On the absence of conformally flat slicings of the Kerr spacetime

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This work investigates the possibility of achieving a conformally flat slicing of the Kerr spacetime. We consider a hypersurface of the form $t = F(r, \theta, a)$, where (t, r, θ, ϕ) are the Boyer-Lindquist coordinates; solve for a vanishing Cotton-York tensor of the induced metric order by order in the spin parameter a; and show that the procedure fails at the fifth order. We also prove that no coordinate change can induce a spatially flat recasting of the Kerr(-de Sitter) metric, beyond linear order in a, adopting a more general ansatz depending on ϕ.

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I. INTRODUCTION AND MOTIVATIONS

From their theoretical discovery by Karl Schwarzschild in 1916 [1] to the first direct observation of their immediate vicinity more than a century later [2], black holes (BHs) have been one of the cornerstones of modern gravitational physics. While a static BH is one of the simplest objects in General Relativity (GR) at least from a theoretical point of view, it is more difficult to study spinning BH spacetimes; their construction was achieved only in 1963 [3]. Those objects possess rich phenomenology; for example, they can trigger the so-called active galactic nuclei; induce Penrose processes; or even, if in binary or triplets, emit detectable gravitational radiation.

The detection of gravitational radiation relies heavily on numerical relativity, notably to estimate the gravitational waveform produced during the merger phase (which is the dominant part of the signal for the binary BH induced events detected by the LIGO/Virgo Collaboration [4]). In order to find initial data for these numerical studies, it has been common to rely on conformal flatness, i.e., that the spatial metric induced by such foliations can be written as $\gamma_{ij} = \Omega^2(r, \theta, \phi)\eta_{ij}$, where Ω is a free function of the spatial coordinates and η is any usual flat metric. Classical examples using or being simplified by this assumption include the Misner [5] or Bowen-York [6] initial data, the Isenberg-Wilson-Mathews formulation [7,8], or the puncture framework [9], all of which have been widely used in the literature.[1] In this context, it has been therefore natural to seek for conformally flat foliations of the different BH spacetimes.

Such a conformally flat slicing is trivially realized in the case of static BHs. A static and spatially flat BH solution was found as early as in 1921, independently by P. Painlevé [15] and A. Gullstrand [16]. This solution was recognized as a simple coordinate transformation of the usual Schwarzschild coordinate system by G. Lemaître, 12 years later [17]. In the case of rotating BHs, the game is more involved, and a first approach, conducted by A. Garat and R. H. Price [18], ended up with a no-go result indicating that the Kerr metric does not allow for a conformally flat slicing.[2] However, as discussed in the next section, their no-go result is based on two restrictive assumptions. The main purpose of the present paper is to strengthen the no-go result by relaxing one of their assumptions.

The present work has significant implications also in a different context. In Ref. [26], the authors have proven that any solution of GR that admits a spatially flat slicing is also a solution of an alternative theory of gravity, namely the

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[1]For a review, see Ref. [10] or [11]. Conformally flat initial data have notably been used for the first promising simulation of a binary black hole spacetime [12]. For more recent work using conformal flatness, see, for example, Refs. [13,14].

[2]Conformally flat initial data have nevertheless been commonly used, in particular for slowly spinning black holes. In such a case, one has to put up with spurious "junk" radiation (see, e.g., Refs. [19–22]). See also Refs. [22–25] for binary black hole initial data beyond conformal flatness.

Minimal Theory of Massive Gravity (MTMG) [27,28]. So, it has been shown that MTMG admits static BH solutions, and the next step is naturally to investigate whether it also admits rotating BH solutions. For this reason, it is of physical interest to elucidate whether the Kerr(-de Sitter) spacetime admits spatially flat slicings, which are a subclass of conformally flat slicings, or not. However, while the Schwarzschild solution of MTMG is in the Painlevé-Gullstrand slicing, the previous no-go result of Ref. [18] applies only to those slicings that reduce to the Schwarzschild slicing in the nonspinning limit. The extended no-go result of the present paper is general enough to exclude a conformally flat slicing that in the nonspinning limit reduces to the Painlevé-Gullstrand slicing. A full proof of the no-go result for spatially flat slicings of the Kerr(-de Sitter) spacetime (i.e., via a general change of coordinates) is separately presented in the Appendix of this work. The new no-go results shown in the present paper imply that rotating BHs solutions cannot be implemented in MTMG by the aforementioned procedure. This implies either that MTMG accommodates the Kerr solution in a different way or that rotating BHs should deviate from the Kerr spacetime. In the latter case, BHs would provide a window to distinguish MTMG and GR observationally.

This work is organized as follows. In Sec. II, we expose our strategy to construct a conformally flat slicing of the Kerr metric, and we apply it in Sec. III, up to its failure at the fifth order in the spin parameter a. We then discuss this no-go result and conclude in IV. As we were initially interested in spatially flat slicings, we also set a no-go result on the construction of such foliations by a general coordinate change in Appendix. A. Any lengthy expressions we had to deal with are presented in Appendix B.

II. STRATEGY

We write the Kerr line element in Boyer-Lindquist coordinates (t, r, θ, ϕ) [29]

$$\mathrm{d}s^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta\mathrm{d}\phi^2 - \frac{4aMr\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\phi, \tag{1}$$

with the auxiliary functions

$$\Delta = r^2 - 2Mr + a^2 \quad \text{and} \quad \Sigma = r^2 + a^2\cos^2\theta. \tag{2}$$

In order to seek a possible conformally flat slicing, we introduce a hypersurface[3] specified by $t = F(r, \theta, a)$ while imposing $F(r, \theta, 0) = f_0(r)$. Such a form is motivated by the original symmetries of the Kerr metric: no azimuthal dependency is included so as to preserve axisymmetry, and in the nonspinning limit, it reduces to the Schwarzschild one, that is a spherically symmetric form. Note that this slicing is more general than the one used in Ref. [18], in which $F(r, \theta, 0) = \text{const.}$ was imposed. This restriction[4] reduces drastically the field of possible solutions: it notably prevents solutions that would reduce to the Painlevé-Gullstrand (i.e., spatially flat) coordinates in the limit $a = 0$. On the other hand, in this work, by assuming $F(r, \theta, 0) = f_0(r)$, we keep the possibility of finding a slicing that in the static limit reduces to a non-Schwarzschild slicing, including the Painlevé-Gullstrand one.

Starting from a general a-expanded slicing

$$t = f_0(r) + \sum_{n=1}^{\infty} a^n f_n(r, \theta), \tag{3}$$

we will compute the Cotton-York tensor on the induced spatial hypersurfaces, given by

$$\mathcal{C}^i_j = \varepsilon^{ikl}\nabla_k\left(R_{jl} - \frac{1}{4}R\gamma_{jl}\right), \tag{4}$$

where ∇_i, R_{ij}, and R are respectively the covariant derivative, the Ricci tensor, and the Ricci scalar associated with the induced three-dimensional metric γ_{ij}, and ε^{ijk} is the Levi-Civita tensor. As there is a one-to-one correspondence between conformal flatness and the cancellation of the Cotton-York tensor in three dimensions, we will try to solve the equation $\mathcal{C}^i_j = 0$ order by order in a and thus constrain the functions $f_n(r, \theta)$.

III. PROOF OF NONCANCELLATION OF THE COTTON-YORK TENSOR

The great advantage of our assumption $F(r, \theta, 0) = f_0(r)$ is that the Cotton-York tensor automatically vanishes at the zeroth order in a. This is naturally linked to the fact that the Schwarzschild metric is conformally flat.

At the linear order in a, the $r\phi$-component of the Cotton-York tensor is factorizable as

$$\mathcal{C}^r_\phi = a\frac{\mathcal{A}_1(r)\mathcal{B}_1(r, \theta)}{2r\sin^4\theta(r^2 - (r-2M)^2(f_0')^2)} + \mathcal{O}(a^2), \tag{5}$$

with

$$\mathcal{A}_1 = r\left(1 - \frac{2M}{r}\right)f_0'' + \left(1 - \frac{2M}{r}\right)^2\left(1 - \frac{3M}{r}\right)(f_0')^3 - \left(1 - \frac{5M}{r}\right)f_0', \tag{6a}$$

[3]If one finds a conformally flat hypersurface of this form, then one can easily promote it to a conformally flat foliation, i.e., a family of conformally flat hypersurfaces, by simply adding different constants to $f_0(r)$. This procedure is guaranteed to work since the spacetime is invariant under a constant shift of t.

[4]They also prevent ϕ dependencies as, to quote their own words, "gaining the advantages of conformal flatness while losing axisymmetry would be a Pyrrhic victory".

$$\mathcal{B}_1 = \frac{\partial_\theta^3 f_1 + \cot\theta \partial_\theta^2 f_1 + (1-\cot^2\theta)\partial_\theta f_1}{\sin\theta}. \tag{6b}$$

So, at this level, it is clear that the possible solutions split in two branches[5]: in the first one, f_0 has to be constrained so that $\mathcal{A}_1$ vanishes, whereas in the second one, we have to solve $\mathcal{B}_1 = 0$ with respect to f_1.

A. First branch

1. First order

Solving $\mathcal{A}_1(r) = 0$ in terms of f_0' yields

$$f_0'(r) = \pm \frac{r^{5/2}}{(r-2M)\sqrt{r^3 + \lambda(r-2M)}}, \tag{7}$$

where λ is an integration constant with dimension of squared mass. Note that we present here f_0' and not f_0 as the latter will never appear in the equations due to the time shift symmetry. When this solution is injected, the whole Cotton-York tensor vanishes at the first order in a.

2. Second order

At the second order in a, the rr-component of the Cotton-York tensor reads

$$\mathcal{C}^r_r = -\frac{3Ma^2\sin\theta}{r^4}\left[\partial_\theta^3 f_1 + 5\cot\theta\partial_\theta^2 f_1 + \frac{10\cos^2\theta - 3}{\sin^2\theta}\partial_\theta f_1\right], \tag{8}$$

which, imposing regularity on $]0, 2\pi[$, yields $f_1 = \bar{f}_1(r) + f_{1,s}(r)/\sin^2\theta$. Injecting it in the $r\theta$-component, yields

$$\mathcal{C}^r_\theta = -\frac{6Ma^2}{r^2(r^3 + \lambda(r-2M))} \times \left[\left(2 + \frac{3\lambda}{r^2} - \frac{7M\lambda}{r^3}\right) f_{1,s} - r\left(1 + \frac{\lambda}{r^2} - \frac{2M\lambda}{r^3}\right) f_{1,s}'\right] + \mathcal{O}(a^3), \tag{9}$$

which imposes $f_{1,s} = \lambda_1 r^{7/2}/\sqrt{r^3 + \lambda(r-2M)}$, with λ_1 a constant with dimension of inverse squared mass. One can then express the $\theta\phi$-component as

$$\mathcal{C}^\theta_\phi = \frac{a^2}{\lambda^2 r^2 \sin^{10}\theta (r^3 + \lambda(r-2M))} \times \sum_{n=0}^{5} \Gamma_n(r)\cos^{2n}\theta + \mathcal{O}(a^3). \tag{10}$$

Here, Γ_n $(n = 0, \ldots, 5)$ are functions of r, which we do not need to fully specify for the argument. As $\mathcal{C}^\theta_\phi$ should vanish for any θ, each of the $\Gamma_n(r)$ should vanish by its own. However, we have

$$\Gamma_5(r) = M\lambda\{42M\lambda^2(r-2M) - 3\lambda r^3(r-16M) - 3r^6\}. \tag{11}$$

One cannot simply impose $\lambda \to 0$ as $\mathcal{C}^\theta_\phi$ would blow up, due to the presence of λ^2 in the denominator, so it is impossible to cancel $\mathcal{C}^i_j$ at the second order in a, if f_0 is of the form (7).

B. Second branch

1. First order

In this branch, we have to solve $\mathcal{B}_1 = 0$ in terms of f_1, which yields simply

$$f_1(r,\theta) = \bar{f}_1(r) + f_{1,c}(r)\cos\theta. \tag{12}$$

The slicing at the first order in a is hence parametrized by two undetermined functions of r, $\{\bar{f}_1, f_{1,c}\}$. Note that $\bar{f}_1(r)$ can be reabsorbed in $f_0(r)$. When this solution is injected, the whole Cotton-York tensor vanishes at the first order in a.

2. Second order

The rr-component of the Cotton-York tensor reads

$$\mathcal{C}^r_r = \frac{4Ma^2(3\cos^2\theta - 1)}{r^4} f_{1,c}(r) + \mathcal{O}(a^3), \tag{13}$$

which imposes $f_{1,c} = 0$; we are left with only $f_1 = \bar{f}_1(r)$ that does not appear in $\mathcal{C}^i_j$ at this order. The next non-vanishing component is

$$\mathcal{C}^r_\phi = -\frac{a^2}{2r^3}\frac{\mathcal{A}_1(r)\mathcal{B}_2(r,\theta) + 4\cos\theta\mathcal{S}_2(r)}{1 - (1-\frac{2M}{r})^2(f_0')^2} + \mathcal{O}(a^3), \tag{14}$$

where $\mathcal{A}_1$ is still given by (6a); $\mathcal{B}_2$ is given by (6b), when substituting f_2 to f_1; and the expression of the source term $\mathcal{S}_2$ is given in Appendix B. Solving for f_2, the slicing reads

$$F(r,\theta,a) = f_0(r) + a\bar{f}_1(r) + a^2[\bar{f}_2(r) + f_{2,c}(r)\cos\theta + \hat{f}_2(r)\cos^2\theta] + \mathcal{O}(a^3), \tag{15}$$

[5]The two branches still exist for a parametrization $F(r,\theta,\phi,a)$ with $F(x^i, 0) = f_0(r)$. However, in this case, the ϕ dependence drastically complicates the resolution of the first branch by introducing highly nonlinear partial differential equations (PDEs). Nevertheless, the whole argument of nonexistence of conformally flat slicing is not significantly changed, for the second branch, by this additional dependence in ϕ.

where $\{f_0, \bar{f}_1, \bar{f}_2, f_{2,c}\}$ are four free functions and $\hat{f}_2(r)$ is given in terms of f_0' and f_0'' in Appendix B.

3. Third order

Again, the rr-component of the Cotton-York tensor imposes a first set of conditions on the lower-order functions. To let $\frac{d^3}{d(\cos\theta)^3}\mathcal{C}^r_r$ vanish, one must impose

$$63\frac{M^2}{r^2} - f_0'^2\left(1-\frac{2M}{r}\right)^2\left(5+\frac{2M}{r}+\frac{45M^2}{r^2}\right) + f_0'^4\left(1-\frac{2M}{r}\right)^4\left(5+\frac{12M}{r}\right) + rf_0'f_0''\left(1-\frac{2M}{r}\right)^3\left(5+\frac{9M}{r}\right) = 0, \quad (16)$$

which can be integrated to give $f_0(r) = \hat{f}_0(r)$. Here, the explicit expression of $\hat{f}_0(r)$ is shown in Appendix B. In order to keep a compact notation, we will keep $\hat{f}_0$ as a shorthand in most expressions. This also fully fixes the form of $\hat{f}_2(r)$. With f_0 fixed,

$$\mathcal{C}^r_r = -\frac{12a^3M}{r^4}(3\cos^2\theta - 1)f_{2,c} + \mathcal{O}(a^4) = 0 \quad (17)$$

gives $f_{2,c} = 0$. Focusing on the $\mathcal{C}^r_\phi$ component then gives

$$\frac{9M}{2r}\frac{\frac{7M}{r} - \hat{f}_0'^2(1-\frac{2M}{r})^2(2+\frac{3M}{r})}{1-\frac{2M}{r}}\mathcal{B}_3(r,\theta) + \frac{2\cos\theta}{3r}\mathcal{S}_3(r) = 0, \quad (18)$$

where $\mathcal{B}_3$ is given by (6b), when substituting f_3 to f_1, and where the expression of the source term $\mathcal{S}_3$ is given in Appendix B. This allows one to fix

$$f_3(r,\theta) = \bar{f}_3(r) + f_{3,c}(r)\cos\theta + \hat{f}_3(r)\cos^2\theta, \quad (19)$$

with $\hat{f}_3(r)$ given in terms of $\bar{f}_1(r)$, which leads to

$$F(r,\theta,a) = \hat{f}_0(r) + a\bar{f}_1(r) + a^2\left[\bar{f}_2(r) + \frac{\hat{f}_0'}{3r}\left(1-\frac{2M}{r}\right)\cos^2\theta\right] + a^3[\bar{f}_3(r) + f_{3,c}(r)\cos\theta + \hat{f}_3(r)\cos^2\theta] + \mathcal{O}(a^4). \quad (20)$$

4. Fourth order

Demanding $\frac{d^3}{d(\cos\theta)^3}\mathcal{C}^r_r$ to vanish, one finds the equation

$$\bar{f}_1'' - \bar{f}_1'\frac{63\frac{M^2}{r^2} - 3\hat{f}_0'^4(1-\frac{2M}{r})^4(5+\frac{12M}{r}) + \hat{f}_0'^2(1-\frac{2M}{r})^2(5+\frac{2M}{r}+\frac{45M^2}{r^2})}{\hat{f}_0'^2(1-\frac{2M}{r})^3(5+\frac{9M}{r})r} = 0, \quad (21)$$

which can be solved by $\bar{f}_1(r) = \hat{f}_1(r)$ given in Appendix B. Plugging this solution back and demanding the vanishing of $\mathcal{C}^r_r$, one finds that $f_{3,c} = 0$. The vanishing of $\mathcal{C}^r_\phi$ then translates into a differential equation for f_4,

$$\frac{9M}{2r}\left[\frac{7M}{r} - \left(1-\frac{2M}{r}\right)^2\left(2+\frac{3M}{r}\right)\hat{f}_0'^2\right]\mathcal{B}_4(r,\theta) + \frac{\cos\theta\mathcal{S}_{4,1}(r) + \cos^3\theta\mathcal{S}_{4,3}(r)}{3(1-\frac{2M}{r})(5+\frac{9M}{r})[\hat{f}_0'^2(1-\frac{2M}{r})^2 - 1]\hat{f}_0'r^3} = 0, \quad (22)$$

where $\mathcal{B}_4$ is given by (6b), when substituting f_4 to f_1, and where the source terms $\mathcal{S}_{4,2}$ and $\mathcal{S}_{4,4}$ are given explicitly in Appendix B. Therefore, we obtain the full solution

$$f_4(r,\theta) = \bar{f}_4(r) + f_{4,c}(r)\cos\theta + \hat{f}_{4,2}(r)\cos^2\theta + \hat{f}_{4,4}(r)\cos^4\theta, \quad (23)$$

where $\hat{f}_{4,2}(r)$ and $\hat{f}_{4,4}(r)$ are fully dependent functions of r and can, for example, be given in terms of $\hat{f}_0$, $\hat{f}_1$, and $\bar{f}_2$. At the fourth order, we thus have

$$F(r,\theta,a) = \hat{f}_0(r) + a\hat{f}_1(r) + a^2\left[\bar{f}_2(r) + \frac{\hat{f}_0'}{3r}\left(1-\frac{2M}{r}\right)\cos^2\theta\right] + a^3\left[\bar{f}_3(r) + \frac{\hat{f}_1'}{3r}\left(1-\frac{2M}{r}\right)\cos^2\theta\right] + a^4[\bar{f}_4(r) + f_{4,c}(r)\cos\theta + \hat{f}_{4,2}(r)\cos^2\theta + \hat{f}_{4,4}(r)\cos^4\theta] + \mathcal{O}(a^5), \quad (24)$$

which lets the Cotton-York tensor completely vanish at this order and which depends on four free functions $\{\bar{f}_2, \bar{f}_3, \bar{f}_4, f_{4,c}\}$ and four integration constants (within $\hat{f}_0$ and $\hat{f}_1$).

5. Fifth order

Finally, by considering

$$\frac{d^5}{d(\cos\theta)^5}\mathcal{C}^r_r = -\frac{7a^5}{54r^6(5+\frac{9M}{r})^3}\frac{\mathcal{N}_1(r)\mathcal{N}_2(r)}{\mathcal{D}_1(r)(\mathcal{D}_2(r))^{1/2}} + \mathcal{O}(a^6), \quad (25)$$

with the fully (up to an integration constant) determined $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{D}_1$, and $\mathcal{D}_2$ given in Appendix B, one finds that the Cotton-York tensor cannot identically vanish under the assumptions made above. This concludes our argument.

IV. CONCLUSION

Conformally flat slicings are of prime importance when dealing with realistic spacetimes, as they allow one to efficiently obtain initial data for numerical computations. While the conformally flat slicings of static BHs have been known for a long time, it has been impossible to find their equivalent for rotating BHs. A. Garat and R. H. Price showed that no slicing of the form $t = F(r, \theta, a)$ with $F(r, \theta, 0) = \text{const.}$ could support conformal flatness [18]. In this work, we followed their steps and relaxed one of their assumptions by taking $t = F(r, \theta, a)$ with $F(r, \theta, 0) = f_0(r)$, which has notably the advantage of including the Painlevé-Gullstrand coordinate change. Even under this weaker restriction, we have demonstrated that it is not possible to find conformally flat hypersurfaces. The next steps would naturally be to either relax the assumption that $F(r, \theta, 0) = f_0(r)$ or/and examine a parametrization $F(r, \theta, \phi, a)$.[6] These extensions could then potentially yield a stronger no-go result for the construction of a conformally flat slicing of the Kerr spacetime, but at the cost of hiding its original symmetries. Finally, note that our results agree, and may be possibly further connected, with the findings of Ref. [30] in an expansion at infinity.

The nonexistence of conformally flat slicings of the Kerr spacetime can be linked to the failure of mimicking the exterior of a Kerr BH with ordinary matter. In general relativity, a well-known theorem due to Jebsen [31] and Birkhoff [32] states that the exterior solution of all spherically symmetric matter content is the Schwarzschild one. But no such theorem exists in the spinning case, and usually the multipole moments created by a system of spinning ordinary matter will only asymptotically agree with the Kerr ones. Nevertheless, some attempts to recreate an external Kerr geometry with matter were made, but they always involve exotic matter (see, e.g., Refs. [33–35]). But when dealing with gravitational radiation for an ordinary spinning matter system, the metric perturbation around a Minkowskian background is usually gauged as

$$h_{00} \simeq \frac{2M}{r}, \qquad h_{0i} \simeq \frac{\epsilon_{ijk} S^j x^k}{r^3}, \qquad h_{ij} \simeq \Omega^2 \delta_{ij} + h_{ij}^{\text{rad}}, \tag{26}$$

with S^i the spin vector. One can naturally see that for $h_{ij}^{\text{rad}} = 0$ this perturbation is conformally flat. Thus, the fact that the Kerr geometry cannot be conformally flat sliced is in agreement with the fact there is *a priori* no ordinary matter system generating it.

The present work was originally motivated by the study of black hole solutions in the minimal theory of massive gravity. Indeed, this theory has been shown to admit as solutions all spatially flat general relativistic spacetimes. The question of the existence of a flat slicing of the Kerr solution, which we show not to exist in Appendix A, was a motivation to search more generically for conformally flat slicings. The present work leaves open the possibility that, within MTMG, rotating black hole solutions be found in a completely different fashion.

[6] A straightforward analysis shows that this possibility fails at the fifth order for the second branch. Indeed, the structure of the argument doesn't change and the same component $\mathcal{C}^r_r$, at the fifth-order in a, can be used to conclude the argument.

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APPENDIX A: ABSENCE OF FLAT SLICINGS IN KERR-DE SITTER SPACTIME

In this Appendix, we demonstrate that there are no spatially flat slicings of the Kerr-de Sitter spactime. We will proceed by performing a general coordinate change. This is equivalent to adopting an ansatz that is more general than the one in the main text used for the proof of the absence of conformally flat slicings.

1. Strategy

Let us recall the Kerr-de Sitter line element written in Boyer-Lindquist-like coordinates

$$\begin{aligned} \mathrm{d}s^2 &= \bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\frac{\tilde{\Delta} - a^2\zeta\sin^2\theta}{\Xi}\mathrm{d}t^2 + \frac{\Sigma}{\tilde{\Delta}}\mathrm{d}r^2 + \frac{\Sigma}{\zeta}\mathrm{d}\theta^2 \\ &\quad + \frac{(r^2+a^2)^2\zeta - a^2\tilde{\Delta}\sin^2\theta}{\Xi}\sin^2\theta \mathrm{d}\phi^2 \\ &\quad - \frac{2a[6Mr - \Lambda(r^2+a^2)\Sigma]}{3\Xi}\mathrm{d}t\mathrm{d}\phi, \end{aligned} \tag{A1}$$

where Λ is the cosmological constant and

$$\tilde{\Delta} = (r^2 + a^2)\left(1 + \frac{\Lambda r^2}{3}\right) - 2Mr, \qquad \Sigma = r^2 + a^2\cos^2\theta,$$

$$\Xi = \Sigma\left(1 - \frac{\Lambda a^2}{3}\right)^2, \quad \text{and} \quad \zeta = 1 - \frac{\Lambda a^2\cos^2\theta}{3}. \tag{A2}$$

This metric reduces to (1) in the $\Lambda = 0$ case. We will hereafter perform a general coordinate change $x^\mu \to \chi^\mu(a, x^\nu) = \{\tau, \rho, \vartheta, \varphi\}$ and ask that the spatially induced metric be flat,

$$\gamma_{ij} \equiv \bar{g}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\chi^i}\frac{\mathrm{d}x^\nu}{\mathrm{d}\chi^j} = \hat{\delta}_{ij}, \tag{A3}$$

where $\hat{\delta}$ is the usual three-dimensional Euclidean metric[7]

$$\hat{\delta}_{ij}\mathrm{d}\chi^i\mathrm{d}\chi^j = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\vartheta^2 + \rho^2\sin^2\vartheta\mathrm{d}\varphi^2. \tag{A4}$$

We will also require the change of coordinates to be invertible, namely that the Jacobian of the transformation be nonvanishing,

$$\mathcal{J} \equiv \left|\frac{\partial\chi^\alpha}{\partial x^\mu}\right| \neq 0. \tag{A5}$$

Note that we already know the result when $a = 0$; in this case, the Boyer-Lindquist-like coordinates reduce to the Schwarzschild-de Sitter ones, and thus the transformation to apply is an extended Painlevé-Gullstrand one (see, e.g., Ref. [26]),

$$t = \tau + \int^\rho \mathrm{d}u\frac{\sqrt{2u\mu(u)}}{u - 2\mu(u)}, \qquad x^i = \chi^i, \tag{A6}$$

where the effective mass is given by $\mu(r) = M - \Lambda r^3/6$. Starting from this zeroth-order solution, we expand the coordinate change in a as

$$t = \tau + \int^\rho \mathrm{d}u\frac{\sqrt{2\mu u}}{u - 2\mu} + \sum_{n=1}^\infty a^n T^{(n)}(\rho, \vartheta, \varphi), \tag{A7a}$$

$$r = \rho + \sum_{n=1}^\infty a^n R^{(n)}(\rho, \vartheta, \varphi), \tag{A7b}$$

$$\theta = \vartheta + \sum_{n=1}^\infty a^n \Theta^{(n)}(\rho, \vartheta, \varphi), \tag{A7c}$$

$$\phi = \varphi + \sum_{n=1}^\infty a^n \Phi^{(n)}(\rho, \vartheta, \varphi) \tag{A7d}$$

[7]Starting from a rotating spacetime, it is natural to aim for a Minkowskian metric written in Born coordinates, which has a purely flat spatial sector.

and solve order by order the equation $\mathcal{E}_{ij} \equiv \gamma_{ij} - \hat{\delta}_{ij} = \sum_{n=1}^\infty a^n \mathcal{E}_{ij}^{(n)} = 0$. Denoting $F_\mu^{(n)}$ the collection $\{T^{(n)}, R^{(n)}, \Theta^{(n)}, \Phi^{(n)}\}$, one can decompose at any order $\mathcal{E}_{ij}^{(n)} = \mathcal{O}_{ij}[F_\mu^{(n)}] + \mathcal{S}_{ij}^{(n)}$, where $\mathcal{S}_{ij}^{(n)}$ is a source term (depending only on $F_\mu^{(m)}$ with $1 \le m \le n-1$) and the linear operator $\mathcal{O}_{ij}[F^{(n)}]$ is given by

$$\mathcal{O}_{ij}[F_\mu^{(n)}] = \left(g_{\mu\nu}^{\mathrm{SdS}}\frac{\partial F_\nu^{(0)}}{\partial\chi^i}\frac{\partial}{\partial\chi^j} + g_{\mu\nu}^{\mathrm{SdS}}\frac{\partial F_\nu^{(0)}}{\partial\chi^j}\frac{\partial}{\partial\chi^i} + \frac{\partial g_{\nu\lambda}^{\mathrm{SdS}}}{\partial x^\mu}\frac{\partial F_\nu^{(0)}}{\partial\chi^i}\frac{\partial F_\lambda^{(0)}}{\partial\chi^j}\right)F_\mu^{(n)}, \tag{A8}$$

where $g_{\mu\nu}^{\mathrm{SdS}}$ is the usual Schwarzschild-de Sitter metric. Explicitly, it reads

$$\mathcal{O}_{\rho\rho} = 2\left(\partial_\rho A + \frac{3M + \Lambda\rho^3}{6M - \Lambda\rho^3}\frac{A - R}{\rho}\right), \tag{A9a}$$

$$\mathcal{O}_{\rho\vartheta} = \partial_\vartheta A + \rho^2\partial_\rho\Theta, \tag{A9b}$$

$$\mathcal{O}_{\rho\varphi} = \partial_\varphi A + \rho^2\sin^2\vartheta\partial_\rho\Phi, \tag{A9c}$$

$$\mathcal{O}_{\vartheta\vartheta} = 2\rho^2\left(\partial_\vartheta\Theta + \frac{R}{\rho}\right), \tag{A9d}$$

$$\mathcal{O}_{\vartheta\varphi} = \rho^2(\partial_\varphi\Theta + \sin^2\vartheta\partial_\vartheta\Phi), \tag{A9e}$$

$$\mathcal{O}_{\varphi\varphi} = 2\rho^2\left(\partial_\varphi\Phi + \frac{R}{\rho} + \Theta\cot\vartheta\right)\sin^2\vartheta, \tag{A9f}$$

where we have introduced the auxiliary function $A(\rho, \vartheta, \varphi) \equiv -\sqrt{\frac{2\mu(\rho)}{\rho}}T(\rho, \vartheta, \varphi) + \frac{\rho}{\rho - 2\mu(\rho)}R(\rho, \vartheta, \varphi)$. So, at a given order, we have to solve a system of coupled linear differential equations; the most general solution will be given by the sum of a homogeneous solution of $\mathcal{O}_{ij}[F_\mu^{(n)}] = 0$ and a particular solution.

2. Homogeneous solution

Let us first find a general solution of the system $\mathcal{O}_{ij} = 0$, where $\mathcal{O}_{ij}$ is defined in (A9). Eliminating all but Θ-dependencies in the angular equations gives

$$\frac{1}{2\rho^2}\partial_\vartheta\left(\mathcal{O}_{\vartheta\vartheta} - \frac{\mathcal{O}_{\varphi\varphi}}{\sin^2\vartheta}\right) + \partial_\varphi\left(\frac{\mathcal{O}_{\vartheta\varphi}}{\rho^2\sin^2\vartheta}\right) = \partial_\vartheta\left[\sin\vartheta\partial_\vartheta\left(\frac{\Theta}{\sin\vartheta}\right)\right] + \frac{1}{\sin^2\vartheta}\partial_\varphi^2\Theta = 0. \tag{A10}$$

Imposing periodicity in φ and regularity in ϑ, the solution reads $\Theta = \Theta_s(\rho)\sin\vartheta + \Theta_c(\rho)\cos\vartheta\sin[\varphi - \varphi_0(\rho)]$. Plugging back in Eqs. (A9d), (A9e), and (A9f) the solution reads $R = -\rho\Theta_s(\rho)\cos\vartheta + \rho\Theta_c(\rho)\sin\vartheta\sin[\varphi - \varphi_0(\rho)]$, and $\Phi = \Phi_0(\rho) + \Theta_c(\rho)\cos[\varphi - \varphi_0(\rho)]/\sin\vartheta$. Last but not least, $\partial_\vartheta\mathcal{O}_{\rho\varphi} - \partial_\varphi\mathcal{O}_{\rho\vartheta} = \rho^2\sin(2\vartheta)\Phi_0'$ forces Φ_0 to be constant. Injecting those solutions in Eqs. (A9b) and (A9c) yields $A = A_0(\rho) + \rho^2\Theta_s'\cos\vartheta - \rho^2\sin\vartheta\partial_\rho(\Theta_c\sin[\varphi - \varphi_0(\rho)])$. Finally, Eq. (A9a) gives

$$A_0' + \lambda\frac{A_0}{\rho} + (\rho^2\Theta_s'' + (2+\lambda)\rho\Theta_s' + \lambda\Theta_s)\cos\vartheta - (\rho^2\tilde{\Theta}_c'' + (2+\lambda)\rho\tilde{\Theta}_c' + \lambda\tilde{\Theta}_c)\sin\vartheta = 0, \tag{A11}$$

where we have shortened $\lambda = (3M + \Lambda\rho^3)/(6M - \Lambda\rho^3)$ and $\tilde{\Theta}_c = \Theta_c(\rho)\sin[\varphi - \varphi_0(\rho)]$. This imposes $A_0 = -T_0\sqrt{2\mu/\rho}$, $\Theta_s = (\kappa_1 + \kappa_2\int du\sqrt{2\mu/u})/\rho$ and $\Theta_c = 0$, where T_0, κ_1, and κ_2 are constants of integration. Let us note that in the $\Lambda = 0$ limit, the solution reads $\Theta_s = \kappa_1/\rho + 2\kappa_2\sqrt{2M/\rho}$. Turning back to the original variables (i.e., expressing T in terms of A and R), it finally comes

$$T_h = T_0 + \left\{\kappa_1\frac{\sqrt{2\rho\mu}}{\rho - 2\mu} + \kappa_2\left(\rho + \frac{\sqrt{2\rho\mu}}{\rho - 2\mu}\int du\sqrt{\frac{2\mu}{u}}\right)\right\}\cos\vartheta, \tag{A12a}$$

$$R_h = \left\{\kappa_1 + \kappa_2\int du\sqrt{\frac{2\mu}{u}}\right\}\cos\vartheta, \tag{A12b}$$

$$\Theta_h = -\frac{1}{\rho}\left\{\kappa_1 + \kappa_2\int du\sqrt{\frac{2\mu}{u}}\right\}\sin\vartheta, \tag{A12c}$$

$$\Phi_h = \Phi_0. \tag{A12d}$$

We can easily recognize that T_0 and Φ_0 are respectively accounting for staticity and axisymmetry of the Schwarzschild-de Sitter metric. The two other constants are associated with the two remaining generators of the group of isometries of the Schwarzschild-de Sitter spacetime.

3. At linear order

At the linear order in a, the only nonvanishing source term is $\mathcal{S}^{(1)}_{\rho\varphi}$, leading to the equation

$$A^{(1)}_\varphi + \left[\rho^2\Phi^{(1)}_\rho - \left(\frac{2\mu}{\rho}\right)^{3/2}\frac{\rho}{\rho - 2\mu}\right]\sin^2\vartheta = 0, \tag{A13}$$

which is easily solved by imposing $\Phi^{(1)}_p = \int^\rho \frac{du}{u}\left(\frac{2\mu}{u}\right)^{3/2}\frac{1}{u - 2\mu}$. Together with the previously found homogeneous solution (A12), the general solution at linear order reads

$$t = \tau + aT_0 + \int^\rho du\frac{\sqrt{2Mu}}{u - 2M} + a\left\{\kappa_1\frac{\sqrt{2\rho\mu}}{\rho - 2\mu} + \kappa_2\left(\rho + \frac{\sqrt{2\rho\mu}}{\rho - 2\mu}\int du\sqrt{\frac{2\mu}{u}}\right)\right\}\cos\vartheta + \mathcal{O}(a^2), \tag{A14a}$$

$$r = \rho + a\left\{\kappa_1 + \kappa_2\int du\sqrt{\frac{2\mu}{u}}\right\}\cos\vartheta + \mathcal{O}(a^2), \tag{A14b}$$

$$\theta = \vartheta - \frac{a}{\rho}\left\{\kappa_1 + \kappa_2\int du\sqrt{\frac{2\mu}{u}}\right\}\sin\vartheta + \mathcal{O}(a^2), \tag{A14c}$$

$$\phi = \varphi + a\varphi_0 + a\int^\rho\frac{du}{u}\left(\frac{2\mu}{u}\right)^{3/2}\frac{1}{u - 2\mu} + \mathcal{O}(a^2). \tag{A14d}$$

Note that $\mathcal{J} = 1 + a(\kappa_1 - \kappa_2\sqrt{2\mu\rho} + \kappa_2\int du\sqrt{\frac{2\mu}{u}})\frac{\cos\vartheta}{\rho} + \mathcal{O}(a^2)$ cannot vanish except in localized points.

4. At second order

At the second order, introducing $\mathcal{K} = \kappa_1 + \kappa_2\int^\rho du\sqrt{\frac{2\mu}{u}}$, the source term is slightly more complicated,

$$\begin{aligned}\mathcal{S}^{(2)}_{\rho\rho} = {} & \frac{(\rho + 2\mu)\cos(2\vartheta)}{2\rho^3} - \frac{\rho^2 - 20\mu^2}{2\rho^3(\rho - 2\mu)} - \frac{2M}{\rho^3}\frac{\rho^2 + 6\rho\mu - 8\mu^2}{(\rho - 2\mu)^2} \\ & - \left[\left(\frac{(3M - 2\mu)^2(\rho^2 - 12\rho\mu - 12\mu^2)}{2\mu\rho(\rho - 2\mu)} + \rho - 6\mu\right)\frac{\rho\cos^2\vartheta}{(\rho + 2\mu)^2} - 1\right]\frac{\mathcal{K}^2}{\rho^2} \\ & + 2\frac{\sqrt{2\mu}}{\rho^{3/2}}\left(\frac{4\mu^2(\rho + 2\mu) + 3M\rho(\rho - 6\mu)}{2\mu(\rho - 2\mu)^2}\cos^2\vartheta - 1\right)\kappa_2\mathcal{K} + \frac{2\mu - (\rho - 2\mu)\cos^2\vartheta}{\rho}\kappa_2^2,\end{aligned} \tag{A15a}$$

$$\mathcal{S}^{(2)}_{\rho\vartheta} = \left[\frac{\rho(\rho - 6\mu) + 3M(\rho + 2\mu)}{\rho(\rho - 2\mu)^2}\mathcal{K}^2 - \frac{3M}{\sqrt{2\rho\mu}}\kappa_2\mathcal{K} + \kappa_2^2\rho\right]\cos\vartheta\sin\vartheta, \tag{A15b}$$

$$\mathcal{S}^{(2)}_{\rho\varphi} = 2\mu\left[\frac{2\mu(\rho + 2\mu) - 3M(3\rho - 2\mu)}{\sqrt{2\mu}\rho^{3/2}}\mathcal{K} + (\rho - 2\mu)\kappa_2\right]\cos\vartheta\sin^2\vartheta, \tag{A15c}$$

$$\mathcal{S}^{(2)}_{\vartheta\vartheta} = \left[1+\frac{\Lambda\rho^2}{3}-2\mathcal{K}^2\right]\cos^2\vartheta + \left[\left(\kappa_1+\left(\int \mathrm{d}u\sqrt{\frac{2\mu}{u}}-\sqrt{2\rho\mu}\right)\kappa_2\right)^2-\kappa_2^2\rho^2\right]\sin^2\vartheta, \tag{A15d}$$

$$\mathcal{S}^{(2)}_{\vartheta\varphi} = \frac{2\mu}{\rho-2\mu}\left[\sqrt{\frac{2\mu}{\rho}}\mathcal{K}+(\rho+2\mu)\kappa_2\right]\sin^3\vartheta, \tag{A15e}$$

$$\mathcal{S}^{(2)}_{\varphi\varphi} = \left[1+\frac{\Lambda\rho^2}{3}+\frac{2M}{\rho}\sin^2\vartheta-(1+\cos^2\vartheta)\mathcal{K}^2\right]\sin^2\vartheta. \tag{A15f}$$

Let us first focus on the angular part. The combination $\partial_\vartheta(\mathcal{E}^{(2)}_{\vartheta\vartheta}-\frac{\mathcal{E}^{(2)}_{\varphi\varphi}}{\sin^2\vartheta})+2\partial_\varphi(\frac{\mathcal{E}^{(2)}_{\vartheta\varphi}}{\sin^2\vartheta})$ yields

$$\partial_\vartheta\left[\sin\vartheta\partial_\vartheta\left(\frac{\Theta^{(2)}_p}{\sin\vartheta}\right)\right]+\frac{\partial^2_\varphi\Theta^{(2)}_p}{\sin^2\vartheta} = \left[1+\frac{2M}{\rho}+\frac{\Lambda\rho^2}{3}-2\mathcal{K}^2+2\sqrt{2\rho\mu}\,\kappa_2\mathcal{K}+\kappa_2^2\rho(\rho-2\mu)\right]\times\frac{\sin(2\vartheta)}{2\rho^2}. \tag{A16}$$

This is notably solved by

$$\Theta^{(2)}_p = -\left[1+\frac{2M}{\rho}+\frac{\Lambda\rho^2}{3}-2\mathcal{K}^2+2\sqrt{2\rho\mu}\kappa_2\mathcal{K}+\kappa_2^2\rho(\rho-2\mu)\right]\times\frac{\sin(2\vartheta)}{4\rho^2}, \tag{A17}$$

and thus, when injected back in $\mathcal{E}^{(2)}_{\vartheta\vartheta}-\frac{\mathcal{E}^{(2)}_{\varphi\varphi}}{\sin^2\vartheta}$ and $\mathcal{E}^{(2)}_{\vartheta\varphi}$, one obtains

$$\Phi^{(2)}_p = \left[\kappa_2+\sqrt{\frac{2\mu}{\rho}}\frac{\mathcal{K}}{\rho-2\mu}\right]\frac{2\mu\cos\vartheta}{\rho^2}. \tag{A18}$$

Then, $\mathcal{E}^{(2)}_{\vartheta\vartheta}$ gives

$$R^{(2)}_p = \left(\frac{M}{\rho^2}+\frac{\rho-2\mu}{2}\kappa_2^2+\sqrt{\frac{2\mu}{\rho}}\kappa_2\mathcal{K}\right)\cos^2\vartheta + \left(\frac{\mu-2M}{\rho^2}+\frac{\mathcal{K}^2-1}{2\rho}\right)\sin^2\vartheta. \tag{A19}$$

Turning to the radial part of the system, $\mathcal{E}^{(2)}_{\rho\varphi}$ gives

$$A^{(2)}_p = \frac{6M\kappa_2\cos\vartheta\sin^2\vartheta}{2\rho}\varphi+f_2(\rho,\vartheta), \tag{A20}$$

which, plugged into $\partial_\varphi\mathcal{E}^{(2)}_{\rho\vartheta}$, imposes that $\kappa_2=0$. $\mathcal{E}^{(2)}_{\rho\vartheta}$ then yields

$$A^{(2)}_p = \left[\frac{3M+\rho}{\rho}-\frac{\rho^2-2\rho\mu+8\mu^2-3M(\rho+2\mu)}{(\rho+2\mu)^2}\right]\frac{\cos(2\vartheta)}{4\rho}. \tag{A21}$$

But it remains

$$\partial_\vartheta\mathcal{E}^{(2)}_{\rho\rho} = \frac{9M^2\sin(2\vartheta)}{2\rho^3\mu} \tag{A22}$$

that cannot vanish. Thus, it is impossible to achieve a spatially flat slicing of the Kerr-de Sitter spacetime.

APPENDIX B: LENGTHY EXPRESSIONS OF THE SECOND BRANCH

In this Appendix, we present the lengthy expressions of Sec. III B in terms of the variable $z=\cos\theta$. So, for example, $\partial_z=-1/\sin\theta\partial_\theta$. Let us also recall our notation

$$\mathcal{A}_1 = r\left(1-\frac{2M}{r}\right)f_0''+\left(1-\frac{2M}{r}\right)^2\left(1-\frac{3M}{r}\right)(f_0')^3 - \left(1-\frac{5M}{r}\right)f_0'. \tag{B1}$$

1. Second order

The explicit form of the source term in the numerator of $\mathcal{C}^r_\phi$ [see Eq. (14)] is

$$\mathcal{S}_2 = \frac{1}{r-2M}\left[\frac{21M^2}{r^2}+r\left(1-\frac{2M}{r}\right)^3\left(1+\frac{3M}{r}\right)f_0'f_0'' + \left(1-\frac{2M}{r}\right)^4\left(1+\frac{6M}{r}\right)(f_0')^4 - \left(1-\frac{2M}{r}\right)^2\left(1+\frac{4M}{r}+\frac{15M^2}{r^2}\right)(f_0')^2\right], \tag{B2}$$

which is canceled by the contribution of for $f_2=\bar{f}_2(r)+f_{2,c}(r)\cos\theta+\hat{f}_2(r)\cos^2\theta$, with

$$\hat{f}_2(r) = \frac{21M^2r^3 + (r-2M)^2[r^2(r-2M)(r+3M)f_0'' + (r-2M)^2(r+6M)(f_0')^3 - r(15M^2+4Mr+r^2)f_0']f_0'}{2r^5(r-2M)\mathcal{A}_1}. \quad \text{(B3)}$$

2. Third order

From $\mathcal{C}^r_r = 0$, one obtains the differential equation (16) for f_0, solved by $\hat{f}_0$. Here, we give this solution in terms of its first derivative,

$$\hat{f}_0' = \frac{C_{0,1}r^2 + \frac{7875M^2}{r^2} + \frac{34020M^3}{r^3} + \frac{51030M^4}{r^4} + \frac{26244M^5}{r^5}}{(\frac{2M}{r}-1)\sqrt{(C_{0,1}r^2 + 1250 + \frac{6500M}{r} + \frac{14175M^2}{r^2} + \frac{14580M^3}{r^3} + \frac{5832M^4}{r^4})(C_{0,1}r^2 + \frac{7875M^2}{r^2} + \frac{34020M^3}{r^3} + \frac{51030M^4}{r^4} + \frac{26224M^5}{r^5})}}, \quad \text{(B4)}$$

where $C_{0,1}$ is an integration constant. Next, considering $\mathcal{C}^r_\phi$ gives the differential equation (18) for $f_3(r,\theta)$. The explicit form of the source term in it is

$$\mathcal{S}_3 = \frac{1}{1-(1-\frac{2M}{r})^2\hat{f}_0'^2}\left\{189\frac{M^2}{r^2}\left(1+\frac{3M}{r}\right)\bar{f}_1' + \left(1-\frac{2M}{r}\right)^2\left[25+\frac{91M}{r}+9\frac{M^2}{r^2}\left(47+\frac{45M}{r}\right)\right]\hat{f}_0'^2\bar{f}_1' \right.$$
$$\left. -3\left(1-\frac{2M}{r}\right)^4\left(1+\frac{3M}{r}\right)\left(25+\frac{42M}{r}\right)\hat{f}_0'^4\bar{f}_1' - \left(1-\frac{2M}{r}\right)^3\left(5+\frac{9M}{r}\right)^2\hat{f}_0'^2\bar{f}_1''r\right\}. \quad \text{(B5)}$$

3. Fourth order

From $\mathcal{C}^r_r = 0$, one obtains the differential equation (21) for $\bar{f}_1$, solved by $\hat{f}_1$. Here, we give this solution in terms of its first derivative,

$$\hat{f}_1' = \frac{C_{1,1}(5+\frac{9M}{r})}{r(C_{0,1}r^2 + 1250 + \frac{6500M}{r} + \frac{14175M^2}{r^2} + \frac{14580M^3}{r^3} + \frac{5832M^4}{r^4})^{3/2}(C_{0,1}r^2 + \frac{7875M^2}{r^2} + \frac{34020M^3}{r^3} + \frac{51030M^4}{r^4} + \frac{26224M^5}{r^5})^{1/2}}, \quad \text{(B6)}$$

where $C_{1,1}$ and $C_{0,1}$ are integration constants.

At the fourth order, there are also notably two source terms in $\mathcal{C}^r_\phi$, $\mathcal{S}_{4,1}$, and $\mathcal{S}_{4,3}$, the full equation (22) being read as a differential equation for $f_4(r,\theta)$. Their explicit form (which does not vanish when replacing the functions $\hat{f}_0$ and $\hat{f}_1$) is

$$\mathcal{S}_{4,3} = -\frac{23814M}{r} - \frac{9M^2}{r^2}\left(1-\frac{2M}{r}\right)\left[475 - \frac{247M}{r} + \frac{162M^2}{r^2}\left(-21+\frac{38M}{r}\right)\right]\hat{f}_0'^2$$
$$+\frac{M}{r}\left(1-\frac{2M}{r}\right)^4\left[1775 + \frac{14576M}{r} + \frac{81M^2}{r^2}\left(87 - \frac{314M}{r} + \frac{324M^2}{r^2}\right)\right]\hat{f}_0'^4$$
$$+\left(1-\frac{2M}{r}\right)^6\left\{-625 - \frac{3235M}{r} + \frac{6M^2}{r^2}\left[-470 + \frac{9M}{r}\left(146+\frac{135M}{r}\right)\right]\right\}\hat{f}_0'^6$$
$$-\frac{54M}{r}\left(1-\frac{2M}{r}\right)^8\left(2+\frac{3M}{r}\right)\left(5+\frac{12M}{r}\right)\hat{f}_0'^8, \quad \text{(B7)}$$

$$\mathcal{S}_{4,1} = \frac{2r^2\hat{f}_1'\hat{f}_0'^2(1-\frac{2M}{r})^4(5+\frac{9M}{r})^2}{\hat{f}_0'^2(1-\frac{2M}{r})^2 - 1}\mathfrak{S}_1(r) + \frac{1}{3}[\mathfrak{S}_0(r) + \mathfrak{S}_2(r)], \quad \text{(B8)}$$

with

$$\mathfrak{S}_1 = \hat{f}_1'\left[5 + \frac{2M}{r} + \frac{171M^2}{r^2} - 5\left(1-\frac{2M}{r}\right)^2\left(5+\frac{14M}{r} - \frac{9M^2}{r^2}\right)\hat{f}_0'^2\right]$$
$$- r\hat{f}_1''\left(5 - \frac{M}{r} - \frac{18M^2}{r^2}\right)\left[1 + \left(1-\frac{2M}{r}\right)\hat{f}_0'^2\right], \quad \text{(B9)}$$

$$\mathfrak{S}_0 = \frac{23814M^4}{r^4} - 9\hat{f}_0'^2\left(1-\frac{2M}{r}\right)\frac{M^2}{r^2}\left(1925+\frac{11037M}{r}+\frac{13924M^2}{r^2}-\frac{9936M^3}{r^3}+\frac{14256M^4}{r^4}\right)$$
$$-\hat{f}_0'^4\left(1-\frac{2M}{r}\right)^3\frac{M}{r}\left(1775+\frac{4726M}{r}-\frac{56665M^2}{r^2}-\frac{72684M^3}{r^3}+\frac{121824M^4}{r^4}\right)$$
$$+\hat{f}_0'^6\left(1-\frac{2M}{r}\right)^5\left(625+\frac{3185M}{r}+\frac{4870M^2}{r^2}+\frac{9804M^3}{r^3}+\frac{36342M^4}{r^4}+\frac{26244M^5}{r^5}\right)$$
$$+\hat{f}_0'^8\left(1-\frac{2M}{r}\right)^8\frac{M}{r}\left(2+\frac{3M}{r}\right)\left(5+\frac{12M}{r}\right), \tag{B10}$$

$$\mathfrak{S}_2 = 6\hat{f}_0' r^2\left(1-\frac{2M}{r}\right)^2\left(5+\frac{9M}{r}\right)\left\{r\bar{f}_2''\hat{f}_0'^2\left(1-\frac{2M}{r}\right)^3\left(5+\frac{9M}{r}\right)^2+\bar{f}_2{}'\left[-\frac{189M^2}{r^2}\left(1+\frac{3M}{r}\right)\right.\right.$$
$$\left.\left.-\hat{f}_0'^2\left(1-\frac{2M}{r}\right)^2\left(25+\frac{91M}{r}+\frac{423M^2}{r^2}+\frac{405M^3}{r^3}\right)+3\hat{f}_0'^4\left(1-\frac{2M}{r}\right)^4\left(25+\frac{117M}{r}+\frac{126M^2}{r^2}\right)\right]\right\}. \tag{B11}$$

4. Fifth order

At this order, it can be shown that a slicing $F(r,\theta,a)$ with $F(r,\theta,0)=f_0(r)$ cannot let the Cotton-York tensor $\mathcal{C}^i_j$ vanish since

$$\frac{d^5}{d(\cos\theta)^5}\mathcal{C}^r_r = -\frac{7a^5}{54r^6(5+\frac{9M}{r})^3}\frac{\mathcal{N}_1(r)\mathcal{N}_2(r)}{\mathcal{D}_1(r)(\mathcal{D}_2(r))^{1/2}}+\mathcal{O}(a^6), \tag{B12}$$

which we recall from (25), does not identically vanish, this for any $C_{0,1}$. The explicit forms of the different terms are

$$\mathcal{N}_1 = C_{0,1}r^2+\frac{7875M^2}{r^2}+\frac{34020M^3}{r^3}+\frac{51030M^4}{r^4}+\frac{26244M^5}{r^5}, \tag{B13}$$

$$\mathcal{D}_1 = -C_{0,1}r^2+\frac{4375M}{r}+\frac{23625M^2}{r^2}+\frac{51030M^3}{r^3}+\frac{51030M^4}{r^4}+\frac{19683M^5}{r^5}, \tag{B14}$$

$$\mathcal{N}_2 = 25C_{0,1}^2r^4+10C_{0,1}r^2\frac{M}{r}(9C_{0,1}r^2-6175)-450\frac{M^2}{r^2}(941C_{0,1}r^2-331250)-270\frac{M^3}{r^3}(4698C_{0,1}r^2-5571875)$$
$$-9477\frac{M^4}{r^3}(288C_{0,1}r^2-685625)-7290\frac{M^5}{r^5}(513C_{0,1}r^2-2167025)-26244\frac{M^6}{r^6}(81C_{0,1}r^2-895325)$$
$$+21379871430\frac{M^7}{r^7}+10979571060\frac{M^8}{r^8}+2448880128\frac{M^9}{r^9}, \tag{B15}$$

$$\mathcal{D}_2 = C_{0,1}r^2(C_{0,1}r^2+1250)+6500C_{0,1}\frac{M}{r}r^2+3150\frac{M^2}{r^2}(7C_{0,1}r^2+3125)+900\frac{M^3}{r^3}(54C_{0,1}r^2+104125)$$
$$+243\frac{M^4}{r^4}(234C_{0,1}r^2+1631875)+2916\frac{M^5}{r^5}(9C_{0,1}r^2+329750)+1435874850\frac{M^6}{r^6}+1314430740\frac{M^7}{r^7}$$
$$+680244480\frac{M^8}{r^8}+153055008\frac{M^9}{r^9}. \tag{B16}$$

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