

Curvature perturbations in the effective field theory of inflation

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We discuss the difference between various gauge-invariant quantities typically used in single-field inflation, namely synchronous ζ_s , comoving ζ_c , and unitary ζ_u curvatures. We show that conservation of ζ_c outside the horizon is quite restrictive on models as it leads to conservation of ζ_s and ζ_u , whereas the reverse does not hold. We illustrate the consequence of these differences with two inflationary models: ultra-slow-roll (USR) and braiding-ultra-slow-roll (BUSR). In USR, we show that out of the three curvatures, only ζ_s is conserved outside the horizon, and we connect this result to the concepts of a separate universe and the usage of the δN formalism. We find that even though ζ_s is conserved, there is still a mild violation of the separate universe approximation in the continuity equation. Nevertheless, the δN formalism can still be applied to calculate the primordial power spectrum of some gauge-invariant quantities such as ζ_u , although it breaks down for others such as the uniform-density curvature. In BUSR, we show that both ζ_u and ζ_s are conserved outside the horizon, but take different values. Additionally, since $\zeta_u \neq \zeta_c$, we find that the prediction for observable curvature fluctuations after inflation does not reflect ζ_c at the horizon crossing during inflation and moreover involves not just ζ_u at that epoch but also the manner in which the braiding phase ends.

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I. INTRODUCTION

Observations of the cosmic microwave background temperature anisotropies [1,2] are in excellent agreement with an early universe with primordial perturbations that are adiabatic and have a nearly scale-invariant power spectrum. Currently, the most compelling explanation for these initial perturbations is the inflationary paradigm, where the universe expanded nearly exponentially fast and matter perturbations were seeded by the vacuum fluctuations of one or more fields. The simplest scenario is described by a single scalar field that slowly rolls down a potential well toward the end of inflation, leading afterwards to reheating and the production of the Standard Model particles.

Predictions on the power spectra and bispectra of inflationary models can be obtained using different gauge-invariant variables, such as the unitary curvature ζ_u , comoving curvature ζ_c , synchronous curvature ζ_s , or uniform-density curvature ζ_ρ , among others. The usage of each one of these variables has its own advantages. For instance, unitary curvature describes the spatial curvature of the spacetime in a frame where the scalar field evolution

provides a clock that breaks temporal but preserves spatial diffeomorphism invariance. In this frame, it is easy to generically describe single-field inflationary models with the effective field theory of inflation [3], where the only explicit perturbation fields come from the spacetime metric. In addition, synchronous curvature is useful for the concept of a local background or separate universe [4], which can be used to straightforwardly estimate the effects of long-wavelength perturbations on the local universe. The separate universe concept itself is closely related to the δN technique which is often used to calculate non-Gaussianity from inflation using the e -folding of a local background [5–10]. Finally, comoving curvature and uniform-density curvature are common variables used to connect inflationary fluctuations to observables.

In typical slow-roll inflationary models found in the literature, all the previous variables mentioned above either coincide or have the same qualitative behavior, and therefore they all provide the same information and we can simply choose the most convenient one. However, in general single-field models this is not the case. In this paper, we exploit this difference and analyze one of the main features of typical inflationary models—conservation of curvature outside the sound horizon—for ζ_u , ζ_c , and ζ_s . We find that, in general, conservation of ζ_c is the most restrictive condition as it will imply conservation of both ζ_s and ζ_u , but the reverse will not hold. In addition, we

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identify sufficient conditions on inflationary models in order to have conserved curvatures given that either ζ_u or ζ_s is conserved. Throughout this paper, we focus on single-field inflation models, which usually predict adiabatic perturbations and conserved curvatures, whereas multifield models will not predict a conserved curvature in general.

In order to illustrate the difference between these three curvatures, we explicitly discuss two inflationary models. First, we consider ultra-slow-roll (USR) inflation [11], in which $\zeta_c = \zeta_u$ grows in time outside the horizon, but ζ_s is conserved. Second, we build a new model dubbed braiding ultra-slow-roll (BUSR) inflation, in which ζ_u and ζ_s are conserved (but different), whereas ζ_c grows outside the horizon.

Furthermore, we discuss some conceptual and observational consequences of having different curvatures for USR and BUSR. First, we discuss the concept of a separate universe, where superhorizon perturbations can be reabsorbed into the background equations such that the total perturbed universe still looks homogeneous and isotropic in a local Hubble-sized patch [6]. As shown in [4,12], a separate universe is valid when synchronous observers see a local approximate Friedman-Robertson-Walker (FRW) universe, which requires conservation of ζ_s outside the horizon. Here we show that USR does have ζ_s conserved but still violates a separate universe via the continuity equation. In addition, we discuss the δN formalism. This formalism is typically assumed to require the validity of a separate universe, although here we show that it can still be used in USR for obtaining observables in terms of appropriate variables, but subtleties can arise when using the formalism for other variables. In particular, we find that the standard δN prescription (see [13]) yields the correct value for ζ_u but the incorrect value for ζ_ρ in USR.

Finally, we explore the observational consequence of BUSR in detail. In general, inflationary models provide initial conditions for the matter distribution of the universe during early times, which are typically in turn provided by the value of ζ_u at the horizon crossing. This value then determines the value of $\zeta_c(t_{\text{end}})$ at the end of inflation, which is then propagated forward to radiation and matter domination outside the horizon, ultimately becoming the initial condition for structure formation. This translation between unitary and comoving curvature is straightforwardly done when both curvatures are conserved and take the same value outside the horizon. However, in BUSR we have that $\zeta_c \neq \zeta_u$ due to a nontrivial coupling called braiding, which corresponds to derivative interactions between the metric and the inflationary field. In this paper, we analyze the difference between ζ_u at the horizon crossing and $\zeta_c(t_{\text{end}})$ in BUSR. We consider a realistic scenario where braiding vanishes before the end of inflation in order to avoid spoiling the subsequent reheating process [14,15]. In this case, we will have that $\zeta_u(t_{\text{end}}) = \zeta_c(t_{\text{end}})$; however, for some wavelengths ζ_u evolves outside the

horizon, and hence $\zeta_c(t_{\text{end}})$ will not be given by ζ_u at the horizon crossing. In particular, we find that for sufficiently superhorizon perturbations at the time the braiding vanishes, ζ_u remains frozen and its value at the horizon crossing becomes $\zeta_c(t_{\text{end}})$. Meanwhile, for wavelengths that have only been outside the horizon for a few e -folds before this epoch, ζ_u evolves, which ultimately leads to a suppression of the power spectrum at the end of inflation for these and smaller scales.

This paper is structured as follows. In Sec. II we give a general fluid description for generic inflationary models, then we introduce some relevant gauge-invariant fields that are typically used for inflation, and finally we review the concept of a separate universe. In Sec. III we discuss the general relationship between the conservation of the curvature in unitary, comoving, and synchronous gauge in single-field inflationary models. In Sec. IV we give examples of inflationary models that illustrate the differences in curvatures discussed in the previous section. In Sec. V we discuss the conceptual and observational consequences of having curvatures evolving differently for the models presented in Sec. IV, including the consequence for a separate universe, the δN formalism, and the evolution of curvatures outside the horizon. Finally, in Sec. VI we summarize our results and discuss their relevance. Throughout this paper we will be using Planck units, with $c = 1$ and $8\pi G = 1$.

II. GENERAL DESCRIPTION

A. Effective fluid decomposition

Let us start by considering an inflationary model in a spatially flat cosmological FRW background, with small scalar perturbations. In this case we can write the metric as

$$ds^2 = -(1 + 2\Phi)dt^2 + 2B_{,i}dx^i dt + a(t)^2[(1 - 2\Psi)\delta_{ij} + 2E_{,ij}]dx^i dx^j, \quad (1)$$

where t corresponds to the cosmic time and x^i are the three-dimensional spatial Cartesian coordinates. Also, here subscript commas denote derivatives throughout, $a(t)$ is the background scale factor, whereas Φ , Ψ , B , and E are the four metric perturbations. We shall often refer to Ψ as the curvature perturbation in the $3 + 1$ slicing defined by the lapse perturbation Φ and the shift $B_{,i}$. All these perturbation fields depend on space and time. Even though there may be nontrivial interactions between the fields driving inflation and the metric, we can always write the equations of motion in an Einstein-like form:

$$G^\mu{}_\nu = T^\mu{}_\nu, \quad (2)$$

where $G^\mu{}_\nu$ is the Einstein tensor and $T^\mu{}_\nu$ is an effective stress tensor from all the possible inflationary fields. Similar to Eq. (1) we can decompose $T^\mu{}_\nu$ into a background and perturbative part in the following fluidlike way:

$$\begin{aligned}
 T^0_0 &= \rho + \delta\rho, \\
 T^i_0 &= -(\rho + p)\bar{g}^{ij}\partial_j(v - B), \\
 T^0_i &= (\rho + p)\partial_iv, \\
 T^i_j &= (p + \delta p)\delta^i_j - p\left(\bar{g}^{il}\partial_l\partial_j - \frac{1}{3}\delta^i_j\bar{g}^{lk}\partial_k\partial_l\right)\pi, \quad (3)
 \end{aligned}$$

where $\bar{g}_{ij} = a^2\delta_{ij}$ is the spatial background metric. Here we have retained linear terms in the perturbations when raising and lowering indices. Here, ρ and p are some effective background energy density and pressure. In addition, $\delta\rho$, v , δp , and π are the four effective fluid perturbations describing the energy density, velocity potential, pressure, and anisotropic stress, respectively.

According to the decomposition given in Eqs. (1) and (3), the background equations will be given by the (00) and trace (ij) components of Eq. (2):

$$3H^2 = \rho, \quad (4)$$

$$2\dot{H} = -(\rho + p), \quad (5)$$

where $H = \dot{a}/a$ is the Hubble rate, where dots denote derivatives with respect to t . Similarly, the perturbed equations of motion in Fourier space will be given by the (00), (0*i*), and combinations of the trace-free and trace (ij) components:

$$2\frac{k^2}{a^2}[\Psi + H(a^2\dot{E} - B)] + 6H(\dot{\Psi} + H\Phi) = -\delta\rho, \quad (6)$$

$$2\dot{\Psi} + 2H\Phi = -(\rho + p)v, \quad (7)$$

$$\Psi - \Phi + (a^2\dot{E} - B) + H(a^2\dot{E} - B) = -p\pi, \quad (8)$$

$$2\dot{\Psi} + 2H\dot{\Phi} + 4\dot{H}\Phi + 6H(\dot{\Psi} + H\Phi) = \delta p + \frac{2k^2}{3a^2}p\pi. \quad (9)$$

Here, all the perturbation fields depend on the comoving wave number k and time t , although this explicit dependence has been omitted.

B. Gauge

In single-field inflation, temporal diffeomorphism invariance is broken by the time evolution of the scalar field in the background, leaving a preferred temporal foliation called unitary slicing where the field is spatially unperturbed. The curvature perturbation generated during inflation in this slicing is the dynamical quantity that controls adiabatic fluctuations after but does not generally correspond to the curvature fluctuations seen by specific observers whose clocks are synchronized differently. To relate observables in other frames, let us consider the following change of coordinates:

$$x^\mu \rightarrow x^\mu + \xi^\mu(x), \quad (10)$$

where ξ^μ is an arbitrary infinitesimal function of x^μ . Under Eq. (10), the linear metric perturbations transform as

$$\begin{aligned}
 \Phi &\rightarrow \Phi - \dot{\xi}^0, & \Psi &\rightarrow \Psi + H\xi^0, \\
 E &\rightarrow E - \xi, & B &\rightarrow B + \xi^0 - a^2\dot{\xi}, \quad (11)
 \end{aligned}$$

where we have defined $\xi^i = \partial^i\xi$. Fluid perturbations, including the effective fluid defined in the previous section, transform as

$$\begin{aligned}
 \delta\rho &\rightarrow \delta\rho - \xi^0\dot{\rho}, & \delta p &\rightarrow \delta p - \xi^0\dot{p}, \\
 v &\rightarrow v + \xi^0, & \pi &\rightarrow \pi. \quad (12)
 \end{aligned}$$

Similarly, the linear perturbation of a scalar field such as the inflaton φ transforms as

$$\delta\varphi \rightarrow \delta\varphi - \dot{\bar{\varphi}}\xi^0, \quad (13)$$

where $\bar{\varphi}(t)$ describes the background value of the scalar field and $\delta\varphi$ is its linear perturbation.

For single-field inflationary models coupled to a massless spin-2 field metric (as in general relativity), there will be only one physical scalar degree of freedom propagating that can be taken to be the unitary curvature. It is convenient to define observables such as unitary curvature in terms of gauge-invariant combinations of the variables that are then valid in any gauge. The gauge-invariant form of the various curvature observables that we shall use below are given by

- (i) Unitary curvature: $\zeta_u = \Psi - Hv_\varphi$,
- (ii) Comoving curvature: $\zeta_c = \Psi - Hv$,
- (iii) Synchronous curvature: $\zeta_s = \Psi - Hv_m$,
- (iv) Uniform density curvature: $\zeta_{\rho_i} = \Psi + \delta\rho_i/\rho'_i$,

where the velocity potentials are as follows: v for the total effective fluid, $v_\varphi \equiv -\delta\varphi/\dot{\bar{\varphi}}$ for the scalar field, and v_m for nonrelativistic test particles that are initially at rest with respect to the background expansion. Note that, for any kind of matter component i present in the system, one could define an associated uniform-density curvature ζ_{ρ_i} , where ρ_i is the energy density of the given matter species i [as opposed to some effective energy density, as the one defined in Eq. (6)].

Here and throughout primes denote derivatives with respect to the background e -folds, that is, $' = d/d\ln a = H^{-1}d/dt$. Note that the sign of the curvature fluctuation is opposite to [16] and much of the literature.

All these definitions are constructed in such a way that they describe the spatial curvature perturbation Ψ in a given coordinate system. The unitary curvature ζ_u describes the spatial curvature as seen by observers that follow the time slicings determined by the perturbations of the scalar field, and thus see $\delta\varphi = 0$. The comoving curvature ζ_c describes the spatial curvature as seen by observers that comove with

the total effective fluid velocity and hence see $v = 0$. Analogously, ζ_s describes the curvature seen by observers that trace nonrelativistic matter and see $v_m = 0$. Finally ζ_{ρ_i} is the curvature on surfaces of spatially uniform density in some matter species i . Since ζ_{ρ_i} depends on the matter species in question and is mainly used in cases where there are multiple fields, we do not consider it further in this section. Its conservation requires a small non-adiabatic stress and velocity divergence in the component (see, e.g., [6]).

The relationships among the remaining three curvatures, for any single-field inflationary model, are given by

$$\zeta_c = \zeta_u - H(v - v_\phi), \quad (14)$$

$$\zeta_c = \zeta_s - H(v - v_m). \quad (15)$$

In addition, we also use a spatially flat gauge, a time slicing where the spatial metric fluctuations vanish: $\Psi_f = E_f = 0$. The dynamical field is then given by the scalar field perturbation $\delta\phi_f$. This choice is widely used in inflation and, as we will see later, it is particularly useful to calculate the inflationary primordial bispectrum of perturbations using the so-called δN formalism. The relationship between the unitary curvature ζ_u and the scalar field in spatially flat gauge $\delta\phi_f$ is generically at linear order given by

$$\zeta_u = \frac{\delta\phi_f}{\bar{\phi}'}. \quad (16)$$

Next, we show that whereas typical calculations for inflation are performed using ζ_u or ζ_c , the concept of a separate universe is defined using ζ_s .

C. Separate universe

In the separate universe approach, each super-Hubble sized region of the universe can be considered as a separate FRW universe, with a different effective matter content but locally homogeneous. In particular, long-wavelength perturbations can be absorbed into the background so that the perturbed equations of motion take the same form as those for an FRW universe, and thus the local effect of long-wavelength cosmological perturbations reduces to a simple change in the background cosmological parameters.

We can define a separate universe condition that determines when the local universe looks close enough to an actual FRW universe, so that we can apply the separate universe approach. In order to do that, we recast the perturbations in terms of their associated local quantities. Without loss of generality, we can define a local scale factor a_W , effective density ρ_W , and spatial curvature K_W such that the (00) equation in (2) including the background and linear perturbation contributions takes the form

$$\begin{aligned} G^0_0 &= \bar{G}^0_0 + \delta G^0_0 \equiv -3(H_W^2 + K_W/a_W^2) \\ &= -\rho_W = -(\rho + \delta\rho). \end{aligned} \quad (17)$$

In this case, the local Hubble factor will be defined as $H_W = d \ln a_W / d\tau$, with $d\tau = (1 + \Phi)dt$. Explicitly, $H_W = H + \delta H$, where $(\delta H/H) = -\Psi' - \Phi - \Sigma/3$ is the deviation from the background Hubble factor, with $\Sigma = (k/a)^2 (a^2 E' - B/H)$ quantifying the effective shear of the perturbed expansion of the universe [6,12]. In addition, the curvature is given by

$$K_W = -\frac{2}{3}k^2\Psi. \quad (18)$$

Similarly, we rewrite the trace of the G^i_j equation as

$$\begin{aligned} G^i_i - G^0_0 &= 3 \left[-\frac{2}{a_W} \frac{d^2 a_W}{d\tau^2} - \frac{2}{3} \left(\frac{k}{a} \right)^2 \Phi \right] \\ &= (\rho_W + 3p_W), \end{aligned} \quad (19)$$

where we have defined $p_W = p + \delta p$. Also, we rewrite the matter continuity equation as

$$\frac{1}{a_W^3} \frac{d(a_W^3 \rho_W)}{d \ln a_W} + 3p_W = (\rho + p)x^2 H v, \quad (20)$$

where we have defined the horizon-to-wavelength ratio

$$x \equiv \frac{k}{aH}. \quad (21)$$

Equations (17)–(20) are completely general but we see that in the case of infinitely long wavelengths, i.e., $k \rightarrow 0$, we have that $K_W \rightarrow 0$, $(k/a)^2 \Phi \rightarrow 0$ and $(k/a)^2 H v \rightarrow 0$, and hence the perturbed equations take the same form as a set of Friedmann equations (4) and (5). In this case, we can interpret H_W and ρ_W as the local windowed average Hubble rate and energy density of the universe on scales much smaller than $2\pi/k$. Therefore, this effective local universe with an infinitely long-wavelength perturbation would look like a homogeneous and isotropic universe. For finite k though, we can define a precise separate universe condition to be that freely falling observers that are initially at rest with respect to the background expansion see an approximate local FRW effective universe. This frame definition coincides with our definition of the synchronous gauge, where $v_m = 0$, which implies that $\Phi_s = 0$, since the matter stress-energy tensor conservation equation $\nabla^\mu T_{\mu\nu,m} = 0$ for a nonrelativistic perfect fluid with $p_m = \delta p_m = 0$ gives

$$H v'_m = -\Phi \quad (22)$$

in any frame. Therefore, for synchronous observers, Eq. (19) looks exactly like one of the Friedmann equations,

whereas Eq. (17) will look like a Friedmann equation if the effective spatial curvature K_W is approximately constant. This places the following condition on $\zeta_s = \Psi_s$:

$$|(\ln \zeta_s)'| \ll 1. \quad (23)$$

Analogously, the continuity equation will approximate to one in FRW when the source term in the right-hand side (RHS) of Eq. (20) becomes negligible. A sufficient condition for this would be that the source term is much smaller than any of the terms on the left-hand side (LHS) of the equation. In particular, we could have

$$|(\rho + p)x^2 H v_s| \ll |\delta \rho_s|, \quad (24)$$

where v_s is the synchronous perturbed effective velocity and $\delta \rho_s$ is the synchronous energy density [which is one contribution to the first term on the LHS of Eq. (20)]. From the equations of motion (6) and (7), this condition can be rewritten as

$$|x^2 \zeta_s'| \ll |x^2 \zeta_s + 3\zeta_s' + \Sigma_s|. \quad (25)$$

As long as there are no cancellations on the RHS of Eq. (25), this condition will be satisfied if (23) is satisfied and when $|\Sigma_s| \lesssim |x^2 \zeta_s|$. Note that the evolution of Σ_s can be determined from Eq. (8), which can be rewritten in synchronous gauge as

$$\Sigma_s' + \left(3 + \frac{H'}{H}\right) \Sigma_s = -x^2 (\zeta_s + p\pi), \quad (26)$$

which is sourced by the effective anisotropic stress π and ζ_s . For minimally coupled single-field inflationary models we have that $\pi = 0$ and hence $|\Sigma_s| \sim |x^2 \zeta_s|$. For standard slow-roll models typically considered in the literature Eq. (25) will then be automatically satisfied whenever Eq. (23) holds. However, in Sec. IV A, we will mention the ultra-slow-roll model, where even though $|\Sigma_s| \sim |x^2 \zeta_s|$, cancellations occur on the RHS of Eq. (25). In this case, Eq. (25) will not follow from Eq. (23), and the source term on the RHS of Eq. (20) will not become negligible. In general, we will say that the separate universe approach holds when (23) holds *and* the source term on the RHS of Eq. (20) is negligible.

III. CURVATURE CONSERVATION

In this section we analyze the difference between the three previously mentioned curvatures for single-field inflation, and we discuss situations where a given curvature observable is conserved, whereas others are not. For concreteness, let us consider the most general diffeomorphism-invariant action for a single scalar field coupled to the metric with second-order derivative equations of motion, known as the Horndeski action [17,18],

$$S = \int d^4x \sqrt{-g} \left\{ \sum_{i=2}^5 \mathcal{L}_i[\phi, g_{\mu\nu}] \right\}, \quad (27)$$

where \mathcal{L}_i are Lagrangians given by

$$\begin{aligned} \mathcal{L}_2 &= G_2, \\ \mathcal{L}_3 &= -G_3 \square \phi, \\ \mathcal{L}_4 &= G_4 R + G_{4,X} [(\square \phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi)], \\ \mathcal{L}_5 &= G_5 G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{1}{6} G_{5,X} [(\square \phi)^3 \\ &\quad - 3(\nabla^\mu \nabla^\nu \phi)(\nabla_\mu \nabla_\nu \phi) \square \phi \\ &\quad + 2(\nabla^\nu \nabla_\mu \phi)(\nabla^\alpha \nabla_\nu \phi)(\nabla^\mu \nabla_\alpha \phi)], \end{aligned} \quad (28)$$

with $G_i(\phi, X)$, $i \in (2, 5)$ as arbitrary functions of ϕ and the kinetic term $X \equiv -\nabla^\nu \phi \nabla_\nu \phi / 2$.

During inflation, we can characterize the expansion of the universe through $H(t)$ by defining the slow-roll parameters: $\epsilon = -H'/H$, and higher derivatives. The equations determining the evolution of the perturbations will be described by four effective parameters $\{\alpha_M, \alpha_T, \alpha_B, \alpha_K\}$, which can be functions of time [19]. In this case, the effective fluid velocity potential v can differ from v_ϕ when the so-called braiding parameter α_B is nonzero, according to the following expression:

$$v_\phi - v = \frac{\alpha_B}{2H\epsilon} (\Phi + H v_\phi'). \quad (29)$$

The braiding parameter is explicitly defined as

$$\begin{aligned} \alpha_B &= [\phi' (X G_{3,X} - G_{4,\phi} - 2X G_{4,\phi X}) \\ &\quad + 4X (G_{4,X} + 2X G_{4,XX} - G_{5,\phi} - X G_{5,\phi X}) \\ &\quad + \phi' X H^2 (3G_{5,X} + 2X G_{5,XX})] \\ &\quad / [G_4 - 2X G_{4,X} + X G_{5,\phi} - \phi' H^2 X G_{5,X}]. \end{aligned} \quad (30)$$

Next, we proceed to rewrite the relationship between the three curvatures for the specific case of Horndeski models. Combining Eqs. (29) and (7), we rewrite Eq. (14) solely in terms of the curvature fields as

$$\zeta_c = \zeta_u - \frac{\Gamma}{\epsilon} \zeta_u'; \quad \Gamma \equiv \frac{\alpha_B}{2 - \alpha_B}. \quad (31)$$

Similarly, we also rewrite Eq. (15) solely in terms of curvatures. We use Eqs. (7) and (5) to rewrite v in terms of the metric perturbations Ψ and Φ . Combining these equations with (22), we find that the difference in velocities ($v - v_m$) can be rewritten in terms of ζ_s' and obtain the following expression:

$$\zeta_c = \zeta_s + \frac{\zeta_s'}{\epsilon}. \quad (32)$$

We emphasize that whereas the relation given in Eq. (32) is valid for any single-field model, the one in Eq. (31) is only valid for Horndeski models, and hence conclusions will be restricted to this class of models.

A. Unitary and comoving

The relation between ζ_c and ζ_u is given by Eq. (31), where α_B is the braiding parameter that depends on the background, and is nonvanishing when the action has kinetic mixing between the metric and the scalar field (e.g., when there is $X \square \phi$). For minimally coupled scalar fields, there is no braiding and therefore comoving and unitary curvatures coincide. Next, we study the relation between these two curvatures for general inflationary Horndeski models when $\alpha_B \neq 0$ in the regime of long-wavelength linear perturbations.

If $\zeta_u(x)$ is conserved for long-wavelength modes, then we can write [20]

$$\frac{\zeta'_u}{\zeta_u} = c_s^2 \mathcal{O}(x^2), \quad (33)$$

where c_s is the sound speed of the scalar field. Note then that for conservation of unitary curvature we require $(c_s x) \ll 1$ instead of just $x \ll 1$. From Eq. (31) we obtain that $\zeta_c(x)$ will thus be given by

$$\frac{\zeta_c}{\zeta_u} = 1 - \frac{\Gamma}{\epsilon} c_s^2 \mathcal{O}(x^2). \quad (34)$$

On the one hand, we see that ζ_c will also be conserved as long as the term $\Gamma = \alpha_B / [(2 - \alpha_B)]$ is order one or higher in the slow-roll parameter ϵ . In that case, both curvatures ζ_c and ζ_u will freeze to the same value outside the horizon. We note that $|\Gamma/\epsilon| \gtrsim 1$ is a sufficient condition to have conservation of ζ_c but it is not necessary.

On the other hand, for $\Gamma \sim 1$ or larger this is not the case. For instance, when $|\alpha_B| \gg 1$, Eq. (34) is given by

$$\frac{\zeta_c}{\zeta_u} = 1 + \frac{c_s^2}{\epsilon} \mathcal{O}(x^2), \quad (35)$$

and since $|\epsilon| \ll 1$ during inflation, we expect ζ_c to freeze out much later than ζ_u . In typical inflationary models, ϵ grows in time, and then if there is a scale at which ζ_u freezes, then ζ_c will also eventually freeze. However, there could be transition regimes in which ϵ decays in time, in which case the difference between ζ_u and ζ_c becomes large, and ζ_c could even grow in time during this regime.

Conversely, we can use Eq. (14) to study the behavior of ζ_u when ζ_c is conserved. Suppose $\zeta_c(x)$ is conserved for long-wavelength modes, that is,

$$\frac{\zeta'_c}{\zeta_c} = \mathcal{O}(x^2). \quad (36)$$

Note that here and below the order counting in x keeps track of the k dependence, although there is generically a k -independent prefactor, typically c_s , which we omit for simplicity. In that case, all conclusions will hold generalizing the Hubble horizon to the sound horizon, and we use the terms interchangeably where no confusion should arise. Next, we analyze whether ζ_u will be conserved as a consequence. In order to do this, we write an explicit expression for ζ_u as a function of ζ_c and ζ'_c . We start by calculating the difference $(v - v_\varphi)$ in terms of the comoving curvature ζ_c . This velocity difference is a gauge-invariant quantity but we can make a gauge choice to simplify its calculation. In particular, in comoving gauge $(v - v_\varphi) = -v_{\varphi,c}$, and $v_{\varphi,c}$ can be obtained from Eqs. (29) and (7) to be

$$v_{\varphi,c} = \frac{1}{u} \int d \ln a u \frac{\zeta'_c}{H}, \quad (37)$$

where $u \equiv \exp(-2 \int d \ln a [\epsilon/\alpha_B])$, and therefore, from Eq. (14) we find

$$\zeta_u = \zeta_c - \frac{H}{u} \int d \ln a u \frac{\zeta'_c}{H}. \quad (38)$$

If comoving curvature is conserved according to Eq. (36), we see that for $|\epsilon/\alpha_B| \ll 1$ we have $u \approx 1$, and therefore we expect $\zeta_u \sim (1 + \mathcal{O}(x^2))\zeta_c$. On the other hand, if $|\epsilon/\alpha_B| \gg 1$, then u decays on the $\delta \ln a \sim |\alpha_B/\epsilon| \approx |\Gamma/\epsilon| \ll 1$ scale compared with the e -fold timescale over which ζ'_c/H varies. Putting together these estimations we get

$$\frac{\zeta_u}{\zeta_c} = 1 + \mathcal{O}(x^2) \times \begin{cases} \Gamma/\epsilon, & |\Gamma/\epsilon| \ll 1 \\ 1, & |\Gamma/\epsilon| \gtrsim 1 \end{cases}. \quad (39)$$

Therefore, if ζ_c is conserved for $x \ll 1$, we conclude that $\zeta_u \approx \zeta_c$, and hence ζ_u will also be conserved for any Γ and ϵ , unlike the converse.

We note that Eq. (37) is valid up to some integration constant that has been ignored, as it depends on the initial velocity, and we have assumed that all perturbations vanish initially, at least in their Hubble time average. We also emphasize that the conditions that we have discussed for conservation are sufficient but not necessary.

B. Comoving and synchronous

In this section we study the relationship between comoving and synchronous curvatures for long-wavelength perturbations. On the one hand, from Eq. (32) we have that if $\zeta_s(x)$ is conserved, that is,

$$\frac{\zeta'_s}{\zeta_s} = \mathcal{O}(x^2), \quad (40)$$

then comoving curvature will be given by

$$\frac{\zeta_c}{\zeta_s} = 1 + \frac{\mathcal{O}(x^2)}{\epsilon}. \quad (41)$$

As a result, as discussed in the previous section, depending on how ϵ behaves in time, ζ_c could even grow. Therefore, conservation of ζ_s does not imply conservation of ζ_c for $x \ll 1$.

On the other hand, if ζ_c is conserved, we can calculate the velocity difference ($v_m - v$) in terms of ζ_c . Similar to the previous section, even though ($v_m - v$) is gauge invariant, we can make a gauge choice to simplify calculations. In comoving gauge, we can combine the momentum conservation equation for nonrelativistic matter (22) together with Eq. (7) to generically obtain that $\zeta'_c = H v'_{m,c}$, which can be generalized to any gauge as

$$v_m - v = \int d \ln a \frac{\zeta'_c}{H}, \quad (42)$$

so that Eq. (15) implies

$$\zeta_s = \zeta_c - H \int d \ln a \frac{\zeta'_c}{H}. \quad (43)$$

From this general expression we see that if comoving curvature is conserved for long-wavelength modes according to Eq. (36), then ζ_s will also be conserved. Therefore, we conclude that conservation of ζ_c generically implies conservation of ζ_s , unlike the converse.

Similar to the previous section, Eqs. (42) and (43) are valid up to some integration constants that have been ignored as we have assumed that all perturbations vanish initially.

C. Unitary and synchronous

In this section, we study the relationship between unitary and synchronous curvatures for long-wavelength modes. We do this by using the results of the previous two sections.

On the one hand, if unitary curvature is conserved, then, according to the results of Sec. III A, comoving curvature will also be conserved as long as $|\Gamma/\epsilon| \lesssim 1$. In this case, according to the results of Sec. III B, synchronous curvature will also be conserved and its value will coincide with ζ_u . This is a sufficient condition that can be generalized. Indeed, combining Eqs. (31) and (32), we have

$$\zeta_s + \frac{\zeta'_s}{\epsilon} = \zeta_u - \frac{\Gamma}{\epsilon} \zeta'_u. \quad (44)$$

From here we see that if we assume that unitary curvature is conserved, then the leading order time dependence on the right-hand side is $(\Gamma/\epsilon)c_s^2 \mathcal{O}(x^2)$. Notice that unlike for ζ_c , the left-hand side contains a term ζ'_s/ϵ . Thus even if ϵ decays quickly, ζ_s will still be conserved in the same way as ζ_u as long as $\Gamma \lesssim 1$. As we will see in the next section,

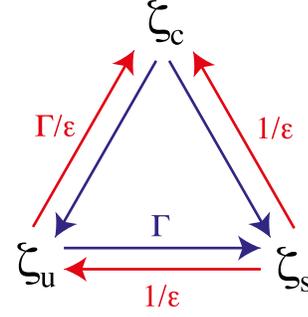


FIG. 1. Summary of Sec. III on conservation relationships between ζ_u , ζ_s , and ζ_c for superhorizon linear perturbations in single-field inflation. Labels on arrows denote additional variables that must be $\lesssim \mathcal{O}(1)$ for conservation of one curvature to suffice for conservation of the other (see text).

although they are then both constant, the values of ζ_s and ζ_u do not necessarily coincide.

On the other hand, if ζ_s is conserved, then, according to Eq. (41), ζ_c will also be conserved if $|1/\epsilon| \ll 1$, and hence ζ_u will also be conserved as shown in Eq. (39), regardless of whether the given model has a braiding interaction (large or small) or not. Note that, again, this is a sufficient but not necessary condition to have ζ_u conserved.

We summarize the results of this section in Fig. 1. Arrows indicate whether conservation of the curvature at the tail suffices to ensure that at the tip. Labels such as Γ/ϵ indicate any additional quantity besides $c_s^2 x^2$, which should be $\lesssim 1$ to establish this sufficient condition.

Next, we show examples of inflationary models that illustrate explicitly the inequivalence between the three curvatures studied in this section.

IV. INFLATIONARY MODELS

In this section we discuss two particular examples of single-field inflationary models that highlight the difference between the three spatial curvatures previously discussed: ζ_u , ζ_c , and ζ_s . In Sec. IV A we first describe the ultra-slow-roll model, which exhibits a clear difference in the behavior of ζ_s and $\zeta_u = \zeta_c$. In this case, ζ_s is conserved for superhorizon perturbations, whereas ζ_u is not conserved and even grows. In Sec. IV B we present a new model dubbed braiding-ultra-slow-roll inflation, which has ζ_u and ζ_s conserved outside the horizon (although their values differ), whereas ζ_c grows in time (or even becomes undefined).

In order to find the evolution of linear perturbations in both models, we start by solving the Mukhanov-Sasaki equation for Horndeski models,

$$\frac{d^2 u}{d\tau^2} + \left(c_s^2 k^2 - \frac{1}{z} \frac{d^2 z}{d\tau^2} \right) u = 0, \quad (45)$$

where τ is conformal time related to physical time by $a d\tau = dt$ and u describes the physical scalar degree of freedom

given by $u \equiv z\zeta_u$, where $z \equiv a\sqrt{2Q_s}$ with Q_s describing an effective Planck mass for scalar perturbations. Also, c_s describes the sound speed of waves for u . Explicitly, for Horndeski models with $\alpha_T = \alpha_M = 0$ (which will be the case for both USR and BUSR models), the factors Q_s and c_s^2 are given by

$$Q_s = \frac{(2\alpha_K + 3\alpha_B^2)}{(2 - \alpha_B)^2}, \quad (46)$$

$$c_s^2 = \frac{(2 - \alpha_B)(2\epsilon + \alpha_B) + 2\alpha'_B}{(2\alpha_K + 3\alpha_B^2)}, \quad (47)$$

where α_B is given by Eq. (30) and α_K is defined as

$$\begin{aligned} \alpha_K = & [(X/H^2)(G_{2,X} + 2XG_{2,XX} - 2G_{3,\varphi} - 2XG_{3,X\varphi}) \\ & + 6\varphi'X(G_{3,X} + XG_{3,XX} - 3G_{4,X\varphi} - 2XG_{4,XX\varphi}) \\ & + 6X(G_{4,X} + 8XG_{4,XX} + 4X^2G_{4,XXX}) \\ & - 6X(G_{5,\varphi} + 5XG_{5,X\varphi} + 2X^2G_{5,XX\varphi}) \\ & + 2\varphi'XH^2(3G_{5,X} + 7XG_{5,XX} + 2X^2G_{5,XXX})] \\ & / [G_4 - 2XG_{4,X} + XG_{5,\varphi} - \varphi'H^2XG_{5,X}]. \end{aligned} \quad (48)$$

The solution to Eq. (45) for u inside and outside the horizon can be found explicitly for known z and c_s^2 functions. Generic solutions have also been discussed in [20].

A. Ultra-slow roll

Ultra-slow-roll inflation [11] is a single-field model, where the scalar field is minimally coupled to the metric, and is one of the few examples that violate the standard consistency relations as the curvature in unitary gauge is not conserved for superhorizon modes [21–26]. This model has regained attention lately where it has been considered as a transient phase to generate large non-Gaussianities and possibly primordial black holes [27–29]. Explicitly, USR is a canonical scalar field model with

$$G_2(\varphi, X) = X - V(\varphi), \quad G_4(\varphi, X) = \frac{1}{2}, \quad (49)$$

and $G_3(\varphi, X) = G_5(\varphi, X) = 0$, and a potential $V(\varphi)$. The two independent background equations of motion are given by

$$3H^2 = \frac{\dot{\bar{\varphi}}^2}{2} + V(\bar{\varphi}), \quad (50)$$

$$\ddot{\bar{\varphi}} + 3H\dot{\bar{\varphi}} + V_{\bar{\varphi}} = 0. \quad (51)$$

In USR, the potential is driving inflation in Eq. (50), but it is extremely flat so that its contribution to Eq. (51) is negligible. The solution to these equations is then such that $\dot{\bar{\varphi}} \propto a^{-3}$ and the slow-roll parameter is $\epsilon = \dot{\bar{\varphi}}^2/2 \propto a^{-6}$.

USR then describes an inflationary model that quickly approaches exact de Sitter.

Next, we solve the evolution for perturbations using Eq. (45). The effective parameters of this model are $\alpha_K = 2\epsilon$ and $\alpha_B = \alpha_T = \alpha_M = 0$, and thus $c_s^2 = 1$ and $Q_s = \epsilon$. The solution for H almost constant and in a Bunch-Davies vacuum is given by

$$u = \frac{1}{\sqrt{2k}} \left(1 + \frac{i}{x} \right) e^{ix}. \quad (52)$$

Unitary curvature is then given by $\zeta_u = u/(a\bar{\varphi}')$. In the superhorizon limit, where $x \ll 1$, the solution approximates to

$$\begin{aligned} \zeta_u &\approx \frac{1}{\bar{\varphi}'} \frac{iH}{\sqrt{2k^{3/2}}} \left[1 + \frac{1}{2}x^2 + \mathcal{O}(x^3) \right], \\ \zeta'_u &\approx \zeta_u [3 - x^2 + \mathcal{O}(x^3)]. \end{aligned} \quad (53)$$

We then see that $\zeta_u \propto a^3$ and $\zeta'_u \propto a^3$ outside the horizon. Note that since the braiding parameter vanishes in this model, then $\zeta_u = \zeta_c$.

Next, from Eq. (43) we can analyze the behavior of synchronous curvature ζ_s . Since both ζ_u and ζ'_u grow in the same way in time, both of these terms will contribute equally to ζ_s , and their leading orders in x will actually cancel out. For this reason, it is convenient to rewrite ζ_s as

$$\zeta_s = H \int d \ln a e^{\frac{\zeta_u}{H}}. \quad (54)$$

This integral can be performed explicitly using the full solution (52) and obtain for superhorizon modes

$$\zeta_s \approx -8i\epsilon x^{-3} \zeta_u(x=0) [1 + \mathcal{O}(x^3)], \quad (55)$$

and therefore its amplitude is approximately constant in time, with $\zeta'_s = e(\zeta_u - \zeta_s) \approx e\zeta_u \propto a^{-3}$. We therefore conclude that in USR $\zeta_c = \zeta_u$ grows in time outside the horizon, whereas ζ_s is conserved.

B. Braiding ultra-slow roll

As previously shown, the difference between ζ_u and ζ_c is determined by the quantity Γ/ϵ . We will then construct a model that gives a large ratio Γ/ϵ , which will hence exemplify a clear case where $\zeta_u \neq \zeta_c$. In order to do this, we use the method described in [30,31] to construct a single-field model with a de Sitter phase, i.e., with $\epsilon \rightarrow 0$ and a nonvanishing braiding parameter α_B . This can be achieved by the following choice of Horndeski functions:

$$\begin{aligned}
 G_2(\varphi, X) &= -\left(\Lambda(\varphi) - \frac{15}{8}H_i^2\hat{\alpha}_B\right) \\
 &\quad + \frac{3}{2}H_i^2(-3\hat{\alpha}_{B3}(\varphi)X + \hat{\alpha}_B X^2), \\
 G_3(\varphi, X) &= H_i\hat{\alpha}_{B3}(\varphi)X, \quad G_4 = \frac{1}{2}, \quad (56)
 \end{aligned}$$

and $G_5(\varphi, X) = 0$. Here, $\Lambda(\varphi)$ is an arbitrary potential term, $H_i > 0$ is an arbitrary parameter that will later determine the Hubble rate of the de Sitter solution, $\hat{\alpha}_B$ is a free constant, and $\hat{\alpha}_{B3}(\varphi)$ is the function that determines the value of the braiding parameter α_B . Note that the kinetic term X in G_2 is not canonically normalized.

For this model, the background equations of motion are given by

$$\begin{aligned}
 \left(\frac{H}{H_i}\right)^2 &= \left(\frac{\Lambda(\bar{\varphi})}{3H_i^2} - \frac{5}{8}\hat{\alpha}_B\right) - \frac{3}{4}\hat{\alpha}_{B3}(\bar{\varphi})\dot{\bar{\varphi}}^2 \\
 &\quad + \frac{H}{H_i}\hat{\alpha}_{B3}(\bar{\varphi})\dot{\bar{\varphi}}^3 + \frac{1}{8}\left(3\hat{\alpha}_B - \frac{4\hat{\alpha}_{B3,\varphi}(\bar{\varphi})}{3H_i}\right)\dot{\bar{\varphi}}^4, \quad (57)
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{H}{H_i}\right)^2 \epsilon &= \frac{3}{4}\dot{\bar{\varphi}}^2[\dot{\bar{\varphi}}^2\hat{\alpha}_B - 3\hat{\alpha}_{B3}(\bar{\varphi})] - \frac{1}{2}\frac{\hat{\alpha}_{B3,\varphi}(\bar{\varphi})}{H_i}\dot{\bar{\varphi}}^4 \\
 &\quad + \frac{1}{2}\frac{H}{H_i}\hat{\alpha}_{B3}(\bar{\varphi})\dot{\bar{\varphi}}^3\left(3 - \frac{H_i}{H}\frac{\ddot{\bar{\varphi}}}{H_i\dot{\bar{\varphi}}}\right). \quad (58)
 \end{aligned}$$

We will start by considering a particular phase of this model, where $\hat{\alpha}_{B3} = \hat{\alpha}_B$ and Λ are both constant. In this case, it is useful to write explicitly the equation of motion for the scalar field as well, in order to illustrate the behavior of the model. Combining Eqs. (57) and (58) we obtain

$$\frac{\ddot{\bar{\varphi}}}{H_i\dot{\bar{\varphi}}} = \frac{3(2(H/H_i)\dot{\bar{\varphi}} + \dot{\bar{\varphi}}^2 - 3)(-2(H/H_i) + \hat{\alpha}_B\dot{\bar{\varphi}}^3)}{-6 + 8(H/H_i)\dot{\bar{\varphi}} + 6\dot{\bar{\varphi}}^2 + 2\hat{\alpha}_B\dot{\bar{\varphi}}^4}. \quad (59)$$

We can also solve Eq. (57) to get two solutions of Hubble parameter H_{\pm} ,

$$\frac{H_{\pm}}{H_i} = \frac{\dot{\bar{\varphi}}^3\hat{\alpha}_B}{2} \pm \sqrt{\frac{\dot{\bar{\varphi}}^6\hat{\alpha}_B^2}{4} + \frac{\Lambda}{3H_i^2} + \frac{\hat{\alpha}_B}{8}(3\dot{\bar{\varphi}}^4 - 6\dot{\bar{\varphi}}^2 - 5)}. \quad (60)$$

If the sum of the last two terms inside the square root is positive, H_+ will always be positive, while H_- will be negative. A negative H corresponds to a contracting universe in which we are not interested. Also note that the solution H_+ is no less than H_- . If H_+ is negative, then so is H_- , so in the following we will focus on the H_+ solution.

In the phase that we are interested in, we see that the equations of motion are independent of $\bar{\varphi}$ itself, and hence only $\dot{\bar{\varphi}}$ determines the evolution of the system. We find that $\dot{\bar{\varphi}} = 0$ is always an attractor solution as we can see from Eq. (59), but it is trivial because the scalar field will stop

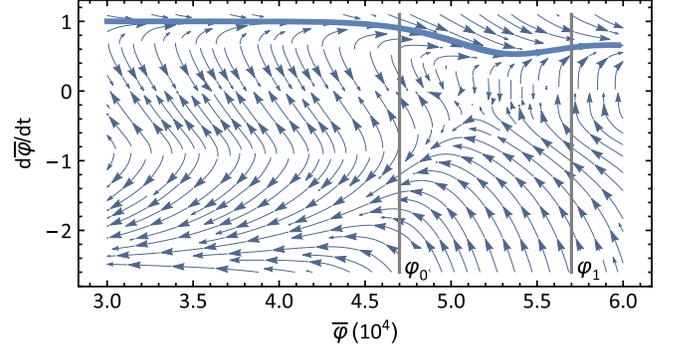


FIG. 2. The phase space evolution of the BUSR model through the transition of $G_3(\varphi, X)$. The blue streams show the direction of the time evolution. The thick blue line shows the attractor solution starting with $\dot{\bar{\varphi}} = 1$. The two vertical gray lines indicate the position of φ_0 and φ_1 .

evolving in time once it hits this solution. Additionally, if we choose $\Lambda = 3H_i^2$, we find another attractor solution $\dot{\bar{\varphi}} = 1$, which can be checked by substituting $\dot{\bar{\varphi}} = 1$ into Eq. (59). In this case, for $\hat{\alpha}_B < 2$, $H_+ = H_i$ and the solution gives a de Sitter background with $\ddot{\bar{\varphi}} = 0$, so that the scalar field velocity stays constant if initially set to $\dot{\bar{\varphi}}_I = 1$.

For a more concrete example, we choose $\hat{\alpha}_B = 1$, and we find three attractor solutions $\dot{\bar{\varphi}} = 1, 0, -\sqrt{6}$. Note that the number and values of real solutions depend on the choice of $\hat{\alpha}_B$. There are also two repellers $\dot{\bar{\varphi}} \approx 0.677, -0.898$. For this parameter choice, the three attractors are shown in Fig. 2, where the behavior of this model can be seen for $\bar{\varphi} < \varphi_0$ (larger values of $\bar{\varphi}$ describe the behavior of an extended model that will be introduced in Sec. VB). In this case, we see that solutions with $\dot{\bar{\varphi}} = 1$ evolve toward the right with increasing field values, whereas solutions with $\dot{\bar{\varphi}} = -\sqrt{6}$ evolve toward the left with decreasing field values.

In what follows, we focus on the $\dot{\bar{\varphi}} = 1$ attractor and the $H = H_i$ branch. In this case, the evolution is such that $\epsilon \propto a^{-3}$, for any value of $\hat{\alpha}_B$. This behavior in ϵ is similar to that of USR inflation but in this case it relies on the presence of the braiding parameter, and thus we dub this model BUSR inflation.

Next, we analyze the evolution of linear perturbations outside the horizon. This model always has $\alpha_M = \alpha_T = 0$, and on the de Sitter attractor we additionally find $\alpha_K = 6\hat{\alpha}_B$ and $\alpha_B = \hat{\alpha}_B$. Therefore, we solve the Mukhanov-Sasaki equation using

$$Q_s = \frac{3\hat{\alpha}_B(4 + \hat{\alpha}_B)}{(2 - \hat{\alpha}_B)^2}, \quad c_s^2 = \frac{1}{3} \frac{(2 - \hat{\alpha}_B)}{(4 + \hat{\alpha}_B)}. \quad (61)$$

Note that in order to avoid ghost and gradient instabilities we will need $Q_s > 0$ and $c_s^2 > 0$ [19], and thus we will impose $0 < \hat{\alpha}_B < 2$. Both of these coefficients are constants and thus $z \propto a$; the solution for u will have the same form as Eq. (52), simply generalizing k to $c_s k$. We then

obtain the unitary curvature as $\zeta_u = (u/a)(2Q_s)^{-1/2}$ and see that it will be conserved outside the sound horizon, that is, for $c_s x \ll 1$,

$$\zeta_u \approx \frac{1}{\sqrt{Q_s}} \frac{iH_i}{2(c_s k)^{3/2}} \left[1 + \frac{1}{2}(c_s x)^2 + \mathcal{O}(x^3) \right], \quad (62)$$

$$\zeta'_u \approx -\zeta_u [(c_s x)^2 + \mathcal{O}(x^3)]. \quad (63)$$

We emphasize that since this model has a finite braiding parameter α_B and $\epsilon \propto a^{-3}$, then we see from Eq. (31) that the unitary curvature will grow as $\zeta_c \propto \zeta'_u/\epsilon \propto a$. Therefore, this inflationary model offers an extreme case where the difference between ζ_u and ζ_c can be arbitrarily large.

Next, we analyze the evolution of ζ_s . From Eq. (44) we know that in the limit of de Sitter BUSR, where $\epsilon \rightarrow 0$, we have that

$$\zeta'_s \approx -\Gamma \zeta'_u, \quad (64)$$

and therefore $\zeta_s \approx -\Gamma \zeta_u(x=0)[1 + \mathcal{O}(\epsilon/\Gamma)]$ (up to some boundary terms). We thus conclude that for superhorizon modes $\zeta'_s \propto \Gamma(c_s x)^2$ and ζ_s will be conserved outside the sound horizon. Note that, as found in the previous section, if $\Gamma \lesssim 1$, then ζ_s will freeze out at the same time or before ζ_u .

Finally, we calculate the dimensionless power spectrum for modes that have left the horizon already. We present here a standard quantity quoted in the literature; the power spectrum of ζ_u at the horizon crossing. Since ζ_u is conserved outside the horizon, this result will hold until the end of inflation. From Eq. (62) we find that

$$\mathcal{P}_{\zeta_u}(k) = \frac{k^3}{2\pi^2} |\zeta_u|^2 = \frac{1}{8\pi^2} \frac{H_i^2}{Q_s c_s^3}, \quad (65)$$

which is perfectly scale invariant. We can rewrite this in terms of $\hat{\alpha}_B$ as

$$\mathcal{P}_{\zeta_u} = \frac{H_i^2 \sqrt{3(2 - \hat{\alpha}_B)(4 + \hat{\alpha}_B)}}{8\pi^2 \hat{\alpha}_B}, \quad (66)$$

and therefore for $\hat{\alpha}_B \approx 1$ we have that H_i^2 determines the amplitude of this power spectrum. For an amplitude of order $\mathcal{P}_{\zeta_u} \sim 10^{-9}$ as constrained by temperature anisotropies of the cosmic microwave background, we would need $H_i \sim 10^{-4}$ (corresponding to an energy scale of $\sqrt{H_i} \sim 10^{16}$ GeV). Lower or higher energy scales can be achieved by adjusting the value of $\hat{\alpha}_B$. For instance, for $H_i \sim 10^{-6}$ (energy scale 10^{15} GeV) then we must have $\hat{\alpha}_B \sim 10^{-5}$. We also note that from the general equations of motion (57) and (58) we see that we can always rescale H_i to get an appropriate value for the dimensionless power spectrum $\mathcal{P}_{\zeta_u}(k)$, and the evolution of the system in terms

of e -folds does not change for the same initial condition $\dot{\phi} = 1$, as long as we also rescale $\hat{\alpha}_{B3,\phi}$ appropriately. Note that in order to obtain a physically meaningful quantity, we must calculate ζ_c at the end of inflation, as this quantity is generically used to set the initial conditions for matter inhomogeneities after inflation. In Sec. VB we will consider a realistic model and calculate this quantity.

V. OBSERVABLES

In this section, we discuss the consequences of the previous results on the inflationary primordial power spectrum and bispectrum.

A. Separate universe and δN formalism

Here we discuss the relationship between the separate universe concept introduced in Sec. IIC and the so-called δN formalism, which we illustrate with the USR model discussed in Sec. IVA.

The δN formalism [5–10] is a common technique to calculate the inflationary bispectrum of spatial curvature in a given hypersurface and obtain consistency relations for single-field inflationary models (although it can be extended to more general cases with multiple fields [32–34] or anisotropic backgrounds [35,36]). In this formalism, superhorizon perturbations can be reabsorbed into the background equations such that the total perturbed universe still looks homogeneous and isotropic in a local Hubble-sized patch. In this case, the spatial curvature can be calculated as a change in e -folds between the effective local and the background universes. Although the usage of the δN formalism is typically associated with the concept of a separate universe, we show that, in some models, violations of a separate universe in the matter continuity equation may be present. In this case, as long as the local Hubble rate approximates that of a Friedmann universe, then the δN formula can still be used to accurately compute the power spectrum or bispectrum for appropriate gauge-invariant quantities but subtleties can arise when calculating δN to a uniform-density surface.

Formally, we can always write the difference in spatial curvature between some initial and final time slices as [5]

$$\Psi(t) - \Psi(t_i) = -\delta N(t; t_i) - \frac{1}{3} \Sigma_I(t; t_i), \quad (67)$$

where we have defined δN such that

$$\delta N(t; t_i) \equiv N_W(t_i, t) - N(t_i, t), \quad (68)$$

$$N_W(t_i, t) \equiv \int_{t_i}^t H_W(1 + \Phi) dt', \quad (69)$$

$$N(t_i, t) \equiv \int_{t_i}^t H dt', \quad (70)$$

where H_W is the total window-averaged Hubble expansion that includes the background and linear perturbations, whereas H is the background Hubble expansion. Similarly, we have defined Σ_I as

$$\Sigma_I(t, t_i) \equiv \int_{t_i}^t dt' H \Sigma, \quad (71)$$

where Σ is the effective shear of the perturbed expansion of the universe, defined in Sec. II C.

Equation (67) can be used to easily calculate the spatial curvature in terms of background quantities, and hence without solving the perturbed equations of motion. This can be done by first considering an initial time t_i that corresponds to a spatially flat hypersurface and thus by definition $\Psi(t_i) = 0$, whereas the final time can correspond to a more general hypersurface A of interest (e.g., unitary, comoving, or uniform density). In addition, we will assume that the shear is negligible in these slices so that we simply obtain that

$$\Psi_A(t) = \zeta_A(t) = -\delta N(t; t_i), \quad (72)$$

where A is a subscript for the arbitrary hypersurface A that is usually chosen to be a unitary ($A = u$) or comoving ($A = c$) hypersurface. In most single-field models studied in the literature, the shear in spatially flat and other slices decays in time and can indeed be neglected. Finally, when spatially flat observers see an effective local (i.e., in Hubble-sized patches) homogeneous and isotropic background, it means that superhorizon perturbations can be reabsorbed in such a way that the full perturbed equations take the same form as FRW equations of motion in a spatially flat gauge (this is known as the spatial-gradient expansion where the leading order perturbations in $x \ll 1$ follow FRW-like equations of motion). In this case, superhorizon perturbations can be treated as a homogeneous perturbation to a fiducial background universe on each Hubble horizon scale, leading to the intuitive idea of a separate universe where each Hubble-sized patch evolves as an independent effective FRW universe. Computationally, N_W can be calculated using just the background equations but changing the initial conditions to reflect the presence of long-wavelength perturbations. Note that these initial conditions do not necessarily result in the same solutions as the global background since they can span a more general class of homogeneous and isotropic background models. Specifically, for second-order derivative single-field theories, the spatial curvature can be expressed as

$$\zeta_A(t) = -N(\varphi_i, \dot{\varphi}_i; t_A) + N(\bar{\varphi}_i, \dot{\bar{\varphi}}_i; t_A), \quad (73)$$

where $\bar{\varphi}_i$ and $\dot{\bar{\varphi}}_i$ denote the initial conditions for the background scalar field and its derivative, whereas φ_i and $\dot{\varphi}_i$ denote the initial conditions for the total scalar field

including perturbations and its derivative in a spatially flat gauge. Furthermore, N denotes the number of e -folds between the initial and final times according to the Friedmann equation, and both e -folds are calculated to a final time t_A such that one reaches the desired hypersurface (e.g., for obtaining the uniform-density curvature, t_A is chosen such that both N quantities reach the same desired value of the energy density at that time).

Equation (73) is known as the δN formula that can be used at all orders in perturbations and allow us to calculate the nonlinear evolution of ζ_A by knowing only the background evolution of the theory. In particular, it can be used to obtain ζ_A at quadratic order for small perturbations. For instance, for models where $\delta\dot{\varphi}$ is negligible, we can Taylor expand the formula and obtain

$$\zeta_A(t) = -N_\varphi \delta\varphi_i - \frac{1}{2} N_{\varphi\varphi} \delta\varphi_i^2 - \mathcal{O}(\delta\varphi^3), \quad (74)$$

which can then be used to calculate, e.g., the squeezed bispectrum for the spatial curvature on comoving slices ζ_c [9].

Overall, the validity of the δN formula relies on neglecting the shear as well as having the ability to express the equations of motion for perturbations in an FRW-like form so that N_W can be calculated using the background equations of motion.

Next, we use the USR model to illustrate how the δN formula can still be used to calculate the spatial curvature in some slices even though the separate universe condition is technically broken. Nevertheless, the δN formula will indeed break for other choices of slices.

Let us suppose we are interested in calculating the spatial curvature in unitary slices, that is, ζ_u . First, we check the conditions of negligible shear. In USR, we find that in a spatially flat gauge at linear order $|\Sigma_f| \sim \epsilon \zeta_u^1$ for superhorizon modes, and since $\epsilon \propto a^{-6}$ we conclude that indeed the shear is negligible in the sense that $|\Sigma_f/\zeta_u| \ll 1$. Similarly, we also find $|\Sigma_u/\zeta_u| \sim x^2 \ll 1$. This shows that the first conditions for the validity of a separate universe are indeed satisfied for USR. We also mention that in the case of uniform-density curvature we obtain similar results: $|\Sigma_\rho/\zeta_\rho| \sim x^2 \ll 1$ and $|\Sigma_f/\zeta_\rho| \sim \epsilon x^{-2} \propto a^{-4}$.

On the other hand, we note that the condition of a separate universe defined in synchronous gauge is closely related to the ability of reabsorbing the superhorizon perturbations into the background in a spatially flat gauge.

¹By definition $\Sigma_f = -x^2(HB_f)$. Also, from Eqs. (7) and (29) we have $\Phi_f = -\epsilon H v_{\varphi,f} = \epsilon \zeta_u$ for USR and, due to Eq. (52), we thus have the exact solution of Φ_f in USR [recall $\zeta_u = u/(a\dot{\varphi}')$]. Since $\dot{B}_f + HB_f = -\Phi_f$ from Eq. (8), we can perform an explicit integration with Φ_f as a source, which contributes mainly at the horizon crossing, and obtain for superhorizon modes that $x^2 HB_f \approx 3\Phi_f$, and therefore $|\Sigma_f| \sim |\epsilon \zeta_u|$.

In Sec. IV A we found that USR does satisfy the condition $|\dot{\zeta}_s/(H\zeta_s)| \ll 1$ but the local continuity equation (20) does not take an FRW-like form as the source term is not negligible. Indeed, it is possible to check that the leading order contribution of the terms in the RHS of Eq. (25) for superhorizon modes cancel out, hence breaking this condition. In particular, we obtain that the source term evolves as $x^2\zeta'_s \propto a^{-5}$ and the perturbed energy density has the same scaling $\delta\rho_s \propto a^{-5}$, which is different from what we would have expected on the background of this model, as $\rho \propto a^{-6}$. We find the same behavior in a spatially flat gauge, where there is a technical violation such that in a spatially flat gauge the continuity equation cannot be written in an FRW-like form, whereas both the (00) equation in (17) and the acceleration equation (19) can indeed be written as FRW. In this gauge, we again find that $\delta\rho_f \propto a^{-5}$.

As a consequence, in USR we find that the δN formula does yield the correct result when calculating ζ_u (as $N_W \approx N$ with an error falling as $\delta H_f/H \propto \delta\rho_f \propto a^{-5}$) but the incorrect result for the spatial curvature in uniform-density slices ζ_ρ . The latter happens because the uniform-density condition for the final surface is misapplied in the separate universe approximation, whereas the expansion itself is still FRW to good approximation.

The result for ζ_u (same as ζ_c in this model) can be found in [21]. Here we calculate ζ_ρ to linear order in perturbations in USR to explicitly illustrate the problem with the δN formula. On the one hand, from the result in Eq. (52), it is known from a gauge transformation that $\zeta_\rho = \delta\rho_f/\rho' \approx -H^2 x^2 (\bar{\varphi}'/\rho')(u/a)$ for superhorizon modes. This expression is always valid for infinitesimal transformations and yields a growing curvature $\zeta_\rho \propto a$, which is the correct result for USR [37]. On the other hand, if we use the δN formula with the background FRW solutions, we would absorb the spatially flat $\delta\rho_f$ into the background, a constant ρ surface, and evolve to another constant ρ surface leaving $\delta N = 0 = \zeta_\rho$.

The results presented so far in this section show that the δN formalism can sometimes be used even if a separate universe is violated. Nevertheless, due to this violation, the formalism will indeed break down when applied to certain gauge-invariant quantities and will return the wrong value. We used USR to illustrate this, as this model violates a separate universe via the continuity equation, which causes the formalism to give the wrong value for the uniform-density curvature ζ_ρ . However, we also show that the formalism does return the correct value for the unitary-curvature ζ_u .

Finally, for completeness, we comment on the observable consequence of nonconservation of ζ_u . On the one hand, if ζ_u is indeed conserved outside the horizon, then the resulting squeezed bispectrum will satisfy the standard consistency relation [38,39] given by

$$\lim_{k_1 \rightarrow 0} B_{\zeta_u}(k_1, k_2, k_3) = [n_s(k_3) - 1] P_{\zeta_u}(k_1) P_{\zeta_u}(k_3), \quad (75)$$

where k_i are three wave numbers such that $k_2 \sim k_3 \gg k_1$. Here, $P_{\zeta_u} = (2\pi^2/k^3) \mathcal{P}_{\zeta_u}$ is the power spectrum with a tilt $(n_s - 1)$ defined as

$$n_s(k) - 1 = \frac{d \ln \mathcal{P}_{\zeta_u}(k)}{d \ln k}. \quad (76)$$

On the other hand, if ζ_u is not conserved outside the horizon, the squeezed bispectrum will have a different relation to the power spectrum. In the case of USR, the squeezed bispectrum can be obtained with the δN formalism, and it is given by [21]

$$\lim_{k_1 \rightarrow 0} B_{\zeta_u}(k_1, k_2, k_3) = 6 P_{\zeta_u}(k_1) P_{\zeta_u}(k_3). \quad (77)$$

However, this relation does not necessarily describe the observable squeezed bispectrum after inflation since the USR phase must end. Its ending changes the local e -folds measured by the observer and the bispectrum accordingly through the δN formalism [28,29].

B. Observable curvature power spectrum

In this section, we analyze the observational consequences of inflationary braiding models with a large difference between unitary and comoving curvatures.

In general, inflationary models provide initial conditions for the seeds of structure formation during early times, which are then used to propagate forward the matter evolution and predict observables such as the cosmic microwave background temperature anisotropies and galaxy distributions. For single-field minimally coupled inflationary models, these initial conditions are typically obtained by calculating ζ_u at the horizon crossing, which then determines ζ_c during radiation or matter domination outside the horizon. This ultimately becomes the initial condition for structure formation. This translation between unitary and comoving curvatures is straightforwardly done as both curvatures are the same, and both are usually conserved outside the horizon.

The previously described picture will not be valid anymore in inflationary models with nonminimal couplings if they lead to relevant differences between unitary and comoving curvatures, or if the evolution of these curvatures is not conserved outside the horizon. In such cases, a careful analysis throughout the evolution of inflation until its end must be done to obtain appropriate initial conditions for comoving curvature after inflation.

As we have previously shown, inflationary models with nonvanishing braiding interactions can lead to a large difference between unitary and comoving curvatures. However, a realistic inflationary model must end and lead to reheating but, as it has been discussed in [14,15],

braiding interactions could potentially spoil the reheating process. For this reason, we will consider BUSR described in Sec. IV B and analyze the behavior of unitary curvature in a setting where braiding vanishes before the end of inflation. In this scenario, we have that, by construction, unitary and comoving curvatures will coincide at the end of inflation, but we will show that the time evolution of the braiding parameter may lead to a nontrivial evolution of unitary curvature outside the horizon, and therefore its value at the horizon crossing may not set the appropriate initial conditions for ζ_c after inflation. Additionally, we will allow for a time evolution of the potential interactions of BUSR, in order to allow the scalar field to roll down the potential and end the inflationary period.

We start by considering the full BUSR model given by the Horndeski functions in Eq. (56). We will make $\hat{\alpha}_{B3}$ evolve in time toward zero, while keeping $\hat{\alpha}_B$ constant. In this case, we keep the kinetic term X^2 in G_2 the same as before and change the linear X term together with G_3 . This is to ensure stability of the model throughout its evolution. We will also allow Λ to evolve in time, in order to allow for a potential interaction. In this case, BUSR will evolve toward a model with a vanishing braiding parameter as well as a potential-dominated scalar field, as it is considered in common inflationary models.

In particular, we will consider $\hat{\alpha}_{B3}$ to be a steplike function, starting at $\hat{\alpha}_{B3} = \hat{\alpha}_B$ (as in the model of Sec. IV B) and ending at zero. For concreteness, we model $\hat{\alpha}_{B3}$ as

$$\hat{\alpha}_{B3}(\varphi) = \frac{\hat{\alpha}_B}{2} \left[1 + \tanh\left(\frac{\varphi_0 - \varphi}{d_0}\right) \right], \quad (78)$$

where φ_0 represents the field value where the braiding transition happens and d_0 the width of the transition. Generically, we would also introduce a potential interaction such that

$$\Lambda(\varphi) = V(\varphi) + \frac{15}{8} H_i^2 \hat{\alpha}_B, \quad (79)$$

where the first term can be any appropriate inflationary potential, and the second term is added to cancel out the cosmological constantlike term in $G_2(\varphi, X)$ in Eq. (56). For simplicity, we will model $V(\varphi)$ with the same tanh function but with a different width and field location, such that before the step we have $\Lambda \approx 3H_i^2$ and after the step $V \approx 0$. Explicitly, we consider

$$\frac{\Lambda(\varphi)}{3H_i^2} = \frac{1}{2} \left(1 - \frac{5}{8} \hat{\alpha}_B \right) \left[1 + \tanh\left(\frac{\varphi_1 - \varphi}{d_1}\right) \right] + \frac{5}{8} \hat{\alpha}_B, \quad (80)$$

where φ_1 represents the field value where the transition in V happens and d_1 its width.

Next, we numerically study the evolution of the model starting from BUSR with initial conditions on the attractor

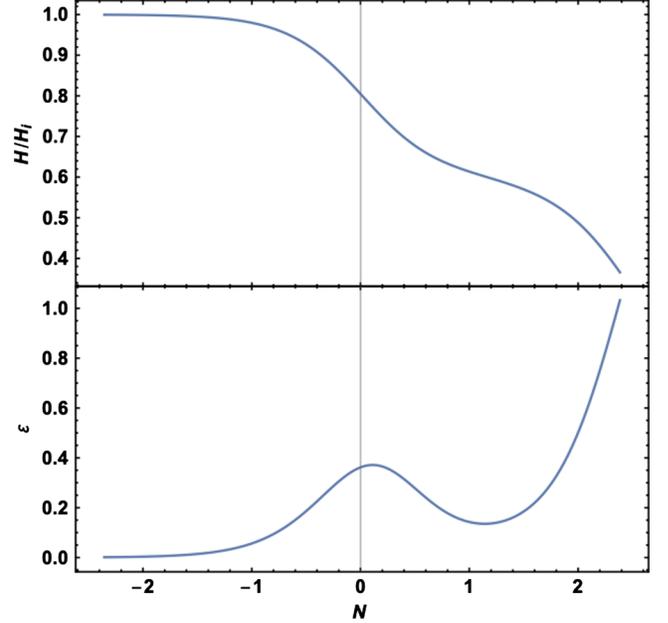


FIG. 3. Evolution of H and $\epsilon = -H'/H$ as a function of e -folds through the transition of $G_3(\varphi, X)$. Here, $N = 0$ corresponds to the time when the transition happens (defined as when $\bar{\varphi} = \varphi_0$).

$\dot{\bar{\varphi}}_i = 1$ and $\bar{\varphi}_i$ chosen appropriately such that $\hat{\alpha}_{B3} \approx \hat{\alpha}_B$. For concreteness, let us fix the parameters of the model to $H_i = 2.1 \times 10^{-4}$, $\hat{\alpha}_B = 1$, $\varphi_0 = 4.7 \times 10^4$, $d_0 = 3.3 \times 10^3$, $\varphi_1 = 5.7 \times 10^4$, and $d_1 = 3.8 \times 10^3$. In this case, $\varphi_1 > \varphi_0$, and thus the braiding interaction will first vanish, and soon afterwards the scalar field will start rolling down the potential until the end of inflation. We make this choice in order to disentangle effects coming from changing the braiding interaction and those from the potential.

In Fig. 2 we show the evolution of the scalar field $\bar{\varphi}$ and its time derivative $\dot{\bar{\varphi}}$ throughout the transition of α_B . Starting near $\dot{\bar{\varphi}} = 1$ on the top left of the plot, we see that flows join the attractor (blue thick curve) as time evolves. Once $\bar{\varphi} \approx \varphi_0$, the transition in G_3 occurs, which makes $\dot{\bar{\varphi}}$ decrease. Then, for $\bar{\varphi} \approx \varphi_1$ we see that the behavior changes as the scalar field starts rolling down the potential and its kinetic energy increases.

In Fig. 3 we show the evolution of the Hubble rate H and the slow-roll parameter ϵ as a function of e -folds from the braiding transition time (defined as when $\bar{\varphi} = \varphi_0$). In the top panel, we see that H starts near the attractor value $H = H_i$. Afterwards, the transition in G_3 alters the evolution, and H decays toward the value $(H/H_i) = \sqrt{(1 - 5/8\hat{\alpha}_B)} \approx 0.6$ for $\hat{\alpha}_B = 1$. This value of H would be a new de Sitter phase if the potential V was constant, and it is obtained from Eq. (57) with $\dot{\bar{\varphi}} \rightarrow 0$ (i.e., a potential dominated phase). Afterwards, the scalar field starts rolling down the potential and H decays even further toward zero.

In the bottom panel, we see the evolution of ϵ , which starts in a de Sitter phase with $\epsilon \approx 0$. Then, the transition in

G_3 causes a quick change in H which makes ϵ grow. Afterwards, ϵ decays again as the system goes toward the aforementioned would-be de Sitter phase with constant potential, but due to the scalar field rolling down the potential V , ϵ grows again reaching $\epsilon = 1$, marking the end of the accelerated expansion of the universe, and hence the end of inflation. Note that for the values chosen here, the step in G_3 is slow enough that the terms involving $\hat{\alpha}_{B3,\varphi}$ are always negligible in the evolution of H and ϵ .

Next, we study the evolution of linear cosmological perturbations in this model. Unitary curvature satisfies Eq. (45). The main coefficients determining the evolution of ζ_u are thus c_s^2 and Q_s . In turn, these two quantities depend on the EFT coefficients α_B and α_K . Whereas α_B decays to 0 by construction, α_K follows closely the behavior of the kinetic energy of the scalar field; that is, it decays during the braiding transition and grows again when the potential dominates the evolution and the scalar field rolls down.

In Fig. 4, we explicitly show the evolution of c_s^2 and Q_s as a function of e -folds when the system undergoes the step in G_3 . In the top panel we see that c_s^2 decays during the step of G_3 . Indeed, the faster the step, the more it decays. Therefore, a fast enough step would lead to a short gradient instability that would potentially spoil the evolution of small-scale perturbations. For this reason, we limit ourselves to cases where the transition in G_3 is slow enough to ensure $c_s^2 > 0$ at all times. After the braiding transition, c_s^2 converges toward the value $c_s^2 = 1/3$ of the final model with $G_2(\varphi, X) = (3/2)H_t^2 \hat{\alpha}_B X^2 - V(\varphi)$. In the bottom

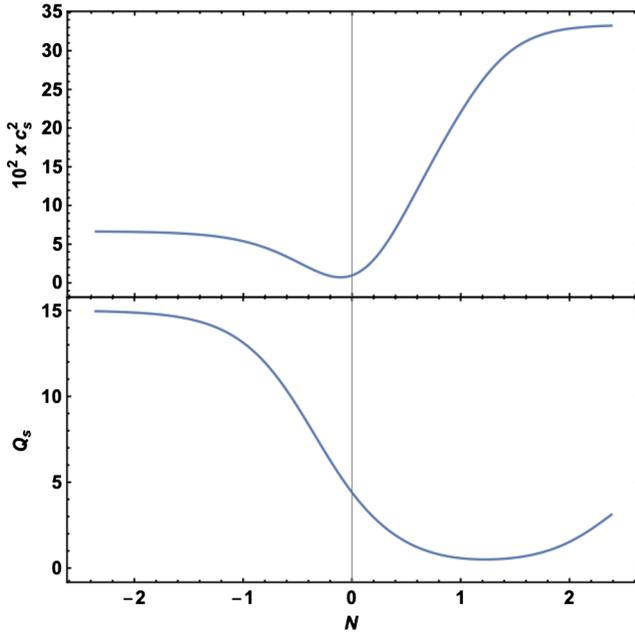


FIG. 4. Evolution of c_s^2 and Q_s as a function of e -folds through the transition of $G_3(\varphi, X)$. Here, $N = 0$ corresponds to the time when the transition happens (defined as when $\bar{\varphi} = \varphi_0$).

panel, we see that Q_s decays during the transition of G_3 and then grows again when the potential dominates. Overall, Q_s follows a similar behavior to the kinetic energy of the scalar field.

Next, we analyze the behavior of ζ_u in more detail. From Eq. (45), the equation of motion for ζ_u is given by

$$\frac{(Ha^3 Q_s \zeta_u)'}{Ha^3 Q_s} + (c_s x)^2 \zeta_u = 0, \quad (81)$$

and thus from here we generically have that

$$\zeta_u' = -\frac{1}{Ha^3 Q_s} \left[\int d \ln a (Ha^3 Q_s) (c_s x^2) \zeta_u + \text{const} \right], \quad (82)$$

where we have added an arbitrary integration constant. While this relation is exact, it only implicitly determines ζ_u . However, as we shall see next, this expression is useful to analyze ζ_u in the superhorizon limit. Here, we see that the evolution depends crucially on the behavior of $(Ha^3 Q_s)$.

Figure 5 shows the time evolution of $(Ha^3 Q_s)^{-1}$ (with an arbitrary normalization) and the evolution of $k^3 |\zeta_u|^2$ for two modes with $x_t = k/(a_t H_t)$ taking values $x_t \approx 0.6$ and $x_t \approx 0.04$, where a and H are evaluated at the time of the braiding transition (corresponding to $N = 0$). In the top panel, we see that $(Ha^3 Q_s)^{-1}$ decays monotonically for the value of d_0 chosen here, although the decay rate decreases during the braiding transition. If the transition in G_3 is faster, that is, for smaller d_0 , then $(Ha^3 Q_s)^{-1}$ may even

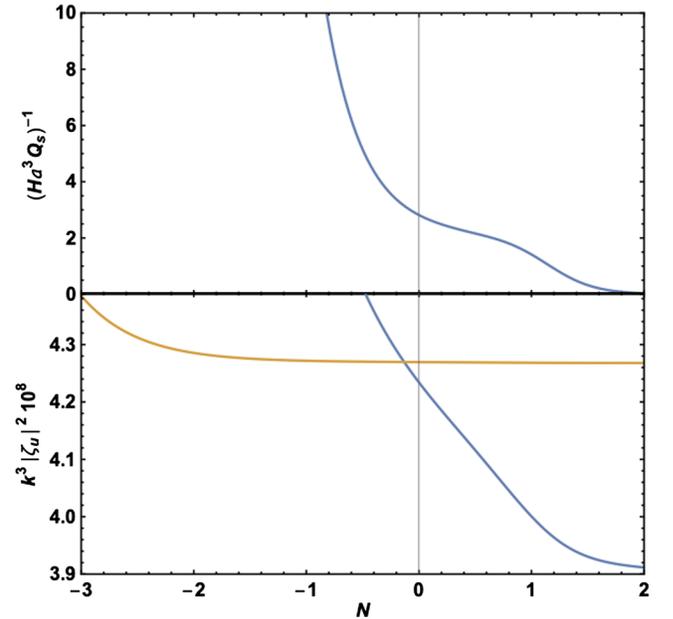


FIG. 5. Evolution of $1/(Ha^3 Q_s)$ and $k^3 |\zeta_u|^2$ for a mode with $x_t \approx 0.6$ (blue line) and a mode with $x_t \approx 0.04$ (yellow line) as a function of e -folds through the transition of $G_3(\varphi, X)$. Here, $N = 0$ corresponds to the time when the transition happens (defined as when $\bar{\varphi} = \varphi_0$).

grow temporarily during the transition. In the bottom panel, we show the evolution of two modes that have already left the horizon before the braiding transition. Whereas very long modes will have a power spectrum at the end of inflation with a value given by Eq. (66) (as shown in the yellow line for a mode with $x_t \approx 0.04$), other modes will keep decaying due to the braiding transition (as shown in the blue line for a mode with $x_t \approx 0.6$). This decay will ultimately cause a suppression on the amplitude of the power spectrum on short scales. In the figure, we show the blue mode that leaves the horizon less than 1 e -fold before the braiding transition, and its amplitude $k^3|\zeta_u|^2$ at the end of inflation is about 10% smaller than that for modes that leave the horizon well before the transition.

We can understand the behavior of ζ_u found in these numerical solutions by using Eq. (82). On the one hand, we see that during the de Sitter BUSR phase, Q_s , c_s , and H are nearly constants, and for modes outside the horizon the integrand of Eq. (82) scales as a and, as a consequence, this integral will be dominant over the integration constant. Thus, we find that $\zeta'_u \approx -(c_s x^2)\zeta_u$, in agreement with our previous calculation in Eq. (63). On the other hand, during the braiding transition, we see that (Ha^3Q_s) becomes roughly constant [this is model dependent as for a given parameter choice (Ha^3Q_s) could be decaying, constant, or growing], and thus the behavior of the integrand will be such that it decays as a^{-2} or even faster. In this case, the total term inside the square brackets in Eq. (82) goes to a constant, and we will get that $\zeta'_u \sim \text{const}$. We can then match these two behaviors at the time t_* where the braiding transition starts (for our figures, t_* would be between $N = -2$ and $N = -1$), and obtain that during the transition, the behavior of ζ_u outside the horizon will be roughly given by $\zeta_u(N) = \zeta_{u*}[1 - (c_{s*}x_*)^2(N - N_*)]$. This result shows that for modes that leave the horizon well before the transition (i.e., $x_* \ll 1$) then the decay rate of ζ_u during the transition will be very small. Similarly, for shorter modes that leave the horizon near the transition (e.g., $x_* \sim 1$), then the decay rate will be much larger. This agrees with the qualitative behavior shown in Fig. 5 but because (Ha^3Q_s) is not exactly constant, the x_* dependence on the slope of $\zeta_u(N)$ is closer to $x_*^{3/2}$ than to x_*^2 .

Finally, we explicitly calculate the power spectrum at the end of inflation from these mode functions. Figure 6 shows \mathcal{P}_{ζ_u} as a function of $x_t \propto k$. Here we see that the power spectrum is scale independent for all modes that left the horizon much earlier than the transition of G_3 , that is, for $x_t \lesssim 10^{-1}$. However, for modes that leave the horizon right before the braiding transition, there is a suppression of the power spectrum that goes up to 15%. From this figure, we find that the decay of \mathcal{P}_{ζ_u} goes roughly as $x_t^{3/2}$, consistent with the mode function evolution described above. Note, however, that this scaling depends on the choice of $\hat{\alpha}_B$ and the width of the braiding transition. Faster steps will lead to

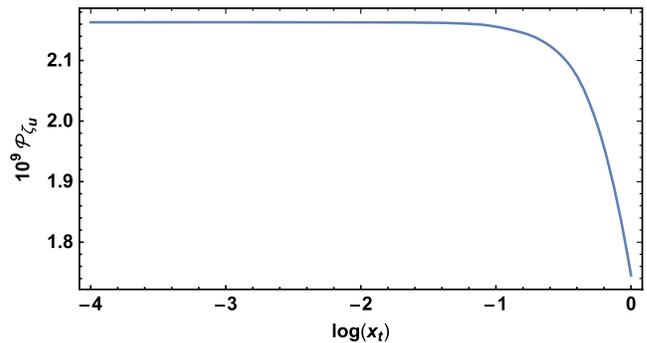


FIG. 6. Dimensionless power spectrum \mathcal{P}_{ζ_u} at the end of inflation as a function of $x_t = k/(a_t H_t)$. $x_t < 1$ represent modes that were outside the horizon at the time of the transition of $G_3(\varphi, X)$.

a faster decay of \mathcal{P}_{ζ_u} . We therefore conclude that most modes that left the sound horizon in the BUSR phase would indeed have the same primordial amplitude except for a small, but model dependent, number of e -folds in k around the sound horizon at the time of the braiding transition.

VI. DISCUSSION

In this paper we analyze the difference between three gauge-invariant quantities typically studied in single-field inflationary models: synchronous curvature ζ_s , unitary curvature ζ_u , and comoving curvature ζ_c . We focus on models with second-order derivative equations of motion and study the evolution of these three curvatures outside the horizon. Generically, we find that conservation of one of these curvatures does not imply conservation of the other two, unless the given model satisfies certain conditions. Conservation of comoving curvature is the most restrictive one, as it will imply conservation of both unitary and synchronous curvatures.

In order to explicitly illustrate the difference of these three curvatures, we provide two specific examples of inflationary models. First, we discuss the USR model and show that here unitary and comoving curvatures are not conserved outside the horizon (and they are equal), but synchronous curvature is. Second, we construct an inflationary model dubbed BUSR, in which there is a nontrivial coupling that corresponds to indirect derivative interactions (called braiding) between the scalar field and the metric. In this model, we show that unitary and synchronous curvatures are conserved outside the horizon (although they are different), but comoving curvature is not.

Finally, we discuss the consequences of having different curvatures conserved. First, we clarify that the concept of a separate universe is strictly associated with the synchronous frame since it requires that freely falling observers cannot distinguish that they live in a long-wavelength inhomogeneity through local measurements. Whereas it is typically assumed that conservation of ζ_s implies a separate universe, we show that this is not always the case.

The USR model is an example where ζ_s is conserved outside the horizon, but the continuity equation of matter does not approximate to that of a Friedmann universe, and therefore the separate universe approximation is technically broken.

In addition, we discuss the fact that a separate universe is typically considered to be a necessary requirement for the validity of the δN formalism, which is in turn used to calculate the primordial power spectrum and bispectrum of inflationary models. We show that although the USR model does mildly violate a separate universe, the δN formalism can still be used to accurately obtain observables for unitary curvature (even though this is not conserved), but the formalism will break down when calculating observables for uniform-density curvature ζ_ρ .

Finally, we discuss the consequence of a difference in unitary and comoving curvatures, as unitary curvature at the horizon crossing is typically used as an initial condition for comoving curvature after inflation. We use the BUSR model and start in a phase where $\zeta_u \neq \zeta_c$ due to braiding interactions, but allow the model to evolve toward a phase where braiding eventually goes away. This process allows inflation to end and the reheating to start with $\zeta_c = \zeta_u$, reflecting a strong evolution of ζ_c both after the horizon crossing and at the transition to vanishing braiding interaction. The transition can also introduce evolution of the unitary

curvature outside the horizon, and thus, at least for some k modes, it would be incorrect to identify ζ_u at the horizon crossing as the initial condition of ζ_c after inflation. In general, one should explicitly track the evolution of all k modes until the end of inflation (which are the actual quantities we are interested in obtaining) instead of naively assuming that all modes will be conserved after the horizon crossing. In the particular model considered here, this effect leads to a suppression of power that is confined to scales that are no larger than a few e -folds compared to the (sound) horizon at the time of the braiding transition, whereas larger modes that froze out well before the transition would be unaffected.

These examples help clarify the concepts of conservation of curvature, separate universe, δN , and gauge transformations that are often conflated in the literature.

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