Bianchi spacetimes as supercurvature modes around isotropic cosmologies

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A powerful result in theoretical cosmology states that a subset of anisotropic Bianchi models can be seen as the homogeneous limit of (standard) linear cosmological perturbations. Such models are precisely those leading to Friedmann spacetimes in the limit of zero anisotropy. Building on previous works, we give a comprehensive exposition of this result, and perform the detailed identification between anisotropic degrees of freedom and their corresponding scalar, vector, and tensor perturbations of standard perturbation theory. In particular, we find that anisotropic models very close to open (i.e., negatively curved) Friedmann spaces correspond to some type of supercurvature perturbations. As a consequence, provided anisotropy is mild, its effects on all types of cosmological observables can always be computed as simple extensions of the standard techniques used in relativistic perturbation theory around Friedmann models. This fact opens the possibility to consistently constrain, for all cosmological observables, the presence of large scale anisotropies on the top of the stochastic fluctuations.

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I. INTRODUCTION

Homogeneous and spatially anisotropic cosmologies, commonly referred to as Bianchi models, have long been the arena for new developments in theoretical cosmology. The interest in these models stems from their unique ability to preserve a high degree of symmetry while remaining phenomenologicaly versatile. Despite the fact that CMB data seems to favor the more restricted class of isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) universes, Bianchi models are possibly the simplest extensions of a maximally symmetric expanding universe, and for that reason they are theoretically (if not observationaly) interesting. However, despite all of their attractiveness, one cannot avoid the feeling that Bianchi models fall in the category of "alternative cosmologies."

Meanwhile, a robust but less known theoretical result states that all nearly isotropic Bianchi models with isotropic limit (namely, models I, VII$_0$, V, VII$_h$ and IX) can be extracted from standard (i.e., FLRW) cosmological perturbations in the limit that these perturbations become homogeneous. This idea, explored in the case of model IX in Refs. [1,2], and fully established for the other models by Pontzen and Challinor in 2010 (henceforth PC10) [3], bridges the gap between FLRW and Bianchi models, forcing us to see the latter as legitimate manifestations of the standard cosmological framework.

The idea that Bianchi models emerge as homogeneous cosmological perturbations on the top of an isotropic universe is quite intuitive. In fact, since linear cosmological perturbations break both translational and rotational isometries of the FLRW background metric, a (suitably defined) homogeneous limit of these perturbations should restore translational invariance while keeping the most general spatial anisotropies compatible with homogeneity. The remaining anisotropic degrees of freedom are, then, nothing else but those describing the subset of Bianchi models—exactly the subset having the initial FLRW metric as their isotropic limits. In fact, by properly defining the isometries of the FLRW metric and demanding rotational invariance to be broken, one can build the isometries of the corresponding Bianchi models from first principles [3]. This idea not only leads to a more intuitive formulation of Bianchi models[1] but, more importantly, sheds light on their connections with linear cosmological perturbations. Given the omnipresence of linear perturbation theory in the cosmologist's toolkit, the exploration of these connections becomes central for a better understanding of the ΛCDM model.

One simple example of this connection is easily illustrated: the spatial anisotropies of Bianchi type I model are

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[1]However, it does not apply to Bianchi models not having FLRW limits, namely, models II, III, IV, VI$_0$, VI$_h$, and VIII. It also excludes the homogeneous and anisotropic Kantowski-Sachs model, which falls outside the usual Bianchi classification anyway.

dynamically equivalent to a gravitational wave of infinite wavelength (i.e., homogeneous) on the top of a spatially flat FLRW universe. Indeed, for small anisotropies, the Bianchi I metric reads

$$\mathrm{d}s^2 = a^2(\eta)[-\mathrm{d}\eta^2 + (\delta_{ij} + 2\beta_{ij}(\eta))\mathrm{d}x^i\mathrm{d}x^j],$$

whereas a FLRW universe with linear and homogeneous gravitational wave is described by

$$\mathrm{d}s^2 = a^2(\eta)[-\mathrm{d}\eta^2 + (\delta_{ij} + 2E_{ij}(\eta))\mathrm{d}x^i\mathrm{d}x^j].$$

Since both the shear β_{ij} and the wave amplitude E_{ij} are symmetric and trace free, both evolve as

$$X''_{ij} + 2\frac{a'}{a}X'_{ij} = 0.$$

While the above analogy could at first look as a happy accident of model I, it is actually not. The general correspondence in the weak field regime was investigated for the five Bianchi types with a FLRW limit in PC10. As hinted by these authors, and explicitly demonstrated here, the connection results in a richer structure than could have been expected from the simple example above, since some Bianchi models arise as *finite* wavelength perturbations over FLRW spacetimes. Indeed, the homogeneous limit of models VII_0, V, VII_h, and IX is not simply given (in Fourier space) by $k \to 0$, but rather by a proper identification of an effective mode ν_m defined in terms of the Fourier mode k as

$$\nu_m^2 = k^2 + (1 + |m|)K,$$

where $|m| = 0, 1, 2$ accounts respectively for scalar, vector, and tensor perturbations, and K is the spatial curvature of the corresponding FLRW model. Furthermore, we show that for models V and VII_h the homogeneous limit corresponds to a complex ν_m or, equivalently, to a perturbation whose wavelength is larger than the curvature scale. In this case, the construction of a proper eigenbasis

TABLE I.   List of main symbols used in this work.

Symbol	Definition	Introduced in
$\{i, j, k, \ldots\}$	spatial indices of noncoordinate (triad) basis	Sec. II A
$\{a, b, c, \ldots\}$	spacetime indices of noncoordinate (tetrad) basis	Sec. II A
$\{\mu, \nu, \lambda, \ldots\}$	abstract space-time indices	Sec. II A
i	imaginary unity	(3.22)
$\boldsymbol{\xi}_i$	Killing vector fields	Sec. II
$\boldsymbol{e}_i$	invariant basis	Sec. II
$\boldsymbol{e}^i$	dual basis (or cobasis) to the invariant basis $\boldsymbol{e}_i$	Sec. II
e^μ	four-velocity of fundamental observers (used to foliate spacetime)	Sec. II
u^μ	timelike fluid four-velocity.	Sec. II C
FLRW quantities		
$\boldsymbol{g}^{\mathrm{MS}}$	metric of maximally symmetric spaces	(3.2)
K	spatial curvature of maximally symmetric spaces	(3.3) and Table II
$\mathcal{K}$	dimensionless spatial curvature	(3.4)
$\mathcal{D}_i$	spatial covariant derivative associated with the FLRW metric	(3.5)
Δ	Laplace-Beltrami operator associated with $\mathcal{D}_i$	(3.6)
$Q_{I_j}^{(jm)}$	tensor plane-wave harmonic in position space	(3.27)
${}_sG_\ell^{(jm)}$	normal modes in the total angular momentum representation	(3.28)
${}_sY_\ell^m$	spin-weighted spherical harmonics	(3.28)
${}_s\alpha_\ell^{(jm)}$	radial functions in the total angular momentum representation	(3.28)
${}^\ell Q_{I_j}^{(jm)}$	tensor harmonic in ℓ-representation	(3.33)
ν_m	modes of the Bianchi-FLRW matching	(5.2) and Table IV
ζ_ℓ^m	pseudo plane-wave weights for the Bianchi-FLRW matching	(5.2) and Table IV
Bianchi quantities		
$\boldsymbol{g}$	metric of Bianchi spacetime	(2.22)
$\boldsymbol{h}$	spatial metric of Bianchi spacetime	(2.23)
γ_{ij}	conformal spatial metric of Bianchi spacetimes	(2.35)
β_{ij}	expansion parameters of spatial anisotropy	(4.15)
D_i	spatial covariant derivative in Bianchi spaces	(2.24)
$q_{ij}^{(m)}$	polarization basis for shear svt modes	(4.25)
N^{ij}, A_i	irreducible components of the constants of structure	(2.6) and (2.37)

for the perturbations requires analytical continuation of radial functions. The implementation and observational consequences of these modes with this method will be explored in a forthcoming publication [4].

The interplay between Bianchi and perturbed FLRW models also has important practical applications: if one wants to derive the dynamical behavior of some cosmological observable in a Bianchi I universe, it is enough to derive the same dynamics for tensor perturbations in flat FLRW universe and take their homogeneous limit. This program was in fact used in [5] to derive theoretical expressions for the weak-lensing signal, and in [6] to derive the direction and redshift time drifts of noninertial observers.

Here, we continue the effort started in PC10 and show that the same program can be applied to all Bianchi models with isotropic limit. In particular, we show how this Bianchi/perturbed-FLRW duality can be used to infer predictions for any observable (as, e.g., CMB radiative transfer, weak gravitational lensing, etc.) in Bianchi models from the well known methods of linearly perturbed FLRW spacetimes [7,8]. Hence, one can use the same theoretical framework by just separating modes describing global anisotropies from the ones describing stochastic perturbations.

From the observational point of view, upper limits on the large scale anisotropy were placed in [9], followed by claims of a Bianchi VII$_h$ pattern in WMAP data [10–12]. Further investigations using the method of [13,14] for radiative transport (see also the related method of [15]) combined with Planck data confirmed, however, that we can only obtain upper limits [16,17] on the level of global anisotropy.

We start this article by recalling some basic definitions of Bianchi spacetimes in Sec. II, where we focus on the subset of models having isotropic limit. We then review, in Sec. III, some key elements of linear perturbation theory in synchronous gauge, focusing on the introduction of scalar, vector, and tensor modes and their decomposition in terms of a complete basis of tensor harmonics. This section summarizes the definition and constructions of the companion paper [18]. Moving forward, we introduce a set of linear modes for small anisotropies of Bianchi spacetimes in Sec. IV; they are then used to find the exact Bianchi/perturbed-FLRW correspondence in Sec. V. Finally, we discuss the cosmological implications of our results in Sec. VI.

Throughout this paper we use metric signature $(-, +, +, +)$ and units where $c = 1$. Coordinate and noncoordinate indices, as well as a list of the main symbols encountered in this work, are defined in Table I.

II. BIANCHI COSMOLOGIES

Let us start with a brief and informal recap of spatially homogeneous (i.e., Bianchi) spacetimes. Detailed and pedagogical introductions can be found in many nice textbooks such as [19–21].

A. General construction

Informally, a three-dimensional space is said to be homogeneous if for any pair of points there exists an isometric (i.e., metric-preserving) path connecting these points in a continuous way. The fields $\boldsymbol{\xi}$ tangent to such paths are Killing vector fields (KVFs) and are defined as

$$\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{g} = 0 \Leftrightarrow \nabla_{(\mu}\xi_{\nu)} = 0, \qquad (2.1)$$

where $\mathcal{L}$ is the Lie derivative and ∇ is the covariant derivative compatible with the metric $\boldsymbol{g}$. In three dimensions, the maximum number of such paths is 6, corresponding to 3 translations and 3 rotations. However, because the space is three dimensional, there can be at most three KVFs which are *everywhere* linearly independent. Any additional vector $\boldsymbol{\zeta}$ obeying (2.1) is necessarily of the form $\boldsymbol{\zeta} = \sum_{i=1}^{3} c_i(x)\boldsymbol{\xi}_i$. Given a point p one can form $\tilde{\boldsymbol{\zeta}} = \sum_{i=1}^{3}[c_i(x) - c_i(p)]\boldsymbol{\xi}_i$ such that $\tilde{\boldsymbol{\zeta}}(p) = 0$, which corresponds to a rotation around point p. As an example, in the Euclidean three-dimensional space, $\boldsymbol{\zeta} = x^1\boldsymbol{\xi}_2 - x^2\boldsymbol{\xi}_1$ corresponds to a rotation around the x^3 axis which keeps the point $p = (0, 0, 0)$ fixed. Thus, homogeneous and spatially anisotropic spaces are represented by KVFs which are everywhere linearly independent. Moreover, since the commutator of any two vectors $\boldsymbol{\xi}_i$ is another KVF, they form a closed algebra given by

$$[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j] \equiv C^k{}_{ij}\boldsymbol{\xi}_k, \qquad (2.2)$$

where the coefficients $C^k{}_{ij}$ are called the constants of structure of the algebra.

We can now build homogeneous spacetimes by simply stacking up homogeneous spaces, each of which is labeled by a continuous time coordinate t and has an orthogonal 1-form

$$\boldsymbol{\omega} = -\mathrm{d}t, \qquad \omega_{\mu} = -\partial_{\mu}t. \qquad (2.3)$$

By construction, the vector $\boldsymbol{e}$ dual to $\boldsymbol{\omega}$ is orthogonal to the KVFs

$$\boldsymbol{e} \cdot \boldsymbol{\xi}_i = e^{\mu}g_{\mu\nu}\xi_i^{\nu} = 0, \qquad (2.4)$$

and is normalized such that

$$\boldsymbol{e} \cdot \boldsymbol{e} = e^{\mu}g_{\mu\nu}e^{\nu} = -1. \qquad (2.5)$$

The task of finding explicit Bianchi spacetimes now consists of finding all constants $C^k{}_{ij}$ which are inequivalent under linear combination of the $\boldsymbol{\xi}_i$. This task is simplified by noting that, since $C^k{}_{ij}$ is antisymmetric in its lower

indexes, it can be written as $C^k_{ij} \equiv \hat{\epsilon}_{ijl}H^{lk}$, where $\hat{\epsilon}_{ijl}$ is the permutation symbol (such that $\hat{\epsilon}_{123} = 1$) and H^{lk} is a general 3×3 matrix. Decomposing the latter in its symmetric ($\hat{N}^{lk}$) and antisymmetric ($\hat{\epsilon}^{lkm}A_m$) parts, we find that[2]

$$C^k{}_{ij} = \hat{\epsilon}_{ijl}\hat{N}^{lk} - A_i\delta^k_j + A_j\delta^k_i. \qquad (2.6)$$

From the Jacobi identity

$$\hat{\epsilon}^{ijk}[\boldsymbol{\xi}_i, [\boldsymbol{\xi}_j, \boldsymbol{\xi}_k]] = 0 \qquad (2.7)$$

the decomposition (2.6) implies

$$\hat{N}^{ij}A_j = 0. \qquad (2.8)$$

By suitable linear transformations of the $\boldsymbol{\xi}_i$ we can diagonalize the matrix $\hat{N}^{ij}$ so that $\hat{N}^{ij} = \text{diag}(N^1, N^2, N^3)$. From property (2.8) we then see that $\boldsymbol{A}$ is either null or an eigenvector of the matrix $\hat{N}$, and so we can set $\boldsymbol{A} = (0, 0, A)$.[3] We are thus left with

$$[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] = +N^3\boldsymbol{\xi}_3,$$
$$[\boldsymbol{\xi}_1, \boldsymbol{\xi}_3] = -N^2\boldsymbol{\xi}_2 + A\boldsymbol{\xi}_1,$$
$$[\boldsymbol{\xi}_2, \boldsymbol{\xi}_3] = +N^1\boldsymbol{\xi}_1 + A\boldsymbol{\xi}_2. \qquad (2.9)$$

We now note that by suitably rescaling the lengths of the KVFs, we can set the components N^1, N^2, N^3, and A to either 0, 1 or −1 (see, e.g., chapter 10 of [20]). We will not adopt this approach here. Instead, since the KVFs have dimensions of inverse length, we will keep these constants with the appropriate dimensions to maintain Eq. (2.2) dimensionally homogeneous. For reasons that will become clear later, the constants A and N^i are associated with curvature and spiral lengths, respectively, so that we introduce

$$A \equiv \ell_c^{-1}, \qquad N^i \equiv \ell_s^{-1}, \qquad (2.10)$$

as two free parameters, except in the Bianchi IX case where $A = 0$ and $N^i \equiv 2\ell_c^{-1}$. We also define the (historical) dimensionless ratio

$$\sqrt{h} \equiv \frac{\ell_s}{\ell_c} \Rightarrow A = \sqrt{h}\ell_s^{-1}. \qquad (2.11)$$

The full set of Bianchi models considered in this work, as well as their underlying isotropic 3-spaces $\mathcal{M}^{(3)}$, are

TABLE II. Bianchi types considered in this work (first column), and their underlying maximally symmetric 3-spaces (last column), namely, the Euclidean ($\mathbb{E}^3$), hyperbolic ($\mathbb{H}^3$), and spherical ($\mathbb{S}^3$) spaces. For comparison, we also give (a combination of) the spatial Ricci scalar appearing in Friedmann equations for each of the 3-spaces $\mathcal{M}^{(3)}$.

Type	A	N^1	N^2	N^3	$\frac{a^{2(3)}R}{6} = K$	$\mathcal{M}^{(3)}$
I	0	0	0	0	0	$\mathbb{E}^3$
V	ℓ_c^{-1}	0	0	0	$-\ell_c^{-2}$	$\mathbb{H}^3$
VII$_0$	0	ℓ_s^{-1}	ℓ_s^{-1}	0	0	$\mathbb{E}^3$
VII$_h$	ℓ_c^{-1}	ℓ_s^{-1}	ℓ_s^{-1}	0	$-\ell_c^{-2}$	$\mathbb{H}^3$
IX	0	$2\ell_c^{-1}$	$2\ell_c^{-1}$	$2\ell_c^{-1}$	$+\ell_c^{-2}$	$\mathbb{S}^3$

summarized in Table II. For future reference, note that models I and V can be obtained from models VII$_0$ and VII$_h$ in the limit $\ell_s \to \infty$.

Next, we define a spacetime basis of invariant vector fields by choosing a set of spatial vectors at a reference point, and Lie dragging them with the KVFs on a given spatial section. That is we define the vector fields on that section by the conditions

$$\boldsymbol{e} \cdot \boldsymbol{e}_i = 0, \qquad (2.12a)$$

$$[\boldsymbol{\xi}_i, \boldsymbol{e}_j] = \mathcal{L}_{\boldsymbol{\xi}_i}\boldsymbol{e}_j = 0. \qquad (2.12b)$$

These spatial vectors are extended throughout the other spatial sections by demanding that

$$[\boldsymbol{e}, \boldsymbol{e}_i] = \mathcal{L}_{\boldsymbol{e}}\boldsymbol{e}_i = 0. \qquad (2.13)$$

Finally, we join the unit normal vector $\boldsymbol{e}$ to the set $\{\boldsymbol{e}_i\}$ to obtain a spacetime basis $\{\boldsymbol{e}_a\}$, with the understanding that $\boldsymbol{e}_0 = \boldsymbol{e}$. From properties (2.1)–(2.4), it is shown that

$$\mathcal{L}_{\boldsymbol{e}}\boldsymbol{\xi}_i = [\boldsymbol{e}, \boldsymbol{\xi}_i] = 0, \qquad (2.14)$$

that is, the normal vector $\boldsymbol{e}$ is also invariant under the action of the KVFs. The Jacobi identity applied to $\boldsymbol{e}$, $\boldsymbol{e}_i$ and $\boldsymbol{\xi}_j$ then shows that (2.13) is consistent. Moreover, the conditions (2.3) and (2.5) imply that $\boldsymbol{e}$ is geodesic ($e^\mu\nabla_\mu e^\nu = 0$).

The commutator of the basis vectors $\boldsymbol{e}_i$ is another vector, and can thus be represented as a linear combination of the basis elements

$$[\boldsymbol{e}_i, \boldsymbol{e}_j] = \tilde{C}^k{}_{ij}\boldsymbol{e}_k, \qquad (2.15)$$

where $\tilde{C}^k{}_{ij}$ are constants. Since we are still free to fix the orientation of the spatial basis $\{\boldsymbol{e}_i\}$ at any point p, we choose $\{\boldsymbol{e}_i\}_p = \{\boldsymbol{\xi}_i\}_p$.[4] However, since our Eq. (2.2) differs with theirs by a minus sign, so does our Eq. (2.16). This then gives

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[2]Our sign convention for A_i agrees with that of PC10 but differs by a minus sign with that of Ref. [13].

[3]Note that our choice differs from the conventional one: $\boldsymbol{A} = (A, 0, 0)$ [13,22–24]. One easily recovers the standard results just replacing the indexes (1, 2 ,3) in our expressions by (2, 3, 1).

[4]This choice agrees with the one made in PC10.

$$[\boldsymbol{e}_i, \boldsymbol{e}_j] = -C^k{}_{ij}\boldsymbol{e}_k, \qquad (2.16)$$

which can be checked by writing $\boldsymbol{e}_i = M_i^j \boldsymbol{\xi}_j$, where M_i^j is a point-dependent matrix obeying $M_i^j(p) = \delta_j^i$, and using Eq. (2.12b). We thus find

$$[\boldsymbol{e}_1, \boldsymbol{e}_2] = -N^3 \boldsymbol{e}_3,$$
$$[\boldsymbol{e}_1, \boldsymbol{e}_3] = +N^2 \boldsymbol{e}_2 - A\boldsymbol{e}_1,$$
$$[\boldsymbol{e}_2, \boldsymbol{e}_3] = -N^1 \boldsymbol{e}_1 - A\boldsymbol{e}_2. \qquad (2.17)$$

Using again the Jacobi identity for $\boldsymbol{e}$, $\boldsymbol{e}_i$, and $\boldsymbol{e}_j$ one can show that

$$\mathcal{L}_{\boldsymbol{e}} C^i{}_{jk} = 0, \qquad (2.18)$$

that is, the constants of structure are really spacetime constants for this invariant basis. Note that our choice of a time-invariant basis contrasts with the more popular choice of a tetrad basis, in which the constants of structure become time dependent [15,22,25].

From the time-invariance property (2.13) and from (2.16), we infer that

$$[\boldsymbol{e}_a, \boldsymbol{e}_b] = -C^c{}_{ab}\boldsymbol{e}_c, \qquad (2.19)$$

where

$$C^0{}_{i0} = C^i{}_{j0} = C^0{}_{ab} = 0, \qquad (2.20)$$

that is, the constants of structure vanish whenever one of the indices is 0.

Next, we define the dual basis $\{\boldsymbol{e}^a\}$ to the basis $\{\boldsymbol{e}_a\}$ from the condition

$$e^a_\mu e^\mu_b = \delta^a_b \qquad (2.21)$$

from where it follows that $e^0_\mu = -\omega_\mu = -e_\mu$. From (2.5) and (2.12a) we deduce that in this dual basis the components of the metric satisfy $g_{00} = -1$ and $g_{0i} = 0$. Since the metric has three spacelike KVFs, it can at most depend on t, so it is of the form

$$\boldsymbol{g} = -\boldsymbol{e}^0 \otimes \boldsymbol{e}^0 + g_{ij}(t)\boldsymbol{e}^i \otimes \boldsymbol{e}^j. \qquad (2.22)$$

It is also convenient to define a spatial metric through

$$\boldsymbol{h} \equiv \boldsymbol{g} + \boldsymbol{e}^0 \otimes \boldsymbol{e}^0 = g_{ij}(t)\boldsymbol{e}^i \otimes \boldsymbol{e}^j, \qquad (2.23)$$

such that $h_{ij} = g_{ij}$. From the covariant derivative ∇ associated with the metric $\boldsymbol{g}$ we can define an induced covariant derivative D associated with the induced spatial metric $\boldsymbol{h}$. For any spatial tensor $\boldsymbol{T}$ it is defined by [25,26]

$$D_\mu T_{\nu_1 \cdots \nu_n} \equiv h_\mu^\sigma h_{\nu_1}^{\lambda_1} \cdots h_{\nu_n}^{\lambda_n} \nabla_\sigma T_{\lambda_1 \cdots \lambda_n}. \qquad (2.24)$$

From the definition (2.21) and the property (2.19), we deduce that the constants of structure also satisfy the property[5]

$$C^c{}_{ab} = 2e_a^\mu e_b^\nu \nabla_{[\mu} e_{\nu]}^c. \qquad (2.25)$$

Next we introduce the connection coefficients through

$$\Gamma^c{}_{ab} \equiv -e_a^\mu e_b^\nu \nabla_\mu e_\nu^c = e_\nu^c e_a^\mu \nabla_\mu e_b^\nu. \qquad (2.26)$$

Comparison with (2.25) then shows that $C^c{}_{ab} = -2\Gamma^c{}_{[ab]}$. In particular

$$\Gamma^0{}_{ij} = -e_i^\mu e_j^\nu \nabla_\mu e_\nu^0 = \frac{1}{2}\mathcal{L}_{\boldsymbol{e}_0} g_{ij} = \frac{1}{2}\dot{g}_{ij}, \qquad (2.27)$$

which is related to extrinsic curvature $K_{\mu\nu} \equiv h_\mu^\alpha h_\nu^\beta \nabla_\alpha e_\beta$ by

$$K_{ij} = \Gamma^0{}_{ij} \equiv \frac{1}{3}\theta g_{ij} + \sigma_{ij}. \qquad (2.28)$$

Here, we have separated its trace (proportional to the volume expansion θ) from its traceless part (given by the expansion shear σ_{ij}).

Using also that the connection is torsionless we can relate its components to the constants of structure by[6]

$$\Gamma_{abc} = \frac{1}{2}[-e_a^\mu \partial_\mu g_{bc} + e_b^\mu \partial_\mu g_{ca} + e_c^\mu \partial_\mu g_{ab}$$
$$+ C_{acb} - C_{bac} + C_{cba}], \qquad (2.29)$$

where we have introduced the definitions

$$\Gamma_{abc} \equiv g_{ad}\Gamma^d{}_{bc}, \qquad C_{abc} \equiv g_{ad}C^d{}_{bc}. \qquad (2.30)$$

For the spatial components we get simply

$$\Gamma_{ijk} = \frac{1}{2}[C_{ikj} - C_{jik} + C_{kji}]. \qquad (2.31)$$

Since the Riemann tensor is associated with the connection and its derivatives, then from (2.31), this implies that the Riemann tensor associated with $\boldsymbol{h}$ can be fully expressed in terms of the constants of structure $C^i{}_{jk}$ (see Appendix A). The Gauss-Codazzi identity [27,28] allows us to relate the Riemann tensor associated with the metric $\boldsymbol{h}$ and its connection D to the spacetime Riemann tensor associated

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[5]Antisymmetrization on n indices is defined with a prefactor $1/n!$, that is $T_{[ij]} = (T_{ij} - T_{ji})/2$. However there is no such prefactor in commutators.

[6]$e_a^\mu \partial_\mu g_{bc} = \Gamma_{cab} + \Gamma_{bac}$ from metric compatibility, and $\Gamma_{c[ab]} = -C_{cab}/2$ from torsionless conditions.

with $\mathbf{g}$ and connection ∇. The most general form of this identity is given by (A1).

Finally, we would like to stress the importance of the invariant basis (2.12b). Indeed, the definition of a homogeneous tensor as any tensor $\mathbf{T}$ such that $\mathcal{L}_{\xi_i}\mathbf{T} = 0$ is natural, since in this basis $\mathbf{T}$ has constant components. In particular, the quantities $C^i{}_{jk}$ can be interpreted as the components of an underlying homogeneous tensor field $\mathbf{C} = C^i{}_{jk}\mathbf{e}_i \otimes \mathbf{e}^j \otimes \mathbf{e}^k$.

B. Conformal parameterization

Using the time coordinate t introduced in (2.3), the metric of a general Bianchi spacetime reads

$$\mathbf{g}^{\text{Bianchi}} = -\mathrm{d}t \otimes \mathrm{d}t + a^2(t)\gamma_{ij}(t)\mathbf{e}^i \otimes \mathbf{e}^j, \qquad (2.32)$$

At this point, it is also convenient to define a conformal time by

$$a(\eta)\mathrm{d}\eta \equiv \mathrm{d}t, \qquad (2.33)$$

which implies that, in conformal time, $e^\mu = a^{-1}\delta_0^\mu$. The volume expansion θ is related to the conformal Hubble rate by

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{a\theta}{3}, \qquad (2.34)$$

where, throughout this work, a prime indicates a derivative with respect to η. The conformal spatial metric γ_{ij} is defined by

$$h_{ij} = a^2\gamma_{ij} \qquad (2.35)$$

such that the Bianchi metric takes the form (2.32). Note that the derivative (2.24) can also be considered as being associated with γ_{ij}.

The conformal shear $\hat{\sigma}_{ij}$ is defined by

$$\hat{\sigma}_{ij} \equiv \frac{1}{2}\gamma'_{ij}. \qquad (2.36)$$

The components of the constants of structure are related to their conformal counterparts by

$$N^{ij} = a^{-3}\hat{N}^{ij}, \qquad N_{ij} = a^1\hat{N}_{ij}, \qquad N_i^j = a^{-1}\hat{N}_i^j,$$
$$A_i = \hat{A}_i, \qquad A^i = a^{-2}\hat{A}^i. \qquad (2.37)$$

We also have that

$$\sigma_{ij} = a\hat{\sigma}_{ij}, \qquad \sigma_i^j = a^{-1}\hat{\sigma}_i^j. \qquad (2.38)$$

This means that in practice the indices of $\hat{\sigma}_{ij}$, $\hat{N}^{ij}$ and $\hat{A}_i$ are raised and lowered by γ_{ij} and γ^{ij}, whereas those of σ_{ij}, N^{ij}

and A_i are raised and lowered by $a^2\gamma_{ij}$ and $a^{-2}\gamma^{ij}$. The Levi-Civita tensor is also decomposed as $\epsilon_{ijk} = a^3\hat{\epsilon}_{ijk}$ with $\hat{\epsilon}_{123} = 1$ such that the combination $\hat{\epsilon}_{ijl}\hat{N}^{lk}$ in the decomposition (2.6) is equal to $\epsilon_{ijl}N^{lk}$.

Since the constants of structure are constant, these definitions ensure that the conformal $\hat{N}^{ij}$ and $\hat{A}_i$ and their related forms with different index placements are constant.

C. Stress-energy tensor

The stress-energy tensor of a fluid with energy density ρ, pressure p, and anisotropic stress $\pi_{\mu\nu}$ is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} + \pi_{\mu\nu}, \qquad (2.39)$$

where u^μ is the (timelike) fluid four-velocity. For simplicity we assume no anisotropic stress, although its inclusion is straightforward. Homogeneity of Bianchi space-times implies that energy density and pressure depend only on time, and therefore we use the notation $\bar{\rho}$ and $\bar{p}$ for the fluid content of Bianchi universes to stress this fact. We stress that the vector u^μ is not necessarily parallel to e^μ, since a homogeneous boost of the fluid is allowed for tilted Bianchi models. Actually, the fluid's four-velocity can be decomposed into components parallel and orthogonal to e^μ so that in the invariant basis we have

$$u^\mu = \Gamma(e^\mu + v^\mu), \quad \text{with} \quad \Gamma = \frac{1}{\sqrt{1 - v_i v^i}}, \qquad (2.40)$$

where $v^\mu e_\mu = 0$. If this homogeneous velocity is not curl free, then this corresponds to a global rotation of the fluid [23,29].

Moving forward, we introduce comoving components for the velocity as follows:

$$v_i = a\hat{v}_i, \qquad v^i = a^{-1}\hat{v}^i, \qquad (2.41)$$

such that the velocity has components

$$u^\mu = \frac{\Gamma}{a}(1, \hat{v}^i), \qquad u_\mu = a\Gamma(-1, \hat{v}_i). \qquad (2.42)$$

Finally, the stress-energy tensor components are

$$-T_\eta^\eta = T_{\mu\nu}e^\mu e^\nu = \bar{\rho} + (\bar{\rho} + \bar{p})(\Gamma^2 - 1),$$
$$T_i^\eta = (\bar{\rho} + \bar{p})\Gamma^2\hat{v}_i,$$
$$T_j^i = [(\bar{\rho} + \bar{p})\Gamma^2\hat{v}^i\hat{v}_j + \bar{p}\delta_j^i]. \qquad (2.43)$$

Because we allow for a tilt, $T_{\mu\nu}$ will not look like (2.39) with $u_\mu \to e_\mu$ for observers following the congruence defined by e^μ. In fact, from (2.43) we check that for those observers the fluid will present an effective momentum density and anisotropic stress. It then follows from Einstein

equations that these components will source spatial anisotropies even in the case of a perfect fluid.

D. Einstein equations

The time-time component of Einstein equation follows by projecting $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ with e^μ. Defining $\kappa = 8\pi G$ and using the relation (A16) we get

$$\mathcal{H}^2 + \frac{a^2}{6}{}^{(3)}R - \frac{1}{6}\hat\sigma^2 = \frac{\kappa a^2}{3} T_{\mu\nu} e^\mu e^\nu \qquad (2.44)$$

where $\hat\sigma^2 \equiv \hat\sigma_{ij}\hat\sigma^{ij}$ and the spatial Ricci scalar is given in terms of constants of structure to be [see (A7) with definitions (2.37)]

$$a^{2\,(3)}R = -6\hat A_i \hat A^i - \hat N_{ij}\hat N^{ij} + \frac{1}{2}(\hat N_i^i)^2. \qquad (2.45)$$

The evolution of the expansion rate is given by the Raychaudhuri equation (A15)

$$3\mathcal{H}' + \hat\sigma^2 = -a^2\kappa\left(T_{\mu\nu}e^\mu e^\nu + \frac{1}{2}T\right), \qquad (2.46)$$

and the dynamics of the shear comes from the traceless part of the spatial Einstein equation

$$\begin{aligned}(\hat\sigma_{ij})' + 2\mathcal{H}\hat\sigma_{ij} &= \hat N_k^k \hat N_{\langle ij\rangle} - 2\hat N_{k\langle i}\hat N_{j\rangle}^k \\ &\quad + 2\hat A^k \hat\epsilon_{kl\langle j}\hat N_{i\rangle}^l + \kappa T_{\langle ij\rangle},\end{aligned} \qquad (2.47)$$

where $T_{\langle ij\rangle} = a^2(\bar\rho + \bar p)\Gamma^2 \hat v_{\langle i}\hat v_{j\rangle}$.

Lastly, there is a constraint equation following from $G_i^\eta = \kappa T_i^\eta$, which is

$$P_i \equiv \kappa a^2 T_i^\eta = 3\hat A^j \hat\sigma_{ji} + \hat\epsilon_{ijk}\hat\sigma^{jl}\hat N_l^k. \qquad (2.48)$$

This is known as the tilt constraint and by definition

$$P_i = \kappa a^2 \Gamma^2(\bar\rho + \bar p)\hat v_i. \qquad (2.49)$$

But this is what is used to deduce $\hat v_i$ that we must then replace in the shear equation because it is inside T_{ij}.

E. Fluid equations

As usual, fluid equations follow from the covariant divergence of the stress-energy tensor. The conservation equation for energy density is just

$$\bar\rho' + (\bar\rho + \bar p)[3\mathcal{H} + (\ln\Gamma)' + D_i\hat v^i] = 0, \qquad (2.50)$$

and from (A11), we get the velocity divergence

$$D_i\hat v^i = -2\hat A_i\hat v^i. \qquad (2.51)$$

When linearizing in the velocity $\hat v_i$, the term $(\ln\Gamma)'$ will behave as a second-order quantity, and thus will not contribute.

The Euler equation is formally just

$$[a^4(\bar\rho + \bar p)\Gamma^2\hat v_i]' + a^4(\bar\rho + \bar p)\Gamma^2 D_j(\hat v^j\hat v_i) = 0. \qquad (2.52)$$

Separating the trace and traceless parts, the second term is handled using (A11). However, when linearizing in the velocity $\hat v_i$ it vanishes hence the Euler equation reduces to

$$[a^4(\bar\rho + \bar p)\hat v_i]' \simeq 0. \qquad (2.53)$$

III. SVT MODES IN FLRW

We now give a brief review of the background geometry of FLRW models and the mathematics behind the standard scalar-vector-tensor (henceforth SVT) decomposition of perturbative modes. This will be needed when comparing perturbations of the FLRW metric with spatial anisotropies in the homogeneous limit.

A. Background FLRW cosmology

Given a cosmic time t which allows us to split spacetime into space and time, all FLRW metrics can be written in the form

$$\boldsymbol{g}^{\mathrm{FLRW}} = -\mathrm{d}t \otimes \mathrm{d}t + a^2(t)\boldsymbol{g}^{\mathrm{MS}} \qquad (3.1)$$

where $a(t)$ is the scale factor of the expansion and $\boldsymbol{g}^{\mathrm{MS}}$ is the metric of maximally symmetric spaces, described by

$$\boldsymbol{g}^{\mathrm{MS}} = [\mathrm{d}\chi^2 + r^2(\chi)\mathrm{d}^2\Omega]. \qquad (3.2)$$

Here, $\mathrm{d}^2\Omega \equiv \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ is the standard line element on the 2-sphere and the function $r(\chi)$ is given by

$$r(\chi) = \begin{cases} \ell_c\sinh(\chi/\ell_c), & K < 0, \\ \chi, & K = 0, \\ \ell_c\sin(\chi/\ell_c), & K > 0. \end{cases} \qquad (3.3)$$

The curvature parameter K differentiates between open ($K < 0$), flat ($K = 0$) and closed ($K > 0$) spatial sections. It is related to the curvature radius ℓ_c of the spatial sections by $\ell_c = 1/\sqrt{|K|}$. It is also useful to introduce the quantity

$$\mathcal{K} \equiv K/|K| = K\ell_c^2, \qquad (3.4)$$

which is either -1, 0 or $+1$ for open, flat, and closed cases respectively.

B. Linear perturbation theory

To complete the matching between Bianchi anisotropies and FLRW perturbations, we will also need equations from

perturbation theory in synchronous gauge (usually fixed completely with an additional comoving condition on cold dark matter) and in conformal time, which will now be briefly summarized. More details can be found in Refs. [30–32].

By definition, only the spatial part of the metric is perturbed in synchronous gauge. In the SVT decomposition, such perturbations can be parametrized as follows:

$$\delta g_{ij} = 2a^2[-\phi g_{ij}^{\mathrm{MS}} + \mathcal{D}_{ij}\psi + \mathcal{D}_{(i}E_{j)} + E_{ij}], \quad (3.5)$$

where $\mathcal{D}_i$ is the covariant derivative compatible with g_{ij}^{MS} and

$$\mathcal{D}_{ij} \equiv \left(\mathcal{D}_i\mathcal{D}_j - \frac{g_{ij}^{\mathrm{MS}}}{3}\Delta\right), \qquad \Delta \equiv \mathcal{D}_i\mathcal{D}^i. \quad (3.6)$$

At linear order, the perturbed components of the Einstein tensor can also be decomposed into scalar, vector, and tensor modes [31]. Working in conformal time [recall Eq. (2.33)], such components are given by

Scalar modes:

$$a^2\delta G_\eta^\eta = 6\mathcal{H}\phi' - (\Delta + 3K)2\phi - \mathcal{D}^i\mathcal{D}^j\mathcal{D}_{ij}\psi, \quad (3.7a)$$

$$a^2\delta G_i^\eta = -2\mathcal{D}_i\phi' - \mathcal{D}^j\mathcal{D}_{ij}\psi', \quad (3.7b)$$

$$a^2\delta G_j^i = \left[\partial_\eta^2 + 2\mathcal{H}\partial_\eta + \frac{1}{3}(\Delta - 6K)\right]\mathcal{D}^i{}_j\psi + \mathcal{D}^i{}_j\phi$$
$$+ 2\delta_j^i\left[\phi'' + 2\mathcal{H}\phi' - \frac{1}{3}(\Delta + 3K)\phi - \frac{1}{6}\mathcal{D}^k\mathcal{D}^l\mathcal{D}_{kl}\psi\right]. \quad (3.7c)$$

Vector modes:

$$a^2\delta G_i^\eta = -\frac{1}{2}(\Delta + 2K)E_i', \quad (3.8a)$$

$$a^2\delta G_j^i = g_{\mathrm{MS}}^{ik}\mathcal{D}_{(k}[E_{j)}'' + 2\mathcal{H}E_{j)}']. \quad (3.8b)$$

Tensor modes:

$$a^2\delta G_j^i = E''^i{}_j + 2\mathcal{H}E'^i{}_j - (\Delta - 2K)E_j^i. \quad (3.9)$$

We also give the Ricci scalar associated with the spatial metric. For isotropic backgrounds and at linear order, it is sourced only by scalar perturbations:

$$a^2\delta(^{(3)}R) = 4(\Delta + 3K)\phi + \frac{4}{3}\Delta(\Delta + 3K)\psi. \quad (3.10)$$

In order to proceed with the identification, we will also need to perturb the energy-momentum tensor, here taken to be that of a perfect fluid for simplicity. As it turns out, the

linearized tensor in synchronous gauge is exactly what one would obtain by setting $\Gamma = 1$ in (2.43) and linearizing ρ and p around their background values (denoted below by an overbar). This leads to

$$-T_\eta^\eta = \bar{\rho} + \delta\rho, \quad (3.11)$$

$$T_i^\eta = (\bar{\rho} + \bar{p})\hat{v}_i, \quad (3.12)$$

$$T_j^i = (\bar{p} + \delta p)\delta_j^i. \quad (3.13)$$

Note that the velocity $\hat{v}_i$ is considered as a first-order perturbation, and as such it is split into scalar and vector parts as

$$\hat{v}_i = \mathcal{D}_i\hat{v} + \hat{V}_i, \qquad \mathcal{D}^i\hat{V}_i = 0. \quad (3.14)$$

We thus have everything needed to write the perturbed Einstein equations. The first of them follows from the time-time component, and corresponds to the (perturbed) Friedmann equation:

$$-6\mathcal{H}\phi' + (\Delta + 3K)2\phi + \mathcal{D}^i\mathcal{D}^j\mathcal{D}_{ij}\psi = \kappa a^2\delta\rho.$$

Then, we have the trace-free components of the space-space Einstein equations. These are independently given for scalar

$$\left[\partial_\eta^2 + 2\mathcal{H}\partial_\eta + \frac{1}{3}(\Delta - 6K)\right]\mathcal{D}^i{}_j\psi + \mathcal{D}^i{}_j\phi = 0, \quad (3.15)$$

vector

$$\mathcal{D}_{(i}E_{j)}'' + 2\mathcal{H}\mathcal{D}_{(i}E_{j)}' = 0, \quad (3.16)$$

and tensor modes

$$E''^i{}_j + 2\mathcal{H}E'^i{}_j - (\Delta - 2K)E_j^i = 0. \quad (3.17)$$

Next, we have two constraint equations for the scalar and vector modes of the velocity perturbation. These are given by

$$-2\mathcal{D}_i\phi' - \mathcal{D}^j\mathcal{D}_{ij}\psi' = \kappa a^2(\bar{\rho} + \bar{p})\mathcal{D}_i\hat{v}, \quad (3.18a)$$

$$-\frac{1}{2}(\Delta + 2K)E_i' = \kappa a^2(\bar{\rho} + \bar{p})\hat{V}_i. \quad (3.18b)$$

Finally, from the conservation of the energy-momentum tensor we have the Euler equation

$$[a^4(\bar{\rho} + \bar{p})\hat{v}_i]' = -a^4\partial_i\delta p, \quad (3.19)$$

and the conservation equation

$$\delta\rho' + (\delta\rho + \delta p)3\mathcal{H} = (\bar{\rho} + \bar{p})(3\phi' - \mathcal{D}_i\hat{v}^i). \quad (3.20)$$

C. Harmonics

Since perturbative modes evolve independently for linear perturbations, it is convenient to expand the metric perturbation on a basis of tensor harmonics. In this section we summarize how these harmonics are built (see [18] for more details).

Let us first introduce the usual orthonormal spatial basis associated with spherical coordinates

$$\begin{aligned}
\boldsymbol{n} &= \partial_\chi, \\
\boldsymbol{n}_\theta &= r^{-1}(\chi)\partial_\theta, \\
\boldsymbol{n}_\phi &= r^{-1}(\chi)\csc\theta\partial_\phi.
\end{aligned} \quad (3.21)$$

They allow us to introduce the standard helicity (vector) basis,

$$\boldsymbol{n}_\pm \equiv \frac{1}{\sqrt{2}}(\boldsymbol{n}_\theta \mp \mathrm{i}\boldsymbol{n}_\phi), \quad (3.22)$$

which is in turn used to define an extended helicity (tensor) basis [18]

$$\hat{n}_{\pm s}^{i_1\cdots i_j}(\boldsymbol{n}) \equiv n_\pm^{\langle i_1}\cdots n_\pm^{i_s}n^{i_{s+1}}\cdots n^{i_j\rangle}. \quad (3.23)$$

In what follows, we shall occasionally use a multi-index notation

$$I_j \equiv i_1\cdots i_j, \quad (3.24)$$

such that the helicity basis is written simply as $\hat{n}_{I_j}^{\pm s}$ or as $\hat{n}_{\pm s}^{I_j}$.

The generalized helicity basis has a series of important properties which are collected in Ref. [18]. For our present purposes, we stress that it is both a complete basis for symmetric and trace-free tensors, as well as a natural basis for separating the angular from the radial dependence of spin-valued tensors.

Cosmological perturbations can be expanded in a basis of spatial eigenfunctions of the Laplace operator Δ. Any tensor-valued perturbation $Q_{I_j}^{(jm)}(\boldsymbol{x};k)$ satisfying [18]

$$[\Delta + k^2 - K(j - |m|)(j + |m| + 1)]Q_{I_j}^{(jm)} = 0 \quad (3.25)$$

will be loosely called a harmonic. Here, j represents the tensorial rank of the harmonic, and m the rank of the primitive tensor from which this harmonic is derived. Thus, for instance, $Q_{ij}^{(2,0)}$ is a rank-2 tensor derived from (two derivatives of) a scalar function. The case $j = |m|$

represents a "fundamental" harmonic, in the sense that it is not derived from lower-rank tensors, and it is also divergence-less. Harmonics with $j > |m|$ are derived from the fundamental ones. In what follows, it will be convenient to give the expression above in terms of the mode ν_m defined as

$$\nu_m^2 = k^2 + (1 + |m|)K \quad (3.26)$$

in terms of which (3.25) becomes

$$\{\Delta + \nu^2 - K[(1 - |m|)(1 + |m|) + j(j + 1)]\}Q_{I_j}^{(jm)} = 0. \quad (3.27)$$

Let us summarize the main definitions and results of [18] in which these harmonics and their decomposition into angular and radial functions are discussed. A point $\boldsymbol{x}$ in space is specified by its distance (χ) and direction $(\boldsymbol{n})$ from a given origin. However, cosmological perturbations are characterized independently by their spatial $(\boldsymbol{x})$ and angular $(\boldsymbol{n})$ dependencies, which requires the use of both orbital and spin eigenfunctions [33,34]. Since what is observed is the total angular dependence, one introduces a set of total angular momentum normal modes [34] which splits perturbations in their effective radial and angular dependencies:

$$_sG_\ell^{(jm)}(\chi,\boldsymbol{n};\nu) \equiv c_{\ell s}\alpha_\ell^{(jm)}(\chi,\nu)_sY_\ell^m(\boldsymbol{n}), \quad (3.28)$$

where

$$c_\ell \equiv \mathrm{i}^\ell\sqrt{4\pi(2\ell + 1)}, \quad (3.29)$$

$_sY_\ell^m(\boldsymbol{n})$ are spherical harmonics of spin s and $_s\alpha_\ell^{(jm)}$ are radial functions. These radial functions are zero whenever one of the following conditions is violated:

$$j \geq \max(|m|, |s|), \qquad \ell \geq \max(|m|, |s|). \quad (3.30)$$

Moreover, these functions are conventionally normalized at origin through the condition

$$_s\alpha_\ell^{(jm)}\big|_{\chi=0} = \frac{1}{2j + 1}\delta_{\ell j}. \quad (3.31)$$

It is also convenient to decompose the radial functions into even (electric) and odd (magnetic) types as

$$_{\pm s}\alpha_\ell^{(jm)} = {}_s\epsilon_\ell^{(jm)} \pm \mathrm{i}_s\beta_\ell^{(jm)}. \quad (3.32)$$

Formal expressions and identities satisfied by the radial functions are collected in [18].

Next we introduce the harmonics $Q_{I_j}^{(jm)}$ in the ℓ-representation. These are constructed by a simple combination of the normal modes with the generalized helicity basis as

$$\ell Q_{I_j}^{(jm)} \equiv \sum_{s=-j}^{j} {}_s g^{(jm)} {}_s G_\ell^{(jm)}(\chi, \boldsymbol{n}; \nu) \hat{n}_{I_j}^s(\boldsymbol{n}), \qquad (3.33)$$

where ${}_s g^{(jm)}$ are numerical coefficients introduced in [18]. Overall, when following reference [18], the reader should make use of the replacements $\chi \to \chi/\ell_c$, $k \to k\ell_c$, and $\nu \to \nu\ell_c$.

D. Plane waves and pseudo plane waves

Finally, we build generalized plane-wave harmonics from summation over ℓ:

$$Q_{I_j}^{(jm)} \equiv \sum_{\ell \geq |m|} \frac{\zeta_\ell^m}{\zeta_j^m} \times \ell Q_{I_j}^{(jm)}(\chi, \boldsymbol{n}; \nu) \qquad (3.34)$$

where ζ_ℓ^m are coefficients that can be fixed up to an overall arbitrary constant (see Appendix E). Likewise, a plane-wave normal mode is

$$_s G^{(jm)} \equiv \sum_{\ell \geq |m|} \frac{\zeta_\ell^m}{\zeta_j^m} \times {}_s G_\ell^{(jm)}(\chi, \boldsymbol{n}; \nu), \qquad (3.35)$$

such that (3.33) also holds without the ℓ indices, that is after summation on ℓ. Usual plane waves correspond to the case $\zeta_\ell^m = \text{const}$, but the previous definitions allow for pseudo plane waves when $\zeta_\ell^m \neq \text{const}$. The plane-waves harmonics were built using the zenith direction as a reference, but we can rotate them so as to define harmonics with respect to the wave vector $\boldsymbol{\nu} = \nu\hat{\boldsymbol{v}}$, as detailed in Sec. 6.2 of [18]. The perturbations defined in (3.5) are expanded on the basis (3.34) as

$$\phi \to \int \frac{\mathrm{d}^3\boldsymbol{\nu}}{(2\pi)^3} H_S^{(0)}(\boldsymbol{\nu}, \eta) Q^{(0,0)}(\boldsymbol{\nu}),$$

$$\mathcal{D}_{ij}\psi \to \int \frac{\mathrm{d}^3\boldsymbol{\nu}}{(2\pi)^3} H_T^{(0)}(\boldsymbol{\nu}, \eta) Q_{ij}^{(2,0)}(\boldsymbol{\nu}),$$

$$\mathcal{D}_{(i}E_{j)} \to \sum_{m=\pm 1} \int \frac{\mathrm{d}^3\boldsymbol{\nu}}{(2\pi)^3} H_T^{(m)}(\boldsymbol{\nu}, \eta) Q_{ij}^{(2,m)}(\boldsymbol{\nu}),$$

$$E_{ij} \to \sum_{m=\pm 2} \int \frac{\mathrm{d}^3\boldsymbol{\nu}}{(2\pi)^3} H_T^{(m)}(\boldsymbol{\nu}, \eta) Q_{ij}^{(2,m)}(\boldsymbol{\nu}). \qquad (3.36)$$

For applications related to observations, most notably those of CMB, it is convenient to work with the propagating direction $\bar{\boldsymbol{n}}$ related to the line-of-sight direction $\boldsymbol{n}$ through

$$\bar{\boldsymbol{n}} \equiv -\boldsymbol{n}. \qquad (3.37)$$

We can also define harmonics, with related normal modes and radial functions, associated with this convention. They are trivially related to the previous ones by

$$\bar{Q}_{I_j}^{(jm)}(\chi, -\bar{\boldsymbol{n}}; \nu) = (-1)^j \times Q_{I_j}^{(jm)}(\chi, \boldsymbol{n}; \nu).$$

The associated normal modes ${}_s \bar{G}^{(jm)}$ and radial functions ${}_{\pm s}\bar{\alpha}_\ell^{(jm)}$ are related to the previous ones as detailed in Sec. 7.1 of [18].

E. Supercurvature modes

Quite generally, square-integrable cosmological perturbations can be constructed by superposing tensor harmonics characterized by $\nu\ell_c \geq 0$. For closed spaces one further requires $\nu\ell_c - 1$ to be an integer larger or equal to $|m|$, see e.g., [18]. In the case of open spaces ($K = -\ell_c^{-2}$), this requires $(k\ell_c)^2 \geq (1 + |m|)$ [see (3.26)]. While the inclusion of modes in this range is enough to describe perturbations that decay at infinity, it has been argued [35] that the most general Gaussian perturbations also require the inclusion of modes having

$$-1 \leq (\nu\ell_c)^2 \leq 0. \qquad (3.38)$$

In the scalar case ($m = 0$) this corresponds to $0 \leq k^2 \leq \ell_c^{-2}$. For this reason, these are known as supercurvature modes.

Supercurvature harmonics are not square integrable. Since they correspond to purely imaginary ν, they can be defined from analytic continuation of the radial functions ${}_s\alpha_\ell^{(jm)}$ of the usual harmonics (i.e., those with $\nu \geq 0$). In [35] only the scalar harmonics were considered, but the procedure of analytic continuation can be followed for all types of harmonics. In fact the analytic continuation is not restricted to (3.38) but can be extended at least to the whole subset of the complex plane defined by

$$-1 \leq \mathrm{Im}(\nu\ell_c) \leq 1 \Rightarrow \mathrm{Re}[(\nu\ell_c)^2] \geq -1. \qquad (3.39)$$

For that, one only needs to know how to formally build the radial functions—see [18]. Hereafter, we call the case $(\nu\ell_c)^2 = -1$ the *maximal supercurvature mode* as it corresponds to $k = 0$ for scalar harmonics.

Given that ν can be complex, the electric and magnetic parts of the radial functions are not necessarily real-valued functions. In particular the relation ${}_s\alpha_\ell^{(jm)\star}(\chi, \nu) = {}_{-s}\alpha_\ell^{(jm)}(\chi, \nu)$, which holds in the flat case, is not valid anymore and we must rely on Eq. (3.35) of [18].

IV. LINEARIZATION OF BIANCHI SPACE-TIMES

Our goal now is to derive the linearized dynamical equations for the Bianchi models in Table II, which will be ultimately matched to the Einstein and fluid equations presented in the last section. Such matching requires contrasting Eqs. (3.1) and (2.32), which in turn depends on the knowledge of the cobasis $\boldsymbol{e}^i$ in a given coordinate system. Since the cobasis is the dual to the invariant basis $\boldsymbol{e}_i$, which is in turn defined by the KVFs through

Eq. (2.12b), we start this section by recalling the KVFs and invariant basis for the selected Bianchi models.

A. KVF and invariant basis

An ingenious method to find the KVFs of Bianchi models with FLRW limit was proposed in PC10, and can be summarized as follows: starting from a maximally symmetric space, one identifies its translational (T_i) and rotational (R_i) KVFs, as well as their commutators. Next, one looks for constants ρ_i^j such that the newly defined vectors

$$\xi_i \equiv T_i + \rho_i^j R_j \qquad (4.1)$$

satisfy Eq. (2.2). From these vectors one then obtains the invariant basis e_i [through Eq. (2.12b)] and their associated cobasis e^i which, once multiplied by a time-dependent tensor $\gamma_{ij}(t)$, and following the prescription of Sec. II A, leads to the metric (2.32).

The beauty of this method is that one naturally sees which Bianchi models can emerge from a given maximally symmetric space. Moreover, coordinate systems for the KVFs are naturally inherited from the coordinate systems of the underlying symmetric space. We now summarize these vectors and their associated invariant basis. Their cobasis can then be obtained from the prescription given in Appendix C.

1. Models I and VII$_0$

Bianchi models I and VII$_0$ are the only models emerging from flat Euclidean space. As such, their Killing vectors and invariant fields can be expressed in terms of natural (Cartesian) coordinates of the flat Euclidean metric:

$$g^{\mathrm{MS}} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \qquad (4.2)$$

The set of KVFs and invariant basis for model I corresponds to simple spatial translations:

$$\xi_i^{(\mathrm{I})} = \partial_i, \qquad e_i^{(\mathrm{I})} = \partial_i. \qquad (4.3)$$

It is trivial to check that $\xi^{(\mathrm{I})}$ satisfies (2.2) with $C^k{}_{ij} = 0$, and that (2.12b) is satisfied automatically.

The isometries of model VII$_0$ correspond to two simple translations, and a translation followed by a rotation. Choosing this rotation to be around the z axis, we then have for the KVFs:

$$\xi_1^{(\mathrm{VII}_0)} = \partial_x, \qquad (4.4a)$$

$$\xi_2^{(\mathrm{VII}_0)} = \partial_y, \qquad (4.4b)$$

$$\xi_3^{(\mathrm{VII}_0)} = \partial_z - \ell_s^{-1}(x\partial_y - y\partial_x). \qquad (4.4c)$$

One can check that these vectors satisfy (2.9) with $N^1 = N^2 = \ell_s^{-1}$ and $A = 0$. The invariant basis which solves (2.12b) is

$$e_i^{(\mathrm{VII}_0)} = M_i^j e_j^{(\mathrm{I})}, \qquad (4.5)$$

where M_i^j are the components of the rotation matrix around the z axis by the angle z/ℓ_s:

$$M = \begin{pmatrix} \cos(z/\ell_s) & -\sin(z/\ell_s) & 0 \\ \sin(z/\ell_s) & \cos(z/\ell_s) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (4.6)$$

Since rotation matrices are orthogonal, we also have $M^i{}_j = M_i{}^j$ such that for the cobasis

$$e^i_{(\mathrm{VII}_0)} = M^i{}_j e^j_{(\mathrm{I})}. \qquad (4.7)$$

2. Models V and VII$_h$

These are the two models emerging from a maximally symmetric open space (i.e., a space with negative curvature). A possible coordinate system for the KVFs and invariant basis are spherical hyperbolic coordinates (χ, θ, ϕ), in terms of which the metric of the underlying space writes

$$g^{\mathrm{MS}} = \mathrm{d}\chi^2 + \ell_c^2 \sinh^2(\chi/\ell_c)\mathrm{d}^2\Omega. \qquad (4.8)$$

In these coordinates, the KVFs and invariant basis for model V are given respectively by

$$\begin{aligned} \xi_1^{(\mathrm{V})} &= \sin\theta\cos\phi\partial_\chi \\ &+ \ell_c^{-1}[\cos\theta\coth(\chi/\ell_c) - 1]\cos\phi\partial_\theta \\ &+ \ell_c^{-1}[\cot\theta - \coth(\chi/\ell_c)\csc\theta]\sin\phi\partial_\phi, \end{aligned} \qquad (4.9a)$$

$$\begin{aligned} \xi_2^{(\mathrm{V})} &= \sin\theta\sin\phi\partial_\chi \\ &+ \ell_c^{-1}[\cos\theta\coth(\chi/\ell_c) - 1]\sin\phi\partial_\theta \\ &+ \ell_c^{-1}[\coth(\chi/\ell_c)\csc\theta - \cot\theta]\cos\phi\partial_\phi, \end{aligned} \qquad (4.9b)$$

$$\xi_3^{(\mathrm{V})} = \cos\theta\partial_\chi - \ell_c^{-1}\coth(\chi/\ell_c)\sin\theta\partial_\theta. \qquad (4.9c)$$

and

$$e_1^{(V)} = \frac{\sin\theta\cos\phi}{\cosh(\chi/\ell_c) - \cos\theta\sinh(\chi/\ell_c)}\partial_\chi$$
$$+ \frac{\ell_c^{-1}[\cos\theta\coth(\chi/\ell_c) - 1]}{\cosh(\chi/\ell_c) - \cos\theta\sinh(\chi/\ell_c)}\cos\phi\partial_\theta$$
$$- \ell_c^{-1}\sin\phi\csc\theta\,\mathrm{csch}(\chi/\ell_c)\partial_\phi,$$

$$e_2^{(V)} = \frac{\sin\theta\sin\phi}{\cosh(\chi/\ell_c) - \cos\theta\sinh(\chi/\ell_c)}\partial_\chi$$
$$+ \frac{\ell_c^{-1}[\cos\theta\coth(\chi/\ell_c) - 1]}{\cosh(\chi/\ell_c) - \cos\theta\sinh(\chi/\ell_c)}\sin\phi\partial_\theta$$
$$+ \ell_c^{-1}\cos\phi\csc\theta\,\mathrm{csch}(\chi/\ell_c)\partial_\phi,$$

$$e_3^{(V)} = \frac{\cos\theta\cosh(\chi/\ell_c) - \sinh(\chi/\ell_c)}{\cosh(\chi/\ell_c) - \cos\theta\sinh(\chi/\ell_c)}\partial_\chi$$
$$- \frac{\ell_c^{-1}\mathrm{csch}(\chi/\ell_c)\sin\theta}{\cosh(\chi/\ell_c) - \cos\theta\sinh(\chi/\ell_c)}\partial_\theta.$$

As emphasized in [36], it proves useful to use a different system of coordinates to recast these expressions in much simpler forms, and we report the detailed expressions in Appendix B.

The other possibility corresponds to model VII$_h$. In this case the KVFs and invariant basis can be related to the expressions above simply as [3]

$$\boldsymbol{\xi}_1^{(VII_h)} = \boldsymbol{\xi}_1^{(V)}, \tag{4.10a}$$

$$\boldsymbol{\xi}_2^{(VII_h)} = \boldsymbol{\xi}_2^{(V)}, \tag{4.10b}$$

$$\boldsymbol{\xi}_3^{(VII_h)} = \boldsymbol{\xi}_3^{(V)} - \ell_s^{-1}\partial_\phi, \tag{4.10c}$$

and

$$\boldsymbol{e}_i^{(VII_h)} = M_i{}^j \boldsymbol{e}_j^{(V)} \tag{4.11}$$

with $\boldsymbol{M}$ given formally by the same expression (4.6), but with z defined as in (B1).

3. Model IX

This is the only model emerging from a closed maximally symmetric space (i.e., a three-dimensional sphere) with metric

$$\boldsymbol{g}^{MS} = \mathrm{d}\chi^2 + \ell_c^2\sin^2(\chi/\ell_c)\mathrm{d}^2\Omega. \tag{4.12}$$

The KVFs and invariant basis are given by

$$\boldsymbol{\xi}_1^{(IX)} = \cos\phi\sin\theta\partial_r$$
$$+ \ell_c^{-1}[\cot(\chi/\ell_c)\cos\theta\cos\phi + \sin\phi]\partial_\theta$$
$$- \ell_c^{-1}[\cot(\chi/\ell_c)\csc\theta\sin\phi - \cos\phi\cot\theta]\partial_\phi,$$

$$\boldsymbol{\xi}_2^{(IX)} = \sin\phi\sin\theta\partial_r$$
$$+ \ell_c^{-1}[\cot(\chi/\ell_c)\cos\theta\sin\phi - \cos\phi]\partial_\theta$$
$$+ \ell_c^{-1}[\cot(\chi/\ell_c)\csc\theta\cos\phi + \sin\phi\cot\theta]\partial_\phi,$$

$$\boldsymbol{\xi}_3^{(IX)} = \cos\theta\partial_r - \ell_c^{-1}[\cot(\chi/\ell_c)\sin\theta\partial_\theta + \partial_\phi],$$

and by

$$\boldsymbol{e}_1^{(IX)} = \cos\phi\sin\theta\partial_r$$
$$+ \ell_c^{-1}[\cot(\chi/\ell_c)\cos\theta\cos\phi - \sin\phi]\partial_\theta$$
$$- \ell_c^{-1}[\cot(\chi/\ell_c)\csc\theta\sin\phi + \cos\phi\cot\theta]\partial_\phi,$$

$$\boldsymbol{e}_2^{(IX)} = \sin\phi\sin\theta\partial_r$$
$$+ \ell_c^{-1}[\cot(\chi/\ell_c)\cos\theta\sin\phi + \cos\phi]\partial_\theta$$
$$+ \ell_c^{-1}[\cot(\chi/\ell_c)\csc\theta\cos\phi - \sin\phi\cot\theta]\partial_\phi,$$

$$\boldsymbol{e}_3^{(IX)} = \cos\theta\partial_r - \ell_c^{-1}[\cot(\chi/\ell_c)\sin\theta\partial_\theta - \partial_\phi],$$

respectively.

Before proceeding, note that the transformation $(\theta, \phi) \to (\pi - \theta, \pi + \phi)$ is such that $(\boldsymbol{\xi}_i^{(IX)}, \boldsymbol{e}_i^{(IX)}) \to (-\boldsymbol{e}_i^{(IX)}, -\boldsymbol{\xi}_i^{(IX)})$, while the metric remains invariant. Thus, the role of KVFs and invariant basis can be reversed in model IX. Such inversion also happens in Bianchi I, as one can trivially check. As we will see, this has practical implications in the identification of homogeneous perturbations with spatial anisotropies in these two models.

B. Linearized Bianchi equations

We now parametrize γ_{ij} in (2.32) as

$$\gamma_{ij}(t) = [e^{2\beta(t)}]_{ij}, \tag{4.13}$$

and we linearize equations in the time-dependent traceless matrix $\beta_{ij}(t)$, and also in $\hat{v}_i$. When find that for the Bianchi types which admit a FLRW limit, the metric reduces to a maximally symmetric space metric when $\beta_{ij} = 0$ (and thus $\gamma_{ij} = \delta_{ij}$), that is we find

$$\boldsymbol{g}^{MS} = \delta_{ij}\boldsymbol{e}^i \otimes \boldsymbol{e}^j. \tag{4.14}$$

Consequently, we also recover in that case that the Ricci scalar takes a FLRW form. This means in particular that the traceless part of the spatial Ricci vanishes, and the value of the Ricci scalar (spatial curvature scalar) indicates to which FLRW type (open, closed, or flat) it corresponds. This is reported in Table II where the values of (A, N^1, N^2, N^3) are given as functions of the two length scales introduced in

(2.10). Neglecting sources of anisotropic stress, the background equations are

$$\mathcal{H}^2 + K = \frac{\kappa a^2}{3}\bar{\rho}, \tag{4.15}$$

$$\bar{\rho}' + 3\mathcal{H}(\bar{\rho} + \bar{p}) = 0, \tag{4.16}$$

where

$$K = -6\hat{A}_i\hat{A}_i - \hat{N}^{ij}\hat{N}^{ij} + \frac{1}{2}\hat{N}^{ii}\hat{N}^{jj}. \tag{4.17}$$

These are formally the same as the dynamical equations for a background FLRW metric. The curvature scale $\ell_c = 1/\sqrt{|K|}$ appears in the FLRW limit of the Ricci scalar whereas the spiral scale ℓ_s (whose meaning will be made clear later) does not.

The shear is given at linear order by

$$\hat{\sigma}_{ij} = \beta'_{ij}, \tag{4.18}$$

and the linear parts of the equations (in β_{ij} and $\hat{v}_i$) are

$$\frac{1}{2}a^2\delta(^{(3)}R) = \kappa a^2\delta\rho, \tag{4.19a}$$

$$\beta''_{ij} + 2\mathcal{H}\beta'_{ij} = \mathcal{S}_{ij}, \tag{4.19b}$$

$$3\hat{A}_j\beta'_{ji} + \hat{\epsilon}_{ijk}\beta'_{jl}\hat{N}_{kl} = P_i \tag{4.19c}$$

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) = -(\bar{\rho} + \bar{p})D_i\hat{v}^i$$
$$= \frac{2\hat{A}^i P_i}{\kappa a^2} \tag{4.19d}$$

$$[a^4(\bar{\rho} + \bar{p})\hat{v}_i]' = 0. \tag{4.19e}$$

The right-hand side of Eqs. (4.19a) and (4.19b) correspond to the isotropic and anisotropic contribution to the spatial curvature due to terms linear in β_{ij}. They are given respectively by

$$a^2\delta(^{(3)}R) = 2(6\hat{A}_i\hat{A}_j - 2\hat{N}^{ik}\hat{N}^{kj} + \hat{N}^{kk}\hat{N}^{ij})\beta_{ij}, \tag{4.20}$$

and by

$$\mathcal{S}_{ij} = \hat{N}^{kk}\hat{N}^{\{ij\}} - 2\hat{N}^{k\{i}\hat{N}^{j\}k} - 2\hat{A}_k\hat{\epsilon}_{lk(i}\hat{N}^{j)l}$$
$$+ 2\left[2\hat{N}^{kk}\beta_{l\{i}\hat{N}^{j\}l} - \frac{1}{3}\hat{N}^{kk}\hat{N}^{ll}\beta_{ij}\right.$$
$$+ \hat{N}^{ij}\beta_{kl}\hat{N}^{kl} - 4\hat{N}^{kl}\hat{N}^{k\{i}\beta_{j\}l}$$
$$- 2\hat{N}^{ri}\hat{N}^{sj}\beta_{rs} + \frac{2}{3}\hat{N}^{kl}\hat{N}^{kl}\beta_{ij}$$
$$+ \left. 2\hat{A}_k\epsilon_{kl(i}\beta_{j)r}\hat{N}^{rl} - 2\hat{A}_l\beta_{kl}\epsilon_{kr(i}\hat{N}^{j)r}\right]. \tag{4.21}$$

Repeated indices are summed, and we have introduced the notation $\{\cdots\}$ for symmetric and trace-free tensors with respect to δ_{ij}, such that all indices on the right-hand side are now manipulated with the Kronecker delta.

C. Homogeneous *svt* modes

In order to identify β_{ij} with homogeneous metric perturbations, we proceed by decomposing the former in a similar fashion to the decomposition of δg_{ij}. Just as the scalar, vector, and tensor modes appearing in (3.5) are defined with respect to their transformation properties under rotations around $\mathbf{k}$—the wave vector of the perturbation—we can introduce scalar, vector, and tensor modes of the shear (henceforth *svt* modes) with respect to their transformations under rotations around some direction $\mathbf{e}_i$ of the invariant basis [3]. For models I and IX it does not matter which direction we choose, since any rotation will preserve $C^k{}_{ij}$ in these models. On the other hand, the constants of structure in models V, VII$_0$, and VII$_h$ have a residual symmetry given by rotations around the vector $\mathbf{A} \propto \mathbf{e}_3$. Since the construction in this section is general to all Bianchi models, we shall omit a subscript in the covectors $\mathbf{e}_i$ to designate the model they belong to, so as to alleviate the notation. We thus choose the latter as a fiducial direction and introduce a complex basis

$$e_i^{(\pm)}(\mathbf{e}_3) \equiv \frac{(e_i^1 \mp i e_i^2)}{\sqrt{2}}, \tag{4.22}$$

in terms of which we introduce the following tensor polarization basis:

$$q_{ij}^{(0)}(\mathbf{e}_3) \equiv (-e_i^3 e_j^3 + \delta_{ij}/3),$$
$$q_{ij}^{(\pm 1)}(\mathbf{e}_3) \equiv \pm e_{(i}^3 e_{j)}^{(\mp)},$$
$$q_{ij}^{(\pm 2)}(\mathbf{e}_3) \equiv -\sqrt{\frac{3}{2}}e_i^{(\mp)}e_j^{(\mp)}. \tag{4.23}$$

In what follows, we will omit the dependence on $\mathbf{e}_3$ whenever there is no chance of confusion. Note also that, because $\mathbf{e}_3$ and $\mathbf{e}^\pm$ form a general triad frame, $q_{ij}^{(m)}$ will in general depend on the spacetime point, whose explicit dependence we shall also omit.

The above polarization tensors allow us to write the shear as

$$\beta_{ij} = \sum_{m=-2}^{2} \beta_{(m)} q_{ij}^{(m)} \tag{4.24}$$

with the values of m corresponding to scalar ($m = 0$), vector ($m = \pm 1$), and tensor ($m = \pm 2$) modes. Note that all modes are constructed such that

$$q_{ij}^{(-m)} = (-1)^m q_{ij}^{(m)\star}, \tag{4.25}$$

and hence the reality of β_{ij} implies

$$\beta_{(-m)} = (-1)^m \beta_{(m)}^{\star}. \tag{4.26}$$

This means that, in practice, we only need to consider physical effects for the case $m > 0$.

Because the Bianchi models we are considering all have a maximally symmetric 3-space as their isotropic limits, the first line of Eq. (4.21) vanishes for all models of Table II, as one can easily check. We can thus write

$$\mathcal{S}_{ij} = \sum_{m=-2}^{2} \mathcal{S}^{(m)} \beta_{(m)} q_{ij}^{(m)}. \tag{4.27}$$

The isotropic spatial curvature, on the other hand, splits into a background plus perturbation as

$$a^{2(3)}R = a^{2(3)}R_{\text{FLRW}} + \delta(a^{2(3)}R), \tag{4.28}$$

where

$$\delta(a^{2(3)}R) = \sum_{m=-2}^{2} \mathcal{R}^{(m)} \beta_{(m)}, \tag{4.29}$$

and

$$\mathcal{R}^{(m)} = \mathcal{R}^{ij} q_{ij}^{(m)},$$
$$\mathcal{R}_{ij} = 2(6\hat{A}_i \hat{A}_j - 2\hat{N}^{ik}\hat{N}^{kj} + \hat{N}^{kk}\hat{N}^{ij}). \tag{4.30}$$

Likewise, the tilt P_i can be decomposed as

$$P_i = \sum_{m=-2}^{2} \beta'_{(m)} \mathcal{P}_i^{(m)}, \tag{4.31}$$

TABLE III. Bianchi *svt* modes and the quantities $\mathcal{R}^{(m)}$, $\mathcal{S}^{(m)}$, and $\mathcal{P}_i^{(m)}$ for all Bianchi models considered in this work. For models I and IX, $\mathcal{P}_i^{(m)} = 0$, and is thus not shown. This reproduces Table II of PC10 up to variations of conventions in definitions.

m:		0	+1	−1	+2	−2
$q_{ij}^{(m)}$:		$\frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}$	$\frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ -1 & i & 0 \end{pmatrix}$	$\sqrt{\frac{3}{8}}\begin{pmatrix} -1 & -i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\sqrt{\frac{3}{8}}\begin{pmatrix} -1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
VII_0	$\ell_s^2 \mathcal{S}^{(m)}$	0	0	0	−4	−4
	$\ell_s^2 \mathcal{R}^{(m)}$	0	0	0	0	0
	$\sqrt{8}\ell_s \mathcal{P}_i^{(m)}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -i \\ -1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
V	$\ell_c^2 \mathcal{S}^{(m)}$	0	0	0	0	0
	$\ell_c^2 \mathcal{R}^{(m)}$	−8	0	0	0	0
	$\sqrt{8}\ell_c \mathcal{P}_i^{(m)}$	$\begin{pmatrix} 0 \\ 0 \\ -4\sqrt{2} \end{pmatrix}$	$\begin{pmatrix} 3 \\ 3i \\ 0 \end{pmatrix}$	$\begin{pmatrix} -3 \\ 3i \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
VII_h	$\ell_c^2 \mathcal{S}^{(m)}$	0	0	0	$-4/h - 4i/\sqrt{h}$	$-4/h + 4i/\sqrt{h}$
	$\ell_c^2 \mathcal{R}^{(m)}$	−8	0	0	0	0
	$\sqrt{8}\ell_c \mathcal{P}_i^{(m)}$	$\begin{pmatrix} 0 \\ 0 \\ -4\sqrt{2} \end{pmatrix}$	$\begin{pmatrix} 3 - i/\sqrt{h} \\ 3i + 1/\sqrt{h} \\ 0 \end{pmatrix}$	$\begin{pmatrix} -3 - i/\sqrt{h} \\ 3i - 1/\sqrt{h} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
I	$\mathcal{S}^{(m)}$	0	0	0	0	0
	$\mathcal{R}^{(m)}$	0	0	0	0	0
IX	$\ell_c^2 \mathcal{S}^{(m)}$	−8	−8	−8	−8	−8
	$\ell_c^2 \mathcal{R}^{(m)}$	0	0	0	0	0

with

$$\mathcal{P}_i^{(m)} = 3\hat{A}_j q_{ji}^{(m)} + \hat{\epsilon}_{ijk} q_{jl}^{(m)} \hat{N}_{kl}. \quad (4.32)$$

The *svt* modes for the tensors $\mathcal{S}_{ij}$, $\mathcal{R}_{ij}$, and $\mathcal{P}_i$ for all Bianchi models are summarized in Table III.

V. BIANCHI AND FLRW MODES RELATED

A. Matching the perturbations

We are now ready to find the correspondence between Bianchi degrees of freedom and FLRW metric perturbations. In other words, we can now identify the modes in the expansion (3.36) which we must consider in (3.5) so as to obtain the matching

$$\delta g_{ij} \rightarrow 2a^2 \beta_{ij}. \quad (5.1)$$

Since β_{ij} is traceless, one can already set $\phi \rightarrow 0$. On the Bianchi side, we have seen that the shear can be decomposed as in Eq. (4.24). As we detail in Appendix E, we show that

$$q_{ij}^{(m)}(\chi, \boldsymbol{n}) = \frac{\xi_2}{\xi_m} Q_{ij}^{(2m)}(\chi, \boldsymbol{n}; \nu_m, \zeta_\ell^m), \quad (5.2)$$

where the constants ξ_m are defined as (see [18] for details)

$$\xi_m \equiv \prod_{i=1}^{m} \frac{k\ell_c}{\sqrt{(\nu_m \ell_c)^2 - \mathcal{K}i^2}}, \quad (5.3)$$

with the understanding that $\xi_0 = 1$. Defining

$$H_T^{(m)} \equiv \beta_{(m)} \frac{\xi_2}{\xi_m} \quad (5.4)$$

implies that the identification (5.1) is made with the discrete sum

$$\frac{\delta g_{ij}}{2a^2} = \sum_{m=-2}^{2} H_T^{(m)}(\nu_m, \eta) Q_{ij}^{(2m)}(\nu_m, \zeta_\ell^m). \quad (5.5)$$

Let us stress already that the identification of model IX is a special case since there are only tensor modes. That is, the Bianchi modes $q_{ij}^{(m)}$ with $m = 0, 1, -1$ do not map to scalar and vector harmonics, but instead to sums of tensor harmonics. We shall treat this case separately.

We still need to specify with which modes ν_m and with which set of constants ζ_ℓ^m the matching (5.2) holds. The matching of the positive and negative m are necessarily related since negative values can be obtained by the reality condition (4.25). Using Eq. (6.8) of [18] for the complex conjugation of a harmonic, we can check that we obtain

$$[Q_{ij}^{(2m)}(\nu_m, \zeta_\ell^m)]^\star = (-1)^m Q_{ij}^{(2,-m)}(\nu_{-m}, \zeta_\ell^{-m})$$

provided that the conditions

$$\nu_{-m} = -\nu_m^\star, \qquad \zeta_\ell^{-m} = (-1)^\ell \zeta_\ell^{m\star} \quad (5.6)$$

are satisfied.

In order to find for each m the mode ν_m and the set of coefficients ζ_ℓ^m which define the pseudo plane wave, we first determined the ν_m by comparing equations (4.19b) and (4.19c) to the linearized Einstein equations given in Sec. III B, and asking the constraints (5.6) to be satisfied. The shear evolution in Bianchi models maps to either tensor, vector, or traceless scalar perturbations in synchronous gauge and we report details of the matching in Appendix D. For the flat and open cases (corresponding to types I, V, VII$_0$, VII$_h$), the modes must be

$$\nu_m = \frac{m}{\ell_s} + \frac{i}{\ell_c} \quad (5.7)$$

with $k_m = \nu_m$ in the I and VII$_0$ cases. For the closed case (Bianchi IX) we obtain

$$\nu_{\pm 2} = \pm \frac{3}{\ell_c}. \quad (5.8)$$

Given that the ν_m can be complex in the open case, it is understood that the radial functions are obtained through analytic continuation corresponding to the presence of a supercurvature mode (see Sec. III E). Technically, this implies that the electric and magnetic radial functions are no more real valued.

We must also specify the coefficients ζ_ℓ^m since some models are matched with pseudo plane waves. In Appendix E we detail how these constants are found for models VII$_0$, V, and VII$_h$, and they are (up to a global constant factor)

$$\zeta_m^{\pm m} = (\pm 1)^m, \quad (5.9a)$$

$$\zeta_\ell^{\pm m} = \zeta_{\ell-1}^{\pm m}(-i) \frac{\sqrt{\ell^2 + (\nu_{\pm m}\ell_c)^2}}{(\ell + 1) \mp mi/\sqrt{h}}. \quad (5.9b)$$

Using the identity (E11), it can be recast in the form

$$\zeta_\ell^{\pm m} = (\pm 1)^m \prod_{p=m+1}^{\ell} (-i)\sqrt{\frac{(p-1) \pm mi/\sqrt{h}}{(p+1) \mp mi/\sqrt{h}}}. \quad (5.10)$$

The case $m = 0$ is special since in (5.9) one finds

$$\zeta_0^0 = 1, \qquad \zeta_{\ell \geq 1}^0 = (-i)^\ell \frac{\sqrt{1 + (\nu\ell_c)^2}}{\sqrt{2\ell(\ell + 1)}}. \quad (5.11)$$

TABLE IV. Summary of modes (k_m or ν_m) and pseudo plane-wave constants (ζ_ℓ^m) found in this section. It is understood that the ζ_ℓ^0 for the V and VII$_h$ models are (5.11).

	ν or k	$\zeta_\ell^{\pm m}$
I	$k_m \to 0$	δ_ℓ^2
VII$_0$	$k_m = \frac{m}{\ell_s}$	$(\pm 1)^\ell$
IX	$\nu_{\pm 2} = \pm \frac{3}{\ell_c}$	δ_ℓ^2, $|m| = 2$ only
V	$\nu_m = \frac{i}{\ell_c}$	$(\pm 1)^m (-i)^{\ell-m} \sqrt{\frac{m(m+1)}{\ell(\ell+1)}}$
VII$_h$	$\nu_m = \frac{m}{\ell_s} + \frac{i}{\ell_c}$	$(\pm)^m \prod_{p=m+1}^\ell (-i) \sqrt{\frac{(p-1)\sqrt{h}\pm mi}{(p+1)\sqrt{h}\mp mi}}$

TABLE V. Transformation rules for harmonics (defined with the zenith direction) under the inversion of a single axis.

	$x \to -x$	$y \to -y$	$z \to -z$
Factor $(-1)^m$		yes	yes
$\zeta_\ell^m \to (-1)^\ell \zeta_\ell^m$			yes
$Q_{I_j}^{(jm)} \to Q_{I_j}^{(j,-m)}$	yes	yes	
$Q_{I_j}^{(jm)}(\nu) \to Q_{I_j}^{(jm)}(-\nu)$			yes

Note that there is a factor $\sqrt{1 + (\nu_0 \ell_c)^2} = 0$ in all ratios ζ_ℓ^0 / ζ_0^0. One could think that this is problematic since the ratios ζ_ℓ^m / ζ_j^m enter the very definition of a pseudo plane wave (3.34). For instance, in $Q_{ij}^{(20)}$ one would encounter $\zeta_0^0 / \zeta_2^0 \sim 1/\sqrt{1 + (\nu_0 \ell_c)^2}$. However the factor $\sqrt{1 + (\nu_0 \ell_c)^2}$ is nothing but k given the relation (3.26), and it appears that there is a compensating divergence in the radial functions. More precisely it can be checked that $_s\alpha_0^{(jm)} \sim \sqrt{1 + (\nu_0 \ell_c)^2}$ for $k \to 0$ (except $_0\alpha_0^{(00)} \to 1$). In practice, one must keep track of all factors of k and take the limit $k \to 0$ at the end. More rigorously, one could have redefined radial functions to be

$$\frac{\zeta_\ell^m}{\zeta_j^m} \, _s\alpha_\ell^{(jm)} \qquad (5.12)$$

as these quantities would never possess any (apparent) divergences.

The results for models VII$_0$ and V can be obtained by letting $\ell_c \to \infty$ (or $h \to 0$) and $\ell_s \to \infty$ (or $h \to \infty$), respectively. For model VII$_0$ this reduces to $\zeta_\ell^{\pm |m|} = (\pm 1)^\ell$. In the Bianchi V case, we also find (5.11) when $m = 0$, and when $m \neq 0$ we get

$$\zeta_\ell^{\pm m} = (\pm 1)^m (-i)^{\ell-m} \sqrt{\frac{m(m+1)}{\ell(\ell+1)}}. \qquad (5.13)$$

As for models I and IX, we find $\zeta_\ell^{\pm m} = \delta_\ell^2$ that is the sum is reduced to the lowest term with $\ell = 2$. In the case of model I, one could alternatively consider that $\zeta_\ell^{\pm m} = (\pm 1)^\ell$ since for $\ell \geq 3$ all radial functions vanish for $k_m = 0$. The pseudo plane waves harmonics needed for the matching in the various models are summarized in Table IV.

Let us comment that the scalar mode ($m = 0$) is special. Indeed we found that the ν_0 and ζ_ℓ^0 are the same for models V and VII$_h$, and also the same for models I and VII$_0$. This is because there is a spiraling structure in models VII$_0$ and

VII$_h$ (with typical spiral scale $2\pi\ell_s/|m|$) which is absent for the scalar mode ($m = 0$).

Note that the prefactor ξ_2/ξ_m is always compensated by an opposite factor ξ_m/ξ_2 in the $_sg^{(2m)}$ which enter in the definition of harmonics from radial functions (3.33). This was expected since this prefactor was precisely added to remove any divergence so as to reach constant matrices for the $q_{ij}^{(m)}$.

B. Transformation properties

When writing the constants of structure in canonical form in Table II, we have used the possibility of a global rotation and of parity inversion. Hence, once we have identified the harmonics corresponding to the *svt* modes, we must explore the effect of global rotations and parity inversion to exhaust all possible FLRW metric perturbations matching Bianchi models.

In Table I of [18] are gathered the transformation rules of the harmonics $^\ell Q_{I_j}^{(jm)}(\nu, \eta_\ell^m)$ (defined with respect to the zenith axis) under inversion of a single individual axis (x, y, or z) axis. For the pseudo plane waves used in the matching (with $j = 2$), the transformation rules can be deduced (see Sec. 6.4 of [18]) and we summarize them in Table V for clarity of the following discussion.

1. Isotropic constants of structure

When the constants of structure of a given model are invariant under arbitrary rotations (such as in models I and IX), any mode s, v, or t (or, equivalently, any basis $q_{ij}^{(m)}$ for $|m| = 0$, 1 or 2) can be constructed from any other by appropriate linear combinations of rotated modes. One can easily verify that

$$q_{ij}^{(0)}(\boldsymbol{e}_3) \propto \sum_{m=\pm 2} q_{ij}^{(m)}(\boldsymbol{e}_1) + q_{ij}^{(m)}(\boldsymbol{e}_2), \qquad (5.14)$$

where we use $e_i^{(\pm)}(\boldsymbol{e}_1) \equiv (-e_i^3 \mp i e_i^2)/\sqrt{2}$ and $e_i^{(\pm)}(\boldsymbol{e}_2) \equiv (-e_i^3 \pm i e_i^1)/\sqrt{2}$. Similar combinations can be used to produce the $q_{ij}^{(\pm 1)}(\boldsymbol{e}_3)$. As a consequence all modes have the same dynamics and this is checked in Table III where the $\mathcal{S}^{(m)}$ are the same for all m in models I and IX. Thus, the anisotropies of every Bianchi I model can be seen as combinations of homogeneous gravitational waves, which

are exhausted by the fives parameters $\beta_{(m)}$ which, given (4.26), are 5 degrees of freedom of the model. Any global rotation leads only to a transformation of the $\beta_{(m)}$ by a constant phase.

The freedom in the point of view about the nature of the modes results from the nonuniqueness in the definition of SVT modes when perturbations do not decay at infinity [37]. However, since Bianchi IX arises as perturbations of a closed FLRW universe, which has compact spatial sections, there is no such freedom of interpretation and the s, v modes *must* be considered as sums of tensors modes. In other words, they are obtained from sums of rotations of $q_{ij}^{(\pm2)}(\nu_{\pm2})$.

In practice, when exploring the implications of Bianchi models on observables, one might choose to lose complete generality and to focus on the effect of a single gravitational wave. Hence, one starts from the tensor mode aligned with the zenith direction $\beta_{(\pm2)}^{\ell=2}Q_{ij}^{(2,\pm2)}(\nu_{\pm2})$ and explores the range of Euler angles for the rotation of the plane-wave axis. When doing so, one does not need to explore the freedom of the γ Euler angle, (i.e., of rotations of the wave around its wave vector ν) since this is degenerate with the phases in the $\beta_{(m)}$.

a. Bianchi I. In the simple I models, one has $\nu_m = 0$ and only $\ell = 2$, so we infer from Table V that the tensor mode with zenith direction is invariant under inversion of the z axis. Also since $\nu_m = \nu_{-m}$ we deduce that an inversion of either the x or the y axis interchanges the m and $-m$ contributions, leaving the whole metric perturbation invariant. Hence the svt modes are invariant under inversion of any axis and in particular of global parity.

b. Bianchi IX. The case of Bianchi IX is different. Indeed, since $\nu_{-2} = -\nu_2$ one loses the parity invariance as seen on Table V. The tensor mode with zenith axis is only invariant by simultaneous inversion of two axes among x, y, z. Physically, this happens because such tensor mode in Bianchi IX is a standing circularly polarized wave [2]; intuitively this is like a spiraling structure in the zenith direction (and which has a spiraling effect on observables [14]), and any rotation which flips this spiraling direction leaves the system invariant. As found in [2] and summarized in Appendix C, the difference between the type IX and type I is related to the fact that the canonical choice of

constants of structure selects only one type of chirality for the circularly polarized wave (one could invert all signs of the constants of structure). The Bianchi IX is a sum of (rotated) tensor harmonics of the same chirality [2]. To exhaust all possible perturbations of the type IX one then needs to consider a global parity transformation which changes the chirality.

2. Anisotropic constants of structure

For anisotropic constants of structure, that is, for models V, VII$_0$, VII$_h$, one has the 5 degrees of freedom of the amplitude $\beta_{(m)}$, on top of which we must also allow for a general direction of the special axis used when writing the constants in canonical form. This brings 2 other degrees of freedom (the angles of the special axis direction) since any rotation around the special axis is degenerate with phases in the $\beta_{(m)}$. It is then instructive to look at the properties under inversion of the axis, since it highlights the effects that modes have on observables. As we now detail, a global parity inversion is needed to explore the whole range of VII$_0$ and VII$_h$ types but this is not needed for the V type.

a. Bianchi V. The associated modes have the property $\nu_{-m} = \nu_m$ and $\zeta_\ell^{-m} = \zeta_\ell^m$. Hence from Table V we find that it is invariant under inversion of either the x or the y axis. However it is not invariant under inversion of the z axis. This is because Bianchi V has a focusing effect on observables in the special z direction [14,36]. This contrasts with the cases of Bianchi VII$_0$ and VII$_h$. Since a rotation of angle π around e.g., the x axis inverts the y and z axis, one deduces that global parity inversion does not bring new models when exploring all possible directions.

b. Bianchi VII$_0$. We still have the property $\zeta_\ell^{-m} = (-1)^\ell \zeta_\ell^m$ but we now have $\nu_{-m} = -\nu_m$. So from Table V we find that the perturbation is invariant under the joint inversion of any two of the three x, y, z axes. This is the same symmetry as the one already found in the IX case. In fact, the Bianchi VII$_0$ modes with $m \neq 0$ are also equivalent to a standing circularly polarized standing wave, whose wavelength is free and controlled by ℓ_s. And as for the IX case, a global parity inversion, that is a change of sign in the constants of structure, inverts the chirality of the standing wave.
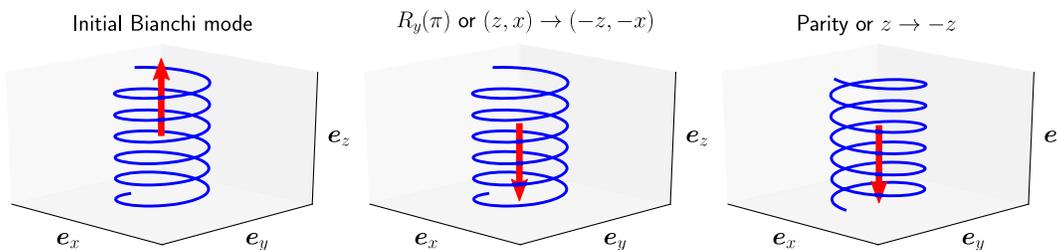

FIG. 1. Schematic representation of the transformation properties of the spiraling and focusing structures, under inversions of some axes. The type VII$_0$ has only a spiraling structure (represented by the blue screw thread), whereas the type V has only a focusing effect (represented by the red arrow). Type VII$_h$ has both features.

c. Bianchi VII_h. The VII_h models possess both the properties of models V (that is a focusing direction) and VII_0 (a spiraling structure from a circularly polarized standing wave in the $m \neq 0$ modes). An inversion of the z axis inverts the focusing direction but not the spiraling structure. An inversion of either the x or y axis inverts the chirality of the spiraling structure but leaves the focusing direction unchanged. Hence a global parity inversion (a change of sign in the constants of structure), combined with a rotation of angle π around the x axis, inverts the chirality of the standing wave while keeping the same focusing direction. The transformation properties of modes in Bianchi VII_h are illustrated in Fig. 1.

VI. COSMOLOGICAL IMPLICATIONS

We now illustrate the power of the identification to compute theoretical predictions for observables in various Bianchi models. Indeed, up to now the effect of a Bianchi space-time on the CMB was computed with independent codes from the ones used for the usual stochastic perturbations around a FLRW background [16,17]. With the identification (5.1) of Bianchi modes as FLRW perturbations in synchronous gauge, it is possible to compute the linear transfer functions from initial conditions to observable multipoles in the same framework. The modes identified as a Bianchi perturbations shall not correlate statistically with other FLRW perturbations since the one-point average of usual fluctuations vanishes. However it should contribute in the three point function since the two-point function average of usual fluctuations are related to the nonvanishing power spectrum. In the next section we review the dynamics of a Bianchi perturbations and recall that only some modes are regular [14] and thus credible as a possible large scale anisotropy. Then we detail the computation of the CMB multipoles and show that it requires only to adapt the usual Boltzmann hierarchy to take into account the fact that we have pseudo plane waves instead of usual plane waves.

A. Bianchi perturbation dynamics

Since our method is a based on a small shear approximation, it is important to discuss the existence of solutions to Eq. (4.19b) which are finite at high redshifts. Such solutions were extensively discussed in Refs. [3,14], and so we will just give a brief summary. The main equation is

$$\beta''_{(m)} + 2\mathcal{H}\beta'_{(m)} - \mathcal{S}^{(m)}\beta_{(m)} = 0 \qquad (6.1)$$

which follows from (4.19b) with (4.24) and (4.27).

Being a homogeneous second-order differential equation, all solutions are linear combinations of two solutions. In the case $\mathcal{S}^{(m)} = 0$, one solution is a pure constant with no observable effect, while the other is diverging at early times. Hence, these modes are considered as being

irregular [14], and they are usually rejected on the basis that they would not pass observational tests at early times. Furthermore, it is also difficult to find a natural mechanism to generate their initial conditions.

In the VII_0 and VII_h models, we have $\mathcal{S}^{(\pm 2)} \neq 0$, hence these modes have a different dynamics. In a matter dominated era, one has $a \propto \eta^2$ and thus $2\mathcal{H} = 4/\eta$. The regular solution of (6.1) for the tensor modes in these models is then

$$\beta_{(\pm 2)} \propto \frac{j_1(\omega_\pm\eta)}{\omega_\pm\eta}, \qquad \omega_\pm \equiv \frac{2}{\ell_s}\sqrt{1 \pm \mathrm{i}(\ell_s/\ell_c)}. \quad (6.2)$$

Given that $\omega_\pm$ is complex valued, this solution has to be understood from the analytic continuation of the spherical Bessel function j_1. When $\ell_s \gg \ell_c$, the solution is regular as long as $\eta \ll \sqrt{\ell_c\ell_s}$, but in that case it is a constant solution and we recover the dynamics of a Bianchi V. In the opposite regime where $\ell_s \ll \ell_c$, the solution is regular for $\eta \ll \ell_c$ (corresponding to the isotropic curvature scale remaining superhorizon) and the solution results in damped oscillations. Type IX also has a regular solution with damped oscillations, since $\mathcal{S}^{(\pm 2)} = -8/\ell_c^2$ and one finds that (6.2) holds with $\omega_\pm = \sqrt{8}/\ell_c$ in a matter dominated era.

Given the matching with FLRW perturbations, we can interpret these regular solutions as frozen tensor modes which become dynamical when their wavelength becomes subhorizon. Note that the nonregular modes could still be interesting for cosmological models with a homogeneous (or very large scale) anisotropic stress which would then source the right-hand side of (6.1), since in this case one could set vanishing initial conditions and still have observational signatures (see, e.g, [38]).

B. CMB

1. Boltzmann hierarchy

We define CMB multipoles exactly like in Sec. 7.2 of [18] (which is also the notation used in [33,34]). Moreover, we use the notation of Sec. 7.3 of [18] for gravitational and collisional terms. When using pseudo plane waves (that is with $\zeta_\ell^m \neq$ const), as is the case for the harmonics which realize the matching with the Bianchi models, the Boltzmann hierarchy [Eqs. (7.30) of [18]], is modified by the rules

$$_s\kappa_j^m \to {}_s\kappa_j^m \frac{\zeta_j^m}{\zeta_{j-1}^m},$$

$$_s\kappa_{j+1}^m \to {}_s\kappa_{j+1}^m \frac{\zeta_j^m}{\zeta_{j+1}^m}. \qquad (6.3)$$

It is also implied that the ν which appears in the definition of $_s\kappa_\ell^m$ [Eq. (3.19) of [18]] is ν_m. The modification (6.3) arises because, for pseudo plane waves, the recursive

relations for the normal modes, e.g., Eq. (7.29) of [18], are now satisfied for the combination $_sG^{(jm)}\zeta_j^m$. Consequently, we find that the integral solutions [Eqs. (7.33) of [18]] are modified by an extra factor $\zeta_j^m/\zeta_{j'}^m$ on their right-hand sides, that is

$$\frac{\Theta_j^m(\eta_0)}{2j+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{j'} ({}^\Theta C_{j'}^m + \mathcal{G}_{j'}^m) \frac{\zeta_j^m}{\zeta_{j'}^m}\, {}_0\bar{\epsilon}_j^{(j'm)}(\chi),$$

$$\frac{E_j^m(\eta_0)}{2j+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{j'} {}^E C_{j'}^m \frac{\zeta_j^m}{\zeta_{j'}^m}\, {}_2\bar{\epsilon}_j^{(j'm)}(\chi),$$

$$\frac{B_j^m(\eta_0)}{2j+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{j'} {}^E C_{j'}^m \frac{\zeta_j^m}{\zeta_{j'}^m}\, {}_2\bar{\beta}_j^{(j'm)}(\chi), \quad (6.4)$$

with $\chi = \eta_0 - \eta$.

The line of sight integral solution (6.4) is only formal because the sources depend on the multipoles themselves and to compute them one needs to rely on the hierarchy. However in practice it is sufficient to solve the Boltzmann hierarchy for a small number of multipoles, since the sources are restricted to multipoles with $j \leq 2$, and to then use the integral solutions for all multipoles.

The case of tensor modes ($m = 2$), which are also the only regular modes, is interesting. The temperature quadrupole is sourced by an integrated Sachs-Wolfe effect (ISW) from tensor modes, and is damped by collision. When Compton scattering is inefficient, the radial functions account for the effect of free streaming, and this reveals the focusing and spiraling patterns of the Bianchi modes if present in the perturbation. Scattering of the temperature quadrupole also generates the electric quadrupole of polarization, and subsequent free streaming feeds higher multipoles of both the electric and magnetic types.

We notice on the structure of the Boltzmann hierarchy that the scalar mode ($m = 0$) of the Bianchi V and VII$_h$ cases is special. Indeed, in that case $_s\kappa_1^0\zeta_1^0/\zeta_0^0 = 0$, which implies that the temperature monopole does not feed the temperature dipole. This is expected since a homogeneous monopole cannot have a divergence. However, there is a homogeneous divergence of radiation velocity which feeds the monopole. This is because these Bianchi models are tilted and the fluid velocity (proportional to $\mathcal{P}_i^{(0)}$) is not normal to the foliation of the hypersurfaces.

2. Comparison with [13]

Our results can be compared with those of [13] using a different method. From (5.9b) [or (5.11) when $m = 0$] and the identity (E11), we get

$$_s\mathcal{F}_{\ell\ell-1}^m \equiv {}_s\kappa_\ell^m \frac{\zeta_\ell^m}{\zeta_{\ell-1}^m}$$

$$= -i[(\ell-1)/\ell_c + mi/\ell_s]\,{}_s\bar{\kappa}_\ell^m, \quad (6.5a)$$

$$_s\mathcal{F}_{\ell\ell+1}^m \equiv {}_s\kappa_{\ell+1}^m \frac{\zeta_\ell^m}{\zeta_{\ell+1}^m}$$

$$= i[(\ell+2)/\ell_c - mi/\ell_s]\,{}_s\bar{\kappa}_{\ell+1}^m, \quad (6.5b)$$

where

$$_s\bar{\kappa}_\ell^m \equiv \sqrt{\frac{(\ell^2 - s^2)(\ell^2 - m^2)}{\ell^2}}. \quad (6.6)$$

With these identities, the coupling between multipoles with neighbor values of ℓ take a simpler form, and the hierarchy has the structure

$$\partial_\eta \Theta_\ell^m = \sum_{\ell'=\ell\pm1} {}_0\mathcal{F}_{\ell\ell'}^m \Theta_{\ell'}^m + \mathcal{G}_\ell^m + {}^\Theta C_\ell^m - \tau'\Theta_\ell^m,$$

$$\partial_\eta E_\ell^m = \sum_{\ell'=\ell\pm1} {}_2\mathcal{F}_{\ell\ell'}^m E_{\ell'}^m - \mathcal{M}_\ell^m B_\ell^m + {}^E C_\ell^m - \tau' E_\ell^m,$$

$$\partial_\eta B_\ell^m = \sum_{\ell'=\ell\pm1} {}_2\mathcal{F}_{\ell\ell'}^m B_{\ell'}^m + \mathcal{M}_\ell^m E_\ell^m - \tau' B_\ell^m, \quad (6.7)$$

with

$$\mathcal{M}_\ell^m \equiv \frac{2m\nu_m}{\ell(\ell+1)} = \frac{2}{\ell(\ell+1)} \left(\frac{m^2}{\ell_s} + \frac{im}{\ell_c}\right).$$

Up to variations in conventions, the hierarchy (6.7) is the same as the one obtained in [13]. Eventually, as we want to decompose the angular dependence of the observed CMB directly on spin-weighted spherical harmonic (without numerical prefactors) the CMB multipoles are related to the previous ones by (7.35) of [18], that is

$$^{\mathrm{CMB}}\Theta_\ell^m(\eta_0) = \Theta_\ell^m(\eta_0)(-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}}. \quad (6.8)$$

For a complete comparison with [13] we also then need a parity inversion which brings factors of $(-1)^\ell$ [respectively $(-1)^{\ell+1}$] in the temperature and electric type multipoles (respectively the magnetic type multipoles) since those authors use multipoles defined with respect to the observed direction.

The difference between our method based on FLRW perturbations, and the method of [13] based on Bianchi spaces directly, is manifest for the Bianchi VII$_h$ and VII$_0$. The invariant basis of these models are related to the invariant basis of the associated FLRW through (4.11) or (4.7). In the method of [13], one works fully in the basis $e_i^{\mathrm{VII}_0}$ or $e_i^{\mathrm{VII}_h}$. Hence the CMB sources are very simple at emission, as they are proportional to the matrices (4.23), but higher order multipoles are populated thanks to the nontrivial propagation of light in this basis. In our point of view, we work in the underlying FLRW basis e_i^{I} (e_i^{V}) when dealing with the case VII$_0$ (VII$_h$). Hence the sources appear

to have some large scale spiraling structure, but propagation is trivial as the direction of propagating photons is constant. To make it short, either the sources are simple but light propagation nontrivial as in [13], or sources are nontrivial but light propagation is simple, eventually leading to the same Boltzmann hierarchy. All these differences disappear for the Bianchi I and IX cases (see Sec. VI B 3) given their very simple structures.

3. Special case of Bianchi I and IX

The case of Bianchi IX is much simpler than other models since we can see that there is no coupling to $\ell = 1$ nor $\ell = 3$. Of course we have $m = 2$ so we need at least $\ell \geq 2$ but we need also $\ell \leq 2$ because $\nu = 3/\ell_c$. So we have a set of equations for the multipoles with $\ell = 2$ only which is

$$\partial_\eta \Theta_2^{\pm 2} = \mathcal{G}_2^{\pm 2} + {}^\Theta\mathcal{C}_2^{\pm 2} - \tau'\Theta_2^{\pm 2},$$
$$\partial_\eta E_2^{\pm 2} = -2\ell_c^{-1} B_2^{\pm 2} + {}^E\mathcal{C}_2^{\pm 2} - \tau' E_2^{\pm 2},$$
$$\partial_\eta B_2^{\pm 2} = 2\ell_c^{-1} E_2^{\pm 2} - \tau' B_2^{\pm 2}. \tag{6.9}$$

The radial functions needed for an integral solution are simply

$$_0\bar{\epsilon}_2^{(2,2)} = \frac{1}{5}, \tag{6.10a}$$

$$_2\bar{\epsilon}_2^{(2,2)} = \frac{1}{5}\cos(2\chi/\ell_c), \tag{6.10b}$$

$$_2\bar{\beta}_2^{(2,2)} = \frac{1}{5}\sin(2\chi/\ell_c). \tag{6.10c}$$

Physically, polarization is rotated with respect to the invariant basis, and a quadrupole in E (generated from scattering out of the temperature quadrupole) converts to a quadrupole in B and back [14].

The Bianchi I case is even simpler, and from the limit $\ell_c \to \infty$, $\ell_s \to \infty$ in (6.5), we also check that there are no couplings to $\ell \pm 1$, and magnetic type multipoles are not fed from free streaming of electric type multipoles. Hence the system of equations is simply

$$\partial_\eta \Theta_2^{\pm m} = \mathcal{G}_2^{\pm m} + {}^\Theta\mathcal{C}_2^{\pm m} - \tau'\Theta_2^{\pm m},$$
$$\partial_\eta E_2^{\pm m} = {}^E\mathcal{C}_2^{\pm m} - \tau' E_2^{\pm m}. \tag{6.11}$$

Also the radial functions needed for the integral solutions (6.4) are pure constants (1/5 for $_s\bar{\epsilon}_2^{(2m)}$) as seen from the limit $\ell_c \to \infty$ in (6.10), in agreement with the simple structure of the previous system which is trivially integrated on η.

C. General cosmological observables

All cosmological observables (weak lensing convergence or shear, lensing field, galaxy number counts, redshift drifts, etc.) are of the form of an integral on the background past light cone. However, when considering the effect of a Bianchi perturbation, one must take into account the fact that it corresponds to a pseudo plane wave, and one must consider the effect of the weights ζ_ℓ^m. Quite similarly to the integral solution (6.4) for CMB, one finds that the general solutions for the multipoles of cosmological observables, when rephrased in terms of integrals on sources multiplied by radial functions [Eq. (7.40) in [18]], need only be modified by

$$_s\alpha_j^{(j'm)} \to \frac{\zeta_j^m}{\zeta_{j'}^m} {}_s\alpha_j^{(j'm)}. \tag{6.12}$$

Hence from the knowledge of the radial functions, which must be computed from analytic continuation given that ν is complex for some Bianchi perturbations, it is immediate that we obtain theoretical predictions for all observables using the same framework as the one for stochastic linear perturbations.

In general, since we use the zenith direction for the reference axis, one needs to allow for a general orientation of that Bianchi special direction, as discussed in Sec. V B. Rotations can be performed directly at the level of the computed observables, that is rotating the angular multipoles of observables. Similarly for models which are not invariant by a global parity transformation (VII$_0$, VII$_h$ and IX), we can perform the parity transformation at the level of multipoles. For even type multipoles (e.g., temperature or electric type polarization), this is a factor $(-1)^\ell$ and for magnetic type ones a factor $(-1)^{\ell+1}$.

Finally, when computing multipoles, one can restrict the computation to $m \geq 0$ since the negative m are constrained by the fact that the metric perturbation must be real [relations (4.25) and (4.26)]. The CMB multipoles satisfy $X_\ell^{-m} = (-1)^m X_\ell^{m\star}$, with $X = \Theta$, E, B, and one encounters the same relations for all cosmological observable multipoles.

VII. CONCLUSION

Bianchi models with isotropic limit, namely, models I, VII$_0$, VII$_h$, V and IX, are not alternative cosmologies. Rather, they are natural manifestations of linear and homogeneous cosmological perturbations in FLRW universes. The exact correspondence between nearly isotropic Bianchi and perturbed FLRW models allows for the computation of all angular multipoles of cosmological observables within the same FLRW framework. The modes ν_m required for the exact correspondence are summarized in Table IV, and our main results can be summarized as follows:

(i) For models I, IX the dynamics involves only $\ell = 2$ multipoles, hence the Boltzmann hierarchy is not really a hierarchy as it reduces to (6.9) and (6.11). Nonetheless one could use existing tools [39,40] for solving the Einstein-Boltzmann set of equations, even though that would amount to use a sledge hammer to kill a fly.

(ii) For the model VII_0, the ν_m are real, and it corresponds to the effect of standing circularly polarized waves. Once again, existing tools are readily usable in that case.

(iii) However, since the correspondence of Bianchi models V and VII_h with perturbed FLRW is through supercurvature modes, that is with complex modes ν_m, one must rely on an analytic continuation of the radial functions needed in the expressions of the normal modes (3.28). In addition, the correspondence is not with usual plane waves, but with pseudo plane waves, which are specified by the weights ζ_ℓ^m in the sum (3.34). This requires one to modify the usual Boltzmann hierarchy of [34] with the rules (6.3). It also modifies the integral solutions for any cosmological observable, but this is simply equivalent to the redefinition (6.12) for radial functions, as seen explicitly on the CMB case in Eqs. (6.4).

The power of the this approach is that one could compute the nonstochastic part due to a Bianchi type perturbation with all the sophistication of the usual linear perturbation theory around FLRW spacetimes. For instance, that would allow one to include consistently the anisotropic stress of photons and neutrinos that should normally enter in the right = -hand side of (6.1). This method would avoid the need to split numerical codes between a part dedicated to Bianchi related effects, and another for stochastic perturbations, as done in [16,17]. Furthermore, using the integral solutions to the CMB multipoles, that is the line of sight method of [33,34], would fasten the computations. Indeed, exactly like it has allowed one to solve the Boltzmann hierarchy of the stochastic component for a limited number of source multipoles, the integral solution method would also allow one to keep only a small number of multipoles when solving for the sources instead of the full hierarchy (typically $\ell_{\max} = 1000$ for the analysis of [16]). More importantly, this allows the possibility of computing consistently the multipoles of various cosmological observables for the same linear Bianchi perturbation, opening the possibility of joint constraints on global anisotropy.

APPENDIX A: 1+3 SPLITTING OF EINSTEIN EQUATIONS

The Gauss-Codazzi relation is [27]

$$R_{\mu\nu\lambda\sigma} = {}^{(3)}R_{\mu\nu\lambda\sigma} + 2K_{\mu[\lambda}K_{\sigma]\nu}$$
$$- 4(D_{[\lambda}K_{\sigma][\mu})e_{\nu]} + 4e_{[\mu}K_{\nu]}{}^{\rho}K_{\rho[\lambda}e_{\sigma]}$$
$$+ 4e_{[\mu}\dot{K}_{\nu][\lambda}e_{\sigma]} - 4(D_{[\mu}K_{\nu][\lambda})e_{\sigma]}, \qquad (A1)$$

where a dot derivative stands for $e^\mu\nabla_\mu$.

For a homogeneous projected tensor, one has [28]

$$D_k T_{i_1\cdots i_p} = -\sum_{j=1}^{p} \Gamma^l{}_{ki_j} T_{i_1\ldots i_{j-1}li_{j+1}\ldots i_p}. \qquad (A2)$$

Hence, using the definition of the Riemann tensor from the commutation of two such derivatives, we infer after using (2.31), the Riemann tensor associated with the spatial metric

$$^{(3)}R_{ij}{}^{kl} = -\frac{1}{2}C^p{}_{ij}C_p{}^{kl} + \frac{1}{2}C_p{}^l{}_iC^{pk}{}_j + C_p{}^l{}_jC_i{}^{kp}$$
$$+ C_p{}^l{}_jC^k{}_i{}^p + C_{ijp}C^{pkl} + \frac{1}{2}C_i{}^l{}_pC_j{}^{kp}$$
$$+ \frac{1}{2}C^l{}_{ip}C^k{}_j{}^p + C^k{}_{jp}C_i{}^{lp}. \qquad (A3)$$

The three-Ricci tensor and three-Ricci scalar can then be deduced, and we obtain:

$$^{(3)}R_{ij} = -\frac{1}{2}C_{kil}C^k{}_j{}^l - \frac{1}{2}C_{kil}C^l{}_j{}^k$$
$$+ \frac{1}{4}C_i{}^{kl}C_{jkl} + C_{(ij)}{}^pC^k{}_{pk}, \qquad (A4)$$

$$^{(3)}R = -\frac{1}{4}C_{ijk}C^{ijk} - \frac{1}{2}C_{ijk}C^{jik} + C^{kj}{}_kC^p{}_{pj}. \qquad (A5)$$

Whenever the placement of indices on the constants of structure is not $C^i{}_{jk}$, it implies that indices are either lowered by h_{ij} or raised by h^{ij}. In particular using the general decomposition (2.6) the Ricci and Ricci scalar are given

$$^{(3)}R_{ij} = h_{ij}\left[-2A_kA^k - N_{kl}N^{kl} + \frac{1}{2}(N_k^k)^2\right]$$
$$+ 2N_i{}^kN_{kj} - N_{ij}N_k^k - 2\epsilon_{kl(j}N_{i)}{}^lA^k, \qquad (A6)$$

$$^{(3)}R = -6A_iA^i - N_{ij}N^{ij} + \frac{1}{2}(N_i^i)^2. \qquad (A7)$$

Note that the traceless part of the Ricci scalar is

$$^{(3)}R_{\langle ij\rangle} = -N_{\langle ij\rangle}N_k^k + 2N_{\langle i}{}^kN_{j\rangle k} - 2\epsilon_{kl(j}N_{i)}{}^lA^k, \qquad (A8)$$

and it vanishes whenever $N^{ij} = 0$.

From (A2) we find that for spatial (projected) symmetric trace-free tensors $T^{\mu_1\ldots\mu_n}$

$$D^k T_{ki_1\ldots i_n} = -(n+2)A^k T_{ki_1\ldots i_n} - n\epsilon^j{}_{l\langle i_1} T_{i_2\ldots i_n\rangle kj}N^{lk},$$
(A9a)

$$\text{curl}T_{i_1\ldots i_n} = -A^j \epsilon_{jk\langle i_1} T_{i_2\ldots i_n\rangle}{}^k - \frac{(n-1)}{2}N^k_k T_{i_1\ldots i_n}$$
$$+ (2n-1)N^k_{\langle i_1} T_{i_2\ldots i_n\rangle k},$$
(A9b)

$$D_{\langle j}T_{i_1\ldots i_n\rangle} = nA_{\langle j}T_{i_1\ldots i_n\rangle} - n\epsilon_{lk\langle i_1}T^l{}_{i_2\ldots i_n}N^k_{j\rangle},$$
(A9c)

where we have introduced the curl in curved space

$$\text{curl}T_{ij\ldots k} = \epsilon_{rs(i}D^r T_{j\ldots k)}{}^s.$$
(A10)

In particular for a homogeneous projector vector V^μ, and a symmetric trace-free homogeneous projected tensor $T^{\mu\nu}$

$$D_j V^j = -2A_i V^i,$$
(A11)

$$D_j T^j_i = -3A_j T^j_i - \epsilon_{ijk}T^j_p N^{pk}.$$
(A12)

This last relation can be used for spatial derivatives on extrinsic curvature (on separating its trace part) in (A1). Furthermore, the Gauss-Codazzi relation is used in practice by also converting the dot derivative to Lie derivative on the extrinsic curvature. In general, the dot derivative of a homogeneous tensor is transformed to a Lie derivative using

$$\mathcal{L}_e T_{i_1\ldots i_p} = \dot{T}_{i_1\ldots i_p} + \sum_{i=1}^{p} K^j{}_{i_i} T_{i_1\ldots i_{i-1}ji_{i+1}\ldots i_p}.$$
(A13)

We need quite often the relations

$$\mathcal{L}_e h_{ij} = \frac{2}{3}\theta h_{ij} + 2\sigma_{ij},$$
(A14a)

$$\mathcal{L}_e h^{ij} = -\frac{2}{3}\theta h^{ij} - 2\sigma^{ij},$$
(A14b)

$$\mathcal{L}_e \sigma_{ij} = \dot{\sigma}_{ij} + \frac{2}{3}\theta\sigma_{ij} + 2\sigma_{ikc}\sigma^k{}_j,$$
(A14c)

$$\mathcal{L}_e \sigma^i_j = \dot{\sigma}^i_j,$$
(A14d)

$$\mathcal{L}_e \sigma^{ij} = \dot{\sigma}^{ij} - \frac{2}{3}\theta\sigma^{ij} - 2\sigma^{ik}\sigma_k{}^j.$$
(A14e)

A contraction of the Gauss-Codazzi relation leads to the Raychaudhuri equation

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \sigma^2 - R_{\mu\nu}e^\mu e^\nu, \quad \text{with} \quad \sigma^2 \equiv \sigma_{ij}\sigma^{ij}$$
(A15)

The general Friedmann equation (constraint) is another contraction of the Gauss-Codazzi relation and is

$$^{(3)}R = -\frac{2}{3}\theta^2 + \sigma^2 + 2G_{\mu\nu}e^\mu e^\nu.$$
(A16)

The tilt constraint is found for yet another contraction leading to

$$h^k_i e^\mu G_{\mu k} = D_j K^j_i = -3A^k\sigma_{kj} - \epsilon_{jkp}\sigma^{kq}N^p_q.$$
(A17)

And finally for the shear evolution we get

$$\dot{\sigma}_{ij} + \theta\sigma_{ij} = N_{\langle ij\rangle}N^k_k - 2N^k_{\langle i}N_{j\rangle k} + 2A^k\epsilon_{kp\langle i}N^p_{j\rangle} + G_{\langle ij\rangle}.$$
(A18)

It can be easily recast with a Lie derivative using Eqs. (A14).

APPENDIX B: OPEN CASE COORDINATES AND BASIS

Here we give expressions for the KVFs and invariant basis of the Bianchi V and VII_h model using the (x, y, z) coordinates of Ref. [36]. These are related to the spherical hyperbolic coordinates in (4.8) by

$$x = \ell_c \exp(+z/\ell_c)\sinh(\chi/\ell_c)\sin\theta\cos\phi,$$
$$y = \ell_c \exp(+z/\ell_c)\sinh(\chi/\ell_c)\sin\theta\sin\phi,$$
$$z = -\ell_c \ln[\cosh(\chi/\ell_c) - \sinh(\chi/\ell_c)\cos\theta].$$
(B1)

In terms of these variables, the metric of the open space becomes

$$ds^2 = dz^2 + \exp(-2z/\ell_c)(dx^2 + dy^2).$$
(B2)

Because these variables are more adapted to the symmetries of the Bianchi V models, the KVFs and invariant basis simplify considerably. The KVFs are

$$\boldsymbol{\xi}^V_x = \partial_x,$$
$$\boldsymbol{\xi}^V_y = \partial_y,$$
$$\boldsymbol{\xi}^V_z = (x/\ell_c)\partial_x + (y/\ell_c)\partial_y + \partial_z.$$
(B3)

These solve (2.9) with $N^i = 0$, as one can check. The corresponding invariant basis is given by

$$e^V_x = \exp(z/\ell_c)\partial_x,$$
$$e^V_y = \exp(z/\ell_c)\partial_y,$$
$$e^V_z = \partial_z.$$
(B4)

As for the VII_h case, the KVFs are

$$\xi_1^{\mathrm{VII}_h} = \partial_x$$

$$\xi_2^{\mathrm{VII}_h} = \partial_y$$

$$\xi_3^{\mathrm{VII}_h} = \left(\frac{x}{\ell_c} + \frac{y}{\ell_s}\right)\partial_x + \left(\frac{y}{\ell_c} - \frac{x}{\ell_s}\right)\partial_y + \partial_z, \qquad \text{(B5)}$$

whereas the invariant basis is trivially found from (B4) and (4.11).

APPENDIX C: INVARIANT COBASIS

In any case, denoting $\mu, \nu \ldots$ components in basis or cobasis associated with coordinates $(x, y, z$ or $\chi, \theta, \phi)$, the invariant cobasis of a given Bianchi type is related to the invariant basis through

$$e^i{}_\mu = g^{ij} g_{\mu\nu} e_j{}^\nu. \qquad \text{(C1)}$$

Here g^{ij} are the components of the inverse metric in the invariant cobasis, and $g_{\mu\nu}$ the components of the metric in the coordinate basis. For Bianchi types I, V, IX, by construction $g^{ij} = \delta^{ij}$. Hence finding the cobasis from the basis is straightforward. For instance from the invariant basis in spherical coordinates in the Bianchi V case (B4), one infers trivially the invariant cobasis. For the Bianchi VII_h case, since the invariant basis is related to the one of V by (4.11), finding the invariant cobasis of VII_h from the one of V is also straightforward from (4.7).

APPENDIX D: MATCHING EQUATIONS (FINDING THE ν_m)

In this section we gather all the details of the identification between FLRW and Bianchi models at the level of equations, allowing to determine which modes ν_m are needed to realize the matching (5.2).

1. Tensors

The matching between tensor modes is the easiest to make. For that we just need to look for tensors such that the definition

$$E_{ij}^{(\pm 2)} \equiv \beta_{(\pm 2)} q_{ij}^{(\pm 2)} \qquad \text{(D1)}$$

holds. The two main equations in this case are Eqs. (3.17) and (4.19b). Their direct comparison leads to

$$\mathcal{S}^{(\pm 2)} = -\nu_{\pm 2}^2 + K, \qquad \text{(D2)}$$

where we have used (3.27) with $j = |m| = 2$. Using the results of Table III, it is now straightforward to find the mode $\nu_{\pm 2}$ which connects these two equations. For flat ($K = 0$) models we find

$$\nu_{\pm 2} = \begin{cases} 0, & \text{(I)}, \\ \pm\frac{2}{\ell_s}, & \text{(VII}_0). \end{cases} \qquad \text{(D3)}$$

For open models, $K = -\ell_c^{-2}$ and we find

$$\nu_{\pm 2} = \begin{cases} \frac{\mathrm{i}}{\ell_c}, & \text{(V)}, \\ \pm\frac{2}{\ell_s} + \frac{\mathrm{i}}{\ell_c}, & \text{(VII}_h). \end{cases} \qquad \text{(D4)}$$

Note that we can obtain models I and V as limits of models VII_0 and VII_h, respectively, by taking ℓ_s to infinity while keeping ℓ_c fixed [see Eq. (2.11)].

For the closed model ($K = \ell_c^{-2}$) we have

$$\nu_{\pm 2} = \pm\frac{3}{\ell_c} \quad \text{(IX)} \qquad \text{(D5)}$$

or $|k| = \sqrt{6}\ell_c^{-1}$. One can show that this corresponds to a tensor wave whose length equals one-half the curvature radius of a closed universe [2]. Finally, we note that in all cases the momentum constraint $\mathcal{P}_i^{(m)} = 0$ is obvious, since tensor perturbations do not induce momentum.

2. Vectors

In principle the matching of the vector modes follows similarly, that is, we introduce modes $E_i^{(\pm 1)}$ satisfying

$$\mathcal{D}_{(i} E_{j)}^{(\pm 1)} \equiv \beta_{(\pm 1)} q_{ij}^{(\pm 1)} \qquad \text{(D6)}$$

and then use it to compare Eqs. (3.18a) and (4.19b), from which we deduce the $\nu_{\pm 1}$ with the help of (3.27). But since the former does not contain a Laplacian term, and the latter has $\mathcal{S}^{ij} = 0$ for flat and open models (the closed case is discussed below), this comparison will lead us nowhere.

We can instead find $\nu_{\pm 1}$ by comparing the constraint equation (3.18a) with the tilt (2.49), provided we have an explicit solution to Eq. (D6). Such solution can be constructed by writing $E_i^{(\pm 1)}$ as a linear combination of the invariant basis and using Eq. (A2) to fix the coefficients. Since $\mathcal{D}_{(i} e_{j)} = 0$ in model I (see Sec. IV A), nontrivial solutions can only be constructed in models VII_0, VII_h, and V. As it turns out, a solution to these models can be written jointly as [3]

$$E_i^{(\pm 1)} = \pm\ell_s \beta_{(\pm 1)} \left(\frac{\ell_s/\ell_c \mp \mathrm{i}}{1 + (\ell_s/\ell_c)^2}\right)(e_{\mathrm{VII}_h}^{(\mp)})_i. \qquad \text{(D7)}$$

One can easily check that model VII_0 is recovered when $\ell_c \to \infty$ (and thus $h \to 0$), and model V in the limit $\ell_s \to \infty$ (and thus $h \to \infty$).

If we now use (4.31) to decompose the right-hand side of (3.18a) and use (3.27) with $j = |m| = 1$, we find

$$\beta'_{(\pm 1)}\mathcal{P}_i^{(\pm 1)} = \left(\frac{(\nu_{\pm 1})^2}{2} - 2K\right)(E_i^{(\pm 1)})'. \quad (D8)$$

This, together with (D7) and the results of Table III, allows us to find the $\nu_{\pm 1}$ for each model. For flat models this gives

$$\nu_{\pm 1} = \begin{cases} 0, & (I), \\ \pm\frac{1}{\ell_s}, & (VII_0). \end{cases} \quad (D9)$$

For open models, on the other hand, we find

$$\nu_{\pm 1} = \begin{cases} \frac{i}{\ell_c}, & (V), \\ \pm\frac{1}{\ell_s} + \frac{i}{\ell_c}, & (VII_h). \end{cases} \quad (D10)$$

From (D8), one notices that models VII_0, V, VII_h are titled, and consequently the fluid velocity has some homogeneous vorticity. The form of this vorticity is deduced using

$$\text{curl}\mathcal{P}_i^{(\pm 1)} = \pm\nu_{\pm 1}\mathcal{P}_i^{(\pm 1)}, \quad (D11)$$

which is consistent with (E4).

Finally, since Bianchi IX have compact spatial sections, the splitting into *svt* modes is unique [2,3]. Thus, all modes of the shear map uniquely to tensor perturbations of FLRW spacetimes, and there are no vector perturbations (see also Appendix E).

3. Scalars

Since the shear is traceless, the correspondence with metric perturbations assumes that $\phi = 0$ in (3.5). Following the same logic as for tensor and vector modes, we now look for modes ψ such that

$$\mathcal{D}_{ij}\psi \equiv \beta_{(0)}q_{ij}^{(0)}. \quad (D12)$$

The relevant equations to compare are Eqs. (3.15) and (4.19b). Using (3.27) for $j - 2 = |m| = 0$, we arrive at

$$\mathcal{S}^{(0)} = \frac{1}{3}[(\nu_0)^2 - K]. \quad (D13)$$

From Table III we see that all open and flat models have $\mathcal{S}^{(0)} = 0$. Thus

$$\nu_0 = \begin{cases} 0 & (I, VII_0), \\ \frac{i}{\ell_c} & (V, VII_h). \end{cases} \quad (D14)$$

For the open models, ν_0 corresponds to the maximal supercurvature mode.

We stress that one will run into difficulties when finding explicitly the harmonic which leads to (D12). Let us consider first the flat case, corresponding to Bianchi types I or VII_0, and for which $k = \nu_0 = 0$. One finds immediately

that $\mathcal{D}^j\mathcal{D}_{ij}\psi = 0$. Hence, the moment constraint (3.18) which reads

$$\begin{aligned} P_i &= -\mathcal{D}^j\mathcal{D}_{ij}\psi' \\ &= -\frac{2}{3}(\Delta + K)\mathcal{D}_i\psi' \end{aligned} \quad (D15)$$

is satisfied. However, this also implies that if (D12) is a harmonic with $j = 2, m = 0$, it cannot be deduced from the STF derivative of a harmonic with $j = 1, m = 0$, as in (2.22) of [18]. The best construction we can find consists in defining

$$\psi(\boldsymbol{x}) = \frac{\beta_{(0)}}{k^2}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad k_i = k(e_{VII_0}^3)_i \quad (D16)$$

from which we can build $\mathcal{D}_{ij}\psi$, and eventually we take the limit $k \to 0$.

For open models, the situation is slightly improved. Indeed, we can define a vector field

$$W_i \equiv -\ell_c\beta_{(0)}(e_{VII_h}^3)_i, \quad (D17)$$

such that

$$\mathcal{D}_{\langle i}W_{j\rangle} = \beta_{(0)}q_{ij}^{(0)}. \quad (D18)$$

One can check by direct covariant differentiation [using Eq. (A2)] that

$$\mathcal{D}^iW_i = 2\beta_{(0)}, \qquad \Delta W_i = -2\ell_c^{-2}W_i. \quad (D19)$$

Since $(\nu_0)^2 = -1/\ell_c^2 = K$ for open models, it follows from the second equation that W_i is a solution of (3.27) for $j - 1 = m = 0$. That is, W_i is a harmonic vector.

Using (D17) in (D15), we obtain

$$\mathcal{P}_i^{(0)} = -\frac{2}{3}[(\nu_0)^2 - 4K]\ell_c(e_{VII_h}^3)_i \quad (D20)$$

where we have once again invoked (3.27), this time for $j - 1 = |m| = 0$. We check that the above expression is satisfied for $\nu_0 = i/\ell_c$, in agreement with (D14). Note that the vector harmonic W_i cannot be deduced from a scalar harmonic using (2.22) of [18] as is usually the case. Indeed, using $\nu_0 = i/\ell_c$ (which corresponds to $k = 0$) in (A.3) of [18], one obtains rather immediately that the scalar harmonics $j = 0, m = 0$ for that mode is a pure constant, and hence any derivative of it vanishes. But contrary to the flat case, one needs not consider a scalar harmonics with $k > 0$ and then form the vector $(j = 1, m = 0)$ and tensor $(j = 2, m = 0)$ harmonics before considering the limit $k \to 0$, since we can start our construction directly from (D17) using (D18).

4. Consistency checks

As a final consistency check, we verify whether the curvature perturbation and the fluid conservation equations are consistently matched with their Bianchi counterparts in the homogeneous limit found in the last sections. Starting with the curvature perturbation, we recall that the identification of the scalar mode assumes $\phi = 0$. It then follows from (3.10) that

$$a^2\delta(^{(3)}R) = \frac{4}{3}\Delta(\Delta + 3K)\psi,$$
$$= 2\mathcal{D}^i\mathcal{D}^j\mathcal{D}_{\langle i}W_{j\rangle}, \tag{D21}$$

where, in going from the first to the second line, we have used the identity

$$\frac{2}{3}\Delta(\Delta + 3K)\psi = \mathcal{D}^i\mathcal{D}^j\mathcal{D}_{ij}\psi,$$
$$= \mathcal{D}^i\mathcal{D}^j\mathcal{D}_{\langle i}W_{j\rangle}.$$

For flat models, $K = 0 = \delta(^{(3)}R)$, which gives $k = \nu_0 = 0$, in agreement with our previous findings. For open models, the second line of (D21) can be rewritten as

$$a^2\delta(^{(3)}R) = -\frac{4}{3}[(\nu_0)^2 - 4K]\mathcal{D}^iW_i,$$
$$= -\frac{8}{3}[(\nu_0)^2 - 4K]\beta_{(0)}, \tag{D22}$$

where we have also used (D19). Using $\nu_0 = i/\ell_c$ from (D14) and the definition (4.29), we find that $\ell_c^2\mathcal{R}^{(0)} = -8$, in accordance with Table III.

As for the fluid conservation equations, the matching follows straightforwardly. Indeed, Eqs. (3.19) with $\partial_i\delta p \to 0$ and (4.19e) are formally the same, whereas Eqs. (3.20) and (4.19d) are formally matched for $\phi = 0$.

APPENDIX E: FINDING THE ζ_ℓ^m

We give here the technical details of the method used in computing the constants ζ_ℓ^m needed in the matching relation (5.2).

1. Method

Before getting into the details, we first note that relations (4.5) and (4.11) allow us to unify the description of the method using only the invariant basis of models I, V, IX. Hence, let us define

$$\tilde{q}_{ij}^{(m)} \equiv q_{kl}^{(m)}\tilde{M}^k{}_i\tilde{M}^l{}_j \tag{E1}$$

with $\tilde{M}^i{}_j = \delta^i_j$ for models I, V and IX and $\tilde{M}^i{}_j = M^i{}_j$ for models VII$_0$ and VII$_h$, where M was introduced in (4.6). In all cases, $\tilde{q}_{ij}^{(0)} = q_{ij}^{(0)}$. In this appendix, indices $i, j, \ldots$

belong to the invariant cobasis of models I, V, IX, which are in turn associated respectively with the flat, open and closed FLRW. For instance, we have

$$q_{ij}^{(m)}e_{\text{VII}_h}^i \otimes e_{\text{VII}_h}^i = \tilde{q}_{ij}^{(m)}e_{\text{V}}^i \otimes e_{\text{V}}^j. \tag{E2}$$

From (E1) we find the convenient property

$$\partial_z\tilde{q}_{ij}^{(m)} = \frac{im}{\ell_s}\tilde{q}_{ij}^{(m)}. \tag{E3}$$

Hence when computing spatial derivatives (D_i) of the $q_{ij}^{(m)}$ in the Bianchi VII$_h$ case, it is convenient to use the right-hand side of (E2). Indeed, the constants of structure associated with the type V invariant basis have $N^{ij} = 0$ and we need only the first lines of Eqs. (A9), while the z dependence of the components $\tilde{q}_{ij}^{(m)}$ is handled simply via (E3). Said differently, we avoid the complication of the terms involving the N^{ij}, as their effect is equivalent to the simple relations (E3). A similar and even simpler method applies for the type VII$_0$ which is related to the type I exactly as in (E2). Since the invariant basis associated with the I type has vanishing constants of structure, we must only consider the derivative of the components (E3). Hereafter we used this method extensively.

Relation (5.2) can now be proven by proving that the relation and all its derivatives (the covariant derivatives $\mathcal{D}_i$ associated with the background metric) hold at $\chi = 0$. Checking that the relation (5.2) holds at $\chi = 0$ is trivial as we chose the $q_{ij}^{(m)}$ precisely on that property. Checking that the derivatives of (5.2) are equal at $\chi = 0$ is less obvious. However, instead of proving that all derivatives are equal at $\chi = 0$, given the general decomposition of a derivative [e.g., Eq. (3.22) of [18]] it is equivalent to check only the equality of STF combinations of derivatives at the origin, in addition to showing that the curl and the successive divergences associated with $\mathcal{D}_i$ (see [18] for definition) of both sides of (5.2) at $\chi = 0$ are equal.

2. Types I, VII$_0$, V, VII$_h$

Let us consider first the I, VII$_0$, V, VII$_h$ cases. We first check that the curl of an harmonic is also an harmonic [see Refs. [18,41] and Eq. (3.16) of [18]], and similarly for plane waves, since

$$\text{curl}^\ell Q_{I_j}^{(jm)}(\nu) = \frac{m\nu}{j} \times {}^\ell Q_{I_j}^{(jm)}(\nu), \tag{E4}$$

where the curl is defined as in (A10), but with D_i replaced by $\mathcal{D}_i$. Hence the same relation is satisfied by pseudo plane waves. Considering the fact that the derivative $\mathcal{D}_i$ associated with the FLRW metric is found from the spatial derivative D_i of the Bianchi metric evaluated at lowest order in β_{ij} (meaning in practice that we can lower and raise

indices of A_i and N^{ij} with δ_{ij} and δ^{ij}), we can use (A9c) to obtain

$$\mathrm{curl} q_{ij}^{(m)} = \frac{m\nu_m}{2} \times \tilde{q}_{ij}^{(m)}, \tag{E5}$$

with the value (5.7) for ν_m. Hence we get

$$\mathrm{curl}\tilde{q}_{ij}^{(m)}\bigg|_{\chi=0} = \frac{\xi_2}{\xi_m} \mathrm{curl} Q_{ij}^{(2m)}\bigg|_{\chi=0}. \tag{E6}$$

Next, we must also check that

$$\mathcal{D}_{\langle i_1}...\mathcal{D}_{i_n}\tilde{q}_{jk\rangle}^{(m)}\bigg|_{\chi=0} = \frac{\xi_2}{\xi_m}\mathcal{D}_{\langle i_1}...\mathcal{D}_{i_n}Q_{jk\rangle}^{(2m)}\bigg|_{\chi=0}. \tag{E7}$$

The right-hand side is evaluated using the very definition of derived harmonics [18] (but modified by the fact that we consider pseudo plane waves)

$$\mathcal{D}_{\langle i_1}...\mathcal{D}_{i_{\ell-2}}Q_{i_{\ell-1}i_\ell\rangle}^{(2m)} = k^{\ell-2}\frac{\zeta_\ell^m}{\zeta_2^m}Q_{I_\ell}^{(\ell m)}, \tag{E8}$$

together with the normalization at origin. More specifically, at $\chi = 0$, only the term $\ell = j$ in the sum (3.34) contributes, and we then use either Eq. (2.39) or Eq. (B.30) of [18]. We then find for the right-hand side of (E7)

$$(\mathrm{i}k)^{\ell-2}\frac{\xi_2\zeta_\ell^m}{\xi_\ell\zeta_2^m}e_{\langle i_1}^z...e_{i_{\ell-2}}^z\tilde{q}_{i_{\ell-1}i_\ell\rangle}^{(m)}\bigg|_{\chi=0}.$$

The left-hand side of (E7) is found by induction using the method detailed in Sec. E 1, and we find

$$\mathcal{D}_{\langle i_1}...\mathcal{D}_{i_{\ell-2}}\tilde{q}_{i_{\ell-1}i_\ell\rangle}^{(m)}\bigg|_{\chi=0}$$

$$= \prod_{p=3}^{\ell}\left(\frac{p-1}{\ell_c} + \frac{m\mathrm{i}}{\ell_s}\right)e_{\langle i_1}^z...e_{i_{\ell-2}}^z\tilde{q}_{i_{\ell-1}i_\ell\rangle}^{(m)}\bigg|_{\chi=0}. \tag{E9}$$

Comparing this with (E8) allows us to check that for $m \neq 0$, (E7) holds if

$$\zeta_\ell^m = \zeta_{\ell-1}^m(-\mathrm{i})\frac{(\ell-1) + m\mathrm{i}/\sqrt{h}}{\sqrt{\ell^2 + (\nu_m\ell_c)^2}}. \tag{E10}$$

Using the identity

$$h[\ell^2 + (\nu_m\ell_c)^2] = [(\ell-1)\sqrt{h} + m\mathrm{i}]$$
$$\times [(\ell+1)\sqrt{h} - m\mathrm{i}] \tag{E11}$$

it is recast in the condition (5.9b). However (E10) is only proven for $\ell - 1 \geq 2$ with this method. To show it is valid for $\ell - 1 = 1$ one must check that divergences are equal. For pseudo plane waves the divergence is

$$\nabla^p Q_{I_{j-1}p}^{(jm)} = -q^{(jm)}\frac{\zeta_{j-1}^m}{\zeta_j^m}Q_{I_{j-1}}^{(j-1,m)}, \tag{E12}$$

where $q^{(jm)}$ is defined in Eq. (2.26) of [18]. A direct computation then shows that

$$\mathcal{D}^j\tilde{q}_{jk}^{(m)}\bigg|_{\chi=0} = \frac{\xi_2}{\xi_m}\mathcal{D}^jQ_{jk}^{(2m)}\bigg|_{\chi=0}, \tag{E13}$$

implying that (E10) can also be used to determine ζ_2^m/ζ_1^m. Finally, for $m = 0$ the equality

$$\mathcal{D}^j\mathcal{D}^k\tilde{q}_{jk}^{(m)}\bigg|_{\chi=0} = \frac{\xi_2}{\xi_m}\mathcal{D}^j\mathcal{D}^kQ_{jk}^{(2m)}\bigg|_{\chi=0} \tag{E14}$$

implies that $\zeta_1^0/\zeta_0^0 = -\mathrm{i}\sqrt{1 + (\nu_0\ell_c)^2}/2$ and thus we find (5.11). Note that since $(\nu_0\ell_c)^2 = -1$, divergences must be handled as discussed in Sec. (5.11). The global factor freedom in the ζ_ℓ^m is fixed if we choose (5.9a) which satisfies manifestly the necessary condition (5.6).

3. Type IX

We now summarize the results of Ref. [2] so as to determine the ζ_ℓ^m in the IX model. Let us consider a homogeneous, symmetric and trace-free tensor $\boldsymbol{T}$ on the closed FLRW background

$$\boldsymbol{T} = T_{ij}\boldsymbol{e}_{\mathrm{IX}}^i \otimes \boldsymbol{e}_{\mathrm{IX}}^j. \tag{E15}$$

We can prove [using e.g., (A9c)] that it must satisfy

$$\mathcal{D}_iT_{jk} = \frac{2}{3}\epsilon_{i(j}^p\mathrm{curl}T_{k)p}, \tag{E16a}$$

$$\mathrm{curl}T_{ij} = \frac{3}{\ell_c}T_{ij}. \tag{E16b}$$

The first of these equations imply that T_{ij} is a Killing tensor, $\mathcal{D}_{(i}T_{jk)} = 0$, and in particular $\mathcal{D}^iT_{ij} = 0$. From the definition of the curl, one can also show that

$$\mathrm{curl}(\mathrm{curl}T_{ij}) = -\Delta T_{ij} + \frac{3}{\ell_c^2}T_{ij} - \frac{3}{2}\mathcal{D}_{i\langle}\mathcal{D}^kT_{j\rangle k}.$$

If we now combine these identities together we finally arrive at

$$\Delta T_{ij} = -\frac{6}{\ell_c^2}T_{ij}. \tag{E17}$$

Hence, exactly as discussed in detail in [2], we find that the homogeneous tensors on a closed FLRW are tensor harmonics with $k^2 = 6/\ell_c^2$ and hence $\nu_m^2 = (3/\ell_c)^2$. Comparison with (E4) then shows that it corresponds to

$m\nu_m = 6/\ell_c$, and thus with $|m| = 2$ the modes needed in the matching are

$$\nu_{\pm 2} = \pm \frac{3}{\ell_c}. \tag{E18}$$

Hence, all modes $q_{ij}^{(m)}$ which define the Bianchi IX as a perturbation to a closed FLRW are tensor harmonics, or linear combination of possibly rotated tensor modes, with $m = 2$ and $\nu_2 = 3/\ell_c$ (along with its complex conjugate which from Table V amounts to adding the harmonic

rotated by an angle π around the y axis so as to form a standing wave of given chirality).

Had we chosen opposite signs for the Bianchi IX constants of structure, there would be a minus sign in the right-hand side of (E16b), and thus an extra minus sign in (E18). We would find that homogeneous tensors correspond instead to (sums of rotations of) $m = -2$ harmonics with $\nu_{-2} = 3/\ell_c$ (with their complex conjugate to form standing waves). One construction is related to the other one by a global parity transformation, and corresponds to switching the KVF with the invariant basis [2,3].

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