Fermionic string theories with deformed dispersion relations

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We study modified fermionic string theories with deformed dispersion relations. We use the square roots of the bosonic string deformed constraints to obtain the whole constraints of these theories, which verify energy dependent closed algebra. We quantize these theories and we find that the characteristics of the spectrum change with respect to the total energy functions. In a subset of these models, the ordinary fermionic string results remain possible, including theories with no ghost, with space-time supersymmetry, and without tachyons after the *GSO* projection.

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I. INTRODUCTION

The canonical quantization procedure successfully combines the special relativity with the quantum mechanics (QM) into the quantum field theory, which is characterized by the speed of light c and the Planck constant $\hbar$. In addition, the general relativity (GR) introduces the gravity constant G, so that one can construct the Planck length L_P, and the Planck energy E_P, etc.

The existence of a universal minimal length makes the border between classical and quantum description of space-time, and also the length contraction in special relativity must be limited by this length. These ideas can imply a modification in the usual canonical quantization for unifying GR and QM [1,2]. The different quantum gravity candidates theories can predict modifications in the physical constants in accordance with energy scales [3]. Some of this information can be encoded in deformed energy dispersion relations. In this stage, the deformed special relativity (DSR) is one of the main generalizations of special relativity, where the Planck length also has an important role as a universal constant similar to the speed of light. In other words, the deformed special relativity supposes the existence of a preferred universal scale (the Planck length, for example), which is the same for all the observers in the same way as the speed of light. The first version of the DSR is proposed by the physicist Giovanni Almelino-Camelia [4–6], and the other one was proposed by Lee Smolin and Joào Magueijo [7]. In this stage, they have given a treatment of the bosonic string theories with deformed dispersion relations [8], which is inspired by the DSR models. The minimal length scale can also lead to a deformation in commutation relations known as the generalized uncertainty principle (GUP) and can result in a modification in the Heisenberg uncertainty relations [9–11].

Instead of deforming the Poisson brackets or the commutation relations, J. Magueijo and L. Smolin introduce a method based on deformation of the ordinary bosonic string constraints by two total energy functions f and g, which gives a modified mass shell condition and reflects an analogy with the DSR and can also give importance to a preferred length in the Universe, for example, the Planck or the string theory length scale. These theories follow the procedure of canonical quantization, with the opportunity to take advantage of the usual string theory successes [12–14]. The main results are the energy independent speed of light and, in appropriate choices of the deformation, the ground state became nontachyonic.

The scope of the present work is the construction of the deformed fermionic string models, which fit the bosonic ones and lead to a fermionic extension of the deformed bosonic string dispersion relations, used in [8]. At this stage, we follow the method and the arguments used in [12,15].

In Sec. II, we begin with the deformed bosonic string constraints and derive the square roots of these constraints. The introduction of fermionic degrees of freedom leads to the local supersymmetry generators in the world sheet. In Sec. III, we use the Poisson brackets and the Hamiltonian to derive the equations of motion. The equation of motion for the space-time bosonic coordinates is in general nonlinear, as in [8]. This equation becomes linear for specific choices of the energy dependent functions. On the other hand, the equations of motion for the fermionic coordinates are linear, independent of f and g functions. The boundary conditions are obtained, through the conservation of the physical generators under variations, then the solutions can be derived. In Sec. IV, we give the energy deformed classical super-Virasoro generators, which are the Fourier modes of the original constraints. In Sec. V, we consider the covariant

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canonical quantization and derive the quantum super-Virasoro algebra, where the central terms are energy dependent. In Sec. VI, we derive the mass spectrum formula and study the first few levels. The massless vector state is similar to the ordinary fermionic string one, while the other states are different, due to the energy dependence of the applied constraints. We also show that the GSO projection is possible, so the first steps toward the supersymmetry in space-time seem to be satisfied. In Sec. VII, we study some interesting examples. In Sec. VIII, we study the Poincaré algebra and show that it is preserved. The last section contains a conclusion and discussion.

II. DEFORMED FERMIONIC STRING CONSTRAINTS

The Hamiltonian formalism has many features. On the one hand, it helps to manipulate the constrained physical systems; note that string theory is a totally constrained system as well as the relativistic particle and general relativity. On the other hand, the canonical quantization is based on this formalism, and it is a very elegant method for going from the classical theory to the quantum one [16–18]. The canonical form of the bosonic string action can be written as

$$S=\int_0^\pi d\sigma d\tau(\dot{X}^\mu\mathcal{P}_\mu-N\mathcal{H}^{\text{bosonic}}-N^\sigma\mathcal{H}_\sigma^{\text{bosonic}}), \quad (1)$$

where, N and N^σ are the lapse and shift functions, respectively, which play the role of Lagrange multipliers. $\mathcal{H}^{\text{bosonic}}$ and $\mathcal{H}_\sigma^{\text{bosonic}}$ are the Hamiltonian and the spatial diffeomorphism constraints, respectively, while $\mathcal{P}^\mu$ is the canonical conjugate of X^μ.

We use the deformation of the two constraints, as those given in [8],

$$\begin{cases}\mathcal{H}^{\text{bosonic}}=\frac{f(E)}{2T}\mathcal{P}_\mu\mathcal{P}^\mu+\frac{Tg(E)}{2}X'_\mu X'^\mu\\ \mathcal{H}_\sigma^{\text{bosonic}}=\sqrt{f(E)g(E)}\mathcal{P}^\mu X'_\mu,\end{cases} \quad (2)$$

where T is the string tension, f and g are total energy functions, and

$$E=\int_0^\pi d\sigma\mathcal{P}^0(\sigma) \quad (3)$$

is the total energy of the string.

One can look to the fermionic string theory as a theory of supergravity in two dimensions. To do this, and in addition to the bosonic variables, one also needs fermionic degrees of freedom, so let us introduce the real anticommuting variables $\psi_a^\mu(\sigma)$ (where $a=1$, 2), which represent spinor fields in the world sheet. After appropriate simplifications [12], one can write the fermionic string action as follows:

$$S_F=\int_0^\pi d\sigma d\tau(\dot{X}^\mu\mathcal{P}_\mu+\bar{\pi}^\mu\dot{\psi}_\mu-N\mathcal{H}-N^\sigma\mathcal{H}_\sigma-\bar{M}\mathcal{S}), \quad (4)$$

where $\pi(\psi_a^\mu)$ is the canonical conjugate momentum of ψ^μ, the constraints $\mathcal{H}$ and $\mathcal{H}_\sigma$ generate the reparametrizations, $\mathcal{S}$ is the fermionic constraint and generates the local supersymmetry in the world sheet, and $\bar{M}$ is the fermionic Lagrange multiplier.

The energy deformed Hamiltonian and spatial diffeomorphism constraints on the fermionic string are

$$\mathcal{H}=\mathcal{H}^{\text{bosonic}}+\frac{i\sqrt{fg}}{8\pi}\left(\psi_1^\mu\frac{d\psi_{1\mu}}{d\sigma}-\psi_2^\mu\frac{d\psi_{2\mu}}{d\sigma}\right) \quad (5)$$

$$\mathcal{H}_\sigma=\mathcal{H}_\sigma^{\text{bosonic}}+\frac{i\sqrt{fg}}{8\pi}\left(\psi_1^\mu\frac{d\psi_{1\mu}}{d\sigma}+\psi_2^\mu\frac{d\psi_{2\mu}}{d\sigma}\right). \quad (6)$$

It is also convenient to define the linear combinations,

$$Q^+(\sigma)=2\pi(\mathcal{H}+\mathcal{H}_\sigma)=P_\mu P^\mu+\frac{i}{2}\sqrt{fg}\psi_1^\mu\frac{d\psi_{1\mu}}{d\sigma} \quad (7)$$

$$Q^-(\sigma)=2\pi(\mathcal{H}-\mathcal{H}_\sigma)=S_\mu S^\mu-\frac{i}{2}\sqrt{fg}\psi_2^\mu\frac{d\psi_{2\mu}}{d\sigma}, \quad (8)$$

where it can be shown that

$$P_\mu(\sigma)=\sqrt{\frac{\pi f}{T}}\mathcal{P}_\mu+\sqrt{\pi Tg}X'_\mu \quad (9)$$

$$S_\mu(\sigma)=\sqrt{\frac{\pi f}{T}}\mathcal{P}_\mu-\sqrt{\pi Tg}X'_\mu. \quad (10)$$

On the other hand, let us define the energy deformed fermionic constraints as follows:

$$\mathcal{S}_1(\sigma)=\psi_1^\mu(\sigma)P_\mu(\sigma) \quad (11)$$

$$\mathcal{S}_2(\sigma)=\psi_2^\mu(\sigma)S_\mu(\sigma). \quad (12)$$

The Poisson brackets in the bosonic part of the phase space are defined by

$$[F,G]=\int_0^\pi d\sigma'\left(\frac{\delta F}{\delta X^\mu(\sigma')}\frac{\delta G}{\delta P_\mu(\sigma')}-\frac{\delta F}{\delta P^\mu(\sigma')}\frac{\delta G}{\delta X_\mu(\sigma')}\right), \quad (13)$$

where F and G are functionals of $X^\mu(\sigma)$ and $\mathcal{P}^\mu(\sigma)$. So

$$[X_\mu(\sigma),\mathcal{P}_\nu(\sigma')]=\delta(\sigma-\sigma')\eta_{\mu\nu}, \quad (14)$$

while, the fermionic variables are described by the definition,

$$[\psi^\mu_a(\sigma), \psi^\nu_b(\sigma')] = -4\pi i\eta^{\mu\nu}\delta_{ab}\delta(\sigma-\sigma'). \tag{15}$$

Next, with the use of the relations,

$$\frac{\delta F[\mathcal{P}^0]}{\delta\mathcal{P}^0(\sigma)} = \frac{dF(E)}{dE} \tag{16}$$

and

$$\frac{\partial}{\partial\sigma'}\delta(\sigma-\sigma') = -\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma'), \tag{17}$$

one can verify that

$$[P_\mu(\sigma), P_\nu(\sigma')] = 2\pi\sqrt{fg}\delta'(\sigma-\sigma')\eta_{\mu\nu} \tag{18}$$

$$[S_\mu(\sigma), S_\nu(\sigma')] = -2\pi\sqrt{fg}\delta'(\sigma-\sigma')\eta_{\mu\nu} \tag{19}$$

$$[P_\mu(\sigma), S_\nu(\sigma')] = 0. \tag{20}$$

Notice that the Eqs. (11) and (12) generate the local supersymmetry transformations in the two dimensions world sheet. Indeed,

$$\begin{aligned}\delta\psi^\mu_1(\sigma) &= \left[\psi^\mu_1(\sigma), \int_0^\pi \epsilon^1(\sigma')\psi^\mu_1(\sigma')P_\mu(\sigma')d\sigma'\right]\\ &= 4\pi i\epsilon^1(\sigma)P^\mu(\sigma),\end{aligned}$$

the same thing for ψ^μ_2,

$$\delta\psi^\mu_2(\sigma) = 4\pi i\epsilon^2(\sigma)S^\mu(\sigma), \tag{21}$$

while

$$\delta X^\mu(\sigma) = \pi\sqrt{2f}\alpha'\epsilon^a(\sigma)\psi^\mu_a(\sigma). \tag{22}$$

With the help of the Poisson brackets (15) and (18), it is easy to verify that

$$[\mathcal{S}_1(\sigma), \mathcal{S}_1(\sigma')] = -4\pi i\left(P_\mu P^\mu + \frac{i}{2}\sqrt{fg}\psi^\mu_1\frac{d\psi_{1\mu}}{d\sigma}\right)\delta(\sigma-\sigma'). \tag{23}$$

The right-hand side of (23) is clearly proportional to Q^+. Using

$$F(\sigma')\delta'(\sigma-\sigma') = F'(\sigma)\delta(\sigma-\sigma') + F(\sigma)\delta'(\sigma-\sigma'), \tag{24}$$

one then arrives at

$$[Q^+(\sigma), \mathcal{S}_1(\sigma')] = 2\pi\sqrt{fg}(2\mathcal{S}_1(\sigma) + \mathcal{S}_1(\sigma'))\delta'(\sigma-\sigma') \tag{25}$$

and

$$[Q^+(\sigma), Q^+(\sigma')] = 4\pi\sqrt{fg}(Q^+(\sigma) + Q^+(\sigma'))\delta'(\sigma-\sigma'), \tag{26}$$

and likewise, for the constraints $\mathcal{S}_2(\sigma)$ and $Q^-(\sigma)$, where the superalgebra is

$$[\mathcal{S}_2(\sigma), \mathcal{S}_2(\sigma')] = -4\pi iQ^-\delta(\sigma-\sigma') \tag{27}$$

$$[Q^-(\sigma), \mathcal{S}_2(\sigma')] = -2\pi\sqrt{fg}(2\mathcal{S}_2(\sigma) + \mathcal{S}_2(\sigma'))\delta'(\sigma-\sigma') \tag{28}$$

$$[Q^-(\sigma), Q^-(\sigma')] = -4\pi\sqrt{fg}(Q^-(\sigma) + Q^-(\sigma'))\delta'(\sigma-\sigma'). \tag{29}$$

The expressions (23), (25), (26), (27), (28), and (29) give a closed system of constraints, which is akin to the first class constraints system with the energy dependent factors which are represented by the functions f and g.

III. THE EQUATIONS OF MOTION AND SOLUTIONS

A. Equations of motion

The dynamics are generated by the Hamiltonian constraint (5), so for the bosonic field X^μ, the equations of motion can be established by the Poisson brackets,

$$\begin{aligned}\dot{X}^0 &= \int_0^\pi d\sigma'[X^0(\sigma), \mathcal{H}(\sigma')]\\ &= \frac{f}{T}\mathcal{P}^0(\sigma) + \int_0^\pi d\sigma'\left(\frac{\mathcal{P}^2}{2T}\frac{df}{dE} + \frac{T}{2}X'^2\frac{dg}{dE}\right.\\ &\quad\left. + \frac{i}{8\pi}\frac{d\sqrt{fg}}{dE}\left(\psi^\mu_1\frac{d\psi_{1\mu}}{d\sigma'} - \psi^\mu_2\frac{d\psi_{2\mu}}{d\sigma'}\right)\right),\end{aligned}$$

and

$$\begin{aligned}\dot{\mathcal{P}}^0(\sigma) &= \int_0^\pi d\sigma'[\mathcal{P}^0(\sigma), \mathcal{H}(\sigma')]\\ &= Tg(E)\partial^2_\sigma X^0.\end{aligned}$$

The same method gives

$$\dot{X}^i(\sigma) = \frac{f(E)}{T}\mathcal{P}^i(\sigma) \tag{30}$$

$$\dot{\mathcal{P}}^i(\sigma) = Tg(E)\partial^2_\sigma X^i, \tag{31}$$

where $i = 1, \ldots, D-1$ denotes the spatial index. So one finds

$$\dot{X}^\mu = \frac{f}{T}\mathcal{P}^\mu + \delta_0^\mu \int d\sigma' \left(\frac{\mathcal{P}^2}{2T}\frac{df}{dE} + \frac{T}{2}X'^2\frac{dg}{dE} + \frac{i}{8\pi}\frac{d\sqrt{fg}}{dE}\left(\psi_1^\mu \frac{d\psi_{1\mu}}{d\sigma'} - \psi_2^\mu \frac{d\psi_{2\mu}}{d\sigma'}\right)\right) \tag{32}$$

$$\dot{\mathcal{P}}^\mu(\sigma) = Tg(E)\partial_\sigma^2 X^\mu. \tag{33}$$

The nonlinear term in (32) can be eliminated if $f = g$. There are, however, other possibilities if

$$\begin{cases} \frac{df}{dE} = hf(E) \\ \frac{dg}{dE} = hg(E), \end{cases} \tag{34}$$

with the solutions,

$$\begin{cases} f = k_f e^{hE} \\ g = k_g e^{hE} \end{cases} \tag{35}$$

where h, k_f, and k_g are constants. The equations of motion became

$$\ddot{X}^\mu - fg\partial_\sigma^2 X^\mu = 0, \tag{36}$$

and from (15), one can write

$$\frac{\partial\psi_1^\mu(\sigma,\tau)}{\partial\tau} = \int_0^\pi d\sigma' [\psi_1^\mu(\sigma,\tau), \mathcal{H}(\sigma')]_{PB} = \sqrt{fg}\frac{\partial\psi_1^\mu(\sigma,\tau)}{\partial\sigma}.$$

So

$$(\partial_\tau - \sqrt{fg}\partial_\sigma)\psi_1^\mu = 0, \tag{37}$$

and the same method leads to

$$(\partial_\tau + \sqrt{fg}\partial_\sigma)\psi_2^\mu = 0. \tag{38}$$

Here, $\sqrt{fg}$ represents the speed of propagation of the fermionic and bosonic waves along the string, as mentioned in [8] for the bosonic string case, and depends on the total energy E of the string.

Thus, with the definition,

$$\sigma^\pm = \sqrt{fg}\tau \pm \sigma, \tag{39}$$

one can write

$$\frac{\partial}{\partial\sigma^\pm} = \frac{1}{2}\left(\frac{1}{\sqrt{fg}}\frac{\partial}{\partial\tau} \pm \frac{\partial}{\partial\sigma}\right); \tag{40}$$

thus,

$$\begin{cases} \partial_-\psi_1^\mu(\tau,\sigma) = 0 \\ \partial_+\psi_2^\mu(\tau,\sigma) = 0, \end{cases} \tag{41}$$

which means that, ψ_1^μ depends only on σ^+, while ψ_2^μ depends on σ^-.

B. The boundary conditions

The next important step for writing the solutions are the boundary conditions. The functionals $\int_0^\pi N\mathcal{H}d\sigma$, $\int_0^\pi N^\sigma\mathcal{H}_\sigma d\sigma$, and $\int_0^\pi \bar{M}\mathcal{S}d\sigma$ must be well defined as generators, but what is the meaning of the generator? In Dirac terminology, the generator must be weakly zero (weak equality) [16–18], and it must be also a constant of motion. Let us remember that the variation of the action equals zero, and this property gives the equations of motion, the constraints, and boundary conditions. The first variation is writing as

$$\delta\int_0^\pi d\sigma N\mathcal{H} = \frac{1}{4\pi}\int_0^\pi d\sigma N\left(\delta P^2 + \delta S^2 + \frac{i}{2}\sqrt{fg}\delta\psi_1^\mu\frac{d\psi_{1\mu}}{d\sigma} - \frac{i}{2}\sqrt{fg}\delta\psi_2^\mu\frac{d\psi_{2\mu}}{d\sigma} + \frac{i}{2}\sqrt{fg}\psi_1^\mu\frac{d\delta\psi_{1\mu}}{d\sigma} - \frac{i}{2}\sqrt{fg}\psi_2^\mu\frac{d\delta\psi_{2\mu}}{d\sigma}\right). \tag{42}$$

So the first part,

$$\frac{N}{4\pi}\delta\left(P^2 + S^2\right) = 2N\left(\frac{f}{2T}\mathcal{P}^\mu\delta\mathcal{P}_\mu + \frac{Tg}{2}X'^\mu\delta X'_\mu\right), \tag{43}$$

gives the boundary term,

$$[2N(X'_\mu\delta X^\mu)]_0^\pi = 0. \tag{44}$$

In general, the physics is independent of the world sheet metric, which is, in the Hamiltonian formalism, represented by the N and N^σ functions. So in the conformal gauge $N = 1$ and $N^\sigma = 0$, at the boundaries, the Eq. (44) is enough to eliminate the surface term, and we can distinguish between two cases,

$$\partial_\sigma X^\mu|_{0,\pi} = 0, \tag{45}$$

which are the Neumann boundary conditions and

$$\delta X^\mu|_{0,\pi} = 0, \tag{46}$$

which are the Dirichlet boundary conditions. Let us now focus on the boundary terms of ψ^μ and M. Here, we follow the same steps given in [12] for the ordinary fermionic open string. The boundary term of ψ^μ is obtained from the second part of the right-hand side of the Eq. (42),

$$[\sqrt{fg}N(\psi_{1\mu}\delta\psi_1^\mu - \psi_{2\mu}\delta\psi_2^\mu)]_0^\pi. \tag{47}$$

The root $\sqrt{fg}$ is a multiplicative factor, and we obtain then the usual Ramond ψ^μ boundary conditions,

$$\psi_1^\mu(0) = \psi_2^\mu(0) \tag{48}$$

$$\psi_1^\mu(\pi) = \psi_2^\mu(\pi), \tag{49}$$

and the Neveu-Schwarz ones,

$$\psi_1^\mu(0) = \psi_2^\mu(0) \tag{50}$$

$$\psi_1^A(\pi) = -\psi_2^\mu(\pi). \tag{51}$$

With the same method, the variation $\delta(N^\sigma\mathcal{H}_\sigma)$ gives the boundary term,

$$[N^\sigma\mathcal{P}_\mu\delta X^\mu]_0^\pi = 0. \tag{52}$$

This surface term vanishes because $N^\sigma = 0$ at the boundaries. One can remember that N and N^σ verify the same boundary conditions, as in the ordinary bosonic string [12], and what remains is the contribution of $\delta\int_0^\pi \bar{M}\mathcal{S}d\sigma = \delta\int_0^\pi(M^1\mathcal{S}_1 + M^2\mathcal{S}_2)d\sigma$, which gives

$$\int_0^\pi d\sigma\left[M^1\left(\delta\psi_1\cdot P + \sqrt{\frac{\pi f}{T}}\psi_1\cdot\delta\mathcal{P} + \sqrt{\pi Tg}\psi_1^\mu\cdot\delta X'\right) + M^2\left(\delta\psi_2\cdot S + \sqrt{\frac{\pi f}{T}}\psi_2\cdot\delta\mathcal{P} - \sqrt{\pi Tg}\psi_2\cdot\delta X'\right)\right].$$

So the surface term can be written after partial integration of the third and the sixth terms in the last expression as follows:

$$\sqrt{\pi Tg}[(M^1\psi_1^\mu - M^2\psi_2^\mu)\delta X_\mu]_0^\pi, \tag{53}$$

which can be vanished if $\delta X_\mu = 0$ in the boundaries or

$$M^1 = M^2 \quad \text{at } \sigma = 0 \tag{54}$$

$$M^1 = M^2(R) \quad \text{or} \quad M^1 = -M^2(NS) \quad \text{at } \sigma = \pi, \tag{55}$$

which are consistent with the R and NS boundary conditions on ψ_1^μ and ψ_2^μ.

C. Solutions

From [8], in the case of the Neumann boundary conditions, the canonical variables are given as

$$X^\mu(\tau,\sigma) = x^\mu + \frac{p^\mu}{\pi T}\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\sqrt{fg}\tau}\cos(n\sigma) \tag{56}$$

$$\mathcal{P}^\mu(\tau,\sigma) = \frac{T}{f}\dot{X}^\mu = \frac{p^\mu}{\pi f} + \sqrt{\frac{gT}{\pi f}}\sum_{n\neq 0}\alpha_n^\mu e^{-in\sqrt{fg}\tau}\cos(n\sigma), \tag{57}$$

while the solutions in Dirichlet-Dirichlet boundary conditions are

$$X^\mu(\tau,\sigma) = x_0^\mu + \frac{1}{\pi}(x_1^\mu - x_0^\mu)\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sqrt{fg}\tau}\sin(n\sigma).$$

From now on, we will consider only the Neumann-Neumann boundary conditions. The Fourier mode expansions of ψ_a^μ, in the Neveu-Schwarz sector, are

$$\psi_1^\mu(\tau,\sigma) = \sqrt{2}\sum_s b_s^\mu e^{-is(\sqrt{fg}\tau+\sigma)} \tag{58}$$

$$\psi_2^\mu(\tau,\sigma) = \sqrt{2}\sum_s b_s^\mu e^{-is(\sqrt{fg}\tau-\sigma)}, \tag{59}$$

with $s = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$

From (15), one can find the following Poisson brackets between modes:

$$[b_s^\mu, b_r^\nu] = -i\eta^{\mu\nu}\delta_{s,-r}. \tag{60}$$

In the Ramond sector, we can write

$$\psi_1^\mu(\tau,\sigma) = \sqrt{2}\sum_n d_n^\mu e^{-in(\sqrt{fg}\tau+\sigma)} \tag{61}$$

$$\psi_2^\mu(\tau,\sigma) = \sqrt{2}\sum_n d_n^\mu e^{-in(\sqrt{fg}\tau-\sigma)}, \tag{62}$$

with $n = 0, \pm 1, \pm 2, \ldots$

and

$$[d_n^\mu, d_m^\nu] = -i\eta^{\mu\nu}\delta_{n,-m}. \tag{63}$$

IV. THE CLASSICAL SUPER-VIRASORO GENERATORS

The super-Virasoro generators are the Fourier modes of the original deformed constraints of the fermionic string,

$$L_n = \frac{1}{4\pi}\int_{-\pi}^{+\pi} d\sigma e^{in\sigma}Q^+(\sigma) \tag{64}$$

$$G_s = \frac{1}{2\pi\sqrt{2}}\int_{-\pi}^{+\pi} d\sigma e^{is\sigma}\mathcal{S}_1(\sigma) \quad (\text{Neveu}-\text{Schwarz}) \tag{65}$$

$$F_n = \frac{1}{2\pi\sqrt{2}}\int_{-\pi}^{+\pi} d\sigma e^{in\sigma}\mathcal{S}_1(\sigma) \qquad (\text{Ramond}). \tag{66}$$

Therefore, these generators in terms of modes can be obtained by using the solutions (56)–(59), (61), and (62), and the definition,

$$\alpha_0^\mu = \frac{p^\mu}{\sqrt{\pi Tfg}}, \tag{67}$$

as follows:

$$L_n = \frac{g}{2}\sum_{m=-\infty}^{+\infty} \alpha_{n-m}\cdot\alpha_m + \frac{\sqrt{fg}}{2}\sum_{s\in\mathbb{Z}+\frac{1}{2}}\left(s+\frac{n}{2}\right)b_{-s}\cdot b_{n+s} \tag{68}$$

$$G_s = \sqrt{g}\sum_{m=-\infty}^{+\infty} b_{m+s}\cdot\alpha_{-m} \tag{69}$$

for the Neveu-Schwars sector, and

$$L_n = \frac{g}{2}\sum_{m=-\infty}^{+\infty} \alpha_{n-m}\cdot\alpha_m + \frac{\sqrt{fg}}{2}\sum_{m\in\mathbb{Z}}\left(m+\frac{n}{2}\right)d_{-m}\cdot d_{n+m} \tag{70}$$

$$F_n = \sqrt{g}\sum_{m=-\infty}^{+\infty} d_{m+n}\cdot\alpha_{-m} \tag{71}$$

for Ramond sector.

V. CANONICAL QUANTIZATION AND SUPER-VIRASORO ALGEBRAS

In terms of the modes, the commutators,

$$[X^\mu(\sigma,\tau),\mathcal{P}^\nu(\sigma',\tau)]_- = i\delta(\sigma-\sigma')\eta^{\mu\nu}, \tag{72}$$

and the anticommutators,

$$[\psi^\mu_a(\sigma,\tau),\psi^\nu_b(\sigma',\tau)]_+ = 4\pi\eta^{\mu\nu}\delta_{ab}\delta(\sigma-\sigma'), \tag{73}$$

are equivalent to the commutation relations,

$$[x^\mu, p_\nu]_- = if\delta^\mu_\nu \tag{74}$$

$$[\alpha^\mu_n,\alpha^\nu_{-n}]_- = \sqrt{\frac{f}{g}}n\eta^{\mu\nu}, \tag{75}$$

and the anticommutation ones,

$$[b^\mu_s, b^\nu_r]_+ = \eta^{\mu\nu}\delta_{s,-r} \tag{76}$$

for the NS sector, and

$$[d^\mu_n, d^\nu_m]_+ = \eta^{\mu\nu}\delta_{n,-m} \tag{77}$$

for the R sector.

Note that the relation (74) gives the energy dependent Planck constant [19], while the choices $f=g$ and (35) can eliminate the energy deformation in (75), and the anticommutators (76) and (77) follow the usual form of fermionic string theory. So with the following redefinitions [8]:

$$\beta^\mu_n = \left(\frac{g}{f}\right)^{\frac{1}{4}}\alpha^\mu_n, \quad \text{for} \quad n\in\mathbb{Z} \tag{78}$$

$$\beta^\mu_0 = \frac{1}{\sqrt{\pi T}}\tilde{p}^\mu, \tag{79}$$

and the help of (67), one readily shows that

$$\tilde{p}^\mu = f^{-\frac{3}{4}}g^{-\frac{1}{4}}p^\mu \tag{80}$$

and finds that

$$[x^\mu,\tilde{p}_\nu]_- = i\left(\frac{f}{g}\right)^{\frac{1}{4}}\delta^\mu_\nu \tag{81}$$

$$[\beta^\mu_n,\beta^\nu_m]_- = n\eta^{\mu\nu}\delta_{n+m}. \tag{82}$$

Notice again that the choices $f=g$ and (35) lead to eliminate the energy deformations from the Eq. (81).

By the use of the quantum version of the generators (68)–(71) and the redefinitions,

$$L_n = \sqrt{fg}\tilde{L}_n, \tag{83}$$

$$G_s = f^{\frac{1}{4}}g^{\frac{1}{4}}\tilde{G}_s \tag{84}$$

$$F_n = f^{\frac{1}{4}}g^{\frac{1}{4}}\tilde{F}_n. \tag{85}$$

The generators $\tilde{L}_n$, $\tilde{G}_s$, and $\tilde{F}_n$ satisfy the ordinary super-Virasoro algebra, so that this latter takes the form,

$$[L_n, L_m]_- = \sqrt{fg}(n-m)L_{n+m} + fg\frac{D}{8}(n^3-n)\delta_{n,-m} \tag{86}$$

$$[G_r, G_s]_+ = 2L_{r+s} + \sqrt{fg}\frac{D}{2}\left(r^2-\frac{1}{4}\right)\delta_{r,-s} \tag{87}$$

$$[L_m, G_r]_- = -\sqrt{fg}\left(r-\frac{m}{2}\right)G_{m+r} \tag{88}$$

for the NS sector.

One can also obtain the following modified super-Virasoro algebra:

$$[L_n, L_m]_- = \sqrt{fg}(n-m)L_{n+m} + fg\frac{D}{8}n^3\delta_{n,-m} \tag{89}$$

$$[F_n, F_m]_+ = 2L_{n+m} + \sqrt{fg}\frac{D}{2}n^2\delta_{n,-m} \tag{90}$$

$$[L_m, F_n]_- = -\sqrt{fg}\left(n-\frac{m}{2}\right)F_{m+n} \tag{91}$$

for the Ramond sector. The superalgebras are deformed by the presence of the functions f and g. For positive and nonvanishing fg, the conventional super-Virasoro algebra with energy independent anomaly can be used to describe the physical states, which are as the usual ones, except that the center of mass energy and momentum are modified. Whereas, for finite p^0 where $fg = 0$, we can see (as in [8]) that the central charges are energy dependent.

VI. THE SPECTRUM AND GSO PROJECTION

From the Hamiltonian relation (5) and with the solutions (56)–(59), (61), and (62), the Hamiltonian constraints, which are identical to L_0 in both sectors (68) and (70), can be obtained as follows:

$$H = H_{\text{bosonic}} + \sqrt{fg}\frac{1}{2}\sum_{s\in\mathbb{Z}+\frac{1}{2}} s:b^{\mu}_{-s}b_{s,\mu}: \tag{92}$$

for the NS sector, and

$$H = H_{\text{bosonic}} + \sqrt{fg}\frac{1}{2}\sum_{n\neq 0} n:d^{\mu}_{-n}d_{n,\mu}: \tag{93}$$

for the R sector, where

$$H_{\text{bosonic}} = \frac{p^2}{2\pi fT} + \frac{g}{2}\sum_{n\neq 0}:\alpha^{\mu}_{n}\alpha_{n,\mu}:. \tag{94}$$

Let us use the same definition of the mass squared as suggested in [8],

$$M^2 = -\frac{p^2}{fg}. \tag{95}$$

In the fermionic string case, the latter allows for the same factor as in the bosonic (in terms of β^{μ}_{n}) and the fermionic parts of the mass squared expression so that we can use the usual results for the spectrum. Indeed,

$$M^2_{NS} = \pi T\left[\sum_{n\neq 0}:\alpha^{\mu}_{n}\alpha_{\mu,n}: + \sqrt{\frac{f}{g}}\sum_{s\in\mathbb{Z}+\frac{1}{2}} s:b^{\mu}_{-s}b_{s,\mu}:\right] \tag{96}$$

$$M^2_{R} = (\pi T)\sum_{n\neq 0}\left[:\alpha^{\mu}_{n}\alpha_{n,\mu}: + \sqrt{\frac{f}{g}}n:d^{\mu}_{-n}d_{n,\mu}:\right] \tag{97}$$

for NS and R sectors, respectively, or with the redefinition (78), one can write

$$M^2_{NS} = \pi T\sqrt{\frac{f}{g}}\left[\sum_{n\neq 0}:\beta^{\mu}_{n}\beta_{n,\mu}: + \sum_{s\in\mathbb{Z}+\frac{1}{2}} s:b^{\mu}_{-s}b_{s,\mu}:\right] \tag{98}$$

$$M^2_{R} = \pi T\sqrt{\frac{f}{g}}\sum_{n\neq 0}[:\beta^{\mu}_{n}\beta_{n,\mu}: + n:d^{\mu}_{-n}d_{n,\mu}:]. \tag{99}$$

We can conclude then that, in terms of the β^{μ}_{n} modes, the above relations have the same form of ordinary fermionic string mass squared in both sectors but with the energy dependent factor $\sqrt{\frac{f}{g}}$.

A. NS sector

In addition to the Hamiltonian condition, the physical states must obey the following equations:

$$G_r|\phi\rangle = 0, \qquad r > 0; \qquad L_n|\phi\rangle = 0, \quad n \geq 1, \tag{100}$$

and the quantum vacuum satisfies

$$\beta^{\mu}_{n}|0,p\rangle = \left(\frac{g}{f}\right)^{\frac{1}{4}}\alpha^{\mu}_{n}|0,p\rangle = 0 \quad \text{for} \quad n \geq 1 \tag{101}$$

$$b^{\mu}_{s}|0,p\rangle = 0 \quad \text{for} \quad n \geq \frac{1}{2}. \tag{102}$$

The spectrum of the NS sector is like a generalization of the bosonic string one. So the first few levels are written in the following general functional:

$$|\phi\rangle = \left(\Phi + ib^{\mu}_{-\frac{1}{2}}A_{\mu} + i\beta^{\mu}_{-1}B_{\mu} + \frac{1}{2}b^{\mu}_{-\frac{1}{2}}b^{\nu}_{-\frac{1}{2}}S_{\mu\nu} + \ldots\right)|0,p\rangle. \tag{103}$$

From the L_0 constraints, the dispersion relation is written as

$$-\frac{p^2}{fg} = M^2_{NS} = (2\pi T)\sqrt{\frac{f}{g}}[N_{NS} + a_{NS}], \tag{104}$$

where

$$N_{NS} = \sum_{n>0}\beta^{\mu}_{-n}\beta_{n,\mu} + \sum_{s\in\mathbb{Z}+\frac{1}{2}>0} sb^{\mu}_{-s}b_{s,\mu}. \tag{105}$$

Let us use the relation (80) which satisfies the expression,

$$\tilde{p}^2 + \tilde{M}^2_{NS} = 0. \tag{106}$$

The above equation has the form of the ordinary mass shell condition, where

$$\tilde{M}^2_{NS} = (2\pi T)[N_{NS} + a_{NS}]. \tag{107}$$

Notice here that the relations (104) and (105) suggest the usual value $a_{NS} = -\frac{1}{2}$, so that the spectrum obtained is inspired by the ordinary fermionic string one.

The application of the Hamiltonian condition gives

$$\left(l^2\partial^2 + f\sqrt{fg}\frac{1}{2}\right)\phi = 0 \tag{108}$$

$$\partial^2 A_{\mu} = 0 \tag{109}$$

$$\left(l^2\partial^2 - f\sqrt{fg}\frac{1}{2}\right)S_{\mu\nu} = \left(l^2\partial^2 - f\sqrt{fg}\frac{1}{2}\right)B_\mu = 0 \qquad (110)$$

where $l^2 = \frac{1}{(2\pi T)}$.

The application of the constraint condition $G_{\frac{1}{2}}|\phi\rangle = 0$ leads to

$$\partial^\mu A_\mu = 0 \qquad (111)$$

$$\frac{\sqrt{2}l}{f^{\frac{3}{4}}g^{\frac{1}{4}}}\partial^\mu S_{\mu\nu} - B_\nu = 0. \qquad (112)$$

From the relations (109) and (111), one can note that the massless vector state seems to be similar to the conventional fermionic string one with $D-2$ independent components, while the other ground and massive states are affected by the energy deformation [see the relations (108), (110), and (112)].

B. Ramond sector

The physical states obey the constraints conditions,

$$L_n|\psi\rangle = 0 \qquad (113)$$

and

$$F_n|\psi\rangle = 0 \qquad (114)$$

for $n \geq 0$, while the ground state is defined by

$$\beta_n^\mu|0,p\rangle_R = \left(\frac{g}{f}\right)^{\frac{1}{4}}\alpha_n^\mu|0,p\rangle_R = 0 \qquad (115)$$

$$d_n^\mu|0,p\rangle_R = 0 \qquad (116)$$

also for $n \geq 0$.

The spinor wave functional is given by

$$|\Phi\rangle_\epsilon = (\lambda_\epsilon(x) + i\beta_{-1}^\mu\Psi_{\mu\epsilon}^1(x) + d_{-1}^\mu\Psi_{\mu\epsilon}^2(x) + \ldots)|0;p\rangle, \qquad (117)$$

where $\epsilon = 1, 2, \ldots, 2^{\frac{D}{2}}$.

The energy dispersion relation is

$$-\frac{p^2}{fg} = M_R^2 = (2\pi T)\sqrt{\frac{f}{g}}N_R, \qquad (118)$$

where

$$N_R = \sum_{n>0}\beta_{-n}^\mu\beta_{n,\mu} + \sum_{m\in\mathbb{Z}>0} m d_{-m}^\mu d_{m,\mu}. \qquad (119)$$

Let us apply the F_0 and F_1 conditions,

$$F_0|\Phi\rangle_\epsilon = \sqrt{g}\left(\alpha_0^\mu\frac{\Gamma_\mu}{\sqrt{2}} + \alpha_{-1}^\mu d_{1\mu} + \alpha_1^\mu d_{-1\mu} + \ldots\right)|\Phi\rangle_\epsilon = 0 \qquad (120)$$

$$F_1|\Phi\rangle_\epsilon = \sqrt{g}\left(\alpha_0^\mu d_{1\mu} + \alpha_1^\mu\frac{\Gamma_\mu}{\sqrt{2}} + \ldots\right)|\Phi\rangle_\epsilon = 0, \qquad (121)$$

where the zero mode $d_0^\mu = \frac{1}{\sqrt{2}}\Gamma^\mu$. We obtain the following equations:

$$\partial\!\!\!/\lambda = 0, \qquad (122)$$

$$\frac{l}{f^{\frac{3}{4}}g^{\frac{1}{4}}}\partial\!\!\!/\Psi_\mu^1 = \Psi_\mu^2, -\frac{l}{f^{\frac{3}{4}}g^{\frac{1}{4}}}\partial\!\!\!/\Psi_\mu^2 = \Psi_\mu^1\ldots, \qquad (123)$$

and

$$\sqrt{2}\frac{l}{f^{\frac{3}{4}}g^{\frac{1}{4}}}\partial^\mu\Psi_\mu^2 - \frac{\Gamma^\mu}{\sqrt{2}}\Psi_\mu^1 = 0, \qquad (124)$$

where $\partial\!\!\!/ = \Gamma^\mu\partial_\mu$.

The massless state seems to be like the ordinary fermionic string one, but the behavior of the other states depends on the energy functions.

C. GSO projection

Let us recall that the mass squared operators in (104) and (118) keep the usual form in terms of β_n and the unchanged fermionic modes, but with the multiplicative factor $\sqrt{\frac{f}{g}}$, so in $D = 10$, the GSO projection appears not to be affected by the deformation, and the two projectors, which act on the NS and R sectors, also have the same form as the ordinary GSO ones. We find the same known results: the possibility of the tachyon elimination, mass levels with the factor $\sqrt{\frac{f}{g}}$ are well defined, and the space-time supersymmetry is preserved with such deformation. Indeed, in our case, the degrees of freedom appears to be similar to the ordinary fermionic string ones. One can see this precisely if we take the nonlinear map (80); we obtain the conventional form of the constraints equations.

After the GSO projection, the ground state of the open fermionic string contains a massless spin-1 state,

$$b_{-\frac{1}{2}}^\mu|0,p\rangle, \qquad (125)$$

and a massless spin-$\frac{1}{2}$ state,

$$|\epsilon,p\rangle_{s,c}, \qquad (126)$$

as a Majorana spinor with a well-defined chirality, while the next state is massive and described by

$$\{\beta_{-1}^\mu, b_{-\frac{1}{2}}^\mu b_{-\frac{1}{2}}^\nu\}|0;p\rangle \qquad (127)$$

in the NS sector, and

$$\{\beta^{\mu}_{-1}, d^{\mu}_{-1}\}|\epsilon, p\rangle_{s,c} \tag{128}$$

in the R sector.

Of course, as in the bosonic case [8], the precedent reasoning supposes that $f\sqrt{fg} > 0$ is nonvanishing and nonsingular; otherwise, the ghosts must not be reintroduced into the theory by the choice of functions.

VII. EXAMPLES

Let us first restrict ourself with the condition $f = g$ and take the example studied in the bosonic string case [8],

$$f^2(p_0) = 1 - (L_P p_0)^2. \tag{129}$$

The dispersion relation can be written as follows:

$$p^2 + f^2 M^2 = 0. \tag{130}$$

In the rest reference frame, we obtain

$$E^2_{NS} = \frac{N_{NS} - \frac{1}{2}}{l^2 + L^2_p (N_{NS} - \frac{1}{2})} \tag{131}$$

for the NS sector and

$$E^2_R = \frac{N_R}{l^2 + L^2_p N_R} \tag{132}$$

for the R sector.

Then let us see the first few levels. First, in the NS sector,

(i) $N_{NS} = 0$

$$E^2_{NS} = \frac{1}{L^2_p - 2l^2}. \tag{133}$$

This state is still a tachyon for $L_P < \sqrt{2}l$, where

$$f^2 = 1 - \frac{1}{1 - \frac{2l^2}{L^2_p}} > 0.$$

(ii) $N_{NS} = \frac{1}{2}$

$$E^2_{NS} = 0. \tag{134}$$

(iii) $N_{NS} = 1$

$$E^2_{NS} = \frac{1}{L^2_p + 2l^2} < \frac{1}{L^2_P}. \tag{135}$$

(iv) $N_{NS} \geq \frac{3}{2}$

$$E^2_{NS} < \frac{1}{L^2_P}. \tag{136}$$

Second, in the R sector,

(i) $N_R = 0$

$$E^2_R = 0. \tag{137}$$

(ii) $N_R = 1$

$$E^2_R = \frac{1}{l^2 + L^2_P} < \frac{1}{L^2_P}. \tag{138}$$

We note that the ground state in the NS sector is a tachyon for $L_P < \sqrt{2}l$, and all states accumulate below the Planck energy in the both sectors. [See (131) and (132), where $\lim_{N\to+\infty} E_{NS} = \lim_{N\to+\infty} E_R = \frac{1}{L_P}$.] On the other hand, the NS ground state has no equivalent state in the R sector, so that, by the use of the GSO projection, we can perform the first steps toward a space-time supersymmetric string theory with the deformed dispersion relation (130) and the function (129).

Other examples concern the cases (35), so in the rest frame, we can write

$$-E^2 + \frac{\lambda}{l^2}\left(N_{NS} - \frac{1}{2}\right)\exp(2hE) = 0 \tag{139}$$

for the NS sector, and

$$-E^2 + \frac{\lambda}{l^2} N_R \exp(2hE) = 0 \tag{140}$$

for the R sector, where λ is a real positive number. Then, we can write the general form of the string energy with the Lambert W function, as follows:

$$E_{nNS} = -\frac{1}{h} W\left(\pm\frac{h}{l}\sqrt{\lambda\left(N_{NS} - \frac{1}{2}\right)}\right) \tag{141}$$

$$E_{nR} = -\frac{1}{h} W\left(\pm\frac{h}{l}\sqrt{\lambda N_R}\right). \tag{142}$$

VIII. THE POINCARE ALGEBRA

Let us consider the Lorentz generators,

$$M^{\mu\nu} = \int_0^{\pi} (X^{\mu}\mathcal{P}^{\nu} - X^{\nu}\mathcal{P}^{\mu}) d\sigma + \frac{1}{4\pi i}\int_0^{\pi} \sum_a \psi^{\mu}_a \psi^{\nu}_a d\sigma \tag{143}$$

in terms of modes. One can write

$$M^{\mu\nu} = J^{\mu\nu} + I^{\mu\nu}, \tag{144}$$

where

$$J^{\mu\nu} = \frac{1}{f}(x^\nu p^\mu - x^\mu p^\nu) - i\sqrt{\frac{g}{f}}\sum_{n>1}\frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\mu \alpha_n^\nu) \tag{145}$$

$$I^{\mu\nu} = -i\sum_{s>0}(b_{-s}^\mu b_s^\nu - b_{-s}^\nu b_s^\mu) \tag{146}$$

for the Neveu-Schwarz sector and

$$I^{\mu\nu} = -i\sum_{n>0}(d_{-n}^\mu d_n^\nu - d_{-n}^\nu d_n^\mu) - i d_0^\mu d_0^\nu \tag{147}$$

for the Ramond sector.

By the use of (72) and (73), one can find the usual Poincare algebra,

$$[p^\mu, p^\nu] = 0 \tag{148}$$

$$[p^\mu, M^{\nu\rho}] = i\eta^{\mu\rho} p^\nu - i\eta^{\mu\nu} p^\rho \tag{149}$$

$$[M^{\mu\nu}, M^{\rho\zeta}] = i\eta^{\mu\rho} M^{\nu\zeta} + i\eta^{\nu\zeta} M^{\mu\rho} - i\eta^{\nu\rho} M^{\mu\zeta} - i\eta^{\mu\zeta} M^{\nu\rho}, \tag{150}$$

where

$$p^\mu = \int_0^\pi d\sigma \mathcal{P}^\mu(\sigma) \tag{151}$$

is the total momentum of the string.

Notice here that, while both the bosonic modes of the string and the center of mass propagate with deformed commutation relation (74) and (75), and the presence of the deformation functions f and g in (145), the Poincaré algebra remains unchanged.

IX. CONCLUSION

We have applied the constraints square root method on the deformed bosonic string ones [8] to obtain the deformed dispersion relations of the modified fermionic string theories. These constraints are also redefined to fit the fermionic model by providing the closure of the whole deformed constraints superalgebra. We have also shown that the world sheet local supersymmetry transformations are energy dependent. We have obtained the equations of motions of the bosonic space-time coordinates and found that they are linear for the two cases $f = g$ or $f' = hf$ and $g' = hg$ where h is a constant, while the fermionic coordinates ones are originally linear. We have also studied the open fermionic string surface terms of the constraints to get the Neumann and Dirichlet boundary conditions and define the R and NS sectors. We performed the canonical quantization procedure; the obtained super-Virasoro algebra has energy dependent central charges.

For $f = g$ where f is real, nonvanishing, and nonsingular, we found that the ordinary fermionic string theory results are still realized; including the tachyonic ground state of the NS sector, and the nontachyonic spectrum in the R sector, it is also possible to use the GSO projection to get something like a theory with space-time supersymmetry.

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