Aspects of the negative mode problem in quantum tunneling with gravity

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Some solutions describing vacuum decay exhibit a catastrophic instability. This, so-called negative mode problem in quantum tunneling with gravity, was discovered 34 years ago and in spite of the fact that in these years many different groups worked on this topic, it has still not been resolved. Here, we briefly summarize the current status of the problem and investigate properties of the bounces, numerically and analytically for physically interesting potentials. In the framework of the Hamiltonian approach we show that for generic polynomial potentials the negative mode problem could arise at energies much lower than the Planck mass, indicating that the negative mode problem is not related to physics at the Planck scale. At the same time we find that for a Higgs like potential, as it appears in the standard model, the problem does not appear at realistic values of the potential's parameters but only at the Planck scale.

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I. INTRODUCTION

Calculating the decay rate of metastable vacua while taking gravitational effects into account, has become more important following the discovery that we might be living in a false vacuum. Using the Euclidean approach [14–16] for calculating the decay rate of metastable vacua to their true value, γ, the Arrhenius formula is given by

$$\gamma = \mathcal{A}e^{-\mathcal{B}}, \tag{1}$$

with

$$\mathcal{B} = S^{(cl)}(\varphi^b) - S^{(cl)}(\varphi^f), \tag{2}$$

where the first term on the rhs is the classical Euclidean action calculated along the bounce solution and the second term is the value of the action evaluated at the false vacuum.

The bounce solution is the lowest-action $O(4)$-symmetric solution to the Euclidean equations of motion that interpolates between false and true vacua (see Fig. 1). Expanding around the bounce solution, gives the preexponential factor $\mathcal{A}$ as a Gaussian integral over the linear perturbations. Proper bounces should have exactly one eigenfunction with a negative eigenvalue in the spectrum of linear perturbations, in order to make the decay picture coherent [17]. While this is always the case in flat space-time, generalizing to curved space-time results in some bounces getting infinitely many negative modes indicating a problem. Note that when gravity is involved, in addition to the basic bounce solution, there are oscillating instantons and an infinite tower of oscillating bounces [6,18,19], which, however, have more than one negative mode [7,10] making their relation to tunneling questionable.

Using new approximate analytic methods and numerical calculations, we aim to clarify the question of whether the negative mode problem is inherently related to Planck-scale physics and highlight differences between the Hamiltonian and Lagrangian approaches to the problem. The paper is organized as follows, In the next section we briefly summarize the negative mode problem. In Sec. III we discuss generic quartic polynomial potentials, while in

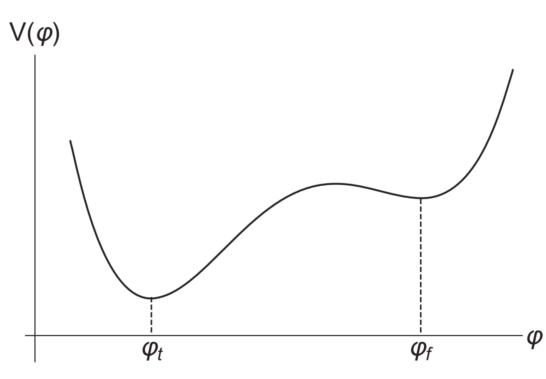

FIG. 1. A typical potential in which false vacuum decay can occur. The bounce solution interpolates between the false vacuum φ_f and true vacuum φ_t.

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Sec. IV we consider a realistic, Higgs-like potential. Finally, the last section contains a summary and concluding remarks.

II. A SHORT SUMMARY OF THE NEGATIVE MODE PROBLEM

Let us consider the theory of a single scalar field minimally coupled to gravity, which is defined by the following Euclidean action:

$$S_E = \int d^4x\sqrt{g}\left(-\frac{1}{2\kappa}R + \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi + V(\varphi)\right), \quad (3)$$

where $\kappa = 8\pi G_N$ is the reduced Newton's gravitational constant. The most general $O(4)$-invariant metric is parametrized as

$$ds^2 = N^2(\eta)d\eta^2 + \rho^2(\eta)d\Omega_3^2, \quad (4)$$

where $N(\eta)$ is the lapse function, $\rho(\eta)$ is the scale factor and $d\Omega_3^2$ is the metric of the unit three-sphere. In proper-time gauge, $N = 1$, the corresponding field equations are

$$\ddot{\varphi} + 3\frac{\dot{\rho}}{\rho}\dot{\varphi} = \frac{\partial V}{\partial\varphi}, \quad (5)$$

$$\ddot{\rho} = -\frac{\kappa\rho}{3}(\dot{\varphi}^2 + V(\varphi)), \quad (6)$$

$$\dot{\rho}^2 = 1 + \frac{\kappa\rho^2}{3}\left(\frac{\dot{\varphi}^2}{2} - V(\varphi)\right), \quad (7)$$

where $\dot{} = d/d\eta$. The leading exponential factor in the decay rate is determined by the bounce: a solution of these equations with appropriate boundary conditions. In order to calculate the preexponential factor $\mathcal{A}$ in Eq. (1) one should consider linear perturbations about the bounce solution. For this purpose we expand the metric and the scalar field over an $O(4)$-symmetric background as follows:

$$ds^2 = (1 + 2A(\eta))d\eta^2 + \rho(\eta)^2(1 - 2\Psi(\eta))d\Omega_3^2,$$
$$\varphi = \varphi(\eta) + \Phi(\eta), \quad (8)$$

where ρ and φ are the background field values and A, Ψ and Φ are small perturbations. Note that under the infinitesimal shift $\eta \to \eta + \alpha$ the gauge transformations are

$$\delta\Psi = -\frac{\dot{\rho}}{\rho}\alpha, \qquad \delta\Phi = \dot{\varphi}\alpha, \qquad \delta A = \dot{\alpha}. \quad (9)$$

In what follows, we will be interested in the lowest (purely η-dependent, "homogeneous") modes and consider only scalar metric perturbations. Expanding the total action to second order in perturbations and using the background equations of motion, we find

$$S = S^{(0)}[\rho, \varphi] + S^{(2)}[A, \Psi, \Phi], \quad (10)$$

where $S^{(0)}$ is the action of the background solution and $S^{(2)}[A, \Psi, \Phi]$ is the quadratic action. An analysis of the equations of motion following from this quadratic action shows [1,4] that there are constraints in this system and only one out of three variables is physical. The unconstrained quadratic action about Coleman–de Luccia bounces was first derived in Ref. [1] using the $\Psi = 0$ gauge in the Lagrangian approach. Integrating out A and expressing the quadratic action in terms of the remaining, physical perturbation Φ, one gets

$$S_L^{(2)} = 2\pi^2\int\rho^3 d\eta\left[\frac{\dot{\rho}^2}{2Q_L}\dot{\Phi}^2 + \frac{1}{2}U_\Phi\Phi^2\right] \quad (11)$$

with the potential being

$$U_\Phi = \frac{\dot{\rho}^2 V''}{Q_L} + \frac{\kappa\rho^2\dot{\rho}^2 V'^2}{3Q_L^2} + \frac{\kappa\rho\dot{\rho}\,\dot{\varphi}\,V'}{3Q_L^2}, \quad (12)$$

where $' \equiv d/d\varphi$. In particular, it was noted that a factor termed Q appears in front of the kinetic term, which in the Lagrangian approach is the following combination of background quantities:

$$Q_L = 1 - \frac{\kappa\rho^2 V(\varphi)}{3} = \dot{\rho}^2 - \frac{\kappa\rho^2\dot{\varphi}^2}{6}. \quad (13)$$

This factor becomes negative for any bounce solution close to the point $\dot{\rho} = 0$. In addition, for some bounces it becomes negative a second time, in a regime where the last term dominates over $\dot{\rho}$. Despite its widespread use, the Lagrangian approach was criticized in Ref. [2] because of poor gauge fixing. Indeed, from the gauge transformations (9) it is clear that we cannot freely transform the variable Ψ. In particular the transformation breaks down at any point where $\dot{\rho} = 0$ making it impossible to impose a nonsingular gauge on Ψ. Unfortunately, there are not many alternatives in the Lagrangian approach since it only involves configuration-space variables. Later, Lee and Weinberg [11] promoted Φ to a gauge-invariant variable

$$\chi = \dot{\rho}\Phi + \rho\dot{\varphi}\Psi, \quad (14)$$

and obtained a pulsation equation, which exactly coincides with the earlier $\Psi = 0$ gauge-fixed approach (see the appendix in Ref. [12]).

Therefore, we will use the Hamiltonian approach in this paper which is more appropriate for constrained dynamical systems. Using a Hamiltonian approach following Dirac the quadratic action has the form [3,12]

$$S_H^{(2)} = \pi^2 \int d\eta \Phi \left[-\frac{d}{d\eta}\left(\frac{\rho^3(\eta)}{Q_H}\frac{d}{d\eta}\right) + \rho^3(\eta) U[\varphi(\eta), \rho(\eta)] \right] \Phi, \tag{15}$$

where the potential U is expressed in terms of the bounce solution as

$$U[\varphi(\eta), \rho(\eta)] \equiv \frac{V''(\varphi)}{Q_H} + \frac{2\kappa\dot{\varphi}^2}{Q_H} + \frac{\kappa}{3Q_H^2}(6\dot{\rho}^2\dot{\varphi}^2 + \rho^2 V'^2(\varphi) - 5\rho\dot{\rho}\dot{\varphi}V'(\varphi)), \tag{16}$$

and again a factor $Q_H \equiv Q$ appears in the quadratic action and this time it reads

$$Q = 1 - \frac{\kappa\rho^2\dot{\varphi}^2}{6}. \tag{17}$$

Unlike the previous prefactor in Eq. (13), this factor is positive definite for a wide class of bounces where one finds exactly one *tunneling* negative mode in the spectrum of the unconstrained action [3–5,12]. When Q becomes negative along the bounce, the pulsation equation is regular and the tunneling negative mode persists, but on top of it one gets an infinite tower of negative modes that has support in the negative-Q region. Furthermore, a negative Q leads to catastrophic particle creation and the instability of the quasiclassical approximation [1].

III. NEGATIVE MODE PROBLEM FOR A POLYNOMIAL POTENTIAL

A. Numerical example of negative Q far from the Planck scale

One might argue that the problematic behavior of Q only appears close to or above the Planck scale where classical general relativity is no longer valid. Here with combined numerical and analytic methods we can show that this is not the case and Q may be negative even far away from the Planck scale. For definiteness we parametrize the quartic potential as

$$V(\varphi) = V_0 + \frac{\lambda}{8}(\varphi^2 - \mu^2)^2 + \frac{\epsilon}{2\mu}(\varphi + \mu) \tag{18}$$

and plot it in Fig. 2. The evolution of the scale factor and scalar field for the Coleman–de Luccia bounce solution and the evolution of the corresponding Q factor is shown in Fig. 3 and we can immediately see that even though the energy scale is significantly below the Planck scale, Q turns negative along the evolution. It might be argued that Q becomes negative because the curvature becomes huge close to the maximal radius of the instanton. However, the four-dimensional Ricci scalar R, given by

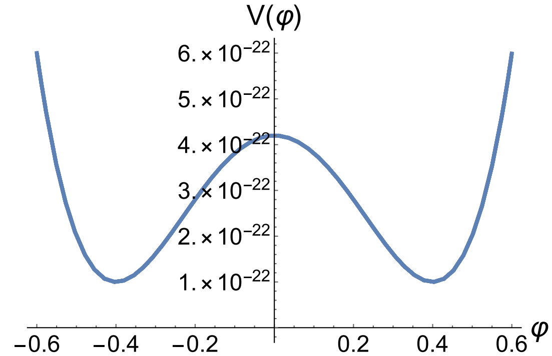

FIG. 2. A plot of the potential (18) for the parameter values $V_0 = 10^{-22}$, $\lambda = 10^{-19}$, $\epsilon = 10^{-30}$, and $\mu = 0.4$. For these parameters $V(\varphi_{\text{top}})$ is 5 orders of magnitude below the Planck scale. The minima for this potential are almost degenerate, a fact that is reflected in the small value for ϵ, but there still is a true and a false vacuum.

$$R = \frac{6}{\rho(\eta)^2}(1 - \dot{\rho}(\eta)^2 - \rho(\eta)\ddot{\rho}(\eta)) \tag{19}$$

is suppressed by a factor of $\frac{1}{\rho^2}$, where the scale factor ρ typically is large in the negative-Q regime. Hence, the curvature is expected to be small as well which is demonstrated for the example above in Fig. 4. In general the intuitive reasoning of φ rolling in the inverted potential gives a good guideline for how to find solutions with negative Q at an arbitrary scale. In particular, taking $V(\varphi_{\text{top}})$ much bigger than $V(\varphi_\pm)$ where $\varphi_\pm$ are the two de Sitter (dS) vacua of the potential will give a fast-rolling field with a large bubble radius which are the exact conditions for negative Q. In the next section we make this argument more precise.

B. Negative Q in the thin-wall approximation

We are interested in a formula for Q that depends only on the parameters of the potential. Critically we note that the smallest value of Q [see Eq. (17)] is obtained when $\rho^2\dot{\varphi}^2$ is maximized which, in the thin-wall limit approximately happens when both ρ and $\dot{\varphi}$ are extremized. Thus, starting with ρ, the general formula for the bubble size [20] is

$$\rho^2 = \frac{\rho_0^2}{1 + 2(\rho_0^2/2\bar{\lambda})^2 + (\rho_0/2\bar{\Lambda})^4}, \tag{20}$$

where ϵ is the separation between the true and false vacuum $\epsilon = V_f - V_t$, ρ_0 is the critical bubble size without gravity and

$$\bar{\lambda}^2 = \frac{3}{\kappa(V_f + V_t)} = \frac{3}{\kappa(2V_f - \epsilon)}, \qquad \bar{\Lambda}^2 = \frac{3}{\kappa(V_f - V_t)}. \tag{21}$$

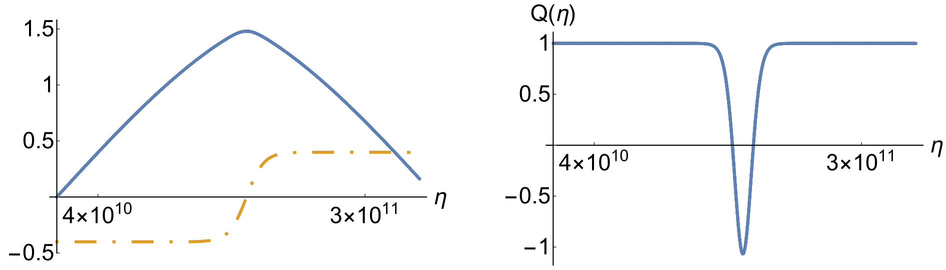

FIG. 3. Left: The evolution of the scale factor $\rho(\eta)/10^{11}$ in blue and the scalar field $\varphi(\eta)$ in orange as a function of Euclidean time η which ranges from 0 to approximately 3.6×10^{11} in this example. Right: The evolution of Q for this instanton clearly demonstrating that it becomes negative along the bounce solution.

This provides a generalization of Coleman and de Luccia's earlier result which can be recovered by setting $\bar{\Lambda}^2/\bar{\lambda}^2 = \pm 1$ corresponding to $V_f = 0$ or $V_t = 0$ respectively. Using the definitions (21), the expression for the bubble size (20) can be written as follows:

$$\rho^2 = \frac{\rho_0^2}{\frac{\kappa\rho_0^2 V_f}{3} + (1 - \frac{\kappa\rho_0^2\epsilon}{12})^2}. \tag{22}$$

This expression shows that in contrast to flat space-time, where the bubble size grows indefinitely when $\epsilon \to 0$, in dS-dS transitions it reaches a maximum size and starts to decrease again. Hence this expression is dramatically simplified by taking a particular value for ϵ, namely

$$\epsilon = \frac{12}{\kappa\rho_0^2} = \frac{3}{4}\kappa\sigma^2, \tag{23}$$

where σ is the bubble tension in the absence of gravity. Due to this choice the bubble size now takes on a particularly simple form

$$\rho^2 = \frac{3}{\kappa V_f}. \tag{24}$$

So far all the calculations have been independent of the particular form of the potential. One can go one step further and obtain a concrete value for ϵ based on the parameters of the potential by choosing

$$V(\phi) = \frac{c^2}{8}(\varphi^2 - \mu^2)^2 + \frac{\epsilon}{2\mu}(\varphi + \mu), \tag{25}$$

where $c^2 > 0, \mu > 0$ and $\epsilon \geq 0$, such that the wall tension σ can be solved for analytically, in the thin-wall approximation

$$\sigma = \int_{\varphi_t}^{\varphi_f} [2(V_s(\varphi) - V_s(\varphi_t))]^{1/2} d\varphi = \frac{2}{3}c\mu^3, \tag{26}$$

where $V_s = \frac{\lambda}{8}(\varphi^2 - \mu^2)^2$ is the symmetric part of the potential and for this potential we have $\varphi_{t,f} = \pm\mu$. This implies that the critical value for ϵ is

$$\epsilon = \frac{1}{3}\kappa c^2 \mu^6. \tag{27}$$

Returning to the definition of Q and making use of the Friedman equation

$$\dot{\rho}^2 = 1 + \frac{\kappa}{3}\rho^2\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right) \tag{28}$$

we obtain

$$Q = 2 - \dot{\rho}^2 - \frac{\kappa}{3}\rho^2 V(\varphi) \tag{29}$$

and consequently, if we restrict ϵ to be of the special form of Eq. (27), we have

$$Q_c = 2 - \dot{\rho}^2 - \frac{V(\varphi)}{V_f} \to Q_c \leq 2 - \frac{V(\varphi)}{V_f}. \tag{30}$$

Hence if we can find a ϕ such that this quantity is negative, we can be sure that Q will be negative somewhere. As a first guess we can take for example $\phi_c = 0$. Numerically we will see that this assumption leaves us very close to the extremal value for Q_c. Writing this in terms of the parameter of the potential given in Eq. (25), we obtain

$$Q_c \leq 2 - \frac{V(\varphi)}{V_f} \approx 2 - \frac{V(0)}{V_f} \tag{31}$$

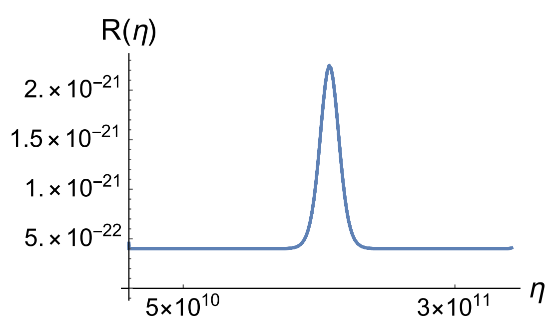

FIG. 4. The four-dimensional Ricci scalar for the instanton solution in Fig. 3.

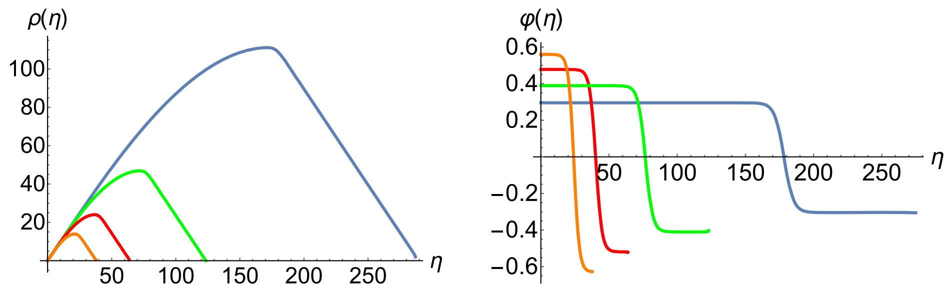

FIG. 5. Plotted here is the evolution of four instantons in the potential given by Eq. (25) but for four different values of μ. The orange, red, green, and blue curves correspond to $\mu = 3/5, 1/2, 2/5$, and $3/10$ respectively. Left: The evolution of the scale factor in terms of Euclidean time η. Right: The evolution of the scalar field.

$$= 2 - \frac{1}{V_f}\left(\frac{c^2}{8}\mu^4 + \frac{\epsilon}{2}\right) \tag{32}$$

$$\approx \frac{3}{2} - \frac{c^2}{8}\frac{\mu^4}{\epsilon} \tag{33}$$

$$= \frac{3}{2}\left(1 - \frac{1}{4\kappa\mu^2}\right) \tag{34}$$

where in the last approximation we took $\varphi_t \approx \mu$ which implies $V_f \approx \epsilon$ and we have plugged in the critical value for epsilon in the second to last line. All this implies that for $\mu^2 < \frac{1}{4\kappa}$ we expect that Q is negative at some point. This confirms our intuition that for steeper potentials we expect Q to be more negative since the scalar field will roll faster in such a potential. Indeed, this choice of ϵ illustrates this beautifully since it eliminates the dependence on the height of the potential. Thus we can find transitions that have the problematic negative prefactor for the kinetic term of the perturbations at *any* scale.

C. Existence of Coleman–de Luccia solutions

It is known [6,21] that for a Coleman–de Luccia bounce solution to exist in a given potential $V(\varphi)$ the following condition should be satisfied:

$$|V''(\varphi_{\rm top})| > 4H^2(\varphi_{\rm top}), \tag{35}$$

where $V''(\varphi) = \frac{d^2V(\varphi)}{d\varphi^2}$ and $H^2(\varphi) = \frac{\kappa V(\varphi)}{3}$. For the quartic potential defined in Eq. (25) we approximate $\varphi_{\rm top} = 0$ and consequently must satisfy

$$\frac{c^2\mu^2}{2} > \frac{2}{3}\kappa\left(\frac{c^2\mu^4}{4} + \epsilon\right). \tag{36}$$

Choosing $\epsilon = \frac{1}{3}\kappa c^2\mu^6$, as above, we find that in order for Coleman–de Luccia instantons to exist we must have

$$\mu^2 < \frac{3}{8\kappa}(\sqrt{17} - 1) \approx \frac{9}{8\kappa}. \tag{37}$$

Hence for $0 < \mu^2 < \frac{1}{4\kappa}$, Coleman–de Luccia solutions exist but are pathological as Q is negative for some part of the instanton. For $\frac{1}{4\kappa} < \mu^2 < \frac{9}{8\kappa}$, the Coleman–de Luccia instantons exist and are perfectly well behaved while for $\mu^2 > \frac{9}{8\kappa}$ no Coleman–de Luccia solutions exist.

D. Comparison with numerics

In deriving the analytic bounds for μ we took several approximations. Therefore it is useful to compare the

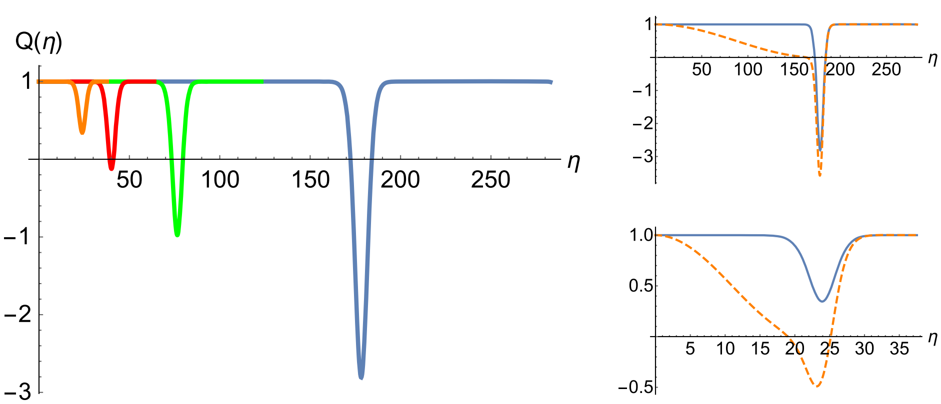

FIG. 6. Left: The kinetic prefactor Q for the bounces shown above. Right: Comparison of Q in blue and Q_L in dashed orange. At the top $\mu = 3/10$ while at the bottom $\mu = 3/5$.

TABLE I. Comparison of various quantities in the analytic expression with the numerics. The ones with a subscript c refer to the values where Q takes its minimum value. ρ_m is the maximum/critical bubble radius and $Q_{\min}$ is the minimum value for Q.

	$\mu = 3/5$		$\mu = 1/2$		$\mu = 2/5$		$\mu = 3/10$	
	Numerics	Analytics	Numerics	Analytics	Numerics	Analytics	Numerics	Analytics
ρ'_c	−0.4901	0	−0.4939	0	−0.4982	0	−0.4976	0
ϕ_c	0.0108	0	0.0037	0	−0.0001	0	0.0002	0
ρ_c	13.266	14.001	23.250	24.132	45.927	47.036	109.852	111.323
ρ_m	13.898	14.001	24.019	24.132	46.916	47.036	111.199	111.323
$Q_{\min}$	0.3457	≤ 0.4583	−0.1242	≤ 0	−0.9768	≤ -0.8437	−2.8087	≤ -2.6667

approximate analytics to the full, numerical solutions. Here we choose $\kappa = c = 1$ for simplicity and without loss of generality and compare the two methods for various values of μ. Note that since ϵ scales like μ^6, the thin-wall approximation is satisfied very rapidly as μ decreases from 1. Four sample geometries are shown in Fig. 5 while their corresponding Q values are plotted in Fig. 6. In Table I we compare the analytics with the numerics, indicating that our approximation yields excellent results. In particular, the approximation of taking $\varphi_c = 0$ is a very good one while the largest uncertainty comes from neglecting the derivative of ρ. From Fig. 6 it is also apparent that the Hamiltonian kinetic prefactor Q and its Lagrangian counterpart Q_L behave in a very similar fashion when μ is large but may differ qualitatively in other situations. In particular since Q_L always develops a negative region, the difference between the two grows as μ shrinks.

These results are still of order one in μ which also corresponds to a field excursion for ϕ of order one which might be considered problematic. On the other hand, the approximations we are using work better for ever smaller values μ, and hence even though it is numerically very hard to find Coleman–de Luccia instantons for these values, we can nevertheless rely on the analytical tools developed to analyze these solutions.

IV. NEGATIVE MODE PROBLEM FOR HIGGS-LIKE POTENTIALS

Taking into account the current experimental bounds of the standard model parameters, the instability scale of the Higgs potential, $\lambda(\mu_\Lambda) = 0$, depends sensitively on the top-quark and Higgs masses. The current bounds at 1σ are [22]

$$1.16 \times 10^9\ \text{GeV} < \mu_\Lambda < 2.37 \times 10^{11}\ \text{GeV} \tag{38}$$

such that the top of the potential barrier lies at about

$$\varphi_{\text{top}} = 4.64 \times 10^{10}\ \text{GeV}, \tag{39}$$

and the barrier height is

$$V_{\text{top}} = 3.46 \times 10^{38}\ \text{GeV}^4 = (4.31 \times 10^9\ \text{GeV})^4. \tag{40}$$

In Planck units $M_{\text{Pl}} = 1/\sqrt{8\pi G} \approx 2.435 \times 10^{18}$ GeV $= 1$, these numbers are

$$4.76 \times 10^{-10} < \mu_\Lambda < 9.73 \times 10^{-8},$$
$$\varphi_{\text{top}} = 1.91 \times 10^{-8}, \qquad V_{\text{top}} = 9.84 \times 10^{-36}. \tag{41}$$

At high energies the Higgs potential can be modeled as [13]

$$V_H = V_0 + \frac{\lambda_H(\varphi)}{4}\varphi^4, \tag{42}$$

$$\lambda_H = q[(\ln\varphi)^4 - (\ln\Lambda)^4], \tag{43}$$

where q is a dimensionless fitting parameter and V_0 is the cosmological constant. A sample potential for specific values of q and Λ is given in Fig. 7. We can further mimic the Higgs potential by choosing $V_0 \ll V_{\text{top}}$ and

(1) $\Lambda = 10^{-9}, q = 10^{-2}$ for the lower-bound value of the instability scale or
(2) $\Lambda = 10^{-7}, q = 10^{-9}$ for the upper-bound value of the instability scale, Eq. (41).

Numerically, we find that for $\Lambda < \Lambda_*$ Q is positive everywhere while for $\Lambda > \Lambda_*$, Q develops a region with $Q < 0$. Choosing the parameters $q = 10^{-7}$ and $V_0 = 10^{-12}$ we find $0.57 < \Lambda_* < 0.6$; see Fig. 8. Therefore for a realistic Higgs-like potential, the negative mode problem shows up only at the Planck-scale values of the instability scale.

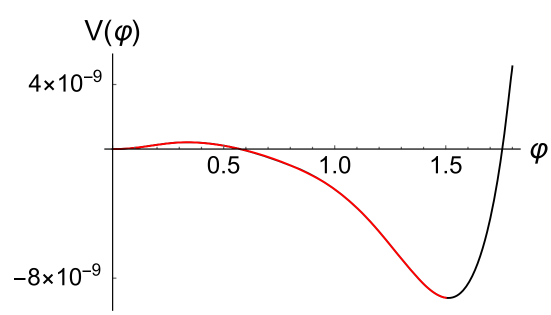

FIG. 7. An example of the Higgs-like potential described in Eq. (42) for $q = 10^{-7}$ and $\Lambda = 0.57$. The bounce solution is marked in red and does not develop a problematic, negative-Q, region.

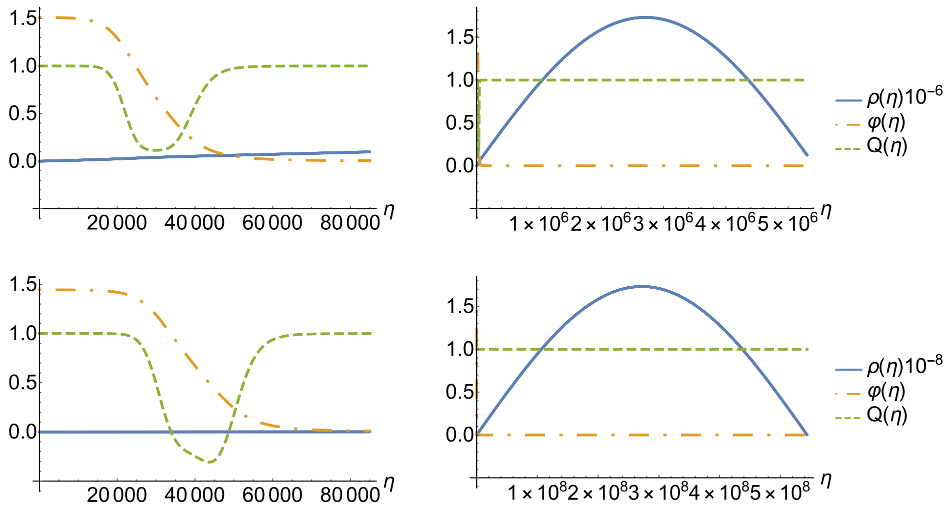

FIG. 8. Here we show the values of the scalar field φ, scale factor ρ and the function Q for the Higgs-like potential (42). The top panel shows the Coleman–de Luccia instanton for $\Lambda = 0.57$ while the bottom one has $\Lambda = 0.6$. The images on the left are zoomed-in versions of the full instantons shown on the right. $M_{\mathrm{Pl}} = 1$ units are used where we zoom in on the part of the instanton where the scalar field tunnels and the problematic behavior of Q might occur.

V. CONCLUDING REMARKS

Using the Hamiltonian approach to false vacuum decay [3,5], we have shown that for generic polynomial potentials the negative mode problem is not related to Planck-scale physics. At the same time we demonstrated that for a Higgs-like potential, a region with $Q < 0$ does not develop for realistic values of the potential's parameters. Instead, the problem only shows up if we assume that the Higgs instability scale is close to the Planck mass.

In the present analysis we used the Hamiltonian reduction scheme, which is based on Dirac's approach to constrained dynamical systems. Within this method, both, gauge-fixed [3] and gauge-invariant [5] approaches, are not problematic and give the same answer. Hence we think this reduction gives a more adequate description of the physical situation than the Lagrangian approach. Note that there is a similar controversy in the counting of the number of negative modes [23,24] of axionic Euclidean wormholes [25,26]. Recently it was advocated that the Hamiltonian approach discussed here, also gives the correct answer in the wormhole case [27]. On the other hand why Lagrangian and Hamiltonian reductions give different kinetic prefactors Q for bounces in false vacuum decay and its physical relevance is still an open, puzzling question. It will be exciting to see if the implementation of a more general framework by not only considering Euclidean but a fully complex lapse as was proposed in Ref. [28] and applied in a cosmological setting in Ref. [29] could resolve this issue. Another interesting issue is to investigate in which realistic cosmological or astrophysical setup a situation with negative Q could occur and what the physical consequences might be. We hope to return to these questions in a future study.

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