Conformality of $1/N$ corrections in Sachdev-Ye-Kitaev-like models

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(Received 1 September 2019; published 6 December 2019)

The Sachdev-Ye-Kitaev (SYK) model is a quantum-mechanical model of N Majorana fermions which displays a number of appealing features—solvability in the strong coupling regime, near-conformal invariance and maximal chaos—which make it a suitable model for black holes in the context of the AdS/CFT holography. In this paper, we show for the colored SYK model and several of its tensor model cousins that the next-to-leading order in the large-N expansion preserves the conformal invariance of the two-point function in the strong-coupling regime, up to the contribution of the pseudo-Goldstone bosons due to the explicit breaking of the symmetry which are already seen in the leading-order four-point function. We also comment on the composite field approach for computing correlation functions in colored tensor models.

DOI: 10.1103/PhysRevD.100.125005

I. INTRODUCTION

In a series of seminal conferences [1–3] Kitaev brought attention to the—now so-called—Sachdev-Ye-Kitaev (SYK) model which displays a set of appealing features in the context of holography, for which a detailed account has been given in Ref. [4]. This model—a simplification of a previous one by Sachdev and Ye [5]—corresponds to a quantum-mechanical system of N Majorana fermions (possibly organized in different families [6]) with an interaction of order q with Gaussian random couplings.

The first key property is that it is solvable at large coupling (or equivalently large time or in the infrared regime) in the large-N limit. This is very precious since systems that are tractable in the large-coupling regime are very scarce. Moreover, in this infrared limit, the system displays an approximate conformal symmetry. Conformal invariance in one dimension is equivalent to reparametrization invariance, and thus is infinite-dimensional which leads to many simplifications; in particular systems at zero and finite temperature are easily related in this regime. This symmetry is spontaneously broken and leads to Goldstone bosons. Since the full action breaks the symmetry explicitly but slightly, these are in fact pseudo-Goldstone bosons, with their dynamics being described by the Schwarzian action. The latter are responsible for the last property of the model: the Lyapunov exponent, which measures the chaos in the system, reaches the maximal bound proposed in Ref. [7] and thus the system is maximally chaotic. All together these properties point toward a (near) AdS_2/CFT_1 interpretation of the model (see Refs. [8–11] for references on near AdS_2). In gravitational theories the maximally chaotic objects are black holes; hence one can expect that the bulk geometry dual of the SYK model corresponds to the near-horizon geometry of black holes. The fact that one can access the strong-coupling regime offers an inestimable window on the quantum properties of black holes.

Another interesting property is its equivalence with random tensor fields theories in the large-N limit, as was pointed in Ref. [12] (some selected references on tensor models include Refs. [13–24]). The Gurau-Witten model [12] is the simplest colored tensor model and consists of a set

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of $q = D + 1$ real fermionic tensor fields with D indices of size N transforming in the fundamental of $\mathrm{O}(N)^{\otimes D}$, the invariance group being $\mathrm{O}(N)^{D(D+1)/2}$ (up to a discrete factor). Two other models of interest are the case of q complex fermionic tensor fields with a $\mathrm{U}(N)$ invariance and the so-called multiorientable model which is given by a complex fermionic field with $D = 3$ indices [25] (there is no q because one is considering an uncolored tensor model [18]). The bosonic zero-dimensional versions of these models have been studied in Refs. [13,14,26–29]. The main simplification in these models occurs because the randomness is moved in the fields and there is a single (fixed) coupling constant. While it is necessary to average over the random couplings by performing the Gaussian integration over them (quenching), implying that one describes a thermodynamical ensemble, the tensor models feature a unique fixed coupling constant and represent a genuine quantum system [12]. Moreover the combinatorics and renormalization properties have been largely studied and one can make use of all the tools already developed.

The disordered and tensor SYK models have been extended in several directions: higher dimensions and lattices [30–38], $N = 1$, 2 supersymmetry [39,40] (see Ref. [41] for a related system with $N = 4$), and nonquenched disorder [42–44]. Various properties have been studied in the last year: spectrum and thermodynamical properties [45–55], correlation functions [45,56–61], the dynamics of the Goldstone bosons [4,60,62,63], the relation with matrix models (for both the disordered and tensor versions) [46,51,64–68], transport properties [30,69,70], and renormalization and phases [52]. Experimental realizations have been proposed in Refs. [71–74].

It was shown in Refs. [4,45] that the next-to-leading-order (NLO) correction in the coupling constant explicitly breaks the conformal invariance in the leading order (LO) in the large-N expansion. The problem we address in this paper is the opposite, i.e., is the conformal symmetry explicitly broken in the NLO in N for the LO in the coupling constant? We consider this question in the models mentioned above: the colored[1] SYK model with disorder, and the real, complex and multiorientable SYK tensor models. We find that in the first three models the NLO two-point function is compatible with conformal symmetry and thus should scale in the same way as the LO two-point function. This means that in the infrared the dimension of the fermions is not modified by the first subleading correction in the large-N expansion. This finding may have some implications for the construction of the bulk dual of the SYK model which has started in Ref. [59] (see also Refs. [4,9,56] and Refs. [75,76] for other proposals). Our method consists in analyzing the transformation properties of the NLO two-point function from the Schwinger-Dyson equation (the Feynman graphs contributing at this order have been studied in Ref. [58]; see also Ref. [28]): this is sufficient to reach our conclusions except for the multiorientable tensor model. In the latter case the conclusion depends on the explicit form of the NLO two-point function and the full analysis is outside the scope of this paper. It is important to note that throughout this paper the divergent contribution to the LO four-point function due to the spontaneous breaking of the conformal symmetry is implicitly excluded (as is implied by any statement in previous works about the conformality of some object) [4]: this contribution can be taken into account only by looking at the NLO in the coupling which regularizes the divergence.

A fruitful approach for computing the correlation functions and determining the structure of the graph appearing at some order in N is to write the action in terms of composite fields—to be identified with the two-point function and self-energy—instead of the fundamental fermions [4,56,57,62]. We briefly discuss in the Appendix how such an approach can be undertaken for the colored tensor models.

The structure of the paper is as follows. In Secs. II to IV we study successively the SYK model, the (real and complex) colored tensor models and the multiorientable tensor models. The results are discussed in Sec. V. The Appendix describes how to perform a composite field analysis for the real colored tensor model.

II. SYK MODEL WITH DISORDER

A. The model

In this section, we consider a specific case of the colored SYK model introduced in Ref. [6]. This has the main advantage of simplifying the study of the combinatorics (which has been done in Ref. [58] in detail) and makes it easier to compare with the Gurau-Witten colored tensor model described later. However, the model we study here keeps all the interesting features of the usual SYK model at leading order.

The colored SYK model we consider is a model of qN real massless fermions ψ_i^c where $c \in \{1\ldots q\}$, $i \in \{1\ldots N\}$, with q being the color index. This model is defined through the following Euclidean space partition function:

$$Z_{N,\lambda}^{\mathrm{SYK}} = \int \mathrm{d}\lambda \, \exp\left(-\frac{N^{q-1}}{2\lambda^2} \sum_{\{i_k\}_{k=1}^{q}}^{N} \lambda_{i_1\ldots i_q}\lambda_{i_1\ldots i_q}\right) \times \int \prod_{c=1}^{q} \mathcal{D}\psi^c \mathrm{e}^{-\int \mathrm{d}t\, L[\psi,\lambda]}, \tag{2.1}$$

where

$$L[\psi,\lambda] = \frac{1}{2}\sum_{c=1}^{q}\sum_{i_c=1}^{N} \psi_{i_c}^c \partial_t \psi_{i_c}^c + \frac{\mathrm{i}^{q/2}}{q!} \sum_{\{i_k\}_{k=1}^{q}}^{N} \lambda_{i_1\ldots i_q} \prod_{c=1}^{q} \psi_{i_c}^c. \tag{2.2}$$

[1]This study is restricted to the colored SYK model (already discussed in Refs. [6,44,58]) because the combinatorics of graphs involving an (anti)symmetric tensor is notoriously difficult and was one of the reasons for the lack of progress in tensor models, until Gurau solved this problem by introducing colors [13].

No particular assumption is made on the symmetry of the random couplings $\lambda_{i_1\ldots i_q}$ and it is convenient to define

$$g = \lambda^2. \tag{2.3}$$

The reason is that there is no need for antisymmetry on the indices here since no color appears twice in the interaction term. Moreover, this simplifies further the combinatorics as this prohibits melonic graphs from contributing to subleading amplitudes in $1/N$ as well.

The free scalar two-point function $G_f(t_1, t_2)$ is defined, after an arbitrary choice of color c_0 (which is kept implicit in the notation), by

$$G_f(t_1, t_2) = \frac{1}{N}\left\langle \sum_i T\psi_i^{c_0}(t_1)\psi_i^{c_0}(t_2)\right\rangle_0 = \frac{1}{2}\mathrm{sign}(t_1 - t_2), \tag{2.4}$$

whose Fourier transform is

$$G_f(\omega) = -\frac{1}{i\omega}. \tag{2.5}$$

We also have that,

$$\langle T\psi_{i_c}^{c}(t_1)\psi_{i_{c'}}^{c'}(t_2)\rangle_0 = \delta_{cc'}\delta_{i_c i_{c'}}G_f(t_1, t_2). \tag{2.6}$$

The exact disorder-averaged two-point function $G_e(t_1, t_2)$ is defined by the following relations:

$$G_e(t_1, t_2) = \frac{1}{N}\left\langle \sum_i T\psi_i^{c_0}(t_1)\psi_i^{c_0}(t_2)\right\rangle, \tag{2.7}$$

$$\langle T\psi_{i_c}^{c}(t_1)\psi_{i_{c'}}^{c'}(t_2)\rangle = \delta_{cc'}\delta_{i_c i_{c'}}G_e(t_1, t_2). \tag{2.8}$$

The Feynman graphs of this model are made up of the following building blocks:

(1) The vertices are $q + 1$ valent.

(2) The edges are of two types. The fermionic edges carry a *color* label $c \in \{1\ldots q\}$. The disorder edges carry a 0 label. Edges are labeled in such a way that no two adjacent edges have the same label (color or disorder).

(3) The faces are cycles made alternatively of edges labeled 0 and c, for some color label.

The free energy of the colored SYK has a $1/N$ expansion of the form

$$F_{N,\lambda}^{\mathrm{SYK}} = \log Z_{N,\lambda}^{\mathrm{SYK}} = \sum_{\ell_m \geq 0} N^{1-\ell_m}F_{[\ell_m]}(\lambda), \tag{2.9}$$

where $\ell_m(G)$ is a characteristic number of the Feynman graph G; more precisely, it is the number of multicolored cycles of the graph $G_{\backslash 0}$ that is obtained from G by contracting all edges labeled 0. From these considerations, we get that the exact two-point function also admits a $1/N$ expansion

$$G_e(t_1, t_2) = \sum_{\ell_m \geq 0} N^{-\ell_m}G_{[\ell_m]}(t_1, t_2). \tag{2.10}$$

B. Leading order

The leading order of the SYK model has been described in several works [4,45], and the colored SYK model has been described in Ref. [6]; therefore, we only give a very brief account and the reader may refer to the excellent presentations mentioned above for more details. The leading-order two-point function ($\ell_m = 0$), $G_{[0]}$ satisfies the following equation:

$$G_{[0]}(t_1, t_2) = G_f(t_1, t_2) + g\int \mathrm{d}t\mathrm{d}t' G_f(t_1, t)\Sigma_{[0]}(t, t')G_{[0]}(t', t_2), \tag{2.11}$$

where $\Sigma_{[0]}$ is the leading-order self-energy. The above relation is easily obtained from the usual relation between the two-point function and the self-energy

$$G_e(t_1, t_2) = (G_f(t_1, t_2)^{-1} - \Sigma(t_1, t_2))^{-1}, \tag{2.12}$$

where the inverse here means that the two-variable functions $G_f(t_1, t_2), \Sigma(t_1, t_2)$ are seen as matrices for the convolution product. The graphs appearing at leading order are the *melonic* graphs, also called melon graphs (see Refs. [12,15] for a description of these graphs). This implies that

$$\Sigma_{[0]}(t, t') = G_{[0]}(t, t')^{q-1}. \tag{2.13}$$

Therefore we have

$$G_{[0]}(t_1, t_2) = G_f(t_1, t_2) + g\int \mathrm{d}t\mathrm{d}t' G_f(t_1, t)G_{[0]}(t, t')^{q-1}G_{[0]}(t', t_2). \tag{2.14}$$

In Fourier space, this equation reads

$$-\mathrm{i}\omega G_{[0]}(\omega) = 1 + g\Sigma_{[0]}(\omega)G_{[0]}(\omega). \tag{2.15}$$

Consequently, in the infrared limit[2] $G_{[0]} \to \bar{G}_{[0]}$, the left-hand side drops out and one has

$$0 = 1 + g\bar{\Sigma}_{[0]}(\omega)\bar{G}_{[0]}(\omega), \tag{2.16}$$

where $\bar{G}_{[0]}$ stands for the infrared limit of $G_{[0]}$ and $\bar{\Sigma}_{[0]}$ stands for the infrared limit of $\Sigma_{[0]}$. In the rest of the paper, any barred quantity denotes the infrared or large-coupling limit of the corresponding unbarred quantity. In position space, we have

$$g\int \mathrm{d}t\bar{G}_{[0]}(t_1, t)^{q-1}\bar{G}_{[0]}(t, t_2) = -\delta(t_1 - t_2). \tag{2.17}$$

[2] One recovers the same results if one considers the large-coupling g limit.

In this regime, the two-point function transforms as

$$\bar{G}_{[0]}(\sigma,\sigma') = \frac{1}{|f'(t)f'(t')|^{1/q}} \bar{G}_{[0]}(t,t') \qquad (2.18)$$

under reparametrizations $\sigma = f(t)$ and $\sigma' = f('t)$.

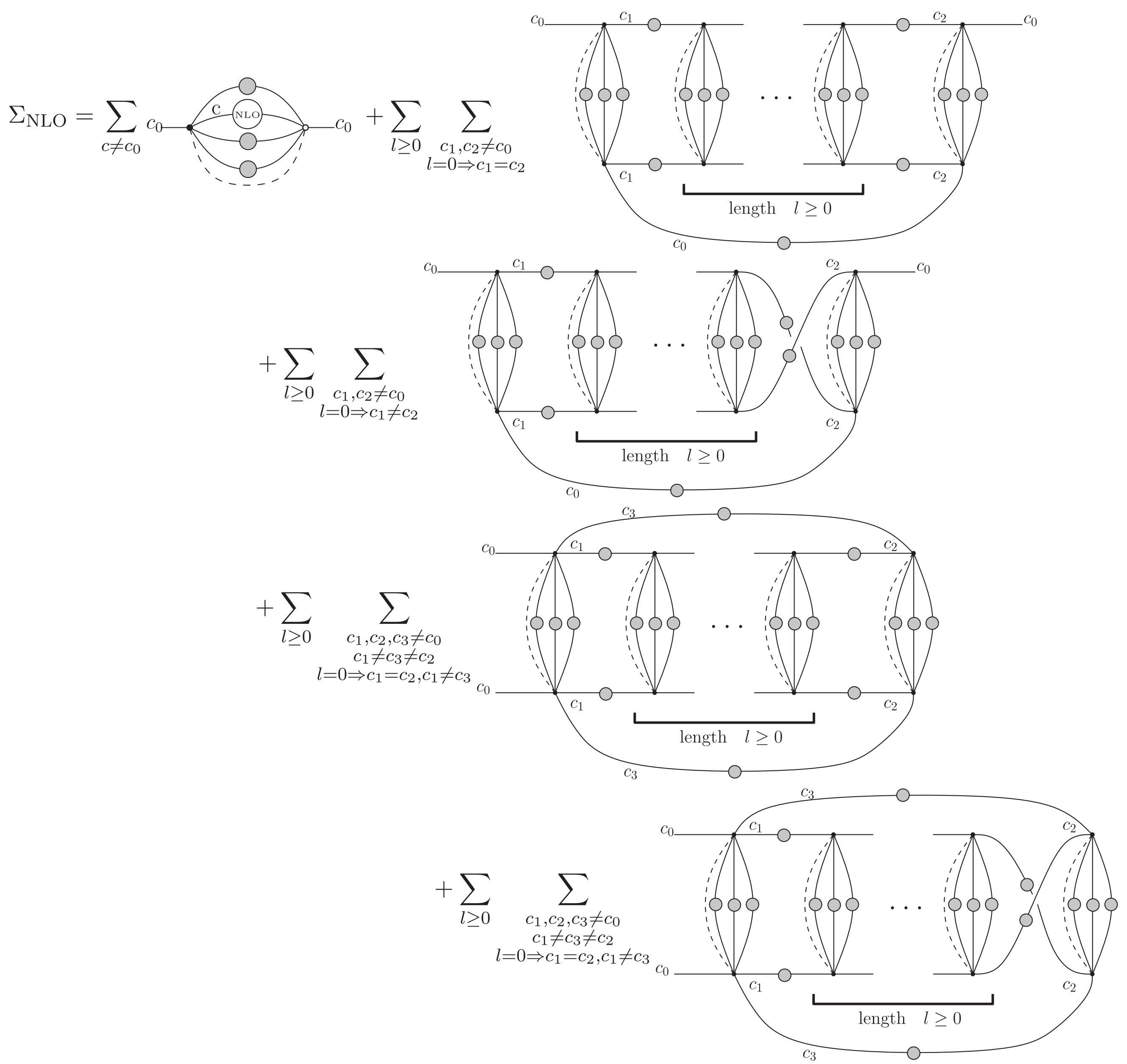
C. The next-to-leading order

In this subsection, we want to investigate the NLO of the colored SYK model.

We want to study the possible corrections to the scaling dimension of the two-point function in the conformal sector. The NLO is given by the graphs with $\ell_m = 1$ in Eq. (2.9), which means that their contracted graphs have one multicolored cycle. To obtain the two-point function G_{NLO} we first need to describe the self-energy Σ_{NLO} at NLO, where we defined

$$G_{\mathrm{NLO}} \coloneqq G_{[1]}, \qquad \Sigma_{\mathrm{NLO}} \coloneqq \Sigma_{[1]} \qquad (2.19)$$

[see Eq. (2.10)]. We use the results from Ref. [58] which classified the NLO graphs. From this work, it is possible to write the self-energy at NLO as



(2.20)

The edges with gray disc insertions represent dressed leading-order propagators.

We give a few indications of the correspondence between these terms and the graphs described in Ref. [58]. In the language of Ref. [58], the two-point function is obtained by cutting an edge of an NLO vacuum graph. NLO vacuum graphs are ladder diagrams that are closed on themselves. They can be closed with an even or odd number of crossings. The even number of crossings class is equivalent to the noncrossing case, while the odd number of crossings class is equivalent to the one-crossing case. As described in Ref. [58] the graphs contributing to G_{NLO} exist in two types A and B, which are themselves separated into two subtypes $\varnothing$ or not $\varnothing$. The second subtype always contributes to the first term of the right-hand side of Eq. (2.20), while the type A, $\varnothing$ (respectively B, $\varnothing$) case accounts for the second and third (fourth and fifth) terms of the right-hand side of Eq. (2.20). These equations can be rewritten using the further defined color space matrix Q. To this aim we first define a matrix in the color space, whose elements $K_{c,c'}(t_1,t_2;t_3,t_4)$ are defined by the equation

$$K_{c,c'}(t_1,t_2;t_3,t_4) = -g(1-\delta_{c,c'})G_{[0]}(t_1,t_3)G_{[0]}(t_2,t_4)G_{[0]}(t_3,t_4)^{q-2}, \tag{2.21}$$

for $c,c' \in \{1,\ldots,q\}$. The analogue of this operator reappears with slight modifications in the tensor model context as well. One defines the matrices Q_0 and Q, whose elements are

$$Q_{0,c,c'}(t_1,t_2;t_3,t_4) = \delta_{c,c'}(G_{[0]}(t_1,t_3)G_{[0]}(t_2,t_4) - G_{[0]}(t_1,t_4)G_{[0]}(t_2,t_3)), \tag{2.22}$$

$$Q_{c,c'}(t_1,t_2;t_3,t_4) = \sum_{n\geq 0}[K^n * Q_0]_{c,c'}(t_1,t_2;t_3,t_4) \tag{2.23}$$

$$= [(\delta^{\otimes 2}\otimes \mathbb{1} - K)^{-1}Q_0]_{c,c'}(t_1,t_2;t_3,t_4) \tag{2.24}$$

where $\delta^{\otimes 2} = \delta(t_1-t_3)\delta(t_2-t_4)$ and the $*$ product here means both matrix and convolution products of the form

$$[K * Q_0]_{c,c'}(t_1,t_2;t_3,t_4) = \sum_{c''}\int \mathrm{d}t\mathrm{d}t' K_{c,c''}(t_1,t_2;t,t')Q_{0,c'',c'}(t,t';t_3,t_4), \tag{2.25}$$

and the powers n of K are taken with respect to this product.

Notice here that Eq. (2.22) is singular if K admits an eigenvector with eigenvalue 1, which is the case in the large-coupling limit. As explained in the Introduction (Sec. I) this signals a spontaneous breaking of the conformal symmetry and for this reason this contribution can be ignored: it is an artifact of the limit which can be handled by including subleading corrections in the coupling constant. Since the latter break the conformal symmetry any statement about the conformal symmetry assumes that one is considering the large-coupling limit with the divergent contribution removed [4].

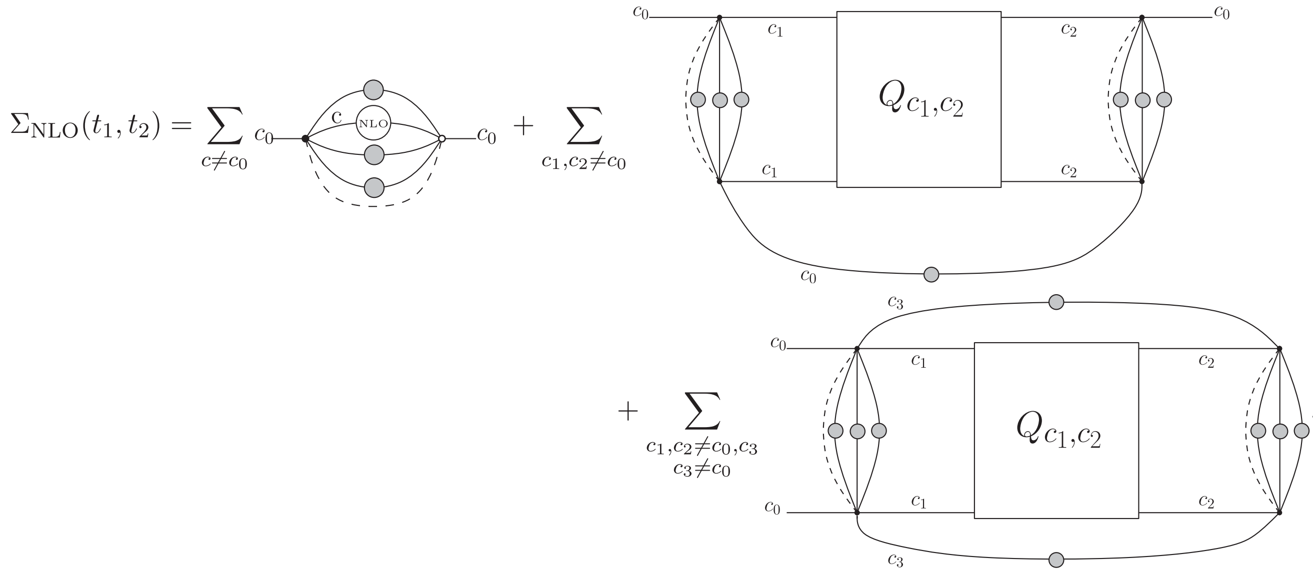
If we consider the one-particle irreducible (1PI) counterpart of Q, written as Γ, we find that it satisfies Schwinger-Dyson-like equations of the form

$$\Gamma(t_1,t_2;t_3,t_4) = \Gamma_0(t_1,t_2;t_3,t_4) + [\Gamma * K](t_1,t_2;t_3,t_4) \tag{2.26}$$

and element-wise Γ_0 reads as

$$\Gamma_{0,c,c'}(t_1,t_2;t_3,t_4) = g(1-\delta_{c,c'})\delta(t_1-t_3)\delta(t_2-t_4)G_{[0]}(t_1,t_2)^{q-2}. \tag{2.27}$$

We can rewrite the equation for Σ_{NLO}:



(2.28)

Equation (2.28) can be rewritten formally as

$$
\begin{aligned}
\Sigma_{\text{NLO}}(t_1,t_2) &= (q-1)gG_{[0]}(t_1,t_2)^{q-2}G_{\text{NLO}}(t_1,t_2) \\
&\quad + g^2\int \mathrm{d}t\mathrm{d}t'\left[\sum_{c_1,c_2\neq c_0} Q_{c_1,c_2}(t_1,t;t_2,t')\right]G_{[0]}(t_1,t)^{q-2}G_{[0]}(t_2,t')^{q-2}G_{[0]}(t,t') \\
&\quad + g^2\int \mathrm{d}t\mathrm{d}t'\left[\sum_{c_3\neq c_0}\sum_{c_1,c_2\neq c_0,f_3} Q_{c_1,c_2}(t_2,t_2;t,t')\right]G_{[0]}(t_1,t_2)^{q-3} \\
&\quad \times G_{[0]}(t_1,t)G_{[0]}(t_2,t')G_{[0]}(t,t')^{q-2} \\
&\coloneqq \Sigma^{(1)}_{\text{NLO}}(t_1,t_2) + \Sigma^{(2)}_{\text{NLO}}(t_1,t_2) + \Sigma^{(3)}_{\text{NLO}}(t_1,t_2).
\end{aligned} \tag{2.29}
$$

We now turn our attention to the NLO two-point function. It can be obtained by rewriting Eq. (2.12) in Fourier space:

$$G_e(\omega) = -\frac{1}{i\omega}\left(1+\frac{\Sigma(\omega)}{i\omega}\right)^{-1} = G_f(\omega)\sum_{p\geq 0}\left(-\frac{\Sigma(\omega)}{i\omega}\right)^p. \tag{2.30}$$

Since $G_e(\omega) = \sum_{\ell_m\geq 0} N^{-\ell_m}G_{[\ell_m]}(\omega)$ and $\Sigma(\omega) = \sum_{\ell_m\geq 0} N^{-\ell_m}\Sigma_{[\ell_m]}(\omega)$, we have

$$G_{\text{NLO}}(\omega) = \left[G_f(\omega)\sum_{qge0}\left(-\frac{\Sigma_{[0]}(\omega)}{i\omega}\right)^q\right]\Sigma_{\text{NLO}}(\omega)\left[G_f(\omega)\sum_{p\geq 0}\left(-\frac{\Sigma_{[0]}(\omega)}{i\omega}\right)^p\right] \tag{2.31}$$

$$= (G_f(\omega)^{-1} - \Sigma_{[0]}(\omega))^{-1}\Sigma_{\text{NLO}}(\omega)(G_f(\omega)^{-1} - \Sigma_{[0]}(\omega))^{-1} \tag{2.32}$$

$$= G_{[0]}(\omega)\Sigma_{\text{NLO}}(\omega)G_{[0]}(\omega). \tag{2.33}$$

Transforming this expression back to position space leads to

$$G_{\text{NLO}}(t_1,t_2) = \int \mathrm{d}t\mathrm{d}t' G_{[0]}(t_1,t)\Sigma_{\text{NLO}}(t,t')G_{[0]}(t',t_2). \tag{2.34}$$

Inserting Eq. (2.29) gives an integral equation for G_{NLO}:

$$G_{\mathrm{NLO}}(t_1,t_2) = (q-1)g\int \mathrm{d}t\mathrm{d}t' G_{[0]}(t_1,t)G_{[0]}(t,t')^{q-2}G_{\mathrm{NLO}}(t,t')G_{[0]}(t',t_2) + \int \mathrm{d}t\mathrm{d}t' G_{[0]}(t_1,t)(\Sigma^{(2)}_{\mathrm{NLO}}(t,t') + \Sigma^{(3)}_{\mathrm{NLO}}(t,t'))G_{[0]}(t',t_2). \tag{2.35}$$

This can be simplified further by recognizing the operator K_{cc}:

$$\int \mathrm{d}t\mathrm{d}t'[\delta(t_1-t)\delta(t_2-t') - K(t_1,t_2;t,t')]G_{\mathrm{NLO}}(t,t') = \int \mathrm{d}t\mathrm{d}t' G_{[0]}(t_1,t)(\Sigma^{(2)}_{\mathrm{NLO}}(t,t') + \Sigma^{(3)}_{\mathrm{NLO}}(t,t'))G_{[0]}(t',t_2). \tag{2.36}$$

Note that $K \coloneqq \sum_c K_{cc}$ which contains a factor $\sum_c \delta_{cc} = q$. Defining the inverse of $(1-K)$ by L

$$\int \mathrm{d}t\mathrm{d}t'[\delta(t_1-t)\delta(t_2-t') - K(t_1,t_2;t,t')]L(t,t';t_3,t_4) = \delta(t_1-t_3)\delta(t_2-t_4), \tag{2.37}$$

the final expression for G_{NLO} reads

$$G_{\mathrm{NLO}}(t,t') = \int \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4 L(t,t';t_1,t_2)G_{[0]}(t_1,t_3)(\Sigma^{(2)}_{\mathrm{NLO}}(t_3,t_4) + \Sigma^{(3)}_{\mathrm{NLO}}(t_3,t_4))G_{[0]}(t_4,t_2). \tag{2.38}$$

We now want to study the scaling dimension of the NLO in the large-coupling limit. The large-coupling limit of Eq. (2.38) follows by adding bars to all quantities[3]:

$$\begin{aligned}
\bar{G}_{\mathrm{NLO}}(t,t') &= \int \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4 \bar{L}(t,t';t_1,t_2)\bar{G}_{[0]}(t_1,t_3)(\bar{\Sigma}^{(2)}_{\mathrm{NLO}}(t_3,t_4) + \bar{\Sigma}^{(3)}_{\mathrm{NLO}}(t_3,t_4))\bar{G}_{[0]}(t_4,t_2). \\
&\quad \times \int \mathrm{d}t\mathrm{d}t'[\delta(t_1-t)\delta(t_2-t') - \bar{K}(t_1,t_2;t,t')]\bar{L}(t,t';t_3,t_4) = \delta(t_1-t_3)\delta(t_2-t_4), \\
\bar{\Sigma}^{(2)}_{\mathrm{NLO}}(t_1,t_2) &= g^2\int \mathrm{d}t\mathrm{d}t'\left[\sum_{c_1,c_2\neq c_0}\bar{Q}_{c_1,c_2}(t_1,t;t_2,t')\right]\bar{G}_{[0]}(t_1,t)^{q-2}\bar{G}_{[0]}(t_2,t')^{q-2}\bar{G}_{[0]}(t,t'), \\
\bar{\Sigma}^{(3)}_{\mathrm{NLO}}(t_1,t_2) &= g^2\int \mathrm{d}t\mathrm{d}t'\left[\sum_{c_3\neq c_0}\sum_{c_1,c_2\neq c_0,f_3}\bar{Q}_{c_1,c_2}(t_2,t_2;t,t')\right]\bar{G}_{[0]}(t_1,t_2)^{q-3} \\
&\quad \times \bar{G}_{[0]}(t_1,t)\bar{G}_{[0]}(t_2,t')\bar{G}_{[0]}(t,t')^{q-2}.
\end{aligned} \tag{2.39}$$

Hence, we need to find the transformation properties of all the objects which appear in these formulas before finding the transformation of G_{NLO}.

To this aim we come back to the equations (2.26) and use Eq. (2.18) repetitively. These imply that, in the conformal sector, the 1PI counterpart of $Q \to \bar{Q}$ has scaling dimension $(q-1)/q$. Indeed, it is easy to check that the terms $\bar{\Gamma}_0$ have scaling dimension $(q-1)/q$:

$$\bar{\Gamma}_{0,c,c'}(\sigma_1,\sigma_2;\sigma_3,\sigma_4) = \frac{\bar{\Gamma}_{0,c,c'}(t_1,t_2;t_3,t_4)}{|f'(t_1)f'(t_2)f'(t_3)f'(t_4)|^{(q-1)/q}}, \tag{2.40}$$

for $\sigma_i = f(t_i)$, $i = 1,\ldots,4$. This follows from

[3]In a previous version, we had assumed incorrectly that the first term in Eq. (2.29) is subleading with respect to the other terms in this regime. Part of the origin of this incorrect statement is a typo in Eq. (51) of Ref. [77]: the equation should read $G_{\mathrm{NLO}} \sim g^2\partial_g G_{\mathrm{LO}}^{q+1}$, whereas the exponent in Ref. [77] was $q+2$. However, this assumption is in fact not necessary and we can proceed differently.

$$\begin{aligned}
&\bar{\Gamma}_{0,c,c'}(\sigma_1,\sigma_2;\sigma_3,\sigma_4)\\
&= (q-2)g(1-\delta_{c,c'})\frac{\delta(t_1-t_3)\delta(t_2-t_4)}{|f'(t_3)f'(t_4)|}\\
&\quad\times\frac{\bar{G}_{[0]}(t_1,t_2)^{q-2}}{|f'(t_1)f'(t_2)|^{(q-2)/q}}\\
&= \frac{|f'(t_3)f'(t_4)|^{1/q}}{|f'(t_1)f'(t_2)|^{1/q}}\frac{\bar{\Gamma}_{0,c,c'}(t_1,t_2;t_3,t_4)}{|f'(t_3)f'(t_4)||f'(t_1)f'(t_2)|^{(q-2)/q}},
\end{aligned}$$

using Eq. (2.18).

Let us consider the terms of the form $[\bar{\Gamma} * K](t_1,t_2;t_3,t_4)$. From the definition of K and Eq. (2.18) we find

$$\bar{K}_{c,c'}(\sigma_1,\sigma_2;\sigma_3,\sigma_4) = \frac{|f'(t_3)f'(t_4)|^{1/q-1}}{|f'(t_1)f'(t_2)|^{1/q}}\bar{K}_{c,c'}(t_1,t_2;t_3,t_4). \tag{2.41}$$

Therefore, using Eqs. (2.40) and (2.41) it is simple to check that if $\bar{\Gamma}(t_1,t_2;t_3,t_4)$ is a solution of Eq. (2.26) in the conformal sector, then the equation satisfied by $\bar{\Gamma}(\sigma_1,\sigma_2;\sigma_3,\sigma_4)$ transforms into Eq. (2.26) for $\sigma_i = f(t_i)$ provided that

$$\bar{\Gamma}(t_1,t_2;t_3,t_4) = |f'(t_1)f'(t_2)f'(t_3)f'(t_4)|^{1-1/q} \times\bar{\Gamma}(\sigma_1,\sigma_2;\sigma_3,\sigma_4). \tag{2.42}$$

One is then interested in the scaling dimension of $\bar{Q}$. We have

$$\begin{aligned}
\bar{Q}(t_1,t_2;t_3,t_4) &= \bar{Q}_0(t_1,t_2;t_3,t_4)\\
&\quad+\int \mathrm{d}t\mathrm{d}t'\mathrm{d}\tau\mathrm{d}\tau'([\bar{G}_{[0]}(\tau,t_3)\bar{G}_{[0]}(\tau',t_4)\\
&\quad-\bar{G}_{[0]}(\tau,t_4)\bar{G}_{[0]}(\tau',t_3)]\\
&\quad\times\bar{G}_{[0]}(t_1,t)\bar{G}_{[0]}(t_2,t')\bar{\Gamma}(t,t';\tau,\tau')),
\end{aligned} \tag{2.43}$$

where the integration is done element-wise. From this last equality, one shows that the scaling dimension of $\bar{Q}(t_1,t_2;t_3,t_4)$ is $1/q$ by using the scaling properties of $\bar{G}_{[0]}$ as well as the ones of $\bar{\Gamma}$. Then, as we know that $\bar{Q}$ has scaling dimension $1/q$, a simple computation shows that $\bar{\Sigma}^{(2)}_{\mathrm{NLO}}$ and $\bar{\Sigma}^{(3)}_{\mathrm{NLO}}$ have scaling dimension $\frac{q-1}{q}$.

Knowing the transformation of $\bar{K}$, we can study that of $\bar{L} = (1-\bar{K})^{-1}$. We start from the definition (2.37) and perform a reparametrization of both sides:

$$\begin{aligned}
\delta(\sigma_1-\sigma_3)\delta(\sigma_2-\sigma_4) &= \frac{\delta(t_1-t_3)\delta(t_2-t_4)}{|f'(t_3)f'(t_4)|}\\
&= \int \mathrm{d}\sigma\mathrm{d}\sigma'[\delta(\sigma_1-\sigma)\delta(\sigma_2-\sigma')-\bar{K}(\sigma_1,\sigma_2;\sigma,\sigma')]\bar{L}(\sigma,\sigma';\sigma_3,\sigma_4)\\
&= \int \mathrm{d}t\mathrm{d}t'|f'(t)f'(t')|\left[\frac{\delta(t_1-t)\delta(t_2-t')}{|f'(t)f'(t')|}-\frac{|f'(t)f'(t')|^{1/q-1}}{|f'(t_1)f'(t_2)|^{1/q}}\bar{K}(t_1,t_2;t,t')\right]\\
&\quad\times\bar{L}(\sigma,\sigma';\sigma_3,\sigma_4)\\
&= \int \mathrm{d}t\mathrm{d}t'\frac{|f'(t)f'(t')|^{1/q}}{|f'(t_1)f'(t_2)|^{1/q}}[\delta(t_1-t)\delta(t_2-t')-\bar{K}(t_1,t_2;t,t')]\bar{L}(\sigma,\sigma';\sigma_3,\sigma_4)
\end{aligned}$$

for $\sigma,\sigma' = f(t), f(t')$. Hence, consistency between both sides leads to the transformation

$$\bar{L}(\sigma_1,\sigma_2;\sigma_3,\sigma_4) = \frac{|f'(t_3)f'(t_4)|^{1/q-1}}{|f'(t_1)f'(t_2)|^{1/q}}\bar{L}(t_1,t_2;t_3,t_4). \tag{2.44}$$

Thanks to these different relations, we can deduce the scaling dimension of $\bar{G}_{\mathrm{NLO}}$:

$$\bar{G}_{\mathrm{NLO}}(\sigma_1,\sigma_2) = \frac{\bar{G}_{\mathrm{NLO}}(t_1,t_2)}{|f'(t_1)f'(t_2)|^{1/q}}. \tag{2.45}$$

This can in turn be used to determine the scaling of the self-energy:

$$\bar{\Sigma}_{\mathrm{NLO}}(\sigma,\sigma') = |f'(t)f'(t')|^{1/q-1}\bar{\Sigma}_{\mathrm{NLO}}(t,t'). \tag{2.46}$$

As a consistency check, every term of the strong-coupling limit of Eq. (2.29) transforms in the same way.[4] As a consequence the scaling dimension of $\bar{G}_{\mathrm{NLO}}$ is $1/q$ in the conformal sector. This is the same scaling dimension as for $\bar{G}_{[0]}$, and thus the conformal symmetry is not altered at NLO in N in the large-coupling limit.

[4]Another derivation would have been to assume that each term must transform and to make the ansatz that G_{NLO} scales with a power law. This would determine the power of the transformation, and self-consistency of the ansatz can be checked by plugging the result into Eq. (2.34).

III. REAL AND COMPLEX COLORED TENSOR SYK MODELS

A. The models

In this part, we consider one-dimensional fermionic quantum field tensor models. The first one is built out of real fermionic fields, while the second one is built from complex fermionic fields. Each field carries a color index c plus D additional indices denoting the component of the tensor.[5] Since this case is very close to that in Sec. II, we focus on the differences and summarize the main ideas.

The real model is the Gurau-Witten model introduced in Ref. [12]. Its partition function reads as

$$Z^{\mathbb{R}}_{N,\lambda} = \int \prod_{c=0}^{D} \mathcal{D}\psi^c \mathrm{e}^{-\int \mathrm{d}t\, L[\psi]} \tag{3.1}$$

where

$$L[\psi] = \frac{1}{2}\sum_{c=0}^{D}\sum_{n_c}\psi^c_{n_c}\partial_t\psi^c_{n_c} + \mathrm{i}^{(D+1)/2}\frac{\lambda}{N^{D(D-1)/4}}\sum_{n}\prod_{c=0}^{D}\psi^c_{n_c}. \tag{3.2}$$

It is also convenient to define

$$g = \lambda^2. \tag{3.3}$$

We now need to explain several points. Let us first start with the notations. As explained above the fermionic fields are tensors. As such they are D-fundamentals of $\mathrm{O}(N)$. The tensors carry a color index c which runs from 0 to D. This means we have a family of $D+1$ fermionic tensor fields $\{\psi^c\}_{c=0}^{D}$. Since each ψ^c is a tensor, its components read $\psi^c_{n^{c0}\ldots n^{cD}}$ for n^{cj} ranging from 1 to N. We call N the size of the tensor, and each field ψ^c has N^D components. Then $\sum_{n_c}\psi^c_{n_c}\partial_t\psi^c_{n_c}$ means

$$\sum_{n_c}\psi^c_{n_c}\partial_t\psi^c_{n_c} \coloneqq \sum_{n^{c0}\cdots n^{cD}\geq 1}\psi^c_{n^{c0}\cdots n^{cD}}\partial_t\psi^c_{n^{c0}\cdots n^{cD}}. \tag{3.4}$$

The interaction-term notation $\sum_n \prod_{c=0}^{d}\psi^c_{n_c}$ contains $\sum_n$ which is a shorthand for the constraint that $n_c = (n^{c(c-1)}\cdots n^{c0}n^{cD}\cdots n^{c(c+1)})$ and that the indices are constrained to $n^{kl} = n^{lk}$.

In this model the free scalar two-point function G_f is the same as Eq. (2.4)

$$G_f(t_1,t_2) = \frac{1}{N^D}\left\langle\sum_{n_i} T\psi^c_{n_i}(t_1)\psi^c_{n_i}(t_2)\right\rangle_0 = \frac{1}{2}\mathrm{sign}(t_1-t_2), \tag{3.5}$$

such that

$$\langle T\psi^{c'}_{n_{c'}}(t_1)\psi^c_{n_c}(t_2)\rangle_0 = \left(\delta_{cc'}\prod_{c_1\neq c}\delta_{n^{c'c_1}n^{cc_1}}\right)G_f(t_1,t_2). \tag{3.6}$$

The exact two-point function G_e satisfies the same relations with $\langle\cdot\rangle_0 \to \langle\cdot\rangle$.

The complex model is very similar to the real one. It is constructed out of $2(D+1)$ complex fermionic tensor fields $\psi^c_{n_c}(t), \bar{\psi}^c_{n_c}(t)$. $c\in[0..D]$ is the color of the tensor, and each subscript n_c is an abbreviation of the form $n_i = \{n^{cc-1},\ldots,n^{c0},n^{cD},\ldots,n^{cc+1}\}$, where each $n^{ij}\in[1..N]$ for some N, again the size of the tensors. The corresponding partition function is

$$Z^{\mathbb{C}}_{N,\lambda,\bar{\lambda}} = \int\prod_{i=0}^{D}\mathcal{D}\psi^i\mathcal{D}\bar{\psi}^i\mathrm{e}^{\int\mathrm{d}t\, L[\psi]}, \tag{3.7}$$

where

$$L[\psi] = \sum_{c=0}^{D}\sum_{n_c}\bar{\psi}^c_{n_c}\partial_t\psi^c_{n_c} + \mathrm{i}^{(D+1)/2}\frac{\lambda}{N^{D(D-1)/4}}\sum_n\prod_{c=0}^{D}\psi^c_{n_c} + \mathrm{i}^{(D+1)/2}\frac{\bar{\lambda}}{N^{D(D-1)/4}}\sum_n\prod_{c=0}^{D}\bar{\psi}^c_{n_c}. \tag{3.8}$$

The definition of the sum in the interaction term is the same as in the real case. Each fermion field is a d-fundamental of $\mathrm{U}(N)$ and we will make use of the notation

$$g = \lambda\bar{\lambda}. \tag{3.9}$$

The two-point functions are defined in similar ways:

$$G_f(t_1,t_2) = \frac{1}{N^D}\left\langle\sum_{n_i}T\bar{\psi}^c_{n_i}(t_1)\psi^c_{n_i}(t_2)\right\rangle_0 = \mathrm{sign}(t_2-t_1), \tag{3.10}$$

$$\langle T\bar{\psi}^{c'}_{n_{c'}}(t_1)\psi^c_{n_c}(t_2)\rangle_0 = \left(\delta_{cc'}\prod_{c_1\neq c}\delta_{n^{c'c_1}n^{cc_1}}\right)G_f(t_1,t_2). \tag{3.11}$$

[5] In this section, D plays the same role as q. We keep the notations different to emphasize which model is being studied.

We make a slight abuse of notation here as we use the same notations for both the complex and real cases. In fact this is to avoid introducing too many notations.

We now describe the Feynman graphs of these models. The Feynman graphs have the following properties:

(1) The vertices are $(D+1)$-valent.
(2) Edges carry a color index c ranging from 0 to D in such a way that no two adjacent edges have the same color index.
(3) The faces of the graphs are the bicolored edge cycles.
(4) In the complex case, the graphs are bipartite.

The free energy of these models has a $1/N$ expansion driven by the degree ϖ,

$$F_{N,\lambda,\bar{\lambda}} = \log Z_{N,\lambda,\bar{\lambda}} = \sum_{\varpi\geq 0} N^{D-\frac{2}{(D-1)!}\varpi} F_{[\varpi]}(\lambda,\bar{\lambda}), \qquad (3.12)$$

where the degree ϖ of a graph $\mathcal{G}$ is computed from the genera of its jackets (see Ref. [13]), its amplitude is then $A(\mathcal{G}) = N^{D-\frac{2}{(D-1)!}\varpi(\mathcal{G})} a(\mathcal{G})$ where $a(\mathcal{G})$ is a reduced amplitude that depends on integral over positions and the coupling constants but not on N. The main difference between the complex and real case is that, *a priori*, the degree in the complex case is an integer because all jackets are ribbon graphs representing surfaces, while in the real case, nonorientable two-dimensional manifolds can appear among the jackets and thus turn the degree into a half-integer. However, it is easy to show that the degree is an integer in both cases.

The fixed-degree free energies $F^{[\varpi]}_{\lambda,\bar{\lambda}}$ can be computed by summing[6] the amplitudes of all vacuum-connected Feynman graphs of degree ϖ.

These considerations imply that the two-point function also has a $1/N$ expansion. This expansion reads in both the real and complex cases

$$G_e(t_1,t_2) = \sum_{\varpi\geq 0} N^{-\frac{2}{(D-1)!}\varpi} G_{[\varpi]}(t_1,t_2). \qquad (3.13)$$

[6]Actually one should be more precise here. By summing all the amplitudes one gets the perturbative free energies. However these free energies are likely to have a finite radius of convergence in the coupling constant, and thus be defined only in a disc-type domain around $\lambda^2 = 0$. As a consequence, if one is interested in large-coupling physics one should find the (possibly many) analytic continuations of these perturbative free energies. Another way to consider the large-coupling case is to find functional equations for the free energies and solve them in the large-coupling regime. These functional equations can sometimes be found using only perturbative arguments; this is exactly what is done for the leading-order two-point function.

B. The leading order

The leading order of the $1/N$ expansion, $\varpi = 0$, is described by melon diagrams. They are graphs of degree 0, meaning that all jackets are planar. Thanks to the structural properties of the melonic graphs, it is easy to infer the equation satisfied by the LO two-point function. Indeed, one has the same relation (2.12) between the self-energy Σ and the exact two-point function:

$$G_e(t_1,t_2) = (G_f(t_1,t_2)^{-1} - \Sigma(t_1,t_2))^{-1}, \qquad (3.14)$$

where the inverse is taken with respect to the matrix-like/convolution product. Then, one finds the same relations as in Sec. II B after setting $D = q-1$.

In particular, one can show that if $\bar{G}_{[0]}(t_1,t_2)$ is a solution, then, $\bar{G}_{[0]}(\sigma_1,\sigma_2)$, where $\sigma_{1,2} = f(t_{1,2})$, is a solution as well, provided that $\bar{G}_{[0]}(t_1,t_2) = |\partial_{t_1} f(t_1)\partial_{t_2} f(t_2)|^{\frac{1}{D+1}} \times \bar{G}_{[0]}(\sigma_1,\sigma_2)$. $\frac{1}{D+1}$ is the scaling dimension of $\bar{G}_{[0]}$.

C. Next-to-leading-order two-point function

We want to study the NLO of the real and complex colored tensor models. The goal is to check whether or not these models display the conformal symmetry property at large coupling. In particular to check if it is true or not, we need to compute the scaling dimension of the two-point function at NLO. We then study the two-point function at NLO.

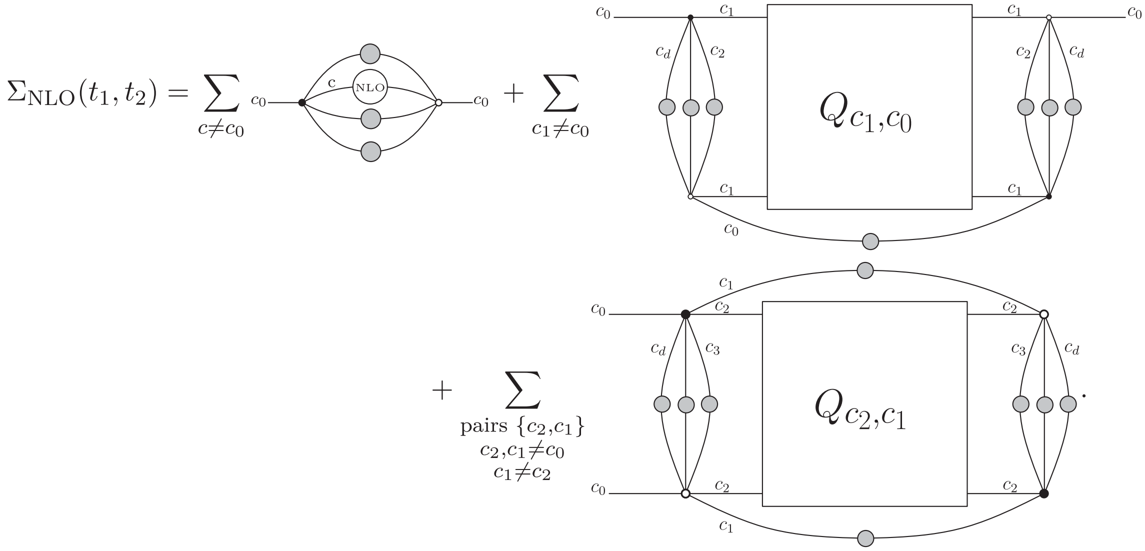
As seen in Ref. [58], the NLOs of the real and complex model are described by the same family of Feynman graphs. This means that nonbipartite graphs do not appear at NLO. This is a specificity of the NLO that is not recovered at all orders. The complex cases have been investigated in the zero-dimensional bosonic tensor model case in Ref. [77]. Following Ref. [77], it is possible to show that the value of the degree at NLO is

$$\varpi_{\mathrm{NLO}} = \frac{(D-1)!}{2}(D-2). \qquad (3.15)$$

The NLO 1PI self-energy and two-point functions are defined by

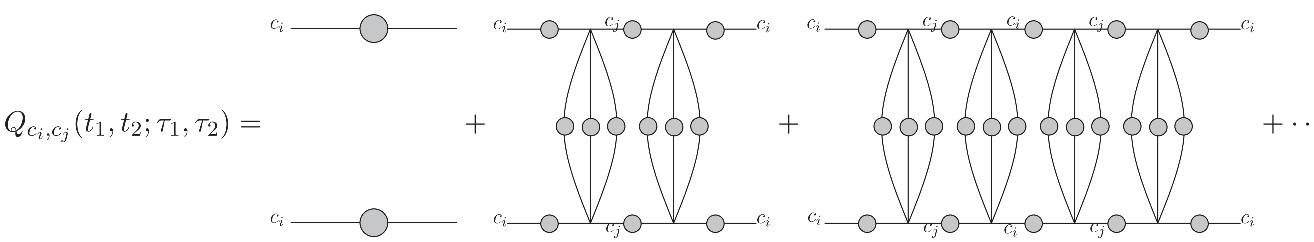
$$G_{\mathrm{NLO}}(t_1,t_2) \coloneqq G_{[\frac{(D-1)!}{2}(D-2)]}(t_1,t_2),$$
$$\Sigma_{\mathrm{NLO}}(t_1,t_2) \coloneqq \Sigma_{[\frac{(D-1)!}{2}(D-2)]}(t_1,t_2). \qquad (3.16)$$

The functional equation for the 1PI self-energy can be written graphically as



(3.17)

The edges with grey-disk insertions represent leading-order two-point functions. The box represents one of the Q_{c_i,c_j}, $c_i \neq c_j$, which are the sum of ladder graphs of even length with ingoing/outgoing color c_i and transmitted colors both c_i and c_j (unbroken chains in the language of Ref. [50]); thus we have



(3.18)

We have the helpful property that $Q_{c_i,c_j}(t_1,t_2;\tau_1,\tau_2) = Q_{c_n,c_m}(t_1,t_2;\tau_1,\tau_2)$ for any choice of c_i, c_j and c_n, c_m. Then we call $Q_{c_i,c_j}(t_1,t_2;\tau_1,\tau_2) = Q(t_1,t_2;\tau_1,\tau_2)$. Equation (3.17) can be formally rewritten as

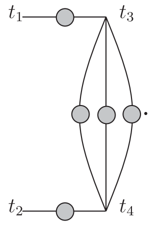
$$\begin{aligned}\Sigma_{\mathrm{NLO}}(t_1,t_2) = {} & DgG_{[0]}(t_1,t_2)^{D-1}G_{\mathrm{NLO}}(t_1,t_2) \\ & + Dg^2\int \mathrm{d}t\mathrm{d}t' G_{[0]}(t_1,t)^{D-1}G_{[0]}(t',t_2)^{D-1}G_{[0]}(t,t')Q(t_1,t;t_2,t') \\ & + \frac{D(D-1)}{2}g^2\int \mathrm{d}t\mathrm{d}t'(G_{[0]}(t_1,t_2)^{D-2}G_{[0]}(t_1,t)G_{[0]}(t_2,t') \\ & \times G_{[0]}(t,t')^{D-1}Q(t_1,t_2;t,t')). \end{aligned} \tag{3.19}$$

We also have

$$G_{\mathrm{NLO}}(t_1,t_2) = \int \mathrm{d}t\mathrm{d}t' G_{[0]}(t_1,t)\Sigma_{\mathrm{NLO}}(t,t')G_{[0]}(t',t_2). \tag{3.20}$$

This is equivalent to Eqs. (2.29) and (2.34) upon setting $D = q-1$.

We now take care of $Q(t_1,t_2;t_3,t_4)$ which appears in Eq. (3.19). $Q(t_1,t_2;t_3,t_4)$ can be constructed from $Q_0(t_1,t_2;t_3,t_4)$ using the operator K defined graphically below,

$$K(t_1,t_2;t_3,t_4) =$$  (3.21)

$Q_0(t_1,t_2;t_3,t_4)$ reads

$$Q_0(t_1,t_2;t_3,t_4) = G_{[0]}(t_1,t_3)G_{[0]}(t_2,t_4). \tag{3.22}$$

The operator K is formally written as

$$K(t_1,t_2,t_3,t_4) = -gG_{[0]}(t_1,t_3)G_{[0]}(t_2,t_4)G_{[0]}(t_3,t_4)^{D-1}. \tag{3.23}$$

We have

$$Q(t_1,t_2;t_3,t_4) = \sum_{n\geq 0} K^{2n}(t_1,t_2;t,t') * Q_0(t,t';t_3,t_4) = (\delta^{\otimes 2} - K*K)^{-1} * Q_0 \tag{3.24}$$

where the (even) powers of K are taken with respect to the convolution product. Again Q is not defined if K possesses eigenvalues ± 1: as explained in Secs. I and II we restrict our discussion to the nondivergent part of Q.

The same argument as in the preceding SYK case applies here and at large g: since all terms on the rhs of Eq. (3.19) scale in the same way and because we only want to find the scaling, we can focus on a single term[7]:

$$\bar{G}_{\mathrm{NLO}}(t_1,t_2) = \int \mathrm{d}\tau\mathrm{d}\tau' \bar{G}_{[0]}(t_1,\tau)\bar{\Sigma}_{\mathrm{NLO}}(\tau,\tau')\bar{G}_{[0]}(\tau',t_2) \tag{3.25}$$

$$\sim Dg^2 \int \mathrm{d}\tau\mathrm{d}\tau'\mathrm{d}t\mathrm{d}t' \bar{G}_{[0]}(t_1,\tau)\bar{G}_{[0]}^{D-1}(\tau,t)\bar{G}_{[0]}^{D-1}(t',\tau') \times \bar{G}_{[0]}(t,t')\bar{Q}(\tau,t;\tau',t')\bar{G}_{[0]}(\tau',t_2). \tag{3.26}$$

In order to get the conformal scaling of $\bar{G}_{\mathrm{NLO}}$ we need to understand how the conformal limit of $\bar{Q}(t_1,t_2;t_3,t_4)$ behaves. To do so we reduce $\bar{Q}$ to its 1PI connected counterpart $\bar{\Gamma}(t_1,t_2;t_3,t_4)$. We have that $\bar{Q} = \bar{G}^{\otimes 2} + \bar{G}^{\otimes 2} * \bar{\Gamma} * \bar{G}^{\otimes 2}$. More precisely,

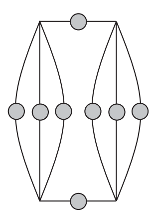
$$\bar{Q}(t_1,t_2;t_3,t_4) = \bar{Q}_0(t_1,t_2;t_3,t_4) + \int \mathrm{d}t\mathrm{d}t'\mathrm{d}\tau\mathrm{d}\tau' (\bar{G}_{[0]}(t_1,t)\bar{G}_{[0]}(t_2,t')\bar{\Gamma}(t,t';\tau,\tau')\bar{G}_{[0]}(\tau,t_3)\bar{G}_{[0]}(\tau',t_4)). \tag{3.27}$$

The scaling dimension of $\bar{Q}_0$ is $\frac{1}{D+1}$ as $\bar{Q}_0$ is written solely in terms of $\bar{G}_{[0]}$.

$\bar{\Gamma}$ satisfies the following Schwinger-Dyson equation:

$$\bar{\Gamma}(t,t';\tau,\tau') = \Gamma_0(t,t';\tau,\tau') + \int \mathrm{d}\eta\mathrm{d}\eta'\mathrm{d}\omega\mathrm{d}\omega' \bar{\Gamma}(t,t';\eta,\eta')\bar{K}(\eta,\eta';\omega,\omega')\bar{K}(\omega,\omega';\tau,\tau'), \tag{3.28}$$

where

$$\bar{\Gamma}_0(t,t';\tau,\tau') =$$  $$= g^2\, \bar{G}_{[0]}(t,t')^{D-1}\bar{G}_{[0]}(t,\tau)\bar{G}_{[0]}(t',\tau')\bar{G}_{[0]}(\tau,\tau')^{D-1} \tag{3.29}$$

and we do not display the colors of the edges as the dependence on the times is not sensitive to it. From Eq. (3.28) we can deduce the scaling of $\bar{\Gamma}$. First notice that

$$\bar{\Gamma}_0(\sigma,\sigma';\zeta,\zeta') = \frac{\bar{\Gamma}_0(t,t';\tau,\tau')}{|f'(t)f'(t')f'(\tau)f'(\tau')|^{\frac{D}{D+1}}} \tag{3.30}$$

for $\sigma,\sigma',\zeta,\zeta' = f(t), f(t'), f(\tau), f(\tau')$. This is obtained from the scaling of $\bar{G}_{[0]}$. Using the explicit expression for $\bar{K}$ we also deduce that

[7]Keeping several terms helps to check the consistency of our computations. But, this should not be seen as an approximation because, according to the discussion in Sec. II, this truncation is not consistent (except to find the scaling).

$$\int \mathrm{d}\beta \mathrm{d}\beta' \bar{K}(\sigma,\sigma';\beta,\beta')\bar{K}(\beta,\beta';\zeta,\zeta') = \int |f'(\omega)f'(\omega')|\mathrm{d}\omega\mathrm{d}\omega' \frac{\bar{K}(\eta,\eta';\omega,\omega')\bar{K}(\omega,\omega';\tau,\tau')}{|f'(\eta)f'(\eta')|^{\frac{1}{D+1}}|f'(\omega)f'(\omega')||f'(\tau)f'(\tau')|^{\frac{D}{D+1}}}$$
$$= \int \mathrm{d}\omega\mathrm{d}\omega' \frac{\bar{K}(\eta,\eta';\omega,\omega')\bar{K}(\omega,\omega';\tau,\tau')}{|f'(\eta)f'(\eta')|^{\frac{1}{D+1}}|f'(\tau)f'(\tau')|^{\frac{D}{D+1}}} \tag{3.31}$$

where we have set $\sigma,\sigma',\zeta,\zeta'$ as before and $\beta,\beta' = f(\omega), f(\omega')$. Consequently the scaling dimension of $\bar{\Gamma}$ is $\frac{D}{D+1}$. Indeed if $\bar{\Gamma}(\sigma,\sigma',\zeta,\zeta')$ is a solution of Eq. (3.28), then the function $\bar{\Gamma}'(t,t';\tau,\tau') = |f'(t)f'(t')f'(\tau)f'(\tau')|^{\frac{D}{D+1}}\bar{\Gamma}(\sigma,\sigma';\tau,\tau')$ with $\sigma,\sigma',\zeta,\zeta' = f(t), f(t'), f(\tau), f(\tau')$ is also a solution.

We now turn to the scaling dimension of $\bar{Q}$ in the conformal sector. We recall its expression in terms of $\bar{\Gamma}$

$$\bar{Q}(\sigma_1,\sigma_2;\sigma_3,\sigma_4) = \bar{Q}_0(\sigma_1,\sigma_2;\sigma_3,\sigma_4) + \int \mathrm{d}\beta\mathrm{d}\beta'\mathrm{d}\gamma\mathrm{d}\gamma'(\bar{G}_{[0]}(\sigma_1,\beta)\bar{G}_{[0]}(\sigma_2,\beta')\bar{\Gamma}(\beta,\beta';\gamma,\gamma')$$
$$\times \bar{G}_{[0]}(\gamma,\sigma_3)\bar{G}_{[0]}(\gamma',\sigma_4)). \tag{3.32}$$

We call the second term of the right-hand side of Eq. (3.32) $\bar{F}(\sigma_1,\sigma_2;\sigma_3,\sigma_4)$,

$$\bar{F}(\sigma_1,\sigma_2;\sigma_3,\sigma_4) = \int \mathrm{d}\beta\mathrm{d}\beta'\mathrm{d}\gamma\mathrm{d}\gamma'\bar{G}_{[0]}(\sigma_1,\beta)\bar{G}_{[0]}(\sigma_2,\beta')\bar{\Gamma}(\beta,\beta';\gamma,\gamma')\bar{G}_{[0]}(\gamma,\sigma_3)\bar{G}_{[0]}(\gamma',\sigma_4). \tag{3.33}$$

We reparametrize $\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4 = f(t_1), f(t_2), f(t_3), f(t_4)$ and $\beta,\beta',\gamma,\gamma' = f(t), f(t'), f(\tau), f(\tau')$ to get the scaling dimension of $\bar{F}$. This leads to

$$\bar{F}(\sigma_1,\sigma_2;\sigma_3,\sigma_4) = \frac{\bar{F}(t_1,t_2;t_3,t_4)}{|f'(t_1)f'(t_2)f'(t_3)f'(t_4)|^{\frac{1}{D+1}}}. \tag{3.34}$$

This tells us that $\bar{F}$ indeed scales and the scaling dimension is $\frac{1}{D+1}$. This together with the fact that $\bar{Q}_0$ has scaling dimension $1/(D+1)$ implies that $\bar{Q}$ has scaling dimension $1/(D+1)$. Let us compute the scaling of the NLO two-point function. In the large-g limit we have that,

$$\bar{G}_{\mathrm{NLO}}(\sigma_1,\sigma_2) \approx Dg^2 \int \mathrm{d}\gamma\mathrm{d}\gamma'\mathrm{d}\beta\mathrm{d}\beta'(\bar{G}(\sigma_1,\gamma)\bar{G}(\gamma,\beta)^{D-1}\bar{G}(\beta',\gamma')^{D-1}$$
$$\times \bar{G}(\beta,\beta')\bar{Q}(\gamma,\beta;\gamma',\beta')\bar{G}(\gamma',\sigma_2)) \tag{3.35}$$

which leads after simplifications to

$$\bar{G}_{\mathrm{NLO}}(\sigma_1,\sigma_2) = \frac{\bar{G}_{\mathrm{NLO}}(t_1,t_2)}{|f'(t_1)f'(t_2)|^{\frac{1}{D+1}}}. \tag{3.36}$$

This shows that the scaling dimension of $\bar{G}_{\mathrm{NLO}}$ is $\frac{1}{D+1}$ as for the leading-order term.

IV. MULTIORIENTABLE SYK TENSOR MODEL

The $\mathrm{U}(N)\times\mathrm{O}(N)\times\mathrm{U}(N)$ model has been introduced in the tensor model literature in Ref. [78]. It was called the multiorientable model. It was then stated that it should be related to a complex-fermion version of the SYK model in Ref. [25]. The model is defined as follows. One considers a pair of complex fermionic tensor fields $\psi,\bar{\psi}$ of rank 3. The partition function of the model reads as

$$Z^{\mathrm{m.o.}}_{\lambda,N} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathrm{e}^{-\int \mathrm{d}t\, L[\psi]} \tag{4.1}$$

where

$$L[\psi] = \sum_n \bar{\psi}_n\partial_t\psi_n + \frac{\lambda}{N^{3/2}}\sum_{i,j,k,i',j',k'}\psi_{ijk}(t)\bar{\psi}_{kj'i'}(t)\psi_{k'ji'}(t)\bar{\psi}_{k'j'i}(t) \tag{4.2}$$

and we also define

$$g = \lambda^2. \tag{4.3}$$

The fields transform under the natural action of $\mathrm{U}(N)\times\mathrm{O}(N)\times\mathrm{U}(N)$ and the action is invariant under this transformation.

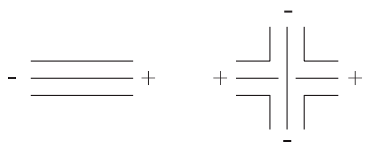


FIG. 1. Propagator and vertex of the multiorientable model.

The Feynman graphs are constructed out of the building blocks represented in Fig. 1 with the condition that a $(+)$ half-edge can only connect to a $(-)$ half-edge. It is also possible to define the notion of jackets for these graphs. This is indeed a nontrivial statement as one can find examples of tensor models for which this is not the case because of the so-called *tadface* graphs; see Refs. [27,29] for a discussion of these topics.

Thanks to this notion of jackets the degree can be generalized in this case and it can be shown that the multiorientable model has a well-defined $1/N$ expansion. Thus for the free energy we have

$$F^{\text{m.o.}}_{\lambda,N} = \log Z^{\text{m.o.}}_{\lambda,N} = \sum_{\varpi\geq 0} N^{3-\varpi} F^{\text{m.o.}}_{[\varpi]}(\lambda). \tag{4.4}$$

In this case however, $\varpi \in \frac{1}{2}\mathbb{N}_{\geq 0}$, where $\mathbb{N}_{\geq 0}$ is the set of integers larger than or equal to zero. We can again define the two-point function. Let us start with the free one,

$$G_f(t_1,t_2) = \frac{1}{N^3}\left\langle \sum_n T\bar{\psi}_n(t_1)\psi_n(t_2)\right\rangle_0 = \text{sign}(t_1 - t_2), \tag{4.5}$$

$$\langle T\bar{\psi}_{ijk}(t_1)\psi_{i'j'k'}(t_2)\rangle_0 = G_f(t_1,t_2)\delta_{ii'}\delta_{jj'}\delta_{kk'}, \tag{4.6}$$

where n here is a multi-index that labels the components of the tensor. For the exact two-point function we have,

$$G_e(t_1,t_2) = \frac{1}{N^3}\left\langle \sum_n T\bar{\psi}_n(t_1)\psi_n(t_2)\right\rangle, \tag{4.7}$$

$$\langle T\bar{\psi}_{ijk}(t_1)\psi_{i'j'k'}(t_2)\rangle = G_e(t_1,t_2)\delta_{ii'}\delta_{jj'}\delta_{kk'}. \tag{4.8}$$

Consequently we have,

$$G_e(t_1,t_2) = \sum_{\varpi\in\frac{1}{2}\mathbb{N}} N^{-\varpi} G_{[\varpi]}(t_1,t_2). \tag{4.9}$$

A. The leading order

As was shown in Ref. [27], the leading order in N is once again dominated by melonic graphs. As a consequence we can write the equation satisfied by the two-point function at leading order,

$$G_{[0]}(t_1,t_2) = G_f(t_1,t_2) + g\int \mathrm{d}t\mathrm{d}t' G_f(t_1,t)G_{[0]}(t,t')^3 G_{[0]}(t',t_2). \tag{4.10}$$

Using now known manipulations we have in the infrared/large-coupling limit the approximated equation

$$g\int \mathrm{d}t\bar{G}_{[0]}(t_1,t)^3\bar{G}_{[0]}(t,t_2) = -\delta(t_1-t_2). \tag{4.11}$$

This equation has a known solution

$$\bar{G}_{[0]}(t_1,t_2) = \left(\frac{\tan(\pi/4)}{4\pi g}\right)^{1/4}\frac{\text{sign}(t_1-t_2)}{|t_1-t_2|^{1/2}}. \tag{4.12}$$

Moreover, the large-coupling equation has the same reparametrization symmetry. If $\bar{G}_{[0]}(t_1,t_2)$ is a solution, then, $\bar{G}_{[0]}(\sigma_1,\sigma_2)$, where $\sigma_{1,2} = f(t_{1,2})$, is a solution as well, provided that $\bar{G}_{[0]}(t_1,t_2) = |\partial_{t_1}f(t_1)\partial_{t_2}f(t_2)|^{\frac{1}{4}}\bar{G}_{[0]}(\sigma_1,\sigma_2)$.

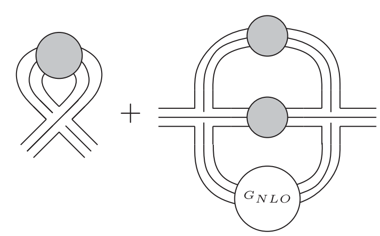
B. The next-to-leading order

The next-to-leading order of the two-point function of the multiorientable model has been studied in Ref. [28]. As the combinatorics is unchanged by the fact that we consider fermionic fields in one-dimensional space we can easily infer the next-to-leading order in this case. The degree at next-to-leading order is

$$\varpi_{\text{NLO}} = \frac{1}{2}. \tag{4.13}$$

The self-energy $\Sigma_{\text{NLO}}(t_1,t_2) \coloneqq \Sigma_{[1/2]}(t_1,t_2)$ at next-to-leading order is written graphically as

$\Sigma_{\text{NLO}}(t_1,t_2) =$



(4.14)

where the gray disks represent insertions of the leading-order two-point function on the edges. This translates into the formal equation

$$\Sigma_{\text{NLO}}(t_1,t_2) = \lambda\delta(t_1-t_2)G_{[0]}(t_1,t_2) + 3gG_{[0]}(t_1,t_2)^2 G_{[1/2]}(t_1,t_2). \tag{4.15}$$

We also have,

$$G_{\mathrm{NLO}}(t_1,t_2) \coloneqq G_{[1/2]}(t_1,t_2) = \int \mathrm{d}t\mathrm{d}t' G_{[0]}(t_1,t)\Sigma_{\mathrm{NLO}}(t,t')G_{[0]}(t',t_2). \tag{4.16}$$

We now use the analogue of the operator[8] $K(t_1,t_3;t_3,t_4)$ introduced in earlier sections. It is here defined as

$$K(t_1,t_3;t_3,t_4) = 3gG_{[0]}(t_1,t_3)G_{[0]}(t_2,t_4)G_{[0]}(t_3,t_4)^2. \tag{4.17}$$

We introduce $G_{[0]}(t_1,t_2) = G_{[0]}(t_2-t_1) = G_{[0]}(\tau)$, with $\tau = t_2 - t_1$. Then, using Eqs. (4.16) and (4.15), we can formally write

$$G_{\mathrm{NLO}}(t_1,t_2) = \lambda \int \mathrm{d}t G_{[0]}(t_1,t)G_{[0]}(0)G_{[0]}(t,t_2) + \int \mathrm{d}t\mathrm{d}t' K(t_1,t_2;t,t')G_{\mathrm{NLO}}(t,t') \tag{4.18}$$

$$= \lambda \int \mathrm{d}t G_{[0]}(t_1,t)G_{[0]}(0)G_{[0]}(t,t_2) + [K * G_{\mathrm{NLO}}](t_1,t_2), \tag{4.19}$$

and thus

$$[(\delta^{\otimes 2} - K) * G_{\mathrm{NLO}}](t_1,t_2) \coloneqq \int \mathrm{d}t\mathrm{d}t'(\delta(t_1-t)\otimes\delta(t_2-t') - K(t_1,t_2;t,t'))G_{\mathrm{NLO}}(t,t') \tag{4.20}$$

$$= \lambda \int \mathrm{d}t G_{[0]}(t_1,t)G_{[0]}(0)G_{[0]}(t,t_2). \tag{4.21}$$

However, since fermionic two-point functions are antisymmetric in the time variables, we have[9] $G_{[0]}(0) = 0$, which then implies

$$[(\delta^{\otimes 2} - K) * G_{\mathrm{NLO}}](t_1,t_2) = 0, \quad \forall \; t_1,t_2. \tag{4.22}$$

This implies that G_{NLO} must lie in the kernel of $(\delta^{\otimes 2} - K)$. This happens as long as $G_{\mathrm{NLO}} = 0$ or G_{NLO} is an eigenvector of K with eigenvalue 1. Since there are such eigenvectors $\bar{G}_{\mathrm{NLO}}$ can be an arbitrary linear combination of them if it does not vanish and, without additional data on the behavior of $\bar{G}_{\mathrm{NLO}}$ it is not possible to make a conclusion about its conformality.

[8]Notice however the difference in sign.
[9]One can also convince oneself by going into Fourier space and defining an appropriate cutoff for the regularization of the integral.

V. DISCUSSION

We have found that the NLO in the large-N expansion does not modify the dependence of the two-point function in the coupling and time in the infrared regime. For this reason the two-point function is still conformally invariant and the IR dimension of the fermions does not receive any correction at this order. Nonetheless higher-order correlation functions may deviate from the conformal field theory (CFT) behavior and this provides an incentive to study their behavior. In any case one can consider the NLO as being a CFT in any context where the corrections to these higher-order functions can be neglected.

This fact may reveal itself to be important in the construction of the bulk dual using the AdS/CFT dictionary [59]: the absence of $1/N$ corrections in the CFT translates into the absence of quantum corrections in the bulk dual. For example if the scaling dimensions of the single-trace operators discussed in Refs. [4,59] are identical at NLO this would be equivalent to the fact that the corresponding bulk field masses do not receive corrections at one loop. Hence our result gives a strong indication that the first quantum correction may be absent and this point calls for a deeper study.

A natural extension of this work would be to determine how the spontaneous breaking of the conformal symmetry appears in the NLO four-point function and what are the effects of incorporating the NLO correction of the coupling constant. Another point of interest is to push the study even further and see if the next-to-NLO continues to preserve the conformal invariance. The method described in this paper can be generalized to study the NLO in other models, such as in the supersymmetric case [39,40]. Finally it would be useful to settle the question of the conformal invariance of the two-point function in the multiorientable tensor model.

ACKNOWLEDGMENTS

We are grateful to Costas Bachas and Ashoke Sen for useful discussions. The work of S. M., performed at LabEx ILP (reference Anr–10–Labx–63), is supported by French state funds managed by the Agence Nationale de la Recherche, as part of the program Investissements d'avenir under the reference Anr–11–Idex–0004–02. S. M. is supported by the Indo-French Centre for the Promotion of Advanced Research under project No. 5204-4.

APPENDIX: COMPOSITE FIELD EFFECTIVE ACTION

The goal of this appendix is to introduce composite fields for colored tensor models. The original motivation was to find an effective action in terms of composite fields to study the IR regime of colored tensor models. However, the naive approach proposed here does not work because it is not suited for an IR approximation (see the discussion at the end of Appendix A 1). Nonetheless, we find it useful to provide the details as an illustration.[10]

We will focus on real tensors for simplicity but the generalization to complex tensors is straightforward.

Recall the action for $D+1$ real fermionic tensor fields (Sec. III)

$$S[\psi^c] = \int \mathrm{d}t \left(\frac{1}{2} \sum_c \psi^c \partial_t \psi^c + \mathrm{i}^{\frac{D+1}{2}} \Lambda \prod_c \psi^c \right), \tag{A1}$$

where the contraction over the tensor indices is implicit and $c = 0, \ldots, D$ and Λ is defined by

$$\Lambda = \frac{\lambda}{N^{\frac{D(D-1)}{4}}}. \tag{A2}$$

The associated partition function is

$$Z = \int \prod_c \mathcal{D}\psi^c \mathrm{e}^{-S[\psi^c]}. \tag{A3}$$

1. Effective action

In order to introduce a composite field[11]

$$G^c_{n_c, n'_c}(t, t') = -\psi^c_{n_c}(t) \psi^c_{n'_c}(t') \tag{A4}$$

where n_c and n'_c are tensor multi-indices, corresponding to the two-point function

$$G^c_e(t, t') = -\frac{1}{N^D} \left\langle \sum_{n_c} \psi^c_{n_c}(t) \psi^c_{n_c}(t') \right\rangle, \tag{A5}$$

one first needs to obtain an action with bilinear terms in each color. In the rest of this section, the tensor indices will be implicit.

This can be achieved by integrating out one of the colors, say ψ^0 which is straightforward since the action is quadratic in this field, with the product

$$\Psi = \mathrm{i}^{\frac{D+1}{2}} \Lambda \prod_{i=1}^{D} \psi^i \tag{A6}$$

acting as a source for ψ^0, where $i = 1, \ldots, D$. Using standard techniques, the effective action obtained after integrating out ψ^0 is

$$S_{\text{eff}}[\psi^i] = \frac{1}{2} \sum_i \int \mathrm{d}t \psi^i \partial_t \psi^i + \mathrm{i}^{D^2+1} \frac{\Lambda^2}{2} \int \mathrm{d}t \mathrm{d}t' S(t, t') \prod_i \psi^i(t) \psi^i(t') \tag{A7}$$

after rearranging the fermions (the signs have been traded for i), where $S(t, t')$ is the Green function for ∂_t.

The next step consists in introducing the bilocal tensor fields $G^i(t, t')$ [Eq. (A4)] and using auxiliary fields $\Sigma^i(t, t')$ such that

$$1 = \int \prod_i \mathcal{D}G^i \delta(G^i(t, t') + \psi^i(t) \psi^i(t')) \tag{A8a}$$

$$= \int \prod_i \mathcal{D}G^i \mathcal{D}\Sigma^i \mathrm{e}^{-S_{\text{aux}}[\psi^i, G^i, \Sigma^i]} \tag{A8b}$$

where

$$S_{\text{aux}}[\psi^i, G^i, \Sigma^i] = -\frac{1}{2} \sum_i \int \mathrm{d}t \mathrm{d}t' \Sigma^i(t, t') (G^i(t, t') + \psi^i(t) \psi^i(t')). \tag{A9}$$

The functional integral (A3) becomes

$$Z = \int \prod_i \mathcal{D}\psi^i \mathcal{D}G^i \mathcal{D}\Sigma^i \mathrm{e}^{-\tilde{S}_{\text{eff}}[\psi^i, G^i] - S_{\text{aux}}[G^i, \Sigma^i]}, \tag{A10}$$

where

$$\tilde{S}_{\text{eff}}[\psi^i, G^i] = \frac{1}{2} \sum_i \int \mathrm{d}t \psi^i \partial_t \psi^i + \mathrm{i}^{(D+1)^2} \frac{\Lambda^2}{2} \int \mathrm{d}t \mathrm{d}t' S(t, t') \prod_i G^i(t, t'). \tag{A11}$$

Performing the quadratic integration over ψ^i yields the effective action for G^i and Σ^i

$$W[G^i, \Sigma^i] = -\frac{1}{2} \sum_i \mathrm{tr} \ln(\partial_t - \Sigma^i) - \frac{1}{2} \sum_i \int \mathrm{d}t \mathrm{d}t' \Sigma^i(t, t') G^i(t, t') + \mathrm{i}^{(D+1)^2} \frac{\Lambda^2}{2} \int \mathrm{d}t \mathrm{d}t' S(t, t') \prod_i G^i(t, t'). \tag{A12}$$

The equations of motion are

$$\frac{\delta W}{\delta G^i} = 0 \Rightarrow \Sigma^i(t, t') = \mathrm{i}^{(D+1)^2} \Lambda^2 S(t, t') \prod_{j \neq i} G^j(t, t'), \tag{A13a}$$

$$\frac{\delta W}{\delta \Sigma^i} = 0 \Rightarrow (\delta(t - t') \mathbb{1}^{\otimes D} \partial_t - \Sigma^i(t, t'))^{-1} - G^i(t, t') = 0 \tag{A13b}$$

[10]Which also motivates the need for the more advanced machinery developed in Ref. [79] which appeared after our paper.

[11]The composite fields are distinguished from the correlation functions by the absence of any lower index.

where $\mathbb{1}^{\otimes D}$ is the tensor identity. The last equation can be rewritten as

$$\delta(t-t'')\mathbb{1}^{\otimes D} = \partial_t G^i(t,t'') + \int \mathrm{d}t' \Sigma^i(t,t') G^i(t',t''), \tag{A14a}$$

$$= \partial_t G^i(t,t'') + \mathrm{i}^{(D+1)^2}\Lambda^2 \int \mathrm{d}t' S(t,t') G^i(t',t'') \prod_{j\neq i} G^j(t,t') \tag{A14b}$$

where the last equality follows from inserting Eq. (A13a).

The computations in this section are exact, which means that the effective action (A12) is exact and leads to the correct Schwinger-Dyson equations.[12] The drawback of our action (and of the corresponding Schwinger-Dyson equations) over the one in Ref. [79] is that it is not suited for studying the IR regime. Indeed, they implicitly include all modes from the ψ^0 field (and thus the UV fluctuations) because it has been exactly integrated out. By contrast in the usual SYK, we integrate out the random coupling constants which are nondynamical (they are pure IR): this explains why there is no difficulty in taking the IR limit afterwards, because there is no hidden UV contribution like what we get when integrating out a dynamical field. In principle, we could correct our action through some kind of renormalization group approach in order to integrate only the IR modes of the field ψ^0.

2. Fluctuations

The solutions to the equations of motion are denoted by $(G_{[0]}, \Sigma_{[0]})$ and they are identical for all colors since the equations are symmetric under exchange of colors

$$\langle G^i \rangle = G_{[0]} \mathbb{1}^{\otimes D}, \qquad \langle \Sigma^i \rangle = \Sigma_{[0]} \mathbb{1}^{\otimes D}, \tag{A15}$$

where $G_{[0]}$ and $\Sigma_{[0]}$ are genuine bilocal fields (not tensors). Powers of $\langle G^i \rangle$ (or $\langle \Sigma^i \rangle$) will be accompanied by factors of N due to the contraction of the identities

$$(\langle G^i \rangle)^k = N^{k+\binom{k}{2}} G_{[0]}^k = N^{\frac{k(k+1)}{2}} G_{[0]}^k. \tag{A16}$$

The saddle-point equations (A13) become

$$\Sigma_{[0]}(t,t') = \mathrm{i}^{(D+1)^2} \lambda^2 S(t,t') G_{[0]}(t,t')^{D-1}, \tag{A17a}$$

$$(\delta(t-t')\partial_t + \Sigma_{[0]}(t,t'))^{-1} - G_{[0]}(t,t') = 0, \tag{A17b}$$

where the relation (A2) between Λ and λ has been used.

Then one can consider fluctuations (g^i, σ^i) around these solutions

$$G^i = G_{[0]} \mathbb{1}^{\otimes D} + g^i, \qquad \Sigma^i = \Sigma_{[0]} \mathbb{1}^{\otimes D} + \sigma^i. \tag{A18}$$

Plugging these expressions into Eq. (A12) yields

$$\begin{aligned} W[g^i,\sigma^i] = {} & \frac{1}{4}\sum_i \int \mathrm{d}t_1 \cdots \mathrm{d}t_4 \sigma^i(t_1,t_2) k(t_1,\ldots,t_4) \sigma^i(t_3,t_4) - \frac{1}{2}\sum_i \int \mathrm{d}t \mathrm{d}t' \sigma^i g^i \\ & + \mathrm{i}^{(D+1)^2} \frac{\lambda^2}{4N^{D-1}} \int \mathrm{d}t\mathrm{d}t' S G_{[0]}^{D-2} \sum_{i,j} g^i g^j + \frac{1}{2}\sum_i \sum_{n\geq 3} \frac{1}{n} \mathrm{tr}(G_{[0]}\sigma^i)^n \\ & + \frac{\mathrm{i}^{(D+1)^2}}{N^{\frac{D(D-1)}{2}}} \sum_{n=3}^{D} \frac{\lambda^2}{2n!} N^{\frac{(D-n)(D-n+1)}{2}} \int \mathrm{d}t\mathrm{d}t' S G_{[0]}^{D-n} \sum_{i_1,\ldots,i_n} g^{i_1}\cdots g^{i_n} \end{aligned} \tag{A19}$$

where the dependence on the time (t,t') has been omitted and the kernel k is

$$k(t_1,\ldots,t_4) = G_{[0]}(t_1,t_3) G_{[0]}(t_2,t_4). \tag{A20}$$

Rescaling the fluctuations such that

$$G^i = G_{[0]}\mathbb{1}^{\otimes D} + |G_{[0]}|^{\frac{1-D}{2}} g^i, \qquad \Sigma^i = \Sigma_{[0]}\mathbb{1}^{\otimes D} + |G_{[0]}|^{\frac{D-1}{2}} \sigma^i \tag{A21}$$

and absorbing the factors inside the kernel gives the symmetric kernel [4]

$$\begin{aligned} K_{\mathrm{sym}}(t_1,\ldots,t_4) = {} & -\lambda^2 |G_{[0]}(t_1,t_2)|^{\frac{D-1}{2}} k(t_1,\ldots,t_4) \\ & \times |G_{[0]}(t_3,t_4)|^{\frac{D-1}{2}} \end{aligned} \tag{A22}$$

which is conjugated to the kernel (3.23).

Truncating the action to the quadratic order, one can obtain an effective action for the g^i only by integrating out σ^i

$$\begin{aligned} W_{\mathrm{eff}}[g^i] = {} & -\frac{\lambda^2}{4}\sum_{i,j} \int \mathrm{d}t_1 \cdots \mathrm{d}t_4 g^i(t_1,t_2) \\ & \times \mathcal{K}^{ij}(t_1,\ldots,t_4) g^j(t_3,t_4) \end{aligned} \tag{A23}$$

[12] This paragraph is an answer to page 2 of Ref. [79] which states that the action derived here is not correct because it gives the "wrong Schwinger-Dyson equations". Note also that the action from Ref. [79] (see also Ref. [80]) is obtained by making approximations, implying that it is not exact.

where

$$\mathcal{K}^{ij}(t_1, \ldots, t_4) = K^{-1}_{\text{sym}}(t_1, \ldots, t_4)\delta^{ij} - \frac{\mathrm{i}^{(D+1)^2}}{N^{D-1}} S(t_1, t_2) \times \delta(t_1 - t_3)\delta(t_2 - t_4). \quad \text{(A24)}$$

The four-point function for the fermions corresponds to the two-point function of the fluctuations $\langle g^i g^j \rangle$. At leading order it can be computed using the above quadratic action and one can see that it is equivalent to the computation done in Sec. III. The additional propagator in the action is a consequence of integrating out one of the colors and it is present to connect vertices, very similar to the way one adds an extra line after averaging over disorder in the standard SYK model [4]. However this time the extra line represents dynamical fields.

[1] A. Y. Kitaev, *Hidden Correlations in the Hawking Radiation and Thermal Noise* (2015), http://online.kitp.ucsb.edu/online/joint98/kitaev/.

[2] A. Y. Kitaev, *A Simple Model of Quantum Holography (Part I)* (2015), http://online.kitp.ucsb.edu/online/entangled15/kitaev/.

[3] A. Y. Kitaev, *A Simple Model of Quantum Holography (Part II)* (2015), http://online.kitp.ucsb.edu/online/entangled15/kitaev2/.

[4] J. Maldacena and D. Stanford, Comments on the Sachdev-Ye-Kitaev model, Phys. Rev. D **94**, 106002 (2016).

[5] S. Sachdev and J. Ye, Gapless Spin-Fluid Ground State in a Random Quantum Heisenberg Magnet, Phys. Rev. Lett. **70**, 3339 (1993).

[6] D. J. Gross and V. Rosenhaus, A generalization of Sachdev-Ye-Kitaev, J. High Energy Phys. 02 (2017) 093.

[7] J. Maldacena, S. H. Shenker, and D. Stanford, A bound on Chaos, J. High Energy Phys. 08 (2016) 106.

[8] A. Almheiri and J. Polchinski, Models of AdS_2 backreaction and holography, J. High Energy Phys. 11 (2015) 014.

[9] K. Jensen, Chaos in AdS_2 holography, Phys. Rev. Lett. **117**, 111601 (2016).

[10] J. Engelsöy, T. G. Mertens, and H. Verlinde, An investigation of AdS_2 backreaction and holography, J. High Energy Phys. 07 (2016) 139.

[11] J. Maldacena, D. Stanford, and Z. Yang, Conformal symmetry and its breaking in two dimensional nearly Anti-de-Sitter space, Prog. Theor. Exp. Phys. **2016**, 12C104 (2016).

[12] E. Witten, An SYK-like model without disorder, J. Phys. A **52**, 474002 (2019).

[13] R. Gurau, The $1/N$ expansion of colored tensor models, Ann. Inst. Henri Poincaré **12**, 829 (2011).

[14] R. Gurau and V. Rivasseau, The 1/N expansion of colored tensor models in arbitrary dimension, Europhys. Lett. **95**, 50004 (2011).

[15] V. Bonzom, R. Gurau, A. Riello, and V. Rivasseau, Critical behavior of colored tensor models in the large N limit, Nucl. Phys. **B853**, 174 (2011).

[16] R. Gurau, A generalization of the Virasoro algebra to arbitrary dimensions, Nucl. Phys. **B852**, 592 (2011).

[17] R. Gurau, The complete 1/N expansion of colored tensor models in arbitrary dimension, Ann. Inst. Henri Poincaré **13**, 399 (2012).

[18] V. Bonzom, R. Gurau, and V. Rivasseau, Random tensor models in the large N limit: Uncoloring the colored tensor models, Phys. Rev. D **85**, 084037 (2012).

[19] R. Gurau and J. P. Ryan, Colored tensor models–A review, SIGMA **8**, 020 (2012).

[20] R. Gurau, The Schwinger Dyson equations and the algebra of constraints of random tensor models at all orders, Nucl. Phys. **B865**, 133 (2012).

[21] V. Bonzom, Revisiting random tensor models at large N via the Schwinger-Dyson equations, J. High Energy Phys. 03 (2013) 160.

[22] V. Rivasseau, The tensor track, III, Fortschr. Phys. **62**, 81 (2014).

[23] R. G. Gurau, *Random Tensors* (Oxford University Press, Oxford, 2016).

[24] V. Rivasseau, Random tensors and quantum gravity, SIGMA **12**, 069 (2016).

[25] I. R. Klebanov and G. Tarnopolsky, Uncolored random tensors, Melon diagrams, and the SYK models, Phys. Rev. D **95**, 046004 (2017).

[26] S. Carrozza and A. Tanasa, $O(N)$ Random tensor models, Lett. Math. Phys. **106**, 1531 (2016).

[27] S. Dartois, V. Rivasseau, and A. Tanasa, The 1/N expansion of multi-orientable random tensor models, Ann. Inst. Henri Poincaré **15**, 965 (2014).

[28] M. Raasakka and A. Tanasa, Next-to-leading order in the large N expansion of the multi-orientable random tensor model, Ann. Inst. Henri Poincaré **16**, 1267 (2015).

[29] A. Tanasa, The multi-orientable random tensor model, A review, SIGMA **12**, 056 (2016).

[30] Y. Gu, X.-L. Qi, and D. Stanford, Local criticality, diffusion and Chaos in generalized Sachdev-Ye-Kitaev models, J. High Energy Phys. 05 (2017) 125.

[31] M. Berkooz, P. Narayan, M. Rozali, and J. Simón, Higher dimensional generalizations of the SYK model, J. High Energy Phys. 01 (2017) 138.

[32] G. Turiaci and H. Verlinde, Towards a 2d QFT analog of the SYK model, J. High Energy Phys. 10 (2017) 167.

[33] S.-K. Jian and H. Yao, Solvable Sachdev-Ye-Kitaev Models in Higher Dimensions: From Diffusion to Many-Body Localization, Phys. Rev. Lett. **119**, 206602 (2017).

[34] M. Berkooz, P. Narayan, M. Rozali, and J. Simón, Comments on the random thirring model, J. High Energy Phys. 09 (2017) 057.
[35] C.-M. Jian, Z. Bi, and C. Xu, A model for continuous thermal metal to insulator transition, Phys. Rev. B **96**, 115122 (2017).
[36] C. Peng, Vector models, and generalized SYK models, J. High Energy Phys. 05 (2017) 129.
[37] P. Narayan and J. Yoon, SYK-like tensor models on the Lattice, J. High Energy Phys. 08 (2017) 083.
[38] D. V. Khveshchenko, Thickening and sickening the SYK model, SciPost Phys. **5**, 012 (2018).
[39] W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, Supersymmetric SYK models, Phys. Rev. D **95**, 026009 (2017).
[40] C. Peng, M. Spradlin, and A. Volovich, A supersymmetric SYK-like tensor model, J. High Energy Phys. 05 (2017) 062.
[41] D. Anninos, T. Anous, and F. Denef, Disordered quivers and cold horizons, J. High Energy Phys. 12 (2016) 071.
[42] B. Michel, J. Polchinski, V. Rosenhaus, and S. J. Suh, Four-point function in the IOP matrix model, J. High Energy Phys. 05 (2016) 048.
[43] T. Nishinaka and S. Terashima, A note on Sachdev-Ye-Kitaev like model without random coupling, Nucl. Phys. **B926**, 321 (2018).
[44] R. Gurau, Quenched equals annealed at leading order in the colored SYK model, Europhys. Lett. **119**, 30003 (2017).
[45] J. Polchinski and V. Rosenhaus, The spectrum in the Sachdev-Ye-Kitaev model, J. High Energy Phys. 04 (2016) 001.
[46] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, Black holes and random matrices, J. High Energy Phys. 05 (2017) 118.
[47] W. Fu and S. Sachdev, Numerical study of Fermion and Boson models with infinite-range random interactions, Phys. Rev. B **94**, 035135 (2016).
[48] Y. Liu, M. A. Nowak, and I. Zahed, Disorder in the Sachdev-Yee-Kitaev model, Phys. Lett. B **773**, 647 (2017).
[49] S. Banerjee and E. Altman, Solvable model for a dynamical quantum phase transition from fast to slow scrambling, Phys. Rev. B **95**, 134302 (2017).
[50] R. Gurau, The complete $1/N$ expansion of a SYK–like tensor model, Nucl. Phys. **B916**, 386 (2017).
[51] A. M. García-García and J. J. M. Verbaarschot, Analytical spectral density of the Sachdev-Ye-Kitaev model at finite N, Phys. Rev. D **96**, 066012 (2017).
[52] Z. Bi, C.-M. Jian, Y.-Z. You, K. A. Pawlak, and C. Xu, Instability of the non-Fermi liquid state of the Sachdev-Ye-Kitaev model, Phys. Rev. B **95**, 205105 (2017).
[53] Y. Chen, H. Zhai, and P. Zhang, Tunable quantum Chaos in the Sachdev-Ye-Kitaev model coupled to a thermal bath, J. High Energy Phys. 07 (2017) 150.
[54] X. Chen, R. Fan, Y. Chen, H. Zhai, and P. Zhang, Competition between Chaotic and Non-Chaotic Phases in a Quadratically Coupled Sachdev-Ye-Kitaev Model, Phys. Rev. Lett. **119**, 207603 (2017).
[55] X.-Y. Song, C.-M. Jian, and L. Balents, A Strongly Correlated Metal Built from Sachdev-Ye-Kitaev Models, Phys. Rev. Lett. **119**, 216601 (2017).
[56] A. Jevicki, K. Suzuki, and J. Yoon, Bi-local holography in the SYK model, J. High Energy Phys. 07 (2016) 007.
[57] A. Jevicki and K. Suzuki, Bi-local holography in the SYK model: Perturbations, J. High Energy Phys. 11 (2016) 046.
[58] V. Bonzom, L. Lionni, and A. Tanasa, Diagrammatics of a colored SYK model and of an SYK-like tensor model, leading and next-to-leading orders, J. Math. Phys. (N.Y.) **58**, 052301 (2017).
[59] D. J. Gross and V. Rosenhaus, The bulk dual of SYK: Cubic couplings, J. High Energy Phys. 05 (2017) 92.
[60] D. Bagrets, A. Altland, and A. Kamenev, Power-law out of time order correlation functions in the SYK model, Nucl. Phys. **B921**, 727 (2017).
[61] R. Gurau, The $\imath\epsilon$ prescription in the SYK model, J. Phys. Commun. **2**, 015003 (2018).
[62] D. Bagrets, A. Altland, and A. Kamenev, Sachdev-Ye-Kitaev model as Liouville quantum mechanics, Nucl. Phys. **B911**, 191 (2016).
[63] D. Stanford and E. Witten, Fermionic localization of the Schwarzian theory, J. High Energy Phys. 10 (2017) 008.
[64] Y.-Z. You, A. W. W. Ludwig, and C. Xu, Sachdev-Ye-Kitaev model and thermalization on the boundary of many-body localized Fermionic symmetry protected topological states, Phys. Rev. B **95**, 115150 (2017).
[65] C. Krishnan, S. Sanyal, and P. N. B. Subramanian, Quantum Chaos and holographic tensor models, J. High Energy Phys. 03 (2017) 056.
[66] T. Li, J. Liu, Y. Xin, and Y. Zhou, Supersymmetric SYK model and random matrix theory, J. High Energy Phys. 06 (2017) 111.
[67] F. Ferrari, The large D limit of planar diagrams, arXiv: 1701.01171.
[68] C. Krishnan, K. V. P. Kumar, and S. Sanyal, Random matrices and holographic tensor models, J. High Energy Phys. 06 (2017) 036.
[69] R. A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen, and S. Sachdev, Thermoelectric transport in disordered metals without quasiparticles: The SYK models andholography, Phys. Rev. B **95**, 155131 (2017).
[70] Y. Gu, A. Lucas, and X.-L. Qi, Energy diffusion and the butterfly effect in inhomogeneous Sachdev-Ye-Kitaev chains, SciPost Phys. **2**, 018 (2017).
[71] L. García-Álvarez, I. L. Egusquiza, L. Lamata, A. del Campo, J. Sonner, and E. Solano, Digital Quantum Simulation of Minimal AdS/CFT, Phys. Rev. Lett. **119**, 040501 (2017).
[72] I. Danshita, M. Hanada, and M. Tezuka, Creating and probing the Sachdev-Ye-Kitaev model with ultracold gases: Towards experimental studies of quantum gravity, Prog. Theor. Exp. Phys. **2017**, 083I01 (2017).
[73] D. I. Pikulin and M. Franz, Black Hole on a Chip: Proposal for a Physical Realization of the SYK Model in a Solid-State System, Phys. Rev. X **7**, 031006 (2017).
[74] A. Chew, A. Essin, and J. Alicea, Approximating the Sachdev-Ye-Kitaev model with Majorana wires, Phys. Rev. B **96**, 121119 (2017).
[75] G. Mandal, P. Nayak, and S. R. Wadia, Coadjoint orbit action of Virasoro group and two-dimensional quantum

gravity dual to SYK/tensor models, J. High Energy Phys. 11 (2017) 46.

[76] S. R. Das, A. Jevicki, and K. Suzuki, Three dimensional view of the SYK/AdS duality, J. High Energy Phys. 09 (2017) 17.

[77] W. Kaminski, D. Oriti, and J. P. Ryan, Towards a double-scaling limit for tensor models: Probing sub-dominant orders, New J. Phys. **16**, 063048 (2014).

[78] A. Tanasa, Multi-orientable group field theory, J. Phys. A **45**, 165401 (2012).

[79] D. Benedetti and R. Gurau, 2PI effective action for the SYK model and tensor field theories, J. High Energy Phys. 05 (2018) 156.

[80] S. Choudhury, A. Dey, I. Halder, L. Janagal, S. Minwalla, and R. Poojary, Notes on Melonic $O(N)^{q-1}$ tensor models, J. High Energy Phys. 06 (2018) 094.