICAL CHARACTERISTIC

In the next section we investigate the area product for black solutions that contain the conical singularity. Hence it is useful to introduce the conical characteristic of these solutions. In general, a solution may contain conical singularity in the space-time. In this case there is a deficit (or excess) of azimuthal angle: $\Delta\phi = 2\pi/\kappa'$. In other words, for $\kappa' > 1$ ($\kappa' < 1$), one obtains conical deficit (excess) in the space-time, while the space-time is regular for $\kappa' = 1$.

In order to find this factor on the horizon, one may expand the $x - \phi$ part (x denotes the polar coordinate) of the horizon metric around $x = x_0$, where the $\phi\phi$ component of the horizon metric vanishes. This part takes the form

$$ds_H^2 = \frac{A}{x - x_0}dx^2 + B(x - x_0)d\phi^2, \tag{7}$$

the values $x_0 = 1, -1$ denote the north and south poles on the horizon metric, respectively. Now using the transformation $x - x_0 = cr^2$, one can rewrite the above part as

$$ds_H^2 = 4cA(dr^2 + \kappa_\pm^2 r^2 d\phi^2), \qquad \kappa_\pm^2 = \frac{B}{4A}. \tag{8}$$

In the above, κ_+ and κ_- denote the conical deficit of the north and south poles on the event horizon, respectively. It is always possible to rescale the ϕ coordinate in a manner that $\kappa_- = 1$ or $\kappa_+ = 1$. This means that the horizon is a "distorted" sphere which is consisting of a regular southern hemisphere (if we set $\kappa_- = 1$) that joined to a conic space in the north pole. The *conical characteristic* of the solution is given by

$$\kappa = \frac{\kappa_+}{\kappa_-}. \tag{9}$$

In the following we see that for the solutions that contain conical singularity, this factor appears in the universality relation of the area product.

IV. UNIVERSALITY AND INNER BLACK HOLE MECHANICS FOR SOLUTIONS WITH CONICAL SINGULARITY

In this section we concentrate on the black hole and black ring solutions with conical singularity. We see that the area product depends on the quantized charges and on the conical characteristic κ of the solutions, but it is independent of mass. In other words, the area product is universal for these solutions. In the following, we also check the first law of black hole thermodynamics on the inner horizon for these solutions.

A. The charged rotating C-metric

The C-metric [35] which describes a pair of uniformly accelerating black holes can be generalized to the black holes containing rotation and charge (the Kerr-Newman black hole) [36–38]. The charged rotating C-metric solution is in the form [38]

$$\begin{aligned} ds^2 = \frac{1}{(1+Axr)^2}\Bigg[&\frac{f(r)+a^2h(x)}{r^2+a^2x^2}dt^2 \\ &-\frac{r^2+a^2x^2}{f(r)}dr^2 + \frac{r^2+a^2x^2}{h(x)}dx^2 \\ &+\frac{a^2(1-x^2)^2 f(r) + (a^2+r^2)^2 h(x)}{r^2+a^2x^2}\Delta_\phi^2 d\phi^2 \\ &+2\frac{a(1-x^2)f(r)+a(a^2+r^2)h(x)}{r^2+a^2x^2}\Delta_\phi dt d\phi\Bigg], \end{aligned} \tag{10}$$

where

$$f(r) = (A^2r^2 - 1)(r - r_+)(r - r_-),$$
$$h(x) = (1 - x^2)(1 + Axr_+)(1 + Axr_-),$$
$$\Delta_\phi = \frac{1}{1 + 2mA + A^2(q^2 + a^2)}. \tag{11}$$

In the above solution there are four parameters m, A, q and a which are related to the mass, acceleration, electric charge and angular momentum of the solution, respectively. Ranges of the coordinates are $-1 \le x \le 1$, $0 \le \phi \le 2\pi$ and the gauge field in this solution is given by

$$A_\mu = \left[-\frac{qr}{r^2 + a^2x^2}, 0, 0, -\frac{-qra(1 - x^2)}{r^2 + a^2x^2}\right]. \tag{12}$$

It is also shown in [36] that the inner and outer horizons are given by

$$r_\pm = m \pm \sqrt{m^2 - q^2 - a^2}. \tag{13}$$

For this solution, the area of outer and inner horizons is

$$\mathcal{A}_\pm = \int \sqrt{\gamma}|_{r=r_\pm} dx d\phi = \int \sqrt{g_{xx}g_{\phi\phi}}|_{r=r_\pm} dx d\phi$$
$$= \frac{4\pi\Delta_\phi(r_\pm^2 + a^2)}{1 - A^2r_\pm^2}, \tag{14}$$

where γ is the determinant of the induced line element obtained by setting $dt = dr = 0$. Rewriting the metric (10) in Arnowitt-Deser-Misner (ADM) form, $ds^2 = -N^2dt^2 + g_{\phi\phi}(d\phi + N^\phi dt)^2 + g_{rr}dr^2 + g_{xx}dx^2$, we calculate temperature on the outer and inner horizons

$$T_+ = \frac{(N^2)'}{4\pi\sqrt{g_{rr}N^2}}\bigg|_{r=r_+} = \frac{(r_+ - r_-)(1 - A^2r_+^2)}{4\pi(r_+^2 + a^2)},$$
$$T_- = \frac{(r_+ - r_-)(1 - A^2r_-^2)}{4\pi(r_-^2 + a^2)}. \tag{15}$$

Now it is straightforward to check that

$$T_+S_+ = T_-S_-, \tag{16}$$

where $S_\pm = \mathcal{A}_\pm/4$ is the entropy. This means that the area product for (10) is universal. The electric charge, angular momentum and mass of the solution can be obtained [38] as

$$Q = \frac{1}{8\pi}\int \epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}dxd\phi = q\Delta_\phi,$$
$$J = \frac{1}{16\pi}\int \epsilon_{\alpha\beta\mu\nu}\nabla^\mu\phi^\nu dxd\phi = am\Delta_\phi^2,$$
$$M = -\frac{1}{8\pi}\int \epsilon_{\alpha\beta\mu\nu}\nabla^\mu\xi^\nu dxd\phi = m\Delta_\phi, \tag{17}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, ϕ^ν and ξ^ν are the rotational and timelike Killing fields ∂_ϕ and ∂_t respectively. Doing the same steps as (7)–(9) for metric (10) one can find the conical characteristic of this solution as

$$\kappa = \frac{1 - 2mA + A^2(q^2 + a^2)}{1 + 2mA + A^2(q^2 + a^2)}. \tag{18}$$

Taking into account these results, one can check that the universality of the area product in this case takes the form

$$\kappa\mathcal{A}_+\mathcal{A}_- = 16\pi^2(4J^2 + Q^4), \tag{19}$$

which is similar to the area product of the Kerr-Newman black hole (3), up to the appearance of conical characteristic κ.

The electric potential and angular velocity on the inner and outer horizon can also be find as

$$\Phi_\pm = -\chi^\mu A_\mu|_{r=r_\pm} = \frac{qr_\pm}{r_\pm^2 + a^2},$$
$$\Omega_\pm = N^\phi|_{r=r_\pm} = \frac{a}{(r_\pm^2 + a^2)\Delta_\phi}. \tag{20}$$

Now one can check that the first law of thermodynamics, $dM = T_+dS_+ + \Omega_+dJ + \Phi_+dQ$, is satisfied. It is easy to check that the Smarr relation and the first law of thermodynamics are also satisfied for the inner horizon:

$$dM = -T_-dS_- + \Omega_-dJ + \Phi_-dQ,$$
$$M = -2T_-S_- + 2\Omega_-J + \Phi_-Q. \tag{21}$$

B. Black ring with rotating S^2

A black ring solution with rotation only along S^2 is introduced in [39]. Note that in the case of black ring solutions ($S^1 \times S^2$ topology), tension of the ring may be canceled by the centrifugal force due to the rotation along the S^1 circle. So the black ring in [39] is unbalanced which means that it contains the conical singularity. The solution is

$$ds^2 = -\frac{H(y,x)}{H(x,y)}\left[dt - \frac{\lambda yR\sqrt{\sigma}(1 - x^2)}{H(y,x)}d\phi\right]^2$$
$$+ \frac{R^2H(x,y)}{(x - y)^2}\left[\frac{dx^2}{(1 - x^2)F(x)} - \frac{dy^2}{(1 - y^2)F(y)}\right.$$
$$\left. - \frac{(1 - y^2)F(x)}{H(x,y)}d\psi^2 + \frac{(1 - x^2)F(y)}{H(y,x)}d\phi^2\right], \tag{22}$$

where

$$F(\xi) = 1 + \lambda\xi + \sigma\xi^2, \quad H(\xi_1, \xi_2) = 1 + \lambda\xi_1 + \sigma(\xi_1\xi_2)^2. \tag{23}$$

In this solution, coordinate ψ parametrizes the S^1 circle and (x, ϕ) shows the S^2. The ring coordinates, x and y, lie in ranges

$$-1 \leq x \leq 1, \qquad -\infty < y \leq -1, \tag{24}$$

with the asymptotic infinity that is located at $x = y = -1$. The solution contains three parameters σ, R and λ, where σ controls the rotation of S^2, R is related to the black ring radius and λ lies in the range $2\sqrt{\sigma} < \lambda < 1 + \sigma$. The conical singularity can be removed at $x = y = -1$ by setting the period of ϕ, ψ coordinates as

$$\Delta\phi = \Delta\psi = \frac{2\pi}{\sqrt{1 - \lambda + \sigma}}. \tag{25}$$

However the conical singularity remains at $x = +1$ and, similar to (7)–(9), one can find the conical characteristic of this solution as

$$\kappa_- = 1, \qquad \kappa = \kappa_+ = \sqrt{\frac{1 + \lambda + \sigma}{1 - \lambda + \sigma}}. \tag{26}$$

Roots of $F(y)$ in (22) correspond to the horizons of this black ring solution. So there are two horizons located at $y_\pm = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\sigma}}{2\sigma}$. Similar to the Sec. IV A, one can find the area of horizon, mass, temperature and angular momentum for this solution as [39]

$$A_+ = \frac{8\pi^2 R^3 \lambda}{(1 - \lambda + \sigma)\sqrt{y_+^2 - 1}}, \qquad M = \frac{3\pi R^2 \lambda}{4(1 - \lambda + \sigma)},$$
$$T_+ = \frac{\sqrt{(\lambda^2 - 4\sigma)(y_+^2 - 1)}}{4\pi\lambda R}, \qquad J^\phi = \frac{\pi R^3 \lambda \sqrt{\sigma}}{(1 - \lambda + \sigma)^{3/2}}. \tag{27}$$

We also calculated the area, temperature and angular velocity of the inner horizon. The result is

$$A_- = \frac{8\pi^2 R^3 \lambda}{(\lambda - 1 - \sigma)\sqrt{y_-^2 - 1}}, \quad T_- = \frac{\sqrt{(\lambda^2 - 4\sigma)(y_-^2 - 1)}}{-4\pi\lambda R},$$
$$\Omega_-^\phi = \frac{(\lambda + \sqrt{\lambda^2 - 4\sigma})\sqrt{1 - \lambda + \sigma}}{-2R\lambda\sqrt{\sigma}}. \tag{28}$$

The inner horizon quantities, as well as the outer horizon ones, satisfy the first law of thermodynamics and the Smarr relation

$$dM = -T_- dS_- + \Omega_-^\phi dJ^\phi, \qquad M = \frac{3}{2}(-T_- S_- + \Omega_-^\phi J^\phi). \tag{29}$$

For this solution $T_+ S_+ = T_- S_-$ is satisfied too and, using (26), one can check the universality of area product as

$$\kappa A_+ A_- = 64\pi^2 J^{\phi 2}. \tag{30}$$

Note again the appearance of the conical characteristic κ in the above.

C. The unbalanced Pomeransky-Sen'kov black ring

A neutral black ring solution in five dimensions characterizes by four parameters relating to its mass, two angular momenta and the ring radius, which is introduced in [40]. In general this solution is unbalanced and contains the conic singularity. The solution is in the form [40]

$$ds^2 = -\frac{H(y,x)}{H(x,y)}[dt - \omega_\phi(x,y)d\phi - \omega_\psi(x,y)d\psi]^2 + \frac{F(y,x)}{H(y,x)}d\phi^2 - 2\frac{J(x,y)}{H(y,x)}d\psi d\phi - \frac{F(x,y)}{H(y,x)}d\psi^2 + \frac{2k^2(1-\mu)^2(1-\nu)H(x,y)}{(1-\lambda)(1-\mu\nu)\Phi\Psi(x-y)^2}\left[\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right], \tag{31}$$

where

$$\Phi = 1 - \lambda\mu - \lambda\nu + \mu\nu, \qquad \Psi = \mu - \lambda\nu + \mu\nu - \lambda\mu^2,$$
$$\Xi = \mu + \lambda\nu - \mu\nu - \lambda\mu^2,$$
$$G(x) = (1 - x^2)(1 + \mu x)(1 + \nu x), \tag{32}$$

also $H(x,y), J(x,y), F(x,y), \omega_\phi(x,y)$ and $\omega_\psi(x,y)$ are messy functions for which we refer the reader to [40]. In this solution, the ring coordinates x, y lie in the ranges $-1 \leq x \leq 1$, $-\infty < y \leq -1$ where the infinity is at $x = y = -1$ and $0 \leq \phi, \psi \leq 2\pi$. There are also four parameters $0 \leq \nu \leq \mu \leq \lambda < 1$ which are dimensionless and $k > 0$ which has the dimension of length and sets the scale of the solution. Doing the same steps as (7)–(9), one can find the conical characteristic for the metric (31) as

$$\kappa_- = 1, \qquad \kappa = \kappa_+ = \frac{1 + \mu}{1 - \mu}\sqrt{\frac{(1 - \lambda)(1 + \nu)\Psi}{(1 + \lambda)(1 - \nu)\Xi}}. \tag{33}$$

By setting $\kappa_+ = 1$ one finds the constraint $\lambda = \frac{2\mu}{1+\mu^2}$ which resolves the conical singularity. Inserting this constraint to the parameters, one can recover the Pomeransky-Sen'kov (balanced) solution [33]. The metric (31) has two horizons [roots of $G(y)$], where the outer one is $y_+ = -1/\mu$ and the inner is located at $y_- = -1/\nu$. The horizon area, Hawking temperature, mass, angular momenta and angular velocities of this solution are given in [40] as

$$A_+ = \frac{16\pi^2 k^3(\mu+\nu)(1-\mu)\Xi}{(1-\lambda)(1+\mu)}\left[\frac{2\lambda(1+\lambda)(1-\nu)}{(1-\mu\nu)^3\Phi\Psi}\right]^{\frac{1}{2}},$$
$$T_+ = \frac{(\mu-\nu)(1-\lambda)(1+\mu)}{8\pi k(\mu+\nu)(1-\mu)\Xi}\left[\frac{2(1-\mu\nu)\Phi\Psi}{\lambda(1+\lambda)(1-\nu)}\right]^{\frac{1}{2}},$$
$$M = \frac{3\pi k^2\lambda(\mu+\nu)(1-\mu)\Phi}{2(1-\lambda)(1-\mu\nu)\Psi},$$
$$J_\phi = \frac{2\pi k^3(\mu+\nu)(1-\mu)}{(1-\mu\nu)^{\frac{3}{2}}}\left[\frac{2\nu\lambda(1+\lambda)\Xi}{(1-\lambda)\Phi\Psi}\right]^{\frac{1}{2}},$$
$$J_\psi = \frac{\pi k^3(\mu+\nu)(1-\mu)[2\nu(1-\lambda)(1-\mu)+(1-\nu)\Phi]}{(1-\lambda)^{\frac{3}{2}}(1-\mu\nu)^{\frac{3}{2}}\Psi^{\frac{3}{2}}} \times\left[\frac{2\lambda(\lambda-\mu)(1+\lambda)(1-\lambda\mu)\Xi}{\Phi}\right]^{\frac{1}{2}},$$
$$\Omega_\psi^+ = \frac{1}{k(1-\mu)}\left[\frac{(\lambda-\mu)(1-\lambda)(1-\lambda\mu)(1-\mu\nu)\Psi}{2\lambda(1+\lambda)\Phi\Xi}\right]^{\frac{1}{2}},$$
$$\Omega_\phi^+ = \frac{1+\mu}{k(\mu+\nu)}\left[\frac{\nu(1-\lambda)(1-\mu\nu)\Psi}{2\lambda(1+\lambda)\Phi\Xi}\right]^{\frac{1}{2}}, \quad (34)$$

which satisfy the Smarr relation $M = \frac{3}{2}[T_+S_+ + \Omega_\phi^+ J_\phi + \Omega_\psi^+ J_\psi]$, with $S = A/4$. We also computed the area, temperature and angular velocities of the inner horizon as

$$A_- = \frac{16\sqrt{2}\pi^2 k^3\nu(\lambda+1)(\mu-1)^2(\mu+\nu)}{(\nu+1)\Psi} \times\left[\frac{\lambda(\nu+1)\Xi}{(\lambda-1)\Phi(\mu\nu-1)^3}\right]^{\frac{1}{2}},$$
$$T_- = \frac{(\nu+1)(-\mu+\nu)\Psi}{4\sqrt{2}\pi k(\lambda+1)(\mu-1)^2(\mu+\nu)(\mu\nu-1)\nu} \times\left[\frac{(\lambda-1)\Phi(\mu\nu-1)^3}{\lambda(\nu+1)\Xi}\right]^{\frac{1}{2}},$$
$$\Omega_\phi^- = \frac{\lambda\mu^2-\lambda\nu^2+\mu\nu^2-\mu}{k(1-\mu)(\mu+\nu)(1+\lambda)}\left[\frac{\Psi(\mu\nu-1)(\lambda^2-1)}{2\nu\lambda\Phi\Xi}\right]^{\frac{1}{2}},$$
$$\Omega_\psi^- = \frac{\sqrt{(\lambda-\mu)(\lambda\mu-1)}}{k\Xi(\mu-1)(1+\lambda)}\left[\frac{\Psi(\mu\nu-1)(\lambda^2-1)}{2\lambda\Phi}\right]^{\frac{1}{2}}. \quad (35)$$

Considering (33)–(35), it is straightforward to check that $T_+S_+ = T_-S_-$ and the universality of area product in this case as

$$\kappa A_- A_+ = 64\pi^2 J_\phi^2. \quad (36)$$

Similar to the previous cases, the conical character κ appears in this relation. One can also check that thermodynamical quantities on the inner horizon for this solution satisfy the first law of thermodynamics and Smarr relation

$$dM = -T_-dS_- + \Omega_\phi^- dJ_\phi + \Omega_\psi^- dJ_\psi,$$
$$M = \frac{3}{2}[-T_-S_- + \Omega_\phi^- J_\phi + \Omega_\psi^- J_\psi]. \quad (37)$$

D. The dipole black ring

A generalization of the single rotating black ring which contains magnetic dipole charge is presented in [34]. The solution is

$$ds^2 = -\frac{F(y)H(x)}{F(x)H(y)}\left[dt + R\sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\frac{1+y}{F(y)}d\psi\right]^2 + \frac{R^2F(x)H(x)H(y)^2}{(x-y)^2}\left[\frac{G(x)}{F(x)H(x)^3}d\phi^2 + \frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} - \frac{G(y)}{F(y)H(y)^3}d\psi^2\right], \quad (38)$$

where the functions F, G, H and the gauge field are

$$F(\xi) = 1+\lambda\xi, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi),$$
$$H(\xi) = 1-\mu\xi, \qquad A_\phi = R\frac{1+x}{H(x)}\sqrt{3\mu(\mu+\nu)\frac{1-\mu}{1+\mu}} + k, \quad (39)$$

in which k is a constant. The solution (38) is characterized by four parameters μ, ν, λ and R that lie in the ranges $0 \le \mu < 1$, $0 < \nu \le \lambda < 1$ and $R > 0$. The x, y are also the ring coordinates similar to the previous cases. In this solution λ, ν are related to the shape and rotation velocity, R sets the scale of the ring and μ controls the dipole charge. The conical characteristic for this solution can be found according to (7)–(9) as

$$\kappa_- = 1, \qquad \kappa = \kappa_+ = \frac{\nu+1}{\nu-1}\sqrt{\frac{(\mu+1)^3(\lambda-1)}{(\mu-1)^3(\lambda+1)}}. \quad (40)$$

The solution has an outer horizon at $y = -1/\nu$, and an inner one at $y = -\infty$. Horizon area, temperature, mass, angular momentum and the dipole charge for this black ring are [34]

$$A_+ = 8\pi^2R^3\frac{(1+\mu)^3(\mu+\nu)^{3/2}\sqrt{\lambda(1-\lambda^2)}}{(1-\nu)^2(1+\nu)},$$
$$T_+ = \frac{\nu(1+\nu)}{4\pi R(\mu+\nu)^{3/2}}\sqrt{\frac{1-\lambda}{\lambda(1+\lambda)}},$$
$$M = \frac{3\pi R^2(1+\mu)^3}{4(1-\nu)}\left[\lambda + \frac{\mu(1-\lambda)}{1+\mu}\right],$$
$$J^\psi = \frac{\pi R^3(1+\mu)^{9/2}}{2(1-\nu)^2}\sqrt{\lambda(\lambda-\nu)(1+\lambda)},$$
$$q = \frac{R(1+\mu)(2\pi)^{1/3}}{(1-\nu)\sqrt{1-\mu}}\sqrt{\mu(\mu+\nu)(1-\lambda)}. \quad (41)$$

One can also calculate the horizon area, temperature, angular velocity and potential of the inner horizon as [9]

$$A_- = 8\pi^2 R^3 \frac{(1+\mu)^3 \mu^{3/2}}{(1-\nu)^2} \sqrt{\lambda(\lambda-\nu)(1-\lambda^2)},$$
$$T_- = \frac{\nu}{4\pi R} \sqrt{\frac{1-\lambda}{\mu^3(1+\lambda)(\lambda-\nu)}},$$
$$\Omega_-^\psi = \frac{1-\nu}{R} \sqrt{\frac{\lambda}{(1+\mu)^3(1+\lambda)(\lambda-\nu)}},$$
$$\Phi_- = \frac{3R(1+\mu)\pi^{2/3}}{2^{4/3}(1-\nu)} \sqrt{\frac{3(\mu+\nu)(1-\mu)(1-\lambda)}{\mu}}. \quad (42)$$

Now it is easy to check that $T_+S_+ = T_-S_-$ and universality of the area product takes the form (note the appearance of κ)

$$\kappa A_- A_+ = 64\pi^2 J q^3. \quad (43)$$

As well as the outer horizon, the first law of black hole inner mechanics and the Smarr relation are satisfied for this solution:

$$dM = -T_- \frac{dA_-}{4} + (\Omega_-^\psi dJ^\psi + \Phi_- dq),$$
$$M = \frac{3}{2}\left[-T_- \frac{A_-}{4} + \Omega_-^\psi J^\psi\right] + \frac{1}{2}\Phi_- q. \quad (44)$$

V. CONCLUSIONS

In this work we investigated the area product for black solutions that contain conical singularity. We considered the charged C-metric, black ring with rotating S^2, the unbalanced double rotating black ring and the dipole black ring. For all these solutions we observed that the area product of inner and outer horizons is mass independent which means that the universality of the area product is true for them.

An interesting point for the solutions with conical singularity is the appearance of the conical characteristic (κ) in the universality relation as $\kappa A_+ A_- = (8\pi)^2 N$, rather than $A_+ A_- = (8\pi)^2 N$ for the regular solutions, where N is related to the quantized charges of the solutions. We observed this behavior for both black hole and black ring solutions. In other words, this property is true for the solutions, regardless of the topology of their horizons.

We also computed the thermodynamical quantities on the inner horizon and checked that the first law of thermodynamics and the Smarr relation are satisfied for the solutions with conical singularity on the inner horizon as well as the outer horizon.

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[1] G. T. Horowitz and A. Strominger, Counting States of Near Extremal Black Holes, Phys. Rev. Lett. **77**, 2368 (1996).
[2] G. T. Horowitz, J. M. Maldacena, and A. Strominger, Nonextremal black hole microstates and U duality, Phys. Lett. B **383**, 151 (1996).
[3] E. Halyo, B. Kol, A. Rajaraman, and L. Susskind, Counting Schwarzschild and charged black holes, Phys. Lett. B **401**, 15 (1997).
[4] F. Larsen, A string model of black hole microstates, Phys. Rev. D **56**, 1005 (1997).
[5] M. Visser, Quantization of area for event and Cauchy horizons of the Kerr-Newman black hole, J. High Energy Phys. 06 (2012) 023.
[6] M. Cvetic and F. Larsen, General rotating black holes in string theory: Grey body factors and event horizons, Phys. Rev. D **56**, 4994 (1997).
[7] P. Galli, T. Ortin, J. Perz, and C. S. Shahbazi, Non-extremal black holes of $N = 2$, $d = 4$ supergravity, J. High Energy Phys. 07 (2011) 041.
[8] M. Cvetic, G. W. Gibbons, and C. N. Pope, Universal Area Product Formulae for Rotating and Charged Black Holes in Four and Higher Dimensions, Phys. Rev. Lett. **106**, 121301 (2011).
[9] A. Castro and M. J. Rodriguez, Universal properties and the first law of black hole inner mechanics, Phys. Rev. D **86**, 024008 (2012).
[10] M. Ansorg and J. Hennig, The Inner Cauchy Horizon of Axisymmetric and Stationary Black Holes with Surrounding Matter in Einstein-Maxwell Theory, Phys. Rev. Lett. **102**, 221102 (2009).
[11] M. Ansorg, J. Hennig, and C. Cederbaum, Universal properties of distorted Kerr-Newman black holes, Gen. Relativ. Gravit. **43**, 1205 (2011).
[12] M. Visser, Area products for stationary black hole horizons, Phys. Rev. D **88**, 044014 (2013).
[13] M. Cvetic, H. Lu, and C. N. Pope, Entropy-product rules for charged rotating black holes, Phys. Rev. D **88**, 044046 (2013).
[14] B. Chen, S. x. Liu, and J. j. Zhang, Thermodynamics of black hole horizons and Kerr/CFT correspondence, J. High Energy Phys. 11 (2012) 017.
[15] B. Chen and J.-j. Zhang, Holographic descriptions of black rings, J. High Energy Phys. 11 (2012) 022.
[16] A. Castro, N. Dehmami, G. Giribet, and D. Kastor, On the universality of inner black hole mechanics and higher curvature gravity, J. High Energy Phys. 07 (2013) 164.

[17] W. Xu, J. Wang, and X. h. Meng, Entropy bound of horizons for charged and rotating black holes, Phys. Lett. B **746**, 53 (2015).

[18] U. Debnath, Entropy bound of horizons for accelerating, rotating and charged Plebanski–Demianski black hole, Ann. Phys. (Amsterdam) **372**, 449 (2016).

[19] V. Faraoni and A. F. Z. Moreno, Are quantization rules for horizon areas universal?, Phys. Rev. D **88**, 044011 (2013).

[20] J. Wang, W. Xu, and X. H. Meng, The "universal property" of horizon entropy sum of black holes in four dimensional asymptotical (anti-)de-Sitter spacetime background, J. High Energy Phys. 01 (2014) 031.

[21] P. Pradhan, Area (or entropy) product formula for a regular black hole, Gen. Relativ. Gravit. **48**, 19 (2016).

[22] M. A. Anacleto, F. A. Brito, and E. Passos, Acoustic black holes and Universal aspects of area products, Phys. Lett. A **380**, 1105 (2016).

[23] R. Emparan and H. S. Reall, A Rotating Black Ring Solution in Five Dimensions, Phys. Rev. Lett. **88**, 101101 (2002).

[24] R. Emparan and H. S. Reall, Black holes in higher dimensions, Living Rev. Relativity **11**, 6 (2008).

[25] M. Guica, T. Hartman, W. Song, and A. Strominger, The Kerr/CFT correspondence, Phys. Rev. D **80**, 124008 (2009).

[26] T. Hartman, K. Murata, T. Nishioka, and A. Strominger, CFT duals for extreme black holes, J. High Energy Phys. 04 (2009) 019.

[27] A. Ghodsi, H. Golchin, and M. M. Sheikh-Jabbari, Dual 2d CFT identification of extremal black rings from holes, J. High Energy Phys. 10 (2013) 194.

[28] A. Ghodsi, H. Golchin, and M. M. Sheikh-Jabbari, More on five dimensional EVH black rings, J. High Energy Phys. 09 (2014) 036.

[29] S. Sadeghian and H. Yavartanoo, Black rings in $U(1)^3$ supergravity and their dual 2d CFT, Classical Quantum Gravity **33**, 095006 (2016).

[30] E. T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, Metric of a rotating, charged mass, J. Math. Phys. (N.Y.) **6**, 918 (1965).

[31] M. Cvetic and D. Youm, Entropy of nonextreme charged rotating black holes in string theory, Phys. Rev. D **54**, 2612 (1996).

[32] M. Cvetic and D. Youm, General rotating five-dimensional black holes of toroidally compactified heterotic string, Nucl. Phys. **B476**, 118 (1996).

[33] A. A. Pomeransky and R. A. Sen'kov, Black ring with two angular momenta, arXiv:hep-th/0612005.

[34] R. Emparan, Rotating circular strings, and infinite non-uniqueness of black rings, J. High Energy Phys. 03 (2004) 064.

[35] W. Kinnersley and M. Walker, Uniformly accelerating charged mass in general relativity, Phys. Rev. D **2**, 1359 (1970).

[36] K. Hong and E. Teo, A new form of the rotating C-metric, Classical Quantum Gravity **22**, 109 (2005).

[37] J. B. Griffiths and J. Podolsky, A new look at the Plebanski-Demianski family of solutions, Int. J. Mod. Phys. D **15**, 335 (2006).

[38] M. Astorino, Thermodynamics of regular accelerating black holes, Phys. Rev. D **95**, 064007 (2017).

[39] P. Figueras, A black ring with a rotating 2-sphere, J. High Energy Phys. 07 (2005) 039.

[40] Y. Chen, K. Hong, and E. Teo, Unbalanced Pomeransky-Sen'kov black ring, Phys. Rev. D **84**, 084030 (2011).