Self-gravitating $SU(5)$ Higgs domain walls as a braneworld

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Five-dimensional domain walls in gauged $SU(5)$ generate a position-dependent symmetry breaking pattern along the additional dimension. We analyze the perturbative stability and the four-dimensional (4D) spectrum of these walls in the self-gravitating case, in terms of diffeomorphism-invariant and Lie algebra gauge-invariant field fluctuations. We show that tachyonic modes are absent, ensuring perturbative stability. As expected, gravitational tensor and vector fluctuations behave like their counterparts in the standard Z_2 domain walls. All the Lie algebra valued fluctuations exhibit towers of 4D massive modes, which propagate in the bulk, with a continuous spectrum starting from zero. All the would-be 4D Nambu-Goldstone fields, which are gravitationally trapped in the case of a global symmetry, are nontrivially absent. However, we find no localizable 4D gauge bosons, either massless or massive. Instead, quasilocalizable discrete 4D massive modes for the gauge field fluctuations are found, along the spontaneously broken directions.

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I. INTRODUCTION

It is well known that field theoretic domain walls, arising in Abelian Z_2-symmetric five-dimensional (5D) Einstein–scalar field theories, provide regularizations of the Randall-Sundrum brane [1] which preserve four-dimensional (4D) gravity on the core of the wall [2–5]. Besides this, localized 4D massless fermion modes appear also *via* their interaction with the scalar field of which the wall is made [6–11]. Localization of gauge fields in these scenarios has been, however, somewhat more elusive [6,12–17].

Perhaps not so familiar is the fact that it is also possible to consider domain walls, generated by a scalar field transforming nontrivially under a non-Abelian group, that break a continuous internal symmetry in addition to the Z_2 symmetry. For instance, in flat space $SU(5)\times Z_2$ theories with a single adjoint scalar $\mathbf{\Phi}$ and symmetry breaking $SU(5)\times Z_2 \to H = SU(3)\times SU(2)\times U(1)/(Z_3\times Z_2)$, there exist perturbatively stable domain walls that interpolate nontrivially between the two disconnected sectors of the vacuum manifold [18–20]. In these walls, the unbroken symmetries far away from the wall, $H_\pm$, and on its core, $H_0 = H_+ \cap H_-$, are such that H_+ and H_-, though isomorphic, are differently embedded in $SU(5)$. Non-Abelian domain walls of this sort are very interesting by themselves as well as in connection to the solitonic nature of fundamental branes [21,22].

Non-Abelian domain walls (rather, their extensions to the gravitating case) may be relevant within the context of braneworlds. In this direction, the idea of a braneworld generated by a domain wall that breaks a gauge symmetry group G in addition to the Z_2 discrete symmetry was put forward in Ref. [23]. Explicit flat space-time realizations have been discussed for a $O(10)$ symmetry in Ref. [24] and, assuming gauge field localization via the Dvali-Shifman mechanism [21], for a E_6-invariant theory in Ref. [25]. The last reference gave also a treatment for dynamical localization of fermions in the model.

Further attempts in which domain wall backgrounds break a grand unified theory with gauge field localization via the Dvali-Shifman mechanism can be seen in Refs. [26,27]. For other examples of braneworlds realized on the 4D core of self-gravitating topological defects formed by the breakdown of a gauge symmetry in more than one extra dimension, see Ref. [28] and references therein (see also Ref. [29]).

Non-Abelian domain walls in theories with gravity have been also considered within the braneworld context. It has been shown [30] that domain wall configurations $(\mathbf{\Phi}^k; g^k_{ab})$ in global $SU(5)\times Z_2$ Einstein–scalar field 5D theories, in which the curvature of the metric g^k_{ab} is a regularization of the curvature of the Randall-Sundrum brane, exist. Analysis of the diffeomorphism-invariant fluctuations of these systems reveals, besides their perturbative stability, an interesting gravitationally trapped content from the point of view of 4D observers [31]. In particular, there are as many normalizable 4D massless scalar modes as there are broken generators (i.e., that do not commute with $\mathbf{\Phi}^k$). Since the domain wall configuration preserves H_0 as the largest global Lie algebra symmetry, the above gravitationally localized 4D massless scalar modes may be identified as (the 4D zero modes of) the Nambu-Goldstone fields associated to the partial breaking $SU(5)\times Z_2 \to H_0$ [31].

Results similar to the above results have been found in Ref. [32] for the self-gravitating versions of the flat space-time $O(10)$ domain wall braneworlds of Ref. [24]. Thus, in domain wall braneworlds in which the domain wall is used to model spontaneous symmetry breaking of continuous global symmetries, the inclusion of gravity leads to massless scalars localized on its core as the Nambu-Goldstone bosons associated to the broken symmetries.

In view of the results of Refs. [31,32] and being the analysis at the reach of a classical perturbative treatment, for gauge couplings sufficiently small, the obvious next step is to look for the fate of the gravitationally trapped 4D Nambu-Goldstone bosons and the behavior of the gauge fields in 5D self-gravitating Higgs domain walls as braneworlds.

To carry out the above program, in Sec. II, we obtain the gauged versions of the self-gravitating global $SU(5)$ domain walls of Ref. [30]. Next, in Sec. III, after a brief discussion on linear perturbations of the Einstein-Yang-Mills-Higgs system and their behavior under diffeomorphisms and Lie algebra gauge transformations, the linearized field equations for the chosen set of diffeomorphism-invariant and Lie algebra gauge-invariant fluctuations around the domain wall backgrounds of Sec. II are derived.

The dimensional reduction and the analysis of the 4D modes is carried out in Sec. IV. There, we show the absence of tachyonic modes for all these gauge-invariant fluctuations and hence the perturbative stability of the domain wall configurations considered. We show the absence of localizable 4D massless scalar modes. No localizable 4D massless nor 4D massive modes for the gauge field fluctuations are found. We show the existence of quasilocalizable discrete 4D massive modes for the gauge field fluctuations along the spontaneously broken gauge sectors. Related issues to the gauge fixing approach are also discussed. A summary and conclusions are given in Sec. V.

II. SELF-GRAVITATING LOCAL $SU(5)$ DOMAIN WALLS

Let us consider the 5D theory[1]

$$S=\int d^4x dy\sqrt{-g}\left[\frac{1}{2}R-g^{ab}\mathrm{Tr}\{\mathbf{D}_a\mathbf{\Phi}\mathbf{D}_b\mathbf{\Phi}\}-V(\mathbf{\Phi})-\frac{1}{2}g^{ac}g^{bd}\mathrm{Tr}\{\mathbf{F}_{ab}\mathbf{F}_{cd}\}\right], \tag{1}$$

[1]We use units in which $G=c=1$.

where R is the scalar curvature of the metric g_{ab}, $g=\det(g_{ab})$, $\mathbf{\Phi}$ is a scalar field that transforms in the adjoint representation of $SU(5)$,

$$\mathbf{D}_a\mathbf{\Phi}=\nabla_a\mathbf{\Phi}+i\mathsf{g}[\boldsymbol{A}_a,\mathbf{\Phi}] \tag{2}$$

is the gauge-covariant derivative of $\mathbf{\Phi}$ with $\nabla_c g_{ab}=0$,

$$\mathbf{F}_{ab}=\nabla_a\boldsymbol{A}_b-\nabla_b\boldsymbol{A}_a+i\mathsf{g}[\boldsymbol{A}_a,\boldsymbol{A}_b] \tag{3}$$

is the field strength tensor of the gauge field $\boldsymbol{A}_a$, and $V(\mathbf{\Phi})$ a sixth-order potential of the form

$$V(\mathbf{\Phi})=V_0-\mu^2\mathrm{Tr}\{\mathbf{\Phi}^2\}+h(\mathrm{Tr}\{\mathbf{\Phi}^2\})^2+\lambda\mathrm{Tr}\{\mathbf{\Phi}^4\}+\alpha(\mathrm{Tr}\mathbf{\Phi}^2\})^3+\beta(\mathrm{Tr}\{\mathbf{\Phi}^3\})^2+\gamma\mathrm{Tr}\{\mathbf{\Phi}^4\}\mathrm{Tr}\{\mathbf{\Phi}^2\}. \tag{4}$$

Besides being invariant under general space-time diffeomorphisms, the theory (1) is invariant under local $SU(5)$ gauge transformations,

$$\mathbf{\Phi}\mapsto\mathbf{U}\mathbf{\Phi}\mathbf{U}^\dagger,\quad \boldsymbol{A}_a\mapsto\mathbf{U}\boldsymbol{A}_a\mathbf{U}^\dagger+(i/\mathsf{g})(\nabla_a\mathbf{U})\mathbf{U}^\dagger,\quad g_{ab}\mapsto g_{ab}$$

where $\mathbf{U}=\exp\{-i\sigma_q\mathbf{T}^q\}$, with $\sigma_q=\sigma_q(x,y)$ finite functions on space-time such that $\mathbf{U}$ tends to the identity at spatial infinite and $\mathbf{T}^q$, $q=1,\ldots,24$, are traceless Hermitian generators of the Lie algebra $\mathfrak{su}(5)$ of $SU(5)$, normalized so that $\mathrm{Tr}\{\mathbf{T}^q\mathbf{T}^p\}=(1/2)\delta^{qp}$. It is also invariant under

$$Z_2{:}\,\mathbf{\Phi}\mapsto-\mathbf{\Phi}\qquad Z_2\notin SU(5)$$

which leave the gravitational and gauge field sectors invariant.

The field equations, following from (1), are given by

$$R_{ab}-\frac{1}{2}g_{ab}R=T_{ab}, \tag{5}$$

where

$$T_{ab}=2\mathrm{Tr}\{\mathbf{D}_a\mathbf{\Phi}\mathbf{D}_b\mathbf{\Phi}\}-g_{ab}(g^{cd}\mathrm{Tr}\{\mathbf{D}_c\mathbf{\Phi}\mathbf{D}_d\mathbf{\Phi}\}+V(\mathbf{\Phi}))+2\mathrm{Tr}\{\mathbf{F}_{ac}\mathbf{F}_b{}^c\}-\frac{1}{2}g_{ab}\mathrm{Tr}\{\mathbf{F}_{cd}\mathbf{F}^{cd}\}, \tag{6}$$

$$g^{ab}\mathbf{D}_a(\mathbf{D}_b\mathbf{\Phi})=\frac{\partial V(\mathbf{\Phi})}{\partial\phi_q}\mathbf{T}^q,\qquad \mathbf{\Phi}=\phi_q\mathbf{T}^q \tag{7}$$

and

$$\mathbf{D}_a\mathbf{F}^{ab}-i\mathsf{g}[\mathbf{\Phi},\mathbf{D}^b\mathbf{\Phi}]=0. \tag{8}$$

Next, assuming that the geometry preserves 4D-Poincaré invariance, the 5D manifold is endowed with a metric of the form

$$g_{ab} = e^{2A(y)}\eta_{\mu\nu}dx_a^\mu dx_b^\nu + dy_a dy_b, \tag{9}$$

with $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$. Now, in the $\{x^\mu, y\}$ coordinate system, we seek for field configurations $(\tilde{\mathbf{\Phi}}^k, \tilde{\boldsymbol{A}}_a^k; g_{ab}^k)$ such that

$$\tilde{\mathbf{\Phi}}^k(x,y) = \mathbf{U}\mathbf{\Phi}^k(y)\mathbf{U}^\dagger, \qquad \tilde{\boldsymbol{F}}_{ab} = 0. \tag{10}$$

Then, $\tilde{\boldsymbol{A}}_a^k$ is given by a pure gauge,

$$\tilde{\boldsymbol{A}}_a^k(x,y) = +\frac{i}{\mathsf{g}}(\partial_a\mathbf{U})\mathbf{U}^\dagger, \tag{11}$$

and (8) requires

$$[\tilde{\mathbf{\Phi}}^k, \tilde{\mathbf{D}}_b\tilde{\mathbf{\Phi}}^k] = \mathbf{U}[\mathbf{\Phi}^k(y), \nabla_b\mathbf{\Phi}^k(y)]\mathbf{U}^\dagger = 0. \tag{12}$$

Indeed, from the family of Lie algebra gauge equivalent domain wall solutions $(\tilde{\mathbf{\Phi}}^k, \tilde{\boldsymbol{A}}_a^k; g_{ab}^k)$, we can choose a gauge such that

$$(\tilde{\mathbf{\Phi}}^k, \tilde{\boldsymbol{A}}_a^k; g_{ab}^k) \mapsto (\mathbf{\Phi}^k, \mathbf{0}_a; g_{ab}^k).$$

However, the search for analytical solutions $(\mathbf{\Phi}^k, \mathbf{0}_a; g_{ab}^k)$ is still a nontrivial task. For these field configurations, we will restrict ourselves to consider only those completely integrable models that were obtained, for special values of the parameters in the Higgs potential (4), in Ref. [30]. These are given by

$$\mathbf{\Phi}^k(y) = \phi_M(y)\mathbf{M} + \phi_P(y)\mathbf{P}, \tag{13}$$

$$\phi_M(y) = v\tanh by, \qquad \phi_P(y) = v\kappa, \tag{14}$$

where $\mathbf{M}$ and $\mathbf{P}$ are two commuting orthogonal diagonal generators of $\mathfrak{su}(5)$, and g_{ab}^k given by (9) with

$$A(y) = -\frac{v^2}{9}\left[2\ln(\cosh\, by) + \frac{1}{2}\tanh^2 by\right]. \tag{15}$$

The space-time is asymptotically 5D anti-deSitter space with cosmological constant $\Lambda = -8b^2v^4/27$. All the couplings which appear in (4) can be written explicitly in terms of v and b. The choice of $\mathbf{M}$ and $\mathbf{P}$ relies on the asymptotic values of $\mathbf{\Phi}^k$ at $y \to \pm\infty$, which are linked to the possible symmetry breaking patterns, and κ in (14) is a numerical constant that depends on this choice.

As discussed in Ref. [30] (see Refs. [18,19] for the flat space case), by imposing the topologically nontrivial boundary conditions

$$\mathbf{\Phi}_\mathbf{A}^k(+\infty) \sim v\,\mathrm{diag}(3,3,-2,-2,-2),$$
$$\mathbf{\Phi}_\mathbf{A}^k(-\infty) \sim v\,\mathrm{diag}(2,2,-3,-3,2), \tag{16}$$

a spatially dependent symmetry breaking pattern is then obtained, and the unbroken symmetries $H_\pm$ (at $y \to \pm\infty$) and H_0 (at $y = 0$) in it are given by

$$H_\pm^\mathbf{A} = \frac{SU(3)_\pm \times SU(2)_\pm \times U(1)_\pm}{Z_3 \times Z_2}, \tag{17}$$

$$H_0^\mathbf{A} = \frac{SU(2)_+ \times SU(2)_- \times U(1)_M \times U(1)_P}{Z_2 \times Z_2}, \tag{18}$$

with the following embeddings:

$$SU(2)_\mp \subset SU(3)_\pm. \tag{19}$$

On the other hand, for $\mathbf{\Phi}^k$ taking the asymptotic values [30]

$$\mathbf{\Phi}_\mathbf{B}^k(+\infty) \sim v\,\mathrm{diag}(1,1,1,1,-4),$$
$$\mathbf{\Phi}_\mathbf{B}^k(-\infty) \sim v\,\mathrm{diag}(-1,-1,-1,4,-1), \tag{20}$$

$SU(5)$ breaks to

$$H_\pm^\mathbf{B} = \frac{SU(4)_\pm \times U(1)_\pm}{Z_4}, \tag{21}$$

$$H_0^\mathbf{B} = \frac{SU(3) \times U(1)_M \times U(1)_P}{Z_3}, \tag{22}$$

where $SU(3)$ is embedded in different manners in $SU(4)_+$ and $SU(4)_-$.

The domain wall configurations $(\tilde{\mathbf{\Phi}}^k, \tilde{\boldsymbol{A}}_a^k; g_{ab}^k)$ provide regularizations of the Randall-Sundrum braneworld, in which the $SU(5)$ gauge symmetry of the theory (1) is *broken* to a spatially dependent subgroup H. On the core of the wall, the gauge group H_0 is an explicit gauge symmetry, while at $y \to \pm\infty$ (as one approaches the AdS horizons), the explicit gauge group is $H_\pm$, with $H_0 = H_+ \cap H_-$ being differently embedded in H_+ and H_-. Table I summarizes the results for $SU(5)$ domain walls.

It should be noted that, while domain walls in Abelian Z_2-symmetric theories are topologically stable, there is no global stability criterium for the non-Abelian ones. This lead us to resort to perturbative analyses [see Refs. [18,19] for flat space and Ref. [31] for the gravitating global $SU(5)$ cases] to establish at least their perturbative stability. Hence, after a domain wall configuration $(\tilde{\mathbf{\Phi}}^k, \tilde{\boldsymbol{A}}_a^k; g_{ab}^k)$ is found, its perturbative stability, which involves second variations of the action and depends explicitly on the field content of the theory, should be addressed. On the other hand, if the original gauge symmetry is spontaneously broken due to this domain wall configuration, we may

TABLE I. A summary of symmetry breaking patterns arising in the $SU(5)$ adjoint-Higgs domain walls. See the main text for definitions of terms.

	Unbroken subgroup H at specified locatio	
Symmetry breaking	$y = \pm\infty$	$y = 0$
A	$H^A_\pm = SU(3)_\pm \times SU(2)_\pm \times U(1)_\pm/(Z_3 \times Z_2)$	$H^A_0 = SU(2)_+ \times SU(2)_- \times U(1)_M \times U(1)_P/(Z_3 \times Z_2)$
B	$H^B_\pm = SU(4)_\pm \times U(1)_\pm/Z_4$	$H^B_0 = SU(3) \times U(1)_M \times U(1)_P/Z_3$

expect that a Higgs mechanism takes place with some imprints on the 4D modes of the field fluctuations.

III. FLUCTUATIONS OF THE DOMAIN WALL CONFIGURATION

A. Diffeomorphisms, Lie algebra gauge transformations, and fluctuations

For the determination of the stability of the domain wall solutions and the analysis of the gravitationally trapped content on their cores, we shall consider perturbative expansions to first order in the fluctuations around the domain wall background.

Let us briefly review the procedure chosen to obtain the perturbation equations, which is applicable to any covariant field theory. Consider the set

$$\mathcal{E}[\mathbf{\Phi}, \mathbf{A}_a; g_{ab}] = 0 \tag{23}$$

of field equations (5)–(8) of the theory (1), and let $({}^0\mathbf{\Phi}, {}^0\mathbf{A}_a; {}^0g_{ab})$ be a solution of the set $\mathcal{E}$. Now, suppose there exists a one-parameter family of solutions $(\mathbf{\Phi}(\lambda), \mathbf{A}_a(\lambda); g_{ab}(\lambda))$,

$$\mathcal{E}[\mathbf{\Phi}(\lambda), \mathbf{A}(\lambda)_a; g_{ab}(\lambda)] = 0, \tag{24}$$

such that $(\mathbf{\Phi}(0), \mathbf{A}(0)_a; g_{ab}(0)) = ({}^0\mathbf{\Phi}, {}^0\mathbf{A}_a; {}^0g_{ab})$. Provided that suitable differentiability conditions for $\mathcal{E}$ and $(\mathbf{\Phi}(\lambda), \mathbf{A}(\lambda)_a; g_{ab}(\lambda))$ hold, we have

$$\frac{d}{d\lambda}\mathcal{E}[\mathbf{\Phi}(\lambda), \mathbf{A}(\lambda)_a; g_{ab}(\lambda)]\bigg|_{\lambda=0} = 0, \tag{25}$$

comprising a set of linear equations for

$$\boldsymbol{\varphi} = \frac{d}{d\lambda}\mathbf{\Phi}(\lambda)\bigg|_{\lambda=0}, \qquad \boldsymbol{\mathcal{A}}_a = \frac{d}{d\lambda}\mathbf{A}_a(\lambda)\bigg|_{\lambda=0} \tag{26}$$

and

$$h_{ab} = \frac{d}{d\lambda}g_{ab}(\lambda)\bigg|_{\lambda=0}, \tag{27}$$

which are the scalar, vector gauge, and metric fluctuations, respectively, around the background given by $({}^0\mathbf{\Phi}, {}^0\mathbf{A}_a; {}^0g_{ab})$.

Now, from (26) and (27), it follows that under an infinitesimal diffeomorphism

$$x^a \mapsto x^a + \epsilon^a \tag{28}$$

we have

$$\boldsymbol{\varphi} \mapsto \boldsymbol{\varphi} + \pounds_\epsilon {}^0\mathbf{\Phi}, \qquad \boldsymbol{\mathcal{A}}_a \mapsto \boldsymbol{\mathcal{A}}_a + \pounds_\epsilon {}^0\mathbf{A}_a \tag{29}$$

and

$$h_{ab} \mapsto h_{ab} + \pounds_\epsilon {}^0g_{ab}, \tag{30}$$

where $\pounds_\epsilon$ is the Lie derivative with respect to the vector field ϵ^a. The full space-time diffeomorphism invariance of the theory (1) implies that $(\boldsymbol{\varphi}, \mathcal{A}_a, h_{ab})$ and $(\boldsymbol{\varphi} + \pounds_\epsilon {}^0\mathbf{\Phi}, \boldsymbol{\mathcal{A}}_a + \pounds_\epsilon {}^0\boldsymbol{\mathcal{A}}_a, h_{ab} + \pounds_\epsilon {}^0g_{ab})$ describe the same physical perturbations.

On the other hand, Eq. (1) is also invariant under Lie algebra gauge transformations. It follows from (26) and (27) that under infinitesimal Lie algebra gauge transformations we have

$$\boldsymbol{\varphi} \mapsto \boldsymbol{\varphi} - i[\boldsymbol{\sigma}, {}^0\mathbf{\Phi}], \qquad \boldsymbol{\mathcal{A}}_a \mapsto \boldsymbol{\mathcal{A}}_a + \frac{1}{\mathsf{g}}{}^0\mathbf{D}_a\boldsymbol{\sigma} \tag{31}$$

and

$$h_{ab} \mapsto h_{ab}, \tag{32}$$

where $\boldsymbol{\sigma}$ is a Lie algebra valued scalar field parametrizing the gauge freedom and ${}^0\mathbf{D}_a$ is the gauge covariant derivative with respect the background gauge field ${}^0\mathbf{A}_a$.

B. (4 + 1) decomposition of the fluctuations

For a background $({}^0\mathbf{\Phi}, {}^0\mathbf{A}_a; {}^0g_{ab})$ that preserves 4D-Poincaré invariance, it is convenient decompose h_{ab} as [33]

$$h_{ab} = 2e^{2A}(h^{TT}_{\mu\nu} + \partial_{(\mu}f_{\nu)} + \eta_{\mu\nu}\psi + \partial_\mu\partial_\nu E)dx^\mu_a dx^\nu_b + e^A(D_\mu + \partial_\mu C)(dx^\mu_a dy_b + dy_a dx^\mu_b) + 2\omega dy_a dy_b, \tag{33}$$

where

$$h^{TT\mu}_\mu = 0, \qquad \partial^\mu h^{TT}_{\mu\nu} = 0 \tag{34}$$

and

$$\partial^\mu f_\mu = 0, \qquad \partial^\mu D_\mu = 0. \tag{35}$$

We may also set

$$\boldsymbol{\mathcal{A}}_a = \boldsymbol{\mathcal{A}}_\mu dx_a^\mu + \boldsymbol{\mathcal{A}}_y dy_a. \tag{36}$$

Now, for an infinitesimal diffeomorphism (28) of the form

$$\epsilon_a = e^{2A}\epsilon_\mu dx_a^\mu + \epsilon_y dy_a, \tag{37}$$

where

$$\epsilon_\mu = \partial_\mu \epsilon + \zeta_\mu, \qquad \partial^\mu \zeta_\mu = 0, \tag{38}$$

we have that (30) induces the transformations

$$\psi \mapsto \psi - A'\epsilon_y, \qquad \omega \mapsto \omega + \partial_y \epsilon_y, \tag{39}$$

$$E \mapsto E - \epsilon, \qquad C \mapsto C - e^A \partial_y \epsilon + e^{-A}\epsilon_y, \tag{40}$$

$$D_\mu \mapsto D_\mu - e^A \partial_y \zeta_\mu, \qquad f_\mu \mapsto f_\mu - \zeta_\mu, \tag{41}$$

and

$$h_{\mu\nu}^{TT} \mapsto h_{\mu\nu}^{TT}, \tag{42}$$

where a prime ($'$) denotes the derivative with respect to y. Indeed, at this point, all these fields depend not only on the point x^μ in the 4-space but also on the coordinate y along the additional dimension.

As follows from (42), $h_{\mu\nu}^{TT}$ is automatically diffeomorphism invariant. The next step is to complete an appropriate set of quantities that are invariant under infinitesimal diffeomorphisms. One may use the above transformations to construct the vector field u^a given by

$$u^a \equiv (\partial^\mu E + f^\mu)\partial_\mu^a + (e^{2A}E' - e^A C)\partial_y^a, \tag{43}$$

which, under an infinitesimal diffeomorphism (28) of the form (37), (38), transforms as

$$u^a \mapsto u^a - \epsilon^a. \tag{44}$$

Hence, since $£_u$ is linear with respect to u^a, the quantities

$$h_{ab}^{\text{inv}} \equiv h_{ab} + £_u{}^0 g_{ab}, \tag{45}$$

$$\boldsymbol{\varphi}^{\text{inv}} \equiv \boldsymbol{\varphi} + £_u{}^0 \boldsymbol{\Phi}, \tag{46}$$

and

$$\boldsymbol{\mathcal{A}}_a^{\text{inv}} \equiv \boldsymbol{\mathcal{A}}_a + £_u{}^0 \boldsymbol{\mathcal{A}}_a \tag{47}$$

are invariant under an infinitesimal diffeomorphism (28), (37), (38).

In particular, from (45), we find that

$$h_{ab}^{\text{inv}} = 2e^{2A}(h_{\mu\nu}^{TT} + \eta_{\mu\nu}\psi^{\text{inv}})dx_a^\mu dx_b^\nu + e^A D_\mu^{\text{inv}}(dx_a^\mu dy_b + dy_a dx_b^\mu) + 2\omega^{\text{inv}} dy_a dy_b, \tag{48}$$

where

$$\psi^{\text{inv}} \equiv \psi - A'(e^{2A}E' - e^A C), \tag{49}$$

$$D_\mu^{\text{inv}} \equiv D_\mu - e^A f_\mu'. \tag{50}$$

and

$$\omega^{\text{inv}} \equiv \omega + (e^{2A}E' - e^A C)'. \tag{51}$$

Notice that in the generalized longitudinal gauge, $E = C = 0$ and $f_\mu = 0$, the freedom of the coordinate transformations (28), (37), (38) is completely fixed and the diffeomorphism-invariant fluctuations coincide with the original ones, i.e.,

$$h_{ab}^{\text{inv}} = h_{ab}, \qquad \boldsymbol{\varphi}^{\text{inv}} = \boldsymbol{\varphi} \qquad \boldsymbol{\mathcal{A}}_a^{\text{inv}} = \boldsymbol{\mathcal{A}}_a.$$

Thus, in the generalized longitudinal gauge, the evolution equations satisfied by the field fluctuations h_{ab}, $\boldsymbol{\varphi}$, and $\boldsymbol{\mathcal{A}}_a$ also hold for the diffeomorphism-invariant fluctuations h_{ab}^{inv}, $\boldsymbol{\varphi}^{\text{inv}}$, and $\boldsymbol{\mathcal{A}}_a{}^{\text{inv}}$. Since only diffeomorphism-invariant fluctuations will be considered, we shall in the following drop the superscript inv on these.

On the other hand, as follows from the Lie algebra gauge invariance of the theory (31), (32), the field fluctuations $(\boldsymbol{\varphi}, \boldsymbol{\mathcal{A}}_a, h_{ab})$ and $(\boldsymbol{\varphi} - i[\boldsymbol{\sigma}, \boldsymbol{\Phi}^k], \boldsymbol{\mathcal{A}}_a + \mathsf{g}^{-1}\nabla_a \boldsymbol{\sigma}, h_{ab})$, with $\boldsymbol{\sigma}$ a Lie algebra valued scalar field parametrizing the gauge freedom, describe the same physical perturbations. When gauge field localization on domain walls is discussed, it is often considered a gauge fixing in which the extra dimension component $\boldsymbol{\mathcal{A}}_y$ of the gauge field $\boldsymbol{\mathcal{A}}_a$ vanishes. Here, we will consider instead field fluctuations that do not change under Lie algebra gauge transformations, since in terms of these the obtained results will be independent of any gauge fixing. In the following, the dependence on the additional coordinate will be expressed in the conformal coordinate z,

$$dy_a = e^{A(z)} dz_a, \tag{52}$$

such that

$$g_{ab}^k = e^{2A(z)}(\eta_{\mu\nu} dx_a^\mu dx_b^\nu + dz_a dz_b). \tag{53}$$

Let $\boldsymbol{\mathcal{A}}_a$ be the gauge vector fluctuation (36) with

$$\boldsymbol{\mathcal{A}}_\mu = e^{-A/2}\boldsymbol{a}_\mu + \boldsymbol{\mathcal{A}}_\mu^L, \tag{54}$$

where

$$\partial^\mu \boldsymbol{a}_\mu = 0, \qquad \boldsymbol{\mathcal{A}}^L_\mu = \partial_\mu \boldsymbol{\chi}. \tag{55}$$

The Lie algebra gauge- and diffeomorphism-invariant fluctuations we use are $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{a}_\mu$, where

$$\alpha \equiv \boldsymbol{\varphi} + i\mathsf{g}[\boldsymbol{\chi}, \boldsymbol{\Phi}^k], \tag{56}$$

and

$$\boldsymbol{\beta} \equiv e^{-A(z)/2}(\boldsymbol{\mathcal{A}}_z - \partial_z \boldsymbol{\chi}), \tag{57}$$

together with $h^{TT}_{\mu\nu}$, D_μ, ψ, and ω, which are unchanged under a Lie algebra gauge transformation (32).

C. Linearized perturbation equations

Making the $(4+1)$ decomposition discussed in the previous subsection, from the set of linearized field equations for the fluctuations $\boldsymbol{\varphi}$, $\boldsymbol{\mathcal{A}}_a$, and h_{ab} around the domain wall background (see the Appendix A), we obtain the field equations for the chosen set of diffeomorphism- and Lie algebra gauge-invariant fluctuations. We find (where now and in the following a prime denotes the derivative with respect to z)

$$(\partial^\rho\partial_\rho + 3A'\partial_z + \partial_z^2)h^{TT}_{\mu\nu} = 0, \tag{58}$$

$$\partial_{(\mu}(\partial_z + 3A')D_{\nu)} = 0, \qquad \partial^\nu\partial_\nu D_\mu = 0, \tag{59}$$

$$-(\partial^\rho\partial_\rho\psi + \partial_z^2\psi + 7A'\partial_z\psi) + (6A'^2 + 2A'')\omega + A'\partial_z\omega = e^{2A}\frac{2}{3}\frac{\partial V(\boldsymbol{\Phi})}{\partial\phi_q}\bigg|_{\boldsymbol{\Phi}^k}\varphi_q, \tag{60}$$

$$-(\partial^\mu\partial_\mu\omega + (6A'^2 + 2A'')\omega + 4A'\partial_z\omega) - 2\phi'_M\partial_z\varphi_M - 4(A'\partial_z\psi + \partial_z^2\psi) = e^{2A}\frac{2}{3}\frac{\partial V(\boldsymbol{\Phi})}{\partial\phi_q}\bigg|_{\boldsymbol{\Phi}^k}\varphi_q, \tag{61}$$

and the two constraints

$$\partial_\mu(\omega + 2\psi) = 0, \qquad \partial_\mu(3A'\omega - 3\partial_z\psi - \phi'_M\varphi_M) = 0. \tag{62}$$

Also, we find

$$e^{-2A}\partial^\mu\partial_\mu\alpha + e^{-5A}\partial_z(e^{3A}\partial_z\alpha) - \frac{\partial^2 V(\boldsymbol{\Phi})}{\partial\phi_p\partial\phi_q}\bigg|_{\boldsymbol{\Phi}^k}\alpha_p\mathbf{T}^q + 4e^{-2A}\partial_z\boldsymbol{\Phi}^k\partial_z\psi - 2e^{-5A}\partial_z(e^{3A}\partial_z\boldsymbol{\Phi}^k)\omega - e^{-2A}\partial_z\boldsymbol{\Phi}^k\partial_z\omega = -2i\mathsf{g}e^{-2A}[\boldsymbol{\beta}, \partial_z\boldsymbol{\Phi}^k] - ige^{-5A}[\partial_z(e^{3A}\boldsymbol{\beta}), \boldsymbol{\Phi}^k], \tag{63}$$

$$e^{-2A}\partial^\mu\partial_\mu\boldsymbol{\beta} - (M^2)^{qp}\beta_p\mathbf{T}^q = i\mathsf{g}[\boldsymbol{\Phi}^k, \partial_z\alpha] + i\mathsf{g}[\boldsymbol{\alpha}, \partial_z\boldsymbol{\Phi}^k], \tag{64}$$

with the constraint

$$e^{-3A}\partial_z(e^A\boldsymbol{\beta}) = -i\mathsf{g}[\boldsymbol{\Phi}^k, \boldsymbol{\alpha}], \tag{65}$$

and

$$e^{-2A}\left(\partial^\mu\partial_\mu\boldsymbol{a}_\nu - \left(\frac{1}{4}A'^2 + \frac{1}{2}A''\right)\boldsymbol{a}_\nu + \partial_z^2\boldsymbol{a}_\nu\right) - (M^2)^{qp}(\boldsymbol{a}_\nu)_p\mathbf{T}^q = 0, \tag{66}$$

where the matrix M^2 is given by

$$(M^2)^{qp} = -2\mathsf{g}^2\mathrm{Tr}\{[\mathbf{T}^q, \boldsymbol{\Phi}^k(y)][\mathbf{T}^p, \boldsymbol{\Phi}^k(y)]\}. \tag{67}$$

As follows from (58), (59), and (66), the fluctuations $h^{TT}_{\mu\nu}$, D_μ and $\mathbf{a}_\mu$ decouple from each other as also from the rest of the field fluctuations, with $h^{TT}_{\mu\nu}$ and D_μ behaving as their corresponding analogous ones in the global $SU(5) \times Z_2$ [31] and standard Abelian Z_2 [33] domain walls. It follows that the tensor perturbation $h^{TT}_{\mu\nu}$ has no tachyonic modes that destabilize the domain wall background, and there is a normalizable massless mode that gives rise to 4D gravity on the core of the wall and a tower of non-normalizable massive Kaluza-Klein (KK) modes that propagate in the bulk. There is no localized vector fluctuation D_μ. We refer the reader to Ref. [31] and references therein for a detailed discussion. In the following, we will restrict ourselves to the analysis of the scalar field fluctuations ψ, ω, α, and $\boldsymbol{\beta}$ and the vector field fluctuation $\boldsymbol{a}_\mu$.

D. Lie algebra decomposition of the fluctuations

Let us consider the Cartan decomposition of the $\mathfrak{su}(5)$ Lie algebra

$$\mathfrak{su}(5) = \mathcal{K} \oplus \mathcal{K}^\perp, \tag{68}$$

where the Lie subalgebra $\mathcal{K}$ is given by

$$\mathcal{K} = \{\mathbf{T}_0\} \oplus \{\mathbf{T}_{\mathrm{br}}\} \tag{69}$$

with the subsets $\{\mathbf{T}_0\}$ and $\{\mathbf{T}_{\mathrm{br}}\}$ defined by

$$[\mathbf{T}^q_0, \boldsymbol{\Phi}^k(y)] = 0, \quad q = 1, \ldots, n_0 \tag{70}$$

and

$$[\mathbf{T}^q_{\mathrm{br}}, \boldsymbol{\Phi}^k(y)] \neq 0, \quad q = 1, \ldots, n_{\mathrm{br}}. \tag{71}$$

The orthogonal complement $\mathcal{K}^\perp$ is given by

$$\mathcal{K}^\perp = \{\mathbf{T}_+\} \oplus \{\mathbf{T}_-\}, \tag{72}$$

where

$$[\mathbf{T}_+^q, \mathbf{\Phi}^k(\infty)] = 0, \quad [\mathbf{T}_+^q, \mathbf{\Phi}^k(-\infty)] \neq 0, \quad q = 1, \ldots, n_+ \tag{73}$$

and

$$[\mathbf{T}_-^q, \mathbf{\Phi}^k(-\infty)] = 0, \quad [\mathbf{T}_-^q, \mathbf{\Phi}^k(\infty)] \neq 0, \quad q = 1, \ldots, n_-. \tag{74}$$

We have $n_0 = 8$, $n_{\rm br} = 8$, and $n_+ = n_- = 4$ for the symmetry breaking **A**. On the other hand, $n_0 = 10$, $n_{\rm br} = 2$, and $n_+ = n_- = 6$ for the symmetry breaking **B** (see Table I).

Let $\partial^2 V(\mathbf{\Phi})/\partial\phi_q^2|_{\mathbf{\Phi}^k}^{\{i\}}$ be the restriction of the Hessian of $V(\mathbf{\Phi})$ at $\mathbf{\Phi}^k$ to the subspace spanned by the subset $\{\mathbf{T}_i\}$. First, let us consider $\partial^2 V(\mathbf{\Phi})/\partial\phi_q^2|_{\mathbf{\Phi}^k}^{\{0\}}$. For $q = M$, we find

$$\left.\frac{\partial^2 V(\mathbf{\Phi})}{\partial{\phi_M}^2}\right|_{\mathbf{\Phi}^k}^{\{0\}} = -2b^2\left(1+\frac{2}{3}v^2\right) + b^2F^2\left(6+\frac{4}{3}v^2\left(4-\frac{5}{3}F^2\right)\right), \tag{75}$$

where

$$F = \tanh by, \qquad y = y(z), \tag{76}$$

and for $q = P$, we have

$$\left.\frac{\partial^2 V(\mathbf{\Phi})}{\partial{\phi_P}^2}\right|_{\mathbf{\Phi}^k}^{\{0\}} = 4b^2\left(1+\frac{4}{9}v^2\right) \equiv \mathcal{M}_H^2, \tag{77}$$

in both symmetry breaking patterns.

For q such that $\mathbf{T}_0^q \neq \mathbf{M}$, $\mathbf{P}$, we find for the symmetry breaking **A**

$$\left.\frac{\partial^2 V(\mathbf{\Phi})}{\partial{\phi_q}^2}\right|_{\mathbf{\Phi}^k}^{\{0\}} = b^2\left(10+\frac{20}{9}v^2\right) - b^2F\left(\frac{4}{3}v^2F \mp \left(6-\frac{4}{9}v^2(F^2+1)\right)\right), \tag{78}$$

where the $\pm$ signs correspond to the two different $SU(2)$ in $H_0^{\mathbf{A}}$. In the symmetry breaking **B**, we find

$$\left.\frac{\partial^2 V(\mathbf{\Phi})}{\partial{\phi_q}^2}\right|_{\mathbf{\Phi}^k}^{\{0\}} = b^2\left(\frac{5}{2}+\frac{5}{4}v^2\right) + b^2F^2\left(\frac{3}{2}+\frac{1}{6}v^2\left(5-\frac{11}{6}F^2\right)\right). \tag{79}$$

Next, along the subset $\{\mathbf{T}_{\rm br}\}$, we find

$$\left.\frac{\partial^2 V(\mathbf{\Phi})}{\partial{\phi_q}^2}\right|_{\mathbf{\Phi}^k}^{\{\rm br\}} = 0, \tag{80}$$

in both symmetry breaking patterns. Along the subsets $\{\mathbf{T}_+\}$ and $\{\mathbf{T}_-\}$, we have

$$\left.\frac{\partial^2 V(\mathbf{\Phi})}{\partial{\phi_q}^2}\right|_{\mathbf{\Phi}^k}^{\{\pm\}} = 2b^2F\left(1+\frac{2}{3}v^2\left(1-\frac{1}{3}F^2\right)\right)(F\pm 1) \tag{81}$$

for both symmetry breaking patterns. The field fluctuations $\boldsymbol{\varphi} \in \{\mathbf{T}_{\rm br}\}$ are 5D Nambu-Goldstone bosons, while the remaining scalar field fluctuations correspond to the 5D massive scalar fields (in general, with y-dependent *masses*) of the spontaneously broken gauge theory.

Now, let $(M^2)^{\{i\}}$ be the restriction of M^2, as given by (67), to the subspace spanned by the subset $\{\mathbf{T}_i\}$. We have

$$(M^2)^{\{i\}} = \mathcal{M}_i^2 \mathbb{I}^{\{i\}}, \tag{82}$$

where $\mathbb{I}^{\{i\}}$ is the identity matrix of dimension $n_i \times n_i$,

$$\mathcal{M}_0^2 = 0, \tag{83}$$

$$\mathcal{M}_{\rm br}^2 = \mathcal{M}_W^2, \qquad \mathcal{M}_W^2 \equiv \frac{5}{2}v^2\mathsf{g}^2, \tag{84}$$

and

$$\mathcal{M}_\pm^2 = \frac{1}{4}\mathcal{M}_W^2(1\mp F)^2, \tag{85}$$

for the symmetry breaking **A**, with essentially the same results for the symmetry breaking **B** differing only in numerical factors. $\mathcal{M}_i^2$ are the gauge boson 5D masses generated through the Higgs mechanism.

IV. DIMENSIONAL REDUCTION

A. Fluctuations along $\{\mathbf{T_0}\}$

The gauge-invariant fluctuations $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{a}_\mu) \in \{\mathbf{T}_0\}$ decouple from each other and $\boldsymbol{\alpha} = \boldsymbol{\varphi}$, i.e., the *wall fluctuations* $\boldsymbol{\varphi}$ are gauge invariant in this sector. Let us recall that the theory considered maintains an explicit H_0 gauge symmetry. On the other hand, the *gravitational* fluctuations ψ and ω mix with α. We find the constraints

$$2\psi + \omega = 0 \qquad 3A'\omega - 3\partial_z\psi - {\phi'}_M\alpha_M = 0. \tag{86}$$

Hence, ψ, ω, and α_M are not independent and correspond to a single physical scalar fluctuation. Note that α_M in (86) is associated to a $U(1)$ factor of the subgroup H_0.

Let Ξ be the scalar fluctuation defined as

$$\Xi \equiv e^{3A/2}\left(\boldsymbol{\alpha} - \frac{\psi}{A'}(\mathbf{\Phi}^k)'\right). \tag{87}$$

From (63), the KK modes $\Xi_q(x,z) \sim e^{ip\cdot x}\Xi_q(z)$, with $\Xi \in \{\mathbf{T}_0\}$, satisfy q-dependent Schrödinger-like equations,

$$(-\partial_z^2 + V_q^{\{0\}})\Xi_q = m^2 \Xi_q, \tag{88}$$

where $V_q^{\{0\}}$ is given by

$$V_q^{\{0\}} = V_{Q_1} + e^{2A} \frac{\partial^2 V(\mathbf{\Phi})}{\partial \phi_q^2}\bigg|_{\mathbf{\Phi}^k}^{\{0\}}, \tag{89}$$

with V_{Q_1} given by

$$V_{Q_1} = \frac{9}{4}A'^2 + \frac{3}{2}A'' \tag{90}$$

and $p_\mu p^\mu = -m^2$.

Now, the Schrödinger operator in (88) can be rewritten as

$$-\partial_z^2 + V_q^{\{0\}} = \left(\partial_z + \frac{3}{2}A'\right)\left(-\partial_z + \frac{3}{2}A'\right) + e^{2A} \frac{\partial^2 V(\mathbf{\Phi})}{\partial \phi_q^2}\bigg|_{\mathbf{\Phi}^k}^{\{0\}}, \tag{91}$$

where the first term of the right-hand side is a non-negative definite operator on normalizable functions. If the second term is never negative, the eigenvalues of (88) are always non-negative, and there are no tachyonic modes.

For $q = M$, $\partial^2 V/\partial \phi_M^2|_{\mathbf{\Phi}^k}^{\{0\}}$ is given by (75), which is no positive definite. However, $V_M^{\{0\}}$ can be written entirely in terms of the derivatives of ϕ_M and A using their equations of motion [31]. In this case, the Schrödinger operator can be rewritten as

$$(-\partial_z^2 + V_M^{\{0\}}) = (\partial_z + (\ln Z)')(-\partial_z + (\ln Z)'), \tag{92}$$

where

$$Z(z) = e^{3A/2} \frac{\phi_M'}{A'}. \tag{93}$$

Hence, the corresponding Schrödinger-like equation admits no mode with $m^2 < 0$, a non-normalizable massless solution $\Xi_M^0(z) \propto Z(z)$ (since $Z(z)$ is not bounded for $z \to 0$) and a tower of continuum states with $m^2 > 0$. Since $V_M^{\{0\}}$ has the shape of a symmetric potential barrier of infinite height, the continuum of massive modes behaves as waves that propagates in the bulk being repelled off the core of the wall. The behavior of the scalar fluctuation Ξ_M close parallels the one of the scalar fluctuations associated to the standard Abelian Z_2 kink [33].

Next, let us consider the scalar perturbations Ξ_q along the generators $\mathbf{T}_0^q$ other than $\mathbf{M}$. For the scalar fluctuation Ξ_P along the generator $\mathbf{P}$, associated to the $U(1)_P$ in H_0, $\partial^2 V/\partial \phi_P^2|_{\mathbf{\Phi}^k}^{\{0\}}$ is a positive definite constant $\mathcal{M}_H^2$ given by (77), and $V_P^{\{0\}}$ is everywhere positive with the shape of a symmetric potential barrier of finite height. Hence, $V_P^{\{0\}}$ does not support bound states with $m^2 \le 0$, while those modes with $m^2 > 0$ behave as scattered waves by the wall.

For $q \neq M$, P, $\partial^2 V/\partial \phi_q^2|_{\mathbf{\Phi}^k}^{\{0\}}$ is given by (78) for the symmetry breaking **A** and (79) for the symmetry breaking **B**. In both symmetry breaking patterns, $\partial^2 V/\partial \phi_q^2|_{\mathbf{\Phi}^k}^{\{0\}}$ is positive definite, and $V_q^{\{0\}}$ has the shape of a symmetric potential barrier of finite height; therefore, Eq. (91) does not support bound states with $m^2 \le 0$.

For $\boldsymbol{\beta}$, from (64) and (65), we find

$$e^{3A}\partial_z(e^{3A/2}\boldsymbol{\beta}) = 0 = e^{-3A/2}\partial^\mu \partial_\mu \boldsymbol{\beta}, \tag{94}$$

i.e., $\boldsymbol{\beta} = 0$. Therefore, $\boldsymbol{\mathcal{A}}_z = \partial_z \boldsymbol{\chi}$.

Finally, let us consider the gauge vector fluctuations $\boldsymbol{a}_\mu$. From (66), it follows that the modes $\boldsymbol{a}_\mu(x,z) \sim e^{ip\cdot x}\boldsymbol{a}_\mu(z)$, with $\boldsymbol{a}_\mu \in \{\mathbf{T}_0\}$, satisfy the Schrödinger-like equation

$$(-\partial_z^2 + \mathcal{V}_1)\boldsymbol{a}_\mu = m^2 \boldsymbol{a}_\mu, \tag{95}$$

where

$$\mathcal{V}_1 = \frac{1}{4}A'^2 + \frac{1}{2}A'' \tag{96}$$

and $p_\mu p^\mu = -m^2$. In this case, the Schrödinger operator can be factorized as

$$\left(-\partial_z^2 + \frac{1}{4}A'^2 + \frac{1}{2}A''\right) = \left(\partial_z + \frac{1}{2}A'\right)\left(-\partial_z + \frac{1}{2}A'\right).$$

Hence, Eq. (95) admits no modes with $m^2 < 0$, a massless mode

$$(\boldsymbol{a}_\mu)_q^0 \sim e^{A/2}\varepsilon_\mu, \qquad p^\mu \varepsilon_\mu = 0, \tag{97}$$

and a tower of massive modes that propagates in the bulk.

Note that the massless mode corresponds to a constant $\boldsymbol{\mathcal{A}}_\mu$ along the additional coordinate. But, as is well known from the Z_2-symmetric standard kink, a constant zero mode of a vector field does not gives rise to a localizable 4D massless vector field [6]. To examine the massive modes, let us suppose a higher enough AdS_5 curvature Λ such that we can approximate $A(z)$ by its thin wall or *brane* limit[2]

$$A(z) \sim -\ln(1 + k|z|), \qquad k \equiv (2v^2/9)|b| = \sqrt{-\Lambda/6}. \tag{98}$$

For $m^2 \neq 0$, in this approximation, we find for the even modes

[2]The thin wall limit can be defined as the limit $b \to \infty$ and $v \to 0$, while $|\Lambda| = 8b^2v^4/27$ is kept finite.

$$(\boldsymbol{a}_\mu)_q \sim \xi^{1/2}\left[Y_1(m\xi) - \frac{Y_0(m/k)}{J_0(m/k)}J_1(m\xi)\right]\varepsilon_\mu, \quad (99)$$

where

$$\xi = \xi(z) = k^{-1} + |z|, \quad (100)$$

with J_ν and Y_ν Bessel functions of order ν. The odd modes are given by

$$(\boldsymbol{a}_\mu)_q \sim \xi^{1/2}\left[Y_1(m\xi) - \frac{Y_1(m/k)}{J_1(m/k)}J_1(m\xi)\right]\varepsilon_\mu, \qquad z > 0, \quad (101)$$

with $\boldsymbol{a}_\mu(z) = -\boldsymbol{a}_\mu(-z)$ for $z < 0$. Since the odd modes have a zero at the brane's position, their derivative is continuous at $z = 0$, and in this sense, they are unaffected by the brane. In the next subsections, whenever the Schrödinger operators are invariant under $z \to -z$, we shall restrict the discussion to the even modes.

Summarizing, for the field fluctuations that lie along $\{\mathbf{T}_0\}$, we find that $\boldsymbol{\beta}$ does not propagate and the absence of modes with masses $m^2 < 0$ for $\boldsymbol{\Xi}$ and $\boldsymbol{a}_\mu$ implies the perturbative stability of the domain wall configuration in this sector. Additionally, we find that the zero modes of Ξ and $\boldsymbol{a}_\mu$ do not generate localizable 4D massless modes on the core of the wall while their massive ones propagate in the bulk being scattered by the wall.

B. Fluctuations along $\{\mathbf{T}_{\mathbf{br}}\}$

As shown in Ref. [31], the $n_{\rm br}$ field fluctuations $\boldsymbol{\varphi} \in \{\mathbf{T}_{\rm br}\}$ would be 5D Nambu-Goldstone bosons if the symmetry were global rather than local, and their 4D massless modes would be gravitationally trapped on the core of the wall. In the gauged model, all these scalar fluctuations can be gauged away by fixing $\boldsymbol{\varphi} = 0$ (see Appendix B), and we are left with $n_{\rm br}$ vector field fluctuations $\boldsymbol{\mathcal{A}}_a \in \{\mathbf{T}_{\rm br}\}$, all with the same 5D mass $\mathcal{M}_W$ (84) generated through a spontaneous gauge symmetry breaking.

Now, for the Lie algebra gauge-invariant fluctuations $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{a}_\mu) \in \{\mathbf{T}_{\rm br}\}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are not independent (65). We find

$$\boldsymbol{\alpha} = i\mathsf{g}(\mathcal{M}_W^2)^{-1}[\boldsymbol{\Phi}^k, e^{-3A}\partial_z(e^{3A/2}\boldsymbol{\beta})]; \quad (102)$$

i.e., they correspond to a single physical perturbation, with the field fluctuation $\boldsymbol{\beta}$ satisfying

$$\partial^\mu\partial_\mu\boldsymbol{\beta} + \left(\frac{3}{2}A'' - \frac{9}{4}A'^2 - e^{2A}\mathcal{M}_W^2\right)\boldsymbol{\beta} + \partial_z^2\boldsymbol{\beta} = 0, \quad (103)$$

where $\mathcal{M}_W^2$ is given by (84). We see that there is a nontrivial mixing between the original fluctuations $\boldsymbol{\varphi}$, $\boldsymbol{\mathcal{A}}_z$, and $\boldsymbol{\chi}$.

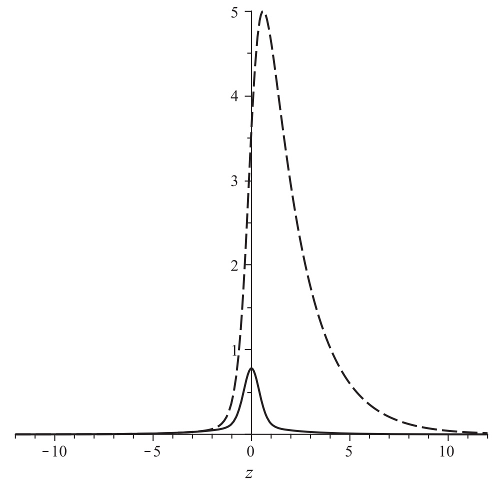

FIG. 1. Schrödinger potentials $\mathcal{V}_2$ (solid line) and $\mathcal{V}_{4+}$ (dashed line) for the modes $\boldsymbol{\beta} \in \{\mathbf{T}_{\rm br}\}$ and $\Omega \in \{\mathbf{T}_+\}$, respectively. $\mathcal{V}_{4-}$ for the modes $\boldsymbol{\Omega} \in \{\mathbf{T}_-\}$ is the mirror image of $\mathcal{V}_{4+}$.

The modes $\boldsymbol{\beta}(x,z) \sim e^{ip\cdot x}\boldsymbol{\beta}(z)$ with $\boldsymbol{\beta} \in \{\mathbf{T}_{\rm br}\}$ satisfy the Schrödinger-like equation

$$(-\partial_z^2 + \mathcal{V}_2)\boldsymbol{\beta} = m^2\boldsymbol{\beta}, \quad (104)$$

where

$$\mathcal{V}_2 = \frac{9}{4}A'^2 - \frac{3}{2}A'' + e^{2A}\mathcal{M}_W^2. \quad (105)$$

The Schrödinger operator in (104) can be rewritten as

$$(-\partial_z^2 + \mathcal{V}_2) = \left(\partial_z - \frac{3}{2}A'\right)\left(-\partial_z - \frac{3}{2}A'\right) + e^{2A}\mathcal{M}_W^2,$$

where the first term is a non-negative definite operator on normalizable functions and the second term is never negative. Therefore, the eigenvalues of (104) are always non-negative, and there are no tachyonic modes. In fact, $\mathcal{V}_2$ is everywhere positive with the shape of a symmetric potential barrier of finite height that vanishes asymptotically at $|z| \to \infty$ (see Fig. 1). Hence, Eq. (104) does not support bound states with $m^2 \leq 0$, while those modes with $m^2 > 0$ behave as scattered waves by the wall. From these results and (102), it follows that $\boldsymbol{\alpha}$ has no zero modes either.

Notice that the eventually large values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as $z \to \pm\infty$ mean that the perturbative theory is not trusted as we move far away from the core of the wall. Indeed, we must specify the behavior of fields at the AdS_5 boundary at $z = \pm\infty$ in order to pick out those solutions that are suitable for the description of the physical situation. For $\boldsymbol{\beta} \to 0$ at large $|z|$, it is not straightforward to prove that $\boldsymbol{\alpha}$ is bounded at $|z| \to \infty$ since (102) is rather involved. However, we can make use of the underlying Lie algebra gauge symmetry of the theory to prove the reliability of the perturbative expansion. Thus, by fixing $\boldsymbol{\varphi} = 0$, it can be

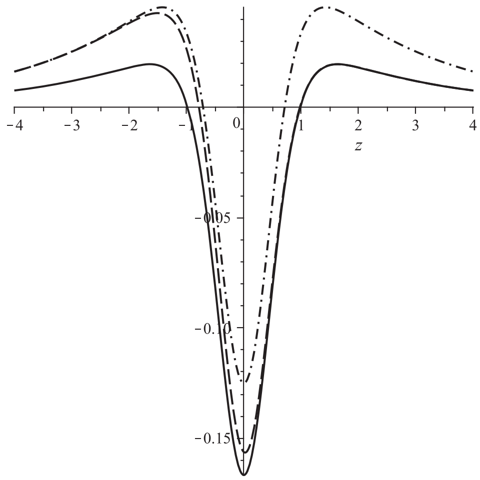

FIG. 2. Schrödinger potentials $\mathcal{V}_1$ (solid line), $\mathcal{V}_3$ (dotted-dashed line), and $\mathcal{V}_{5+}$ (dashed line) for the $\boldsymbol{a}_\mu$ modes along $\{\mathbf{T}_0\}$, $\{\mathbf{T}_{\rm br}\}$, and $\{\mathbf{T}_+\}$, respectively. $\mathcal{V}_{5-}$ for the modes $\boldsymbol{a}_\mu \in \{\mathbf{T}_-\}$ is the mirror image of $\mathcal{V}_{5+}$. As discussed in the main text, $\mathcal{V}_3$, $\mathcal{V}_{5+}$, and $\mathcal{V}_{5-}$ support metastable confinement, while $\mathcal{V}_1$ does not.

shown that the 4D massive modes $(\boldsymbol{\chi}, \boldsymbol{\mathcal{A}}_z) \in \{\mathbf{T}_{\rm br}\}$ go to zero for large $|z|$ (see Appendix B). Since $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ are gauge invariant, we find from the above results and (56) and (57) that the massive modes of $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \{\mathbf{T}_{\rm br}\}$ go to zero also at $|z| \to \infty$.

Next, let us consider the gauge-invariant fluctuations $\boldsymbol{a}_\mu$. We find that the modes $\boldsymbol{a}_\mu(x,z) \sim e^{ip\cdot x}\boldsymbol{a}_\mu(z)$ satisfy

$$(-\partial_z^2 + \mathcal{V}_3)\boldsymbol{a}_\mu = m^2 \boldsymbol{a}_\mu, \tag{106}$$

where

$$\mathcal{V}_3 = \mathcal{V}_1 + e^{2A}\mathcal{M}_W^2, \tag{107}$$

with $\mathcal{V}_1$ given by (96). The Schrödinger operator in (106) can be rewritten as

$$(-\partial_z^2 + \mathcal{V}_3) = \left(\partial_z + \frac{1}{2}A'\right)\left(-\partial_z + \frac{1}{2}A'\right) + e^{2A}\mathcal{M}_W^2,$$

where the first term is a non-negative definite operator and the second term is never negative. Therefore, the eigenvalues of (106) are non-negative definite, and there are no tachyonic modes. For $\mathsf{g}^2 < b^2/15$, $\mathcal{V}_3$ has a volcanolike profile (see Fig. 2), and one might naively expect that it supports a massless mode and a continuum of massive modes that propagates in the bulk. But, as is well known, a bulk mass term for a vector field does not allow for a 4D massless mode localized on the core of the wall [15,16]. Since $\mathcal{V}_3$ vanishes asymptotically at $|z| \to \infty$, there is no gap, and the continuum modes have all possible $m^2 > 0$.

To examine the massive modes, let us suppose that we can approximate $A(z)$ by its brane limit (98).[3] We find for the massive modes

$$(\boldsymbol{a}_\mu)_q \sim \xi^{1/2}[Y_\alpha(m\xi) + C_3 J_\alpha(m\xi)]\varepsilon_\mu, \tag{108}$$

where ξ is given by (100), $\alpha = \sqrt{1 + (\mathcal{M}_W/k)^2}$, and C_3 is a constant, determined by the continuity of $\boldsymbol{a}_\mu$ and the jump condition in $\partial_z \boldsymbol{a}_\mu$ at $z = 0$, given by

$$C_3 = -\frac{Y_\alpha(m/k) + (m/k)Y'_\alpha(m/k)}{J_\alpha(m/k) + (m/k)J'_\alpha(m/k)}. \tag{109}$$

There are no massless nor localized 4D massive modes.

Notice that the disappearance of the would-be gravitationally trapped 4D Nambu-Goldstone modes along $\{\mathbf{T}_{\rm br}\}$ does not generate localizable 4D massive modes for the gauge field fluctuations $\boldsymbol{a}_\mu \in \{\mathbf{T}_{\rm br}\}$. On the other hand, if we set $\mathsf{g} = 0$, the coupling between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ disappears, and $\boldsymbol{\beta} = 0$. The 4D Nambu-Goldstone bosons $\in \{\mathbf{T}_{\rm br}\}$ then reappear in the physical spectrum, and the gauge bosons $\boldsymbol{a}_\mu \in \{\mathbf{T}_{\rm br}\}$ behave as those that lie along $\{\mathbf{T}_0\}$. Obviously, from the 4D observer's point of view, the spectrum along $\{\mathbf{T}_{\rm br}\}$ is discontinuous in the limit $g \to 0$. However, as it will be shown in the following, once the continuum modes are also taken into account, the spectrum is continuous in this limit. Discrete 4D massive vector field fluctuations along $\{\mathbf{T}_{\rm br}\}$ appear, and these are *quasilocalized*.

In the brane approximation (98), Eq. (106) admits also a 4D massive metastable mode with a complex eigenvalue $m^2 = m_0^2 - im_0\Gamma$, when radiative boundary conditions [34] at $z \to \pm\infty$ are imposed.[4] These metastable modes are given by

$$(\boldsymbol{a}_\mu)_q \sim \xi^{1/2} H_\alpha^{(1)}(m\xi)\varepsilon_\mu, \tag{110}$$

where $H_\alpha^{(1)} = J_\alpha + iY_\alpha$ is the first Hankel function, while the continuity of $\boldsymbol{a}_\mu$ and the jump condition in $\partial_z \boldsymbol{a}_\mu$ at $z = 0$ yield the eigenvalue condition

$$\frac{m}{k}\frac{H_{\alpha-1}^{(1)}(m/k)}{H_\alpha^{(1)}(m/k)} = \alpha - 1.$$

For $(\mathcal{M}_W/k)^2 \ll 1$, with $(m_0/\mathcal{M}_W)^2 \ll 1$ and $(\Gamma/m_0)^2 \ll 1$, we find

[3] We cannot take the brane limit without sending the scale of symmetry breaking v, and hence $\mathcal{M}_W^2$, to zero at the same time. However, the spectrum for $v \neq 0$ in the high-curvature regime is expected to be qualitatively similar to the one obtained by approximating $A(z)$ as in (98) for $(\mathcal{M}_W/k)^2 \ll 1 \ll 1/v^2$, for v not zero as long as g is sufficiently small.

[4] This effect is similar to that studied in Ref. [35] for a free 5D massive scalar field in the Randall-Sundrum brane background.

$$m_0^2 = \frac{1}{2}\mathcal{M}_W^2\left(\frac{\mathcal{M}_W}{k}\right)^2, \qquad \frac{\Gamma}{m_0} = \frac{\pi}{2}\left(\frac{\mathcal{M}_W}{k}\right)^2. \tag{111}$$

It is seen from (110) and (111) that now a massive discrete mode with a finite lifetime, the mass and *width* of which are suppressed by $(\mathcal{M}_W/k)^2$, exists. This mode decays into the continuum modes, due to its finite lifetime, and disappears from the spectrum (see Ref. [36] for a discussion on the nature of fermion and scalar resonant states on domain wall braneworlds).

To summarize, for the field fluctuations $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{a}_\mu) \in \{\mathbf{T}_{\rm br}\}$, we find no modes with $m^2 < 0$; i.e., the domain wall configuration is perturbatively stable in this sector. All these field fluctuations exhibit a tower of massive modes that propagate in the bulk, with a continuous spectrum for $m^2 > 0$ and no massless modes. On the other hand, as a consequence of the spontaneous gauge symmetry breaking, the gauge field excitations $\boldsymbol{a}_\mu$ become massive and get quasilocalizable discrete 4D massive modes that decay into the continuum modes due to their finite lifetime. For the gauge fields $\boldsymbol{a}_\mu \in \{\mathbf{T}_0\}$ of the previous subsection, such resonances are indeed absent.

C. Fluctuations along $\mathcal{K}^\perp = \{\mathbf{T}_+\} \oplus \{\mathbf{T}_-\}$

To start with, it should be noted that if the $SU(5)$ symmetry were global rather than gauge the scalars $\boldsymbol{\varphi} \in \mathcal{K}^\perp$ would be associated to fluctuations along the generators that are broken only at one side of the wall [see Eqs. (73) and (74)]. In this case, gravitationally trapped 4D Nambu-Goldstone fields appear corresponding to rotations of $\boldsymbol{\Phi}^k$ within the class described by $H_\pm/H_0$ [31]. On the other hand, in the gauged model, we cannot fix $\boldsymbol{\varphi} = 0$ for $\boldsymbol{\varphi} \in \mathcal{K}^\perp$ (see Appendix B).

Now, for the gauge-invariant fluctuations $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{a}_\mu) \in \mathcal{K}^\perp$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy the constraint

$$\boldsymbol{\alpha} = i\mathrm{g}(\mathcal{M}_\pm^2)^{-1}[\boldsymbol{\Phi}^k, e^{-3A}\partial_z(e^{3A/2}\boldsymbol{\beta})], \tag{112}$$

where $\mathcal{M}_\pm^2$ is given by (85), with the plus sign for $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \{\mathbf{T}_+\}$ and the minus sign for $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \{\mathbf{T}_-\}$. Hence, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are not independent and correspond to a single physical perturbation. The gauge-invariant field fluctuation $\boldsymbol{\beta}$ satisfies

$$\partial^\mu\partial_\mu\boldsymbol{\beta} + \left(\frac{3}{2}A'' - \frac{9}{4}A'^2 - e^{2A}\mathcal{M}_\pm^2\right)\boldsymbol{\beta} + \partial_z^2\boldsymbol{\beta} - (\partial_z \ln \mathcal{M}_\pm^2)\left(\frac{3}{2}A'\boldsymbol{\beta} + \partial_z\boldsymbol{\beta}\right) = 0. \tag{113}$$

As in the sector $\{\mathbf{T}_{\rm br}\}$, we find a non-trivial mixing between the original fluctuations $\boldsymbol{\varphi}$, $\mathcal{A}_z$ and $\boldsymbol{\chi}$ along $\mathcal{K}^\perp$, this time with the background playing a more prominent role.

Let $\boldsymbol{\Omega}$ be defined as

$$\boldsymbol{\Omega} \equiv (\mathcal{M}_\pm)^{-1}\boldsymbol{\beta}. \tag{114}$$

The modes $\boldsymbol{\Omega}(x,z) \sim e^{ip\cdot x}\boldsymbol{\Omega}(z)$ with $\boldsymbol{\Omega} \in \{\mathbf{T}_{\rm br}\}$ satisfy the Schrödinger-like equation

$$(-\partial_z^2 + \mathcal{V}_{4\pm})\boldsymbol{\Omega} = m^2\boldsymbol{\Omega}, \tag{115}$$

where

$$\mathcal{V}_{4\pm} = \frac{9}{4}A'^2 - \frac{3}{2}A'' + e^{2A}\mathcal{M}_\pm^2 + \frac{1}{4}((\ln \mathcal{M}_\pm^2)')^2 - \frac{1}{2}(\ln \mathcal{M}_\pm^2)'' + \frac{3}{2}A'(\ln \mathcal{M}_\pm^2)', \tag{116}$$

with $\mathcal{V}_{4\pm} \to \mathcal{V}_{4\mp}$ under $z \to -z$. The Schrödinger operator in (115) can be rewritten as

$$(-\partial_z^2 + \mathcal{V}_{4\pm}) = \left(\partial_z - \frac{3}{2}Q'_\pm\right)\left(-\partial_z - \frac{3}{2}Q'_\pm\right) + e^{2A}\mathcal{M}_\pm^2, \tag{117}$$

where

$$Q_\pm \equiv A + \frac{1}{3}\ln \mathcal{M}_\pm^2. \tag{118}$$

It follows that the eigenvalues of (115) are non-negative definite and there are no tachyonic modes. In fact, Eq. (115) is just a Schrödinger equation with an asymmetric potential barrier of finite height that vanishes asymptotically at $|z| \to \infty$ (see Fig. 1). Hence, Eq. (115) does not support bound states with $m^2 \leq 0$, while those modes with $m^2 > 0$ behave as scattered waves by the wall. From these results and (112), it follows that $\boldsymbol{\alpha}$ has no tachyonic nor normalizable zero modes, either.

Finally, the modes $\boldsymbol{a}_\mu(x,z) \sim e^{ip\cdot x}\boldsymbol{a}_\mu(z)$ of the gauge-invariant fluctuation $\boldsymbol{a}_\mu$ satisfy

$$(-\partial_z^2 + \mathcal{V}_{5\pm})\boldsymbol{a}_\mu = m^2\boldsymbol{a}_\mu, \tag{119}$$

where

$$\mathcal{V}_{5\pm} = \mathcal{V}_1 + e^{2A}\mathcal{M}_\pm^2, \tag{120}$$

with $\mathcal{V}_1$ given by (96) and $\mathcal{V}_{5\pm} \to \mathcal{V}_{5\mp}$ under $z \to -z$. The Schrödinger operator in (119) can be rewritten as

$$(-\partial_z^2 + \mathcal{V}_{5\pm}) = \left(\partial_z + \frac{1}{2}A'\right)\left(-\partial_z + \frac{1}{2}A'\right) + e^{2A}\mathcal{M}_\pm^2.$$

Therefore, in the spectrum of (119), there are no negative eigenvalues, and we have no tachyonic modes. For $\mathrm{g}^2 < 4b^2/15$, the potential has an asymmetric volcanolike profile that vanishes asymptotically at $|z| \to \infty$ (see Fig. 2). Hence, as in the previous Lie algebra sector, it supports no localizable massless modes and a continuum of massive modes with all the possible $m^2 > 0$. Obviously, there are no localizable 4D massive modes, either.

The absence of the would-be gravitationally trapped 4D Nambu-Goldstone fields if the symmetry were global rather than local [31] and of localizable 4D massive modes for the gauge field suggests the existence of quasilocalizable 4D massive modes of this last one, in order to make continuous the zero gauge coupling limit of the spectrum in this Lie algebra sector of the fluctuations.

To determine the existence of metastable states $\boldsymbol{a}_\mu \in \mathcal{K}^\perp$, we approximate $A(z)$ as in (98) and $\mathcal{M}_\pm^2$ by[5]

$$\mathcal{M}_+^2 \sim \begin{cases} 0, & z > 0 \\ \mathcal{M}_W^2, & z < 0, \end{cases} \qquad \mathcal{M}_-^2 \sim \begin{cases} \mathcal{M}_W^2, & z < 0 \\ 0, & z > 0 \end{cases}. \tag{121}$$

Now, for the radiative boundary problem, $\boldsymbol{a}_\mu(z) \in \{\mathbf{T}_+\}$ is given by

$$(\boldsymbol{a}_\mu)_q \sim \varepsilon_\mu \xi^{1/2} \begin{cases} C_4 H_1^{(1)}(m\xi), & z > 0 \\ C_5 H_\alpha^{(1)}(m\xi), & z < 0, \end{cases} \tag{122}$$

where ξ is given by (100); $\alpha = \sqrt{1 + (\mathcal{M}_W/k)^2}$; and the coefficients C_4 and C_5 are determined, as before, by imposing the continuity of $\boldsymbol{a}_\mu(z)$ and the discontinuity of its first derivative at $z = 0$, which must be $-k\boldsymbol{a}_\mu(0)$. The latter condition leads to the eigenvalue formula

$$\frac{m}{k}\left[\frac{H_{\alpha-1}^{(1)}(m/k)}{H_\alpha^{(1)}(m/k)} + \frac{H_0^{(1)}(m/k)}{H_1^{(1)}(m/k)}\right] = \alpha - 1.$$

For $(\mathcal{M}_W/k)^2 \ll 1$, we find the same massive resonance given by (111). The quasilocalized 4D massive modes $\boldsymbol{a}_\mu \in \{\mathbf{T}_-\}$ can be obtained from (122) under $z \to -z$.

The absence of tachyonic modes for the field excitations $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{a}_\mu) \in \mathcal{K}^\perp$ means that the domain wall configuration is perturbatively stable also in this sector. All these fluctuations exhibit a tower of 4D massive modes that propagate in the bulk, with a continuous spectrum for $m^2 > 0$ and no localized 4D massless modes. As in the Lie algebra sector $\{\mathbf{T}_{\rm br}\}$, we find metastable 4D massive gauge fluctuations in this sector, also.

V. CONCLUSIONS

In terms of diffeomorphism-invariant and Lie algebra gauge-invariant excitations, we have proven the perturbative stability of some topologically nontrivial 5D self-gravitating $SU(5) \times Z_2$ domain wall configurations. As expected, gravitational tensor and vector fluctuations, which are unchanged under Lie algebra gauge transformations, behave like its counterparts in the standard Z_2 domain walls.

The behavior of the Lie algebra valued fluctuations is, of course, much more interesting. All exhibit towers of 4D massive modes that propagate in the bulk, with a continuous spectrum for $m^2 > 0$. All the would-be 4D Nambu-Goldstone excitations associated to the partial breaking $SU(5) \times Z_2 \to H_0$ [gravitationally trapped if the SU(5) symmetry were global [31] rather than gauge] disappear from the physical 4D spectrum. No 4D massless gauge field excitations are found, and the massive ones are not localized. As we have seen, discrete metastable 4D massive mode functions for the gauge field fluctuations exist along the Lie algebra sectors where the 4D Nambu-Goldstone fields appear. Thus, an interesting version of the Higgs phenomenon takes place in these systems, whereby 4D gauge fluctuations along the spontaneously broken gauge sectors acquire masses and then escape from the core of the wall into the bulk.

Self-gravitating Higgs domain walls, of the sort studied here, provide perturbatively stable minimal settings with enhanced symmetry breaking patterns. Depending of the nature of this pattern, these backgrounds are ideal for discussing [30] the Dvali-Shifman mechanism of gauge field localization via bulk confinement [21]. Indeed, if one wishes to construct a phenomenological viable non-Abelian domain wall braneworld, the explicit gauge group H_0 on the core of the wall should be more akin to the standard model group. We hope to return to this and other related issues in the near future.

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APPENDIX A: LINEARIZED EQUATIONS FOR THE FLUCTUATIONS

Following the procedure outlined in Sec. III, it is straightforward to derive perturbative equations to first order for fluctuations $(\boldsymbol{\varphi}, \boldsymbol{\mathcal{A}}_a, h_{ab})$ around a solution $(\boldsymbol{\Phi}, \boldsymbol{A}_a; g_{ab})$ of the field equations (5)–(8). We find

$$\begin{aligned} &-\frac{1}{2}g^{cd}\nabla_c\nabla_d h_{ab} + R^c(ab)^d h_{cd} + R^c_{(a}h_{b)c} + \nabla_{(a}\nabla^c h_{b)c} - \frac{1}{2}\nabla_a\nabla_b(g^{cd}h_{cd}) \\ &= 4\mathrm{Tr}\{\mathbf{D}_{(a}\boldsymbol{\Phi}\mathbf{D}_{b)}\boldsymbol{\varphi}\} + \frac{2}{3}h_{ab}V(\boldsymbol{\Phi}) + \frac{2}{3}\left(\frac{\partial V(\boldsymbol{\Phi})}{\partial\phi_q}\varphi_q\right)g_{ab} + 4i\mathsf{g}\mathrm{Tr}\{\mathbf{D}_{(a}\boldsymbol{\Phi}[\boldsymbol{\mathcal{A}}_{b)}, \boldsymbol{\Phi}]\} + 2h_{cd}\mathrm{Tr}\{\mathbf{F}^c_a\mathbf{F}^d_b\} \\ &\quad + 4g^{cd}\mathrm{Tr}\{\mathbf{F}_{ac}\mathbf{D}_{[b}\boldsymbol{\mathcal{A}}_{d]}) + \mathbf{F}_{bc}\mathbf{D}_{[a}\boldsymbol{\mathcal{A}}_{d]})\} - \frac{1}{3}h_{ab}\mathrm{Tr}\{\mathbf{F}_{cd}\mathbf{F}^{cd}\} - \frac{2}{3}g_{ab}\mathrm{Tr}\{\mathbf{F}^{cd}\mathbf{D}_{[c}\boldsymbol{\mathcal{A}}_{d]}\}, \end{aligned} \tag{A1}$$

[5] See Footnote 3.

where $R^d{}_{abc}$ and R^b_a are the Riemann and Ricci curvatures of g_{ab}, respectively,

$$-\frac{1}{2}g^{ab}g^{cd}(\nabla_a h_{bd}+\nabla_b h_{ad}-\nabla_d h_{ab})\mathbf{D}_c\mathbf{\Phi}+ig^{ab}\mathsf{g}\mathbf{D}_b[\boldsymbol{\mathcal{A}}_a,\mathbf{\Phi}]+ig^{ab}\mathsf{g}[\boldsymbol{\mathcal{A}}_a,\mathbf{D}_b\mathbf{\Phi}]$$
$$-h^{ab}\mathbf{D}_a\mathbf{D}_b\mathbf{\Phi}+g^{ab}\mathbf{D}_a\mathbf{D}_b\boldsymbol{\varphi}=\frac{\partial^2 V(\mathbf{\Phi})}{\partial\phi_p\partial\phi_q}\varphi_p\mathbf{T}^q, \tag{A2}$$

where

$$V(\mathbf{\Phi}+\boldsymbol{\varphi})=V(\mathbf{\Phi})+\frac{\partial V(\mathbf{\Phi})}{\partial\phi_q}\varphi_q+\frac{1}{2}\frac{\partial^2 V(\mathbf{\Phi})}{\partial\phi_p\partial\phi_q}\varphi_p\varphi_q+O(\boldsymbol{\varphi}^3) \tag{A3}$$

and

$$g^{ac}\mathbf{D}_c\boldsymbol{\mathcal{F}}_{ab}+i\mathsf{g}g^{ac}[\boldsymbol{\mathcal{A}}_c,\mathbf{F}_{ab}]+\mathsf{g}^2[\mathbf{\Phi},[\boldsymbol{\mathcal{A}}_b,\mathbf{\Phi}]]-\frac{1}{2}g^{ac}g^{de}((\nabla_c h_{be}+\nabla_b h_{de}-\nabla_e h_{cb})\mathbf{F}_{ad}$$
$$-(\nabla_c h_{ae}+\nabla_a h_{de}-\nabla_e h_{ca})\mathbf{F}_{bd})-h^{ac}D_c\mathbf{F}_{ab}-i\mathsf{g}[\boldsymbol{\varphi},\mathbf{D}_b\mathbf{\Phi}]-i\mathsf{g}[\mathbf{\Phi},\mathbf{D}_b\boldsymbol{\varphi}]=0, \tag{A4}$$

where $\mathcal{F}_{ab}\equiv\mathbf{D}_a\boldsymbol{\mathcal{A}}_b-\mathbf{D}_b\boldsymbol{\mathcal{A}}_a$. Clearly, Eqs. (A1), (A2), and (A4) are Lie algebra gauge covariant with respect to the general background $(\mathbf{\Phi},\boldsymbol{A}_a;g_{ab})$, since $\boldsymbol{A}_a$ appears only in $\mathbf{F}_{ab}$ and in the covariant derivative $\mathbf{D}_a$.

Next, within the gauge-equivalent classes of background domain wall configurations $(\tilde{\mathbf{\Phi}}^k,\tilde{\boldsymbol{A}}^k_a;g^k_{ab})$, for simplicity, we write (A1), (A2), and (A4) in the (Lie algebra) gauge fixed domain wall background $(\mathbf{\Phi}^k,\mathbf{0}_a;g^k_{ab})$. We find

$$-\frac{1}{2}g^{cd}\nabla_c\nabla_d h_{ab}+R^c{}_{(ab)}{}^d h_{cd}+R^c_{(a}h_{b)c}+\nabla_{(a}\nabla^c h_{b)c}-\frac{1}{2}\nabla_a\nabla_b(g^{cd}h_{cd})$$
$$=4\mathrm{Tr}\{\nabla_{(a}\mathbf{\Phi}^k\nabla_{b)}\boldsymbol{\varphi}\}+\frac{2}{3}h_{ab}V(\mathbf{\Phi}^k)+\frac{2}{3}\frac{\partial V(\mathbf{\Phi})}{\partial\phi_q}\bigg|_{\mathbf{\Phi}^k}\varphi_q g_{ab}, \tag{A5}$$

where now $R^d{}_{abc}$ and R^b_a are the Riemann and Ricci curvatures of g^k_{ab}, respectively,

$$-\frac{1}{2}g^{ab}g^{cd}(\nabla_a h_{bd}+\nabla_b h_{ad}-\nabla_d h_{ab})\nabla_c\mathbf{\Phi}^k+ig^{ab}\mathsf{g}\nabla_b[\boldsymbol{\mathcal{A}}_a,\mathbf{\Phi}^k]+ig^{ab}\mathsf{g}[\boldsymbol{\mathcal{A}}_a,\nabla_b\mathbf{\Phi}^k]$$
$$-h^{ab}\nabla_a\nabla_b\mathbf{\Phi}^k+g^{ab}\nabla_a\nabla_b\boldsymbol{\varphi}=\varphi_p\frac{\partial^2 V(\mathbf{\Phi})}{\partial\phi_p\partial\phi_q}\bigg|_{\mathbf{\Phi}^k}\mathbf{T}^q \tag{A6}$$

and

$$g^{ac}\nabla_c(\nabla_a\boldsymbol{\mathcal{A}}_b-\nabla_b\boldsymbol{\mathcal{A}}_a)+\mathsf{g}^2[\mathbf{\Phi}^k,[\boldsymbol{\mathcal{A}}_b,\mathbf{\Phi}^k]]=+i\mathsf{g}[\boldsymbol{\varphi},\nabla_b\mathbf{\Phi}^k]+i\mathsf{g}[\mathbf{\Phi}^k,\nabla_b\boldsymbol{\varphi}], \tag{A7}$$

where

$$\frac{\partial V(\mathbf{\Phi})}{\partial\phi_q}\bigg|_{\mathbf{\Phi}^k}=(\phi''_M+4A'\phi'_M)\delta^{Mq} \tag{A8}$$

and the Hessian of $V(\mathbf{\Phi})$ at $\mathbf{\Phi}^k$, $\partial^2V(\mathbf{\Phi})/\partial\phi_p\partial\phi_q|_{\mathbf{\Phi}^k}$, is a block-diagonal $(5^2-1)\times(5^2-1)$ matrix.

In obtaining (A5), we have used

$$\mathrm{Tr}\{\partial_{(a}\mathbf{\Phi}^k[\boldsymbol{\mathcal{A}}_{b)},\mathbf{\Phi}^k]\}=\delta^y_{(a}\mathrm{Tr}\{\boldsymbol{\mathcal{A}}_{b)}[\mathbf{\Phi}^k(y),\partial_y\mathbf{\Phi}^k(y)]\}=0 \tag{A9}$$

because $[\mathbf{\Phi}^k(y),\partial_y\mathbf{\Phi}^k(y)]=0$. Additionally, if we take the divergence of (A7), we find an integrability condition that can be used to remove some of the degrees of freedom.

Finally, Eq. (A7) is rewritten as

$$\frac{1}{2}\delta^{qp}g^{ac}\nabla_c(\nabla_a\boldsymbol{\mathcal{A}}_b-\nabla_b\boldsymbol{\mathcal{A}}_a)_p-\frac{1}{2}(M^2)^{qp}(\boldsymbol{\mathcal{A}}_b)_p$$
$$=ig\mathrm{Tr}\{\boldsymbol{\varphi}[\partial_b\mathbf{\Phi}^k,\mathbf{T}^q]+\partial_b\boldsymbol{\varphi}[\mathbf{T}^q,\mathbf{\Phi}^k]\}, \tag{A10}$$

where

$$(M^2)^{qp}\equiv-2g^2\mathrm{Tr}\{[\mathbf{T}^q,\mathbf{\Phi}^k(y)][\mathbf{T}^p,\mathbf{\Phi}^k(y)]\}. \tag{A11}$$

APPENDIX B: LIE ALGEBRA GAUGE FIXING FOR $(\boldsymbol{\varphi},\boldsymbol{\mathcal{A}}_a) \notin \{\mathbf{T_0}\}$

Here, we show explicitly some issues related to the Lie algebra gauge fixing for $(\boldsymbol{\varphi},\boldsymbol{\mathcal{A}}_a) \notin \{\mathbf{T}_0\}$ in the symmetry breaking **A** (the symmetry breaking **B** differing only in numerical factors).

Under infinitesimal gauge transformations, we find

$$\varphi_q \mapsto \varphi_q + v\sqrt{\frac{5}{2}}\sigma_{q'}, \tag{B1}$$

where $\mathbf{T}^q$, $\mathbf{T}^{q'} \in \{\mathbf{T}_{\rm br}\}$. For φ_q bounded, it follows that we can choose $\sigma_{q'}$ in order to make $\varphi_q = 0$ whenever $\mathbf{T}^q \in \{\mathbf{T}_{\rm br}\}$. Hence, using these $n_{\rm br}$ gauge degrees of freedom, we can fix $\boldsymbol{\varphi} = 0$ for $\boldsymbol{\varphi} \in \{\mathbf{T}_{\rm br}\}$.

It is instructive to analyze the consequences of adopting the above gauge fixing. In the gauge $\boldsymbol{\varphi} = 0$ for $(\boldsymbol{\varphi},\mathcal{A}_a) \in \{\mathbf{T}_{\rm br}\}$, the integrability condition that follows after taking the divergence of (A7) implies

$$\nabla^a\boldsymbol{\mathcal{A}}_a = e^{-2A(z)}(\partial^\mu\boldsymbol{\mathcal{A}}^L_\mu + 3A'\boldsymbol{\mathcal{A}}_z + \boldsymbol{\mathcal{A}}'_z) = 0; \tag{B2}$$

i.e., $\boldsymbol{\mathcal{A}}_z$ and $\boldsymbol{\mathcal{A}}^L_\mu \equiv \partial_\mu\boldsymbol{\chi}$ are not independent and correspond to a single physical perturbation. Indeed, neither the extra dimension component $\boldsymbol{\mathcal{A}}_z$ nor the longitudinal component $\boldsymbol{\mathcal{A}}^L_\mu$ of the gauge field $\boldsymbol{\mathcal{A}}_a \in \{\mathbf{T}_{\rm br}\}$ can be further eliminated, since the Lie algebra gauge of freedom has been completely fixed to set $\boldsymbol{\varphi} = 0$ for $\boldsymbol{\varphi} \in \{\mathbf{T}_{\rm br}\}$.

Next, let $\boldsymbol{\Upsilon}$ be the fluctuation defined as

$$\boldsymbol{\Upsilon} \equiv e^{-A/2}\boldsymbol{\mathcal{A}}_z. \tag{B3}$$

From (A7) and (B2), in the gauge $\boldsymbol{\varphi} = 0$, it follows that the modes $\boldsymbol{\Upsilon}(x,z) \sim e^{ip\cdot x}\Upsilon(z)$ with $\boldsymbol{\Upsilon} \in \{\mathbf{T}_{\rm br}\}$ satisfy the Schrödinger-like equation

$$(-\partial_z^2 + \mathcal{V}_2)\boldsymbol{\Upsilon} = m^2\Upsilon, \tag{B4}$$

where $\mathcal{V}_2$ is given by (105) and $\mathcal{M}^2_W$ is given by (84). As mentioned in the main text, $\mathcal{V}_2$ (see Fig. 1) does not support bound states with $m^2 \le 0$, while those modes with $m^2 > 0$ behave as waves scattered by the wall. Approximating $A(z)$ as in (98), we find

$$(\boldsymbol{\Upsilon})_q \sim \xi^{1/2}[Y_\alpha(m\xi) + DJ_\alpha(m\xi)], \tag{B5}$$

where ξ is given by (100), $\alpha = \sqrt{1 + (\mathcal{M}_W/k)^2}$, and D is a constant given by

$$D = -\frac{Y_\alpha(m/k) - (m/k)Y'_\alpha(m/k)}{J_\alpha(m/k) - (m/k)J'_\alpha(m/k)}. \tag{B6}$$

Next, from (B2), (B3), and (B5), we find for large mz

$$k(\boldsymbol{\mathcal{A}}_z)_q \sim \sqrt{2/(\pi mz)}[\sin(mz - \alpha/2 - \pi/4) + D\cos(mz - \alpha/2 - \pi/4) + O(|z|^{-1})] \tag{B7}$$

and

$$mk(\boldsymbol{\chi})_q \sim -\sqrt{2/(\pi mz)}[\cos(mz - \alpha/2 - \pi/4) - D\sin(mz - \alpha/2 - \pi/4) + O(|z|^{-1})], \tag{B8}$$

which go to zero as $z \to \infty$.

On the other hand, we cannot fix $\boldsymbol{\varphi} = 0$ for $\boldsymbol{\varphi} \in \mathcal{K}^\perp$. We find

$$\varphi_q \mapsto \varphi_q + v\frac{1}{2}\sqrt{\frac{5}{2}}(F+1)\sigma_{q'}, \tag{B9}$$

where $\mathbf{T}^q$, $\mathbf{T}^{q'} \in \{\mathbf{T}_-\}$ and

$$\varphi_p \mapsto \varphi_p + v\frac{1}{2}\sqrt{\frac{5}{2}}(F-1)\sigma_{p'}, \tag{B10}$$

where $\mathbf{T}^p, \mathbf{T}^{p'} \in \{\mathbf{T}_+\}$, with F given by (76). If φ_q $[\varphi_p]$ is a bounded function that decays to zero faster than approximately $v(F+1)$ $[\sim v(F-1)]$ as $y \to -\infty$ $[y \to +\infty]$, we could choose $\sigma_{q'}$ $[\sigma_{p'}]$ to gauge away φ_q $[\varphi_p]$. But for fluctuations φ_q $[\varphi_p]$ that do not vanish away of the core of the wall, it is clear that this will require a growing $\sigma_{q'}$ $[\sigma_{p'}]$ as $y \to -\infty$ $[y \to +\infty]$, which conflicts with $\sigma_{q'}$ $[\sigma_{p'}]$ being small. It follows that, for general bounded fluctuations $(\boldsymbol{\varphi},\boldsymbol{\mathcal{A}}_a) \in \mathcal{K}^\perp$, we cannot fix to zero the 5D scalar fluctuation $\boldsymbol{\varphi}$ all along the additional dimension.

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