


Implications of a frame dependent dark energy for the spacetime metric, cosmography, and effective Hubble constant

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In earlier papers we showed that a frame dependent effective action motivated by the postulates of three-space general coordinate invariance and Weyl scaling invariance exactly mimics a cosmological constant in Friedmann-Robertson-Walker (FRW) spacetimes but alters the linearized equations governing scalar perturbations around a spatially flat FRW background metric. Here we analyze the implications of a frame dependent dark energy for the spacetime cosmological metric within both a perturbative and a nonperturbative framework. Both methods of calculation give a one-parameter family of cosmologies which are in close correspondence to one another, and which contain the standard FRW cosmology as a special case. We discuss the application of this family of cosmologies to the standard cosmological distance measures and to the effective Hubble parameter, with special attention to the current tension between determinations of the Hubble constant at late time, and the Hubble value obtained through the cosmic microwave background angular fluctuation analysis.

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I. INTRODUCTION

The experimental observation of an accelerated expansion of the universe has been interpreted as evidence for a cosmological term in the gravitational action of the usual form

$$S_{\text{cosm}} = -\frac{\Lambda}{8\pi G} \int d^4x ({}^{(4)}g)^{1/2}, \quad (1)$$

with $\Lambda = 3H_0^2\Omega_\Lambda$ in terms of the Hubble constant H_0 and the cosmological fraction Ω_Λ . This functional form incorporates the usual assumption that gravitational physics is four-space general coordinate invariant, with no frame dependence in the fundamental action.

In a series of papers [1–4], motivated by the frame dependence of the cosmic microwave background (CMB) radiation, and ideas about scale invariance of an underlying prequantum theory, we have studied the implications of the assumption that there is an induced gravitational effective action that is a three-space general coordinate and Weyl scaling invariant, but is not four-space general coordinate invariant. This analysis leads to an alternative dark energy action given by

$$S_{\text{eff}} = -\frac{\Lambda}{8\pi G} \int d^4x ({}^{(4)}g)^{1/2} (g_{00})^{-2}, \quad (2)$$

which in Friedmann-Robertson-Walker (FRW) spacetimes where $g_{00} = 1$ exactly mimics the cosmological constant effective action of Eq. (1).

To set up a phenomenology for testing for the difference between standard and frame-dependent dark energy actions, we made the ansatz that the observed cosmological constant arises from a linear combination of the two of the form

$$S_\Lambda = (1-f)S_{\text{cosm}} + fS_{\text{eff}} \\ = -\frac{\Lambda}{8\pi G} \int d^4x ({}^{(4)}g)^{1/2} [1-f+f(g_{00})^{-2}], \quad (3)$$

so that $f = 0$ corresponds to only a standard cosmological constant, and $f = 1$ corresponds to only an apparent cosmological constant arising from a frame dependent effective action. In [4] we gave detailed results for the scalar perturbation equations around a FRW background arising when dark energy is included through the action of Eq. (3). Our aim in this paper is to analyze the implications of the ansatz of Eq. (3) for the spacetime cosmological metric, with applications to standard cosmological distance measures and the effective Hubble parameter. We will carry out the calculations in two ways, first by using the linearized perturbation equations around a FRW background derived in [4], and then by a nonperturbative method, giving results that are in close correspondence. Both methods yield a one-parameter family of cosmologies, parametrized by the initial value at cosmic time $t = 0$

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of one of the metric components, with the standard FRW cosmology included as a special case.

This paper is organized as follows.¹ In Sec. II we state our metric in nonperturbative and perturbative forms, and explain why when $f \neq 0$ the model of Eq. (3) cannot be reduced to a standard cosmological constant by redefinition of the time variable. We also introduce some notations that will be used frequently later on. In Sec. III we solve for the metric dynamics implied by Eq. (3) using the perturbative formalism. In Sec. IV we repeat this calculation by a nonperturbative method and show that the results closely correspond to those of Sec. III. In Sec. V we give results for the standard cosmographic distance measures and the effective Hubble parameter, and apply our results to the recently much discussed ‘‘Hubble tension.’’ In the Appendix A we give details of the perturbative derivation of the metric dynamical equation, which is also obtained as a by-product of the nonperturbative analysis. In Appendix B we give formulas for comparing our model with results from baryon acoustic oscillation (BAO) measurements.

II. HOMOGENEOUS ISOTROPIC METRIC, WHY THE $f \neq 0$ CASE DOES NOT REDUCE TO $f = 0$, AND SOME NOTATIONS

Our starting point is the observation that the general form for the line element in a homogeneous, isotropic, zero spatial curvature universe in which physics is invariant under three-space general coordinate transformations, but not invariant under four-space general coordinate transformations, is

$$ds^2 = \alpha^2(t)dt^2 - \psi^2(t)d\vec{x}^2 = g_{00}dt^2 + g_{ij}dx^i dx^j, \quad (4)$$

with metric components

$$\begin{aligned} g_{00} &= \alpha^2(t), & g_{ij} &= -\delta_{ij}\psi^2(t), \\ g^{00} &= 1/\alpha^2(t), & g^{ij} &= -\delta_{ij}/\psi^2(t). \end{aligned} \quad (5)$$

A. Why the $f \neq 0$ case cannot be reduced to $f = 0$ by redefining the time variable

An evident feature of Eq. (4) is that if we define a proper time τ by²

$$d\tau = \alpha(t)dt, \quad \tau = \int_0^t du \alpha(u), \quad (6)$$

with inversion $t(\tau)$, and denote $\psi(t)$ as written in terms of τ by $\psi(t(\tau)) = \psi[\tau]$, then Eq. (4) takes the form

$$ds^2 = d\tau^2 - \psi^2[\tau]d\vec{x}^2. \quad (7)$$

This has the same form as the standard FRW metric

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2, \quad (8)$$

with τ replacing t and with $\psi(t) = \psi[\tau]$ replacing the FRW expansion factor $a(t)$. However, this does not mean that Eq. (3) with $f \neq 0$ can be reduced to the $f = 0$ case. To see this, we rewrite Eq. (3) in terms of the metric components $\alpha(t)$ and $\psi(t)$, both assumed positive so that $({}^{(4)}g)^{1/2} = \alpha(t)\psi^3(t)$, giving

$$\begin{aligned} S_\Lambda &= -\frac{\Lambda}{8\pi G} \int dt d^3x \alpha(t)\psi^3(t)[1 - f + f\alpha(t)^{-4}] \\ &= -\frac{\Lambda}{8\pi G} \int d\tau d^3x \psi^3[\tau][1 - f + f\alpha[\tau]^{-4}]. \end{aligned} \quad (9)$$

When $f = 0$ the metric component $\alpha(t)$ is absorbed into the definition of the new time variable, as expected since the standard cosmological action is four-space general coordinate invariant. But when $f \neq 0$, the factor $\alpha[\tau]^{-4}$ cannot be similarly absorbed, reflecting the fact that the f term is only three-space, but not four-space general coordinated invariant.³ So we anticipate that Eq. (3) with $f \neq 0$ will give a more general cosmology than the standard FRW cosmology. However, since when $\alpha(t) \equiv 1$ Eq. (3) reduces back to a standard cosmological term, we also anticipate that this more general dynamics will include the standard FRW cosmology as a special case.

B. Some notations

We record the following notational conventions that will be used throughout.

- (i) The metric components $\alpha(t)$ and $\psi(t)$ form the basis of our nonperturbative treatment. But in both the nonperturbative and perturbative discussions, we will also use the related components $\theta(t)$, $\Phi(t)$, and $\Psi(t)$ defined by⁴

³If one were to define $d\tau = d\alpha(t)[1 - f + f/\alpha(t)^4]$, the Einstein-Hilbert action which is proportional to $\int dt d^3x \alpha(t)\psi^3(t)R$ would be left with a residual $\alpha(t)$ dependence. The same is true if one were to define $d\tau = \alpha(0)dt$.

⁴The factor $1/\theta(0)$ drops out of the equations below from which we calculate $\psi(t)$, since these are homogeneous in ψ . Similarly, it is not relevant for the equations of Sec. III where we calculate $\Phi(t)$ and $\Psi(t)$, since these equations are homogeneous in $a(t)$. But it guarantees that at early times $\psi(t)$ becomes identical to $a(t)$, which is important for the match to the CMB fits, since these are based on small fluctuation equations including \vec{x} dependence that are sensitive to the absolute magnitude of $a(t)$.

¹An earlier version [5] has been merged into this one.

²We have chosen the arbitrary lower limit of the integral in Eq. (6) so that the proper time origin $\tau = 0$ coincides with the coordinate time origin $t = 0$.

$$\begin{aligned}
 \alpha(t) &= 1 + \Phi(t), \\
 \psi(t) &= a(t)\theta(t)/\theta(0), \\
 \theta(t) &= 1 - \Psi(t),
 \end{aligned} \tag{10}$$

with $a(t)$ the standard FRW expansion factor.

- (ii) Since our focus is on the matter-dominated era, we use the following convenient approximate formula [6] for $a(t)$,⁵

$$a(t) \simeq \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} (\sinh(x))^{2/3}, \quad \Omega_m = 1 - \Omega_\Lambda, \tag{11}$$

with x the dimensionless time variable

$$x = \frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t. \tag{12}$$

At the present era $t = t_0$ defined by $a(t_0) = 1$, x_0 takes the value

$$x_0 = \operatorname{arcsinh}((\Omega_\Lambda/\Omega_m)^{1/2}) \simeq 1.169. \tag{13}$$

Equations (12) and (13) then give $H_0 t_0 = (2/3)x_0/\sqrt{\Omega_\Lambda} = 0.946$, in agreement with the Planck values for this product, where we have used the Planck 2018 values [7] $\Omega_\Lambda = 0.679$, $\Omega_m = 0.321$. However, we will see below in Sec. V that H_0 in Eqs. (11) and (12) is rescaled by a factor $\alpha(0)$ from the Planck value $H_0^{\text{Pl}} \simeq 67 \text{ km s}^{-1} \text{ Mpc}^{-1}$, so that $H_0 = \alpha(0)H_0^{\text{Pl}}$.

- (iii) The Hubble parameter $H(t)$ defined by the standard FRW cosmology is

$$H(t) = \frac{da(t)/dt}{a(t)} = \frac{\dot{a}(t)}{a(t)}, \tag{14}$$

which in the matter-dominated era, using the approximate formula of Eq. (11), is

$$\begin{aligned}
 H(t) &= H_0 \sqrt{\Omega_\Lambda} \coth(x) \\
 &= H_0 [\Omega_m (1+z)^3 + \Omega_\Lambda]^{1/2},
 \end{aligned} \tag{15}$$

with $\coth(x) = \cosh(x)/\sinh(x)$ and with the redshift z defined by $1+z = 1/a(t)$. This is to be distinguished from the Hubble parameter $H_{\text{eff}}(t)$ arising from the modified dynamics of Eq. (3), which is given by the proper time derivative

$$H_{\text{eff}}[\tau] = \frac{d\psi[\tau]/d\tau}{\psi[\tau]} = H_{\text{eff}}(t) = \frac{d\psi(t)/dt}{\alpha(t)\psi(t)}. \tag{16}$$

III. METRIC PERTURBATION DERIVATION

A. Setting up the Φ equation

In [4] we gave detailed results for the scalar perturbation equations around a FRW background arising when dark energy is included through the action of Eq. (3). Writing [8]⁶

$$\begin{aligned}
 g_{00} &= 1 + E, \\
 g_{i0} &= -a(t)(\partial_i F + \text{vector}), \\
 g_{ij} &= -a^2(t)[(1+A)\delta_{ij} + \partial_i \partial_j B + \text{vector} + \text{tensor}],
 \end{aligned} \tag{17}$$

we gave formulas in $B = 0$ gauge for the linearized equations governing the scalar perturbations A , E , and F , for the general case in which these are functions of both space and time. For the analysis of this paper, it is more convenient to express the $B = 0$ gauge linear perturbation equations in terms of F and the functions Φ and Ψ that are related to E and A by

$$\begin{aligned}
 E &= 2[\Phi - \partial_t(aF)], \\
 A &= -2[\Psi + \dot{a}F],
 \end{aligned} \tag{18}$$

as given in Appendix A. This form of the perturbation equations corresponds to writing the perturbed line element as

$$ds^2 = [1 + 2\Phi]dt^2 - a^2(t)[1 - 2\Psi]d\vec{x}^2, \tag{19}$$

in agreement to linear order with Eqs. (4) and (10) above when the $1/\theta(0)$ factor is omitted (or is included in the normalization of the expansion factor a).

In the limit that the metric perturbations have no spatial dependence, the perturbation F , which appears in Eq. (17) acted on by spatial derivatives, can be assumed to have the \vec{x} -independent limit $F(t) = 0$ and so can be dropped from the metric perturbation equations.⁷ Also, in the matter-dominated era the anisotropic inertia [8] $\pi^S = 0$, so assuming continuity in the limit of vanishing spatial dependence, the $Y = 0$ part of Eq. (A1) in Appendix A implies that $\Phi(t) = \Psi(t)$. With these simplifications, we

⁵Before the matter-dominated era, when the radiation content of the universe is significant, $a(t)$ is no longer well approximated by Eq. (11). But this does not affect the numerical results given below.

⁶The equations of Sec. III and Appendix A follow [4,8] and do not include the factor $1/\theta(0)$ of Eq. (10), which can be thought of here as residing in the normalization of a . This factor is not relevant for the analysis of this section since the equations that we use to determine Φ and Ψ are homogeneous in a , and so the normalization of a drops out.

⁷This still allows one to make three-space general coordinate transformations, but not four-space general coordinate transformations, which corresponds to the invariance properties of the action of Eq. (3).

show in Appendix A that the perturbation equations governing the time evolution of $\Phi(t)$ can be put into the form (with $\dot{} = d/dt$)

$$\ddot{\Phi} + 4\frac{\dot{a}}{a}\dot{\Phi} + \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\Phi = 4\pi G\delta p + 2\Lambda f\Phi, \quad (20)$$

with δp the pressure perturbation. As noted, this equation is independent of the normalization of $a(t)$. When the term proportional to f is dropped, this agrees with Eq. (7.49) of Mukhanov [9] and Eq. (23c) of Ma and Bertschinger [10] when their conformal time derivatives are converted to time derivatives. In the matter-dominated era, δp vanishes for adiabatic perturbations, so then we can drop the δp term in Eq. (20), giving

$$\ddot{\Phi} + 4\frac{\dot{a}}{a}\dot{\Phi} + \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\Phi = 2\Lambda f\Phi. \quad (21)$$

It is convenient now to use the dimensionless time variable x introduced above in Eq. (12), which when substituted into Eq. (21) gives finally the evolution equation for Φ in terms of x ,

$$\frac{d^2\Phi}{dx^2} + \frac{8}{3}\coth(x)\frac{d\Phi}{dx} = \frac{4}{3}(2f-1)\Phi. \quad (22)$$

B. Large and small x behavior and numerical solution

Before proceeding to the numerical solution of Eq. (22), we examine analytically the large and small x behavior of the solutions. For large x the function $\coth(x)$ approaches unity, and Eq. (22) becomes an equation with constant coefficients solved by exponentials,

$$\Phi(x) = C_1 e^{\mu_+ x} + C_2 e^{\mu_- x},$$

$$\mu_{\pm} = -\frac{2}{3}[2 \pm (6f+1)^{1/2}]. \quad (23)$$

As suggested already by the factor $2f-1$ in Eq. (22), there is a crossover in behavior at $f=1/2$. For $f < 1/2$, both exponents in Eq. (23) are negative, and Φ decays to zero as $x \rightarrow \infty$. For $f = 1/2$, one exponent is negative, while the other is zero, and Φ approaches a constant as $x \rightarrow \infty$. For $f > 1/2$, one exponent remains negative, while one is positive, and Φ becomes infinite as $x \rightarrow \infty$. So for $f = 1$, the case of a scale invariant cosmological action, the metric perturbation Φ grows with time.

We examine next the small x behavior, where the term proportional to $2f-1$ becomes much less important than the terms on the left of Eq. (22). This equation is then approximated by

$$\frac{d^2\Phi}{dx^2} + \frac{8}{3}\frac{1}{x}\frac{d\Phi}{dx} = 0, \quad (24)$$

with the general solution

$$\Phi(x) = C_3 + C_4 x^{-5/3}. \quad (25)$$

This shows that sufficient boundary conditions for getting a unique solution are the requirements that the solution be regular at $x=0$, together with specification of the value $\Phi(0)$.

With this analysis in mind, we rewrite Eq. (22) as an integral equation. Defining the normalized perturbation

$$\hat{\Phi}(x) = \Phi(x)/\Phi(0), \quad (26)$$

$\hat{\Phi}$ obeys the integral equation

$$\hat{\Phi}(x) = 1 + \frac{4}{3}(2f-1) \int_0^x dw (\sinh(w))^{-8/3}$$

$$\times \int_0^w du (\sinh(u))^{8/3} \hat{\Phi}(u), \quad (27)$$

which incorporates both the boundary conditions at $x=0$ and the differential equation of Eq. (22). Starting from an initial assumption $\hat{\Phi}(x) = 1$, and then updating at each evaluation of the right-hand side of Eq. (27), the integral equation converges rapidly to an answer in 5 iterations on an 800 bin mesh⁸ taking integrand values at center of bin. The results for $d\hat{\Phi}/dx$ with $f=1$ are given in Table I, for various values of x ranging from 0 to x_0 , showing that $d\hat{\Phi}/dx|_{x=x_0}$ is positive. Consistent with the large time analysis given above, when we repeat the calculation with $f=0$ we find a negative value of $d\hat{\Phi}/dx|_{x=x_0}$. In the final column of Table I we tabulate the quadratic approximation to $\hat{\Phi}(x)$,

$$\hat{\Phi}(x) \simeq 1 + C(x/x_0)^2, \quad C = 0.244, \quad (28)$$

which is accurate to a few parts per thousand over the entire interval $0 \leq x/x_0 \leq 1$.

IV. NONPERTURBATIVE DERIVATION

A. Einstein tensor and equations, and the dark energy and matter energy momentum tensors

We return now to the general form for the line element given in Sec. II,

$$ds^2 = \alpha^2(t)dt^2 - \psi^2(t)d\vec{x}^2. \quad (29)$$

⁸On this coarse mesh, the value $x_{\text{eq}} < 10^{-5}$ at the transition from radiation domination to matter domination is indistinguishable from 0, so it suffices to take zero as the lower limit of the integrals.

TABLE I. Values of $d\hat{\Phi}/dx$ and $\hat{\Phi}$ versus x/x_0 and redshift z , calculated with $f = 1$, that is, with all of the observed cosmological constant arising from a scale invariant but frame dependent action. The final column gives the fit of $\hat{\Phi}$ to the quadratic $1 + C(x/x_0)^2$, with $C = 0.244$.

x/x_0	z	$d\hat{\Phi}/dx$	$\hat{\Phi}$	$1 + 0.244(x/x_0)^2$
1.0	0.00	0.409	1.244	1.244
0.9	0.10	0.369	1.198	1.197
0.8	0.22	0.331	1.158	1.156
0.7	0.36	0.290	1.121	1.120
0.6	0.54	0.251	1.089	1.088
0.5	0.77	0.209	1.062	1.061
0.4	1.08	0.169	1.040	1.039
0.3	1.55	0.126	1.023	1.022
0.2	2.36	0.0848	1.010	1.0098
0.1	4.36	0.0416	1.0026	1.0024
0.0341	10	0.0131	1.0003	1.000028
0.00122	100	0.000018	1.0000	1.0000

Using a *Mathematica* notebook for general relativity [11], the nonzero Einstein tensor components for this metric are computed to be

$$G_{00} = -3 \frac{\dot{\psi}^2(t)}{\psi^2(t)},$$

$$G_{ij} = \delta_{ij} \frac{2\psi(t)[\alpha(t)\dot{\psi}(t) - \dot{\alpha}(t)\psi(t)] + \alpha(t)\dot{\psi}^2(t)}{\alpha^3(t)} \quad (30)$$

where $\dot{\psi}^2(t) = (d\psi(t)/dt)^2$.

Writing the dark energy action of Eq. (3) as

$$S_\Lambda = -\frac{\Lambda}{8\pi G} \int d^4x ({}^{(4)}g)^{1/2} [1 - f + f/\alpha^4(t)], \quad (31)$$

and varying this equation with respect to g_{ij} , we get the ij component of the dark energy contribution to the energy momentum tensor,

$$T_\Lambda^{ij} = \frac{\Lambda}{8\pi G} [(1-f)g^{ij} + f t^{ij}],$$

$$T_{\Lambda ij} = \frac{\Lambda}{8\pi G} [(1-f)g_{ij} + f t_{ij}], \quad (32)$$

with

$$t^{ij} = -\delta_{ij} \frac{1}{\alpha^4(t)\psi^2(t)}, \quad t_{ij} = -\delta_{ij} \frac{\psi^2(t)}{\alpha^4(t)}. \quad (33)$$

As detailed in [1–4], in order to use this as the source term in the Einstein equations, the remaining components of $t_{\mu\nu}$ must be determined so that covariant conservation with respect to the metric is satisfied. The conserving completion of the g_{ij} part of Eq. (32) is just $g_{\mu\nu}$. Since the metric is diagonal, the conserving completion of t_{ij} in Eq. (33) has

$t_{0i} = t_{i0} = 0$, and t_{00} given by the solution of the covariant conservation equation, which is

$$\dot{t}_{00}(t) + t_{00}(t) \left[3 \frac{\dot{\psi}(t)}{\psi(t)} - 2 \frac{\dot{\alpha}(t)}{\alpha(t)} \right] = 3 \frac{\dot{\psi}(t)}{\alpha^2(t)\psi(t)}. \quad (34)$$

This equation can be readily integrated to give

$$t_{00}(t) = 3 \frac{\alpha^2(t)}{\psi^3(t)} \int_0^t du \frac{\dot{\psi}(u)\psi^2(u)}{\alpha^4(u)}, \quad (35)$$

where we have arbitrarily taken the lower limit of integration as zero (more about this below).

We additionally need the particulate matter energy momentum tensor, for which we take the relativistic perfect fluid form

$$T_{\mu\nu}^{\text{pm}} = (p + \rho)u_\mu u_\nu - p g_{\mu\nu}, \quad (36)$$

with p the pressure, ρ the energy density, and $u_\mu = g_{\mu\nu} dx^\nu/ds$ the four velocity. For the metric of Eq. (4), u_0 is given by

$$u_0 = \alpha^2(t) \frac{dx^0}{ds} = \alpha^2(t) \frac{dx^0}{\alpha(t)dt} = \alpha(t) \frac{dx^0}{dt} = \alpha(t), \quad (37)$$

from which we find

$$T_{00}^{\text{pm}} = \rho \alpha^2(t),$$

$$T_{ij}^{\text{pm}} = \delta_{ij} p \psi^2(t). \quad (38)$$

Covariant conservation of $T_{\mu\nu}^{\text{pm}}$ implies

$$\frac{d(\rho(t)\psi^3(t))}{dt} = -3p(t)\dot{\psi}(t)\psi^2(t), \quad (39)$$

which dividing by $\alpha(t)$ on both sides is equivalent to

$$\frac{d(\rho[\tau]\psi^3[\tau])}{d\tau} = -3p[\tau] \frac{d\psi[\tau]}{d\tau} \psi^2[\tau]. \quad (40)$$

B. Einstein equations

We now have all the ingredients needed to write down the Einstein equations. From the 00 component we get

$$\frac{\dot{\psi}^2(t)}{\alpha^2(t)\psi^2(t)} = \frac{\Lambda}{3} \left[1 - f + f \frac{3}{\psi^3(t)} \int_0^t du \frac{\dot{\psi}(u)\psi^2(u)}{\alpha^4(u)} \right] + \frac{8\pi G}{3} \rho(t), \quad (41)$$

and from the spatial components we get

$$\begin{aligned} & \frac{2\psi(t)[\alpha(t)\dot{\psi}(t) - \dot{\alpha}(t)\psi(t)] + \alpha(t)\dot{\psi}^2(t)}{\alpha^3(t)} \\ &= \psi^2(t) \left[(1-f)\Lambda + \frac{f\Lambda}{\alpha^4(t)} - 8\pi Gp(t) \right]. \end{aligned} \quad (42)$$

As noted above, both of these equations are independent of the normalization of $\psi(t)$.

The three equations Eq. (41), Eq. (42), and Eq. (39) are not independent, by virtue of the Bianchi identities for $G_{\mu\nu}$ and covariant conservation of the energy momentum tensors. Multiplying Eq. (41) through by $\psi^3(t)$, differentiating with respect to time, eliminating $(d/dt)(\psi^3(t)\rho(t))$ by Eq. (39), and dividing by $\dot{\psi}(t)$, we recover Eq. (42). Hence effectively we have only one equation relating the metric components α and ψ , and a second equation will be needed, a topic which we address in the next section.

But first let us address several issues raised by the structure of the above equations, beginning with the arbitrary lower limit in the integral in Eq. (35). In the sequel we will be interested in solving Eqs. (41) and (42) in the matter-dominated era, in which the pressure $p = 0$. Then Eq. (39) can be integrated to give $\rho(t) = C/\psi^3(t)$, with C a constant of integration, and we see that a shift in the lower limit of the integral in Eqs. (35) and (41) can be absorbed into a shift in the integration constant C . So there is no loss of generality in taking the lower limit of integration as zero.⁹ A further property of Eqs. (41) and (42) is that when $\alpha(t) \equiv 1$, they reduce when $\psi(0) = 0$ to the usual FRW equations with $\psi(t)$ playing the role of the FRW expansion factor $a(t)$,

$$\begin{aligned} \frac{\dot{\psi}^2(t)}{\psi^2(t)} &= \frac{\Lambda}{3} + \frac{8\pi G}{3}\rho(t), \\ 2\psi(t)\dot{\psi}(t) + \dot{\psi}^2(t) &= \psi^2(t)[\Lambda - 8\pi Gp(t)]. \end{aligned} \quad (43)$$

These equations have as the solution $\psi(t) = a(t)$, a property that will be relevant for finding a second equation relating $\alpha(t)$ and $\psi(t)$.

C. Finding the second equation

To find a second equation relating $\alpha(t)$ and $\psi(t)$, we generalize to the case where these are also functions of the coordinate \vec{x} , so that we have $\alpha(t, \vec{x})$, $\psi(t, \vec{x})$, and then assume a smooth limit as the coordinate dependence limits to zero. When α and ψ are coordinate dependent, the Einstein tensor component G_{xy} is given by

$$\begin{aligned} \psi^2 \alpha G_{xy} &= -2\alpha \partial_x \psi \partial_y \psi + \alpha \psi \partial_x \partial_y \psi + \psi^2 \partial_x \partial_y \alpha \\ &\quad - \psi (\partial_x \alpha \partial_y \psi + \partial_y \alpha \partial_x \psi), \end{aligned} \quad (44)$$

which is independent of the normalization of $\psi(t, \vec{x})$. Since the energy momentum source tensors have no xy component, G_{xy} must be equated to zero. To proceed further, with no loss of generality so far we write α and ψ as

$$\begin{aligned} \alpha(t, \vec{x}) &= 1 + \Phi(t, \vec{x}), \\ \psi(t, \vec{x}) &= a(t)\theta(t, \vec{x})/\theta(0, \vec{0}), \\ \theta(t, \vec{x}) &= 1 - \Psi(t, \vec{x}), \end{aligned} \quad (45)$$

corresponding to introducing \vec{x} dependence into Eq. (10) above. Substituting into Eq. (44), we get

$$\begin{aligned} 0 &= -2(1 + \Phi)\partial_x \Psi \partial_y \Psi - (1 + \Phi)(1 - \Psi)\partial_x \partial_y \Psi \\ &\quad + (1 - \Psi)^2 \partial_x \partial_y \Phi + (1 - \Psi)(\partial_x \Phi \partial_y \Psi + \partial_y \Phi \partial_x \Psi). \end{aligned} \quad (46)$$

To leading linear order in Φ and Ψ , this reduces to

$$0 = \partial_x \partial_y [\Phi(t, \vec{x}) - \Psi(t, \vec{x})], \quad (47)$$

which assuming a smooth limit to vanishing spatial dependence gives the leading order result

$$\Phi(t) = \Psi(t). \quad (48)$$

We have not found a model-independent way to go beyond leading order, so we proceed by introducing a specific model, from which we abstract a relation which we then use with no further reference to the model. Our model is to assume that Φ and Ψ are dominated by a single plane wave so that

$$\begin{aligned} \Phi(t, \vec{x}) &= \Phi(t)e^{i\vec{k}\cdot\vec{x}}, \\ \Psi(t, \vec{x}) &= \Psi(t)e^{i\vec{k}\cdot\vec{x}}. \end{aligned} \quad (49)$$

With this assumption $\partial_x \equiv ik_x$, $\partial_y \equiv ik_y$, which can then be factored from all terms of Eq. (46). This leaves a purely algebraic relation between $\Phi(t)$ and $\Psi(t)$, which can be reduced to the simple form

$$\Phi(t) = \frac{\Psi(t)}{1 - 2\Psi(t)}, \quad (50)$$

which to first order reproduces Eq. (48). Reexpressing Eq. (50) in terms of $\theta(t)$ and $\alpha(t)$ we find

⁹When Eq. (41) is converted to Eq. (42) by multiplication by $\psi^3(t)$ and differentiation, the constant C drops out, and is replaced by the additional constant of integration needed for a second order differential equation as opposed to a first order equation. This new constant will end up parametrizing the family of solutions of our model.

$$\begin{aligned}\alpha(t) &= \frac{\theta(t)}{2\theta(t) - 1}, \\ \theta(t) &= \frac{\alpha(t)}{2\alpha(t) - 1}, \\ 2\theta(t)\alpha(t) &= \theta(t) + \alpha(t).\end{aligned}\quad (51)$$

These equations give the relation between α and ψ that we will use in our analysis beyond leading order. It has the nice feature that $\alpha(t) \equiv 1$ implies that $\theta(t) \equiv 1$, $\psi(t) \equiv a(t)$, which is the same property found for Eqs. (41) and (42) above. Thus our model relating α and ψ is structurally compatible with the Einstein equations derived in the preceding section.

D. Differential equation for θ , reduction to the linearized case, and large and small time behavior

Since study of the linearized case in Sec. III shows that $\Psi(t) = 1 - \theta(t)$ is slowly varying in comparison to the FRW expansion factor $a(t)$, we turn Eq. (42) into an equation for θ by substituting $\psi(t) = a(t)\theta(t)/\theta(0)$. Setting the pressure term p to zero, using the zeroth order FRW equation $2\ddot{a}/a + (\dot{a}/a)^2 = \Lambda$, changing to the dimensionless variable x of Eq. (12), and using $a'/a = (2/3)\coth(x)$ with $'$ denoting d/dx , we arrive at

$$\begin{aligned}\theta'' &= F[\theta', \theta], \\ F[\theta', \theta] &= \frac{2}{3}\alpha^2[1 - f + f/\alpha^4]\theta - \frac{(2\theta + 1)}{2(2\theta - 1)}\frac{(\theta')^2}{\theta} \\ &\quad - \frac{4}{3}\coth(x)\frac{3\theta - 1}{2\theta - 1}\theta' - \frac{2}{3}\theta,\end{aligned}\quad (52)$$

with α in the first term of F related to θ by $\alpha = \theta/(2\theta - 1)$. To check that this reduces to the linearized case, we set $\theta = 1 - \Psi$ and keep only terms of first order in Ψ , giving after a little algebra

$$\Psi'' + \frac{8}{3}\coth(x)\Psi' = \frac{4}{3}(2f - 1)\Psi, \quad (53)$$

in agreement (using the perturbative relation $\Psi = \Phi$) with Eq. (22) above. We note that since $\theta = 1$ corresponds to $\alpha = 1$, Eq. (52) implies that $F(0, 1) = 0$, with the consequence that initial data $\theta'(0) = 0$ and $\theta(0) = 1$ propagate forward in time unchanged.

Since Eq. (52) is even in x it admits θ to be an even function of x , and so at $x = 0$ there will be a regular solution with $\theta(0)$ an arbitrary constant, and $\theta'(0) = 0$. Making the ansatz that θ becomes infinite for large x , Eq. (52) reduces to an equation with constant coefficients. Substituting $\theta(x) = e^{\lambda x}$ we find the algebraic equation

$$\lambda^2 + \frac{4}{3}\lambda + \frac{4}{9} = \frac{1}{9}(15f + 1), \quad (54)$$

with roots

$$\lambda = -\frac{2}{3} \pm \frac{1}{3}(15f + 1)^{1/2}, \quad (55)$$

to be compared with

$$\lambda = -(2/3)[2 \pm (6f + 1)^{1/2}] \quad (56)$$

found from the linearized equation in Sec. III. When $f = 1$ Eq. (55) gives an exponentially growing solution with $\lambda = 2/3 \simeq 0.67$, somewhat larger than the exponent $(2/3)(\sqrt{7} - 2) \simeq 0.43$ given for $f = 1$ by Eq. (56). For the nonperturbative formula of Eq. (55) the exponentially growing solution appears for $f > 1/5$, whereas for the linearized result of Eq. (56) the exponentially growing solution first appears for the larger value $f > 1/2$.

E. Numerical solution

To solve for θ numerically, we employ stepwise forward integration of Eq. (52) starting from $x = 0$,

$$\begin{aligned}\theta'(i + 1) &= \theta'(i) + F[\theta'(i), \theta(i)]\Delta x, \\ \theta(i + 1) &= \theta(i) + \theta'(i)\Delta x,\end{aligned}\quad (57)$$

with $\Delta x = 1/800$, and then as a check with $\Delta x = 1/1600$. These two calculations differ only in the fifth decimal place. In Table II we give results with $f = 1$ for θ , and for the normalized ratio $\Psi(x)/\Psi(0) = [1 - \theta(x)]/[1 - \theta(0)]$, starting from $\theta(0) = 1.1$, or equivalently $\Psi(0) = -0.1$, and $\theta'(0) = 0$. The normalized ratio $\Psi(x)/\Psi(0)$ is directly comparable to $\hat{\Phi}$ calculated in Sec. III and given in Table I; in the final column of Table II we give $\hat{\Phi}$ recalculated by the stepwise forward integration method used here for solving the θ differential equation, which agrees to within one or two parts per thousand with the iterative integral equation

TABLE II. Values of $\theta(x)$ and $\Psi(x)/\Psi(0)$ calculated for $\theta(0) = 1.1$, and in the final column the corresponding linearized equation result $\hat{\Phi}(x)$, versus x and redshift z , taking $f = 1$, that is, with all of the observed cosmological constant arising from a scale invariant but frame dependent action.

z	x	$\theta(x)$	$\Psi(x)/\Psi(0)$	$\hat{\Phi}(x)$
0	1.169	1.126	1.264	1.243
0.1	1.054	1.122	1.215	1.198
0.2	0.955	1.118	1.177	1.163
0.3	0.868	1.115	1.146	1.135
0.4	0.792	1.112	1.122	1.113
0.5	0.726	1.110	1.103	1.095
0.6	0.668	1.109	1.087	1.080
0.7	0.616	1.107	1.074	1.069
0.8	0.571	1.106	1.064	1.059
0.9	0.530	1.106	1.055	1.051
1.0	0.494	1.105	1.048	1.044

method used in Sec. III.¹⁰ We see that the final two columns of Table II agree to within a couple of percent, indicating that the linear perturbation results of Sec. III suffice at present for phenomenological applications.

We have also checked that with the iteration of Eq. (57), initial data $\theta(0) = 1$ and $\theta'(0) = 0$ propagate forward in time unchanged. For $\theta(0) > 1$, the numerical solution for $\theta(x)$ with $f = 1$ increases with increasing x , while for $\theta(0) < 1$ the numerical solution for $\theta(x)$ decreases with increasing x , corresponding respectively to a universe that expands more rapidly, or more slowly, than the standard FRW solution $a(t)$.

F. Correspondence of the nonperturbative and perturbative solutions

To conclude this section, we see that a nonperturbative treatment of the problem that we addressed by linearized perturbation theory in Sec. III leads again to a one-parameter family of cosmologies, parametrized now by the initial value $\theta(0)$. In the perturbative treatment, the corresponding parameter was $\Phi(0)$, and this parameter is related to the one used in the nonperturbative analysis by $\Phi(0) \simeq \Psi(0) = 1 - \theta(0)$. For $\theta(0) = 1$, $\Phi(0) = 0$, the nonperturbative and perturbative evolutions both have as the solution the standard FRW expansion factor $a(t)$; that is, $\theta(t)$ remains equal to 1 [and $\Phi(t)$ remains 0] for all times t . For $\theta(0) > 1$, $\Phi(0) < 0$ the $f = 1$ nonperturbative and perturbative evolutions both give a universe that expands faster than the standard FRW solution $a(t)$, as discussed further below, and for $\theta(0) < 1$, $\Phi(0) > 0$ the $f = 1$ nonperturbative and perturbative evolutions both give a universe that expands more slowly than the standard FRW solution $a(t)$. There is thus a direct qualitative correspondence between the two methods of treating the model of Eq. (3), and for parameter values $\theta(0)$ close to unity, there is only a small quantitative difference. We have also seen that the nonperturbative treatment gives a relatively straightforward way of rederiving the perturbation equation of Eq. (22).

V. COSMOGRAPHIC EQUATIONS AND APPLICATION TO THE ‘‘HUBBLE TENSION’’

To discuss cosmography we rewrite the metric in terms of proper time τ as in Eq. (7),

$$ds^2 = d\tau^2 - \psi^2[\tau]d\vec{x}^2, \quad (58)$$

which we recall takes the standard FRW form with τ replacing t and with $\psi[\tau] \equiv \psi(t) = a(t)\theta(t)/\theta(0)$ replacing $a(t)$. Thus in comparisons with experiment, it is τ that will be the true physical time, and $\psi[\tau]$ the true expansion rate. We proceed now with a number of applications of the

¹⁰For example, interpolating into Table II we find for $z = 0.54$ that $\hat{\Phi} = 0.6 \times 1.095 + 0.4 \times 1.080 = 1.089$, compared with 1.089 in Table I.

analysis of the previous sections. Where we give perturbative results, we use the first order accurate equality $\Phi(t) \simeq \Psi(t)$, which is equivalent to $\theta(t)\alpha(t) \simeq 1$.

- (i) Substituting the quadratic approximation to $\hat{\Phi}$ from Table I, we get a simple approximate formula for the transformation from coordinate time t to proper time τ ,

$$\begin{aligned} \tau(t) &= \int_0^t \alpha(u)du = \int_0^t du\alpha(0)[\alpha(u)/\alpha(0)] \\ &\simeq \alpha(0) \int_0^t du[1 + \Phi(0)(\hat{\Phi}(u) - \hat{\Phi}(0))] \\ &\simeq \alpha(0) \int_0^t du[1 + \Phi(0)C(u/t_0)^2] \\ &= \alpha(0)t[1 + (\Phi(0)C/3)(t/t_0)^2], \end{aligned} \quad (59)$$

with $C = 0.244$. This has the inversion

$$t[\tau] = (\tau/\alpha(0))[1 - (\Phi(0)C/3)(\tau/\tau_0)^2]. \quad (60)$$

- (ii) We now have to find the transformation that relates H_0 and t_0 with the physical quantities H_0^{Pl} and τ_0^{Pl} measured by Planck [7] (and earlier, by WMAP [12]) from the CMB. We note from Table I that for redshifts z greater than 100, and therefore at and before decoupling, $\Phi(t)$ is very accurately the constant $\Phi(0)$, and thus $\alpha(t)$ is very accurately equal to $\alpha(0)$. So the transformation from coordinate time t to proper time τ is just a constant rescaling $\tau = \alpha(0)t$. Since to first order $\theta(t) \simeq 1/\alpha(t)$, for large redshifts $\theta(t) \simeq \theta(0)$, and thus $\psi[\tau] = a(t)\theta(t)/\theta(0) \simeq a(t) = a(\tau/\alpha(0))$. Hence the relation between H_0^{Pl} and H_0 is just a rescaling by $\alpha(0)$,

$$H_0^{\text{Pl}} = H_0/\alpha(0), \quad H_0 = H_0^{\text{Pl}}\alpha(0), \quad (61)$$

and introducing the age of the universe τ_0^{Pl} as measured by Planck, we have

$$\begin{aligned} \tau_0^{\text{Pl}} &= H_0 t_0 / H_0^{\text{Pl}} = t_0 \alpha(0), \\ H_0 t_0 &= H_0^{\text{Pl}} \tau_0^{\text{Pl}} = \frac{2x_0}{3\sqrt{\Omega_\Lambda}} = 0.946. \end{aligned} \quad (62)$$

We will see that the effect of these rescalings is to make all physical process depend on $\hat{\Phi}(t)$ through the combination $\hat{\Phi}(t) - \hat{\Phi}(0) = C(t/t_0)^2$, so that if the late time increase coefficient C were set to zero, there would be no physical effects. This is essential, because with $C = 0$ the relation between the coordinate time t and the proper time τ of Eq. (59) becomes a constant rescaling for all times, and a change in the units in which time is measured should

have no effect on physical consequences of the equations.

- (iii) The nonperturbative generalizations of the standard cosmological distance measures are obtained by making the substitution $a(t) \rightarrow \psi[\tau] = \psi(t) = a(t)\theta(t)/\theta(0)$ in the usual formulas, with the following results, where the subscript 0 denotes the present time, and 1 denotes the past time at which an object at coordinate distance r_1 emitted a signal.

For waves of wavelength λ and frequency ν , we have

$$1 + z_{\text{eff}} = \frac{\psi[\tau_0]}{\psi[\tau_1]} = \frac{1}{\psi[\tau_1]} = \frac{\theta(0)}{a(t_1)\theta(t_1)}, \quad (63)$$

where in the second equality we used $\psi[\tau_0] = 1$.

The parallax distance d_P , luminosity distance d_L , angular diameter distance d_A , and proper motion or comoving angular diameter distance d_M , are given by

$$\begin{aligned} d_P &= r_1\psi(t_0), \\ d_L &= r_1\psi(t_0)(1 + z_{\text{eff}}), \\ d_A &= r_1\psi(t_1), \\ d_M &= r_1\psi(t_0), \end{aligned} \quad (64)$$

with ratios

$$\begin{aligned} \frac{d_A}{d_L} &= \frac{1}{(1 + z_{\text{eff}})^2}, \\ \frac{d_M}{d_L} &= \frac{1}{1 + z_{\text{eff}}}, \\ \frac{d_A}{d_P} &= \frac{1}{1 + z_{\text{eff}}}. \end{aligned} \quad (65)$$

- (iv) The perturbative formulas needed to evaluate the above expressions are¹¹

$$\begin{aligned} \frac{\theta(0)}{\theta(t)} &= 1 + \Phi(0)[\hat{\Phi}(t) - \hat{\Phi}(0)] \\ &= 1 + \Phi(0)Cx_\tau^2/x_0^2, \\ a(t) &= \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \left(\sinh\left(x_\tau \left[1 - \frac{1}{3}\Phi(0)C\frac{x_\tau^2}{x_0^2}\right]\right)\right)^{2/3}, \\ x_\tau &= \frac{3}{2}\sqrt{\Omega_\Lambda}H_0^{\text{Pl}}\tau. \end{aligned} \quad (66)$$

- (v) As noted in Sec. II, the effective Hubble parameter is defined as the proper time derivative

$$H_{\text{eff}}[\tau] = \frac{d\psi[\tau]/d\tau}{\psi[\tau]}, \quad (67)$$

or in terms of t ,

$$\begin{aligned} H_{\text{eff}}(t) &= \frac{d\psi(t)/dt}{\alpha(t)\psi(t)} = \frac{d(\theta(t)a(t))/dt}{\alpha(t)\theta(t)a(t)} \\ &= \frac{1}{\alpha(t)} \left(\frac{\dot{a}(t)}{a(t)} + \frac{\dot{\theta}(t)}{\theta(t)}\right) \\ &= \frac{1}{\alpha(t)} \left(H(t) + \frac{\dot{\theta}(t)}{\theta(t)}\right). \end{aligned} \quad (68)$$

In the linearized perturbation limit this is (using $\alpha\theta = 1$)

$$H_{\text{eff}}(t) \simeq [1 - \Phi(t)]H(t) - \dot{\Phi}(t). \quad (69)$$

From this and Eq. (B2), working to first order accuracy, we find at the present proper time τ_0 that

$$\begin{aligned} \frac{H_{\text{eff}}[\tau_0]}{H_0^{\text{Pl}}} &\simeq 1 - \Phi(0) \left[(\hat{\Phi}(x_0) - 1) \left(1 + \frac{3}{2}\Omega_m \right) \right. \\ &\quad \left. + \frac{3}{2}\sqrt{\Omega_\Lambda}d\hat{\Phi}/dx|_{x=x_0} \right] \\ &\simeq 1 - 0.867\Phi(0), \end{aligned} \quad (70)$$

with $(\hat{\Phi}(x_0) - 1)(1 + (3/2)\Omega_m)$ contributing 0.3615 and with $\frac{3}{2}\sqrt{\Omega_\Lambda}d\hat{\Phi}/dx|_{x=x_0}$ contributing 0.5055 to the coefficient 0.867. If we use the quadratic approximation to $\hat{\Phi}(x)$ given in the final column of Table I, Eq. (70) becomes

$$\begin{aligned} \frac{H_{\text{eff}}[\tau_0]}{H_0^{\text{Pl}}} &\simeq 1 - \Phi(0)C \left(1 + \frac{3}{2}\Omega_m + 3\sqrt{\Omega_\Lambda}/x_0 \right) \\ &= 1 - 0.877\Phi(0), \end{aligned} \quad (71)$$

in good agreement with Eq. (70).

- (vi) Much recent attention has been paid to a tension between values of the local Hubble constant directly measured from redshifts using supernova samples calibrated with Cepheids [13], or supernova samples calibrated with the tip of the red giant branch [14], versus the value at early cosmic times inferred from the study of fluctuations in the CMB radiation [7,15]. This tension may require recalibration of the distance ladder used in direct redshift measurements [16], or may indicate a need for new physics [17]. To interpret the Hubble tension in terms of new physics via a frame-dependent dark energy, we take $f = 1$ and use Eq. (70) to fit the ratio

¹¹Whenever multiplied by $\Phi(0)$, which is first order in magnitude, other quantities can be taken to zeroth order. Hence $\Phi(0)x_\tau^2 \simeq \Phi(0)x_0^2$, etc.

TABLE III. Results of fits of $\Phi(0)$ to the Hubble tension obtained by the Cepheid [13] and red giant [14] methods. The second column gives the ratio of the local Hubble measurement to the CMB Hubble value $H_0^{\text{Pl}} = 67.27 \text{ km s}^{-1} \text{ Mpc}^{-1}$ quoted in the review [15]. On the Cepheid line [13], the error is statistical. On the red giant line [14], the first error is statistical and the second is systematic. The third column gives the value of $\Phi(0)$ needed to fit the second column using Eq. (70). The fourth column gives the ratio of the Hubble parameter to the corresponding BAO value [18] at $z_{\text{eff}} = 0.51$, as calculated in Appendix B. The fifth column gives the correction to the age of the universe from Eq. (78).

Method	$H_{\text{local}}/H_0^{\text{Pl}}$	$\Phi(0)$	$H_{\text{eff},0.51}/H_{\text{BAO},0.51}$	$\tau_0 - \tau_0^{\text{Pl}}$ (Gyr)
Cepheid	1.100 ± 0.023	-0.114 ± 0.026	1.048 ± 0.021	-0.53 ± 0.12
red giant	$1.038 \pm 0.015 \pm 0.025$	$-0.043 \pm 0.017 \pm 0.028$	$1.018 \pm 0.019 \pm 0.012$	$-0.20 \pm 0.08 \pm 0.13$

$H_{\text{local}}/H_0^{\text{Pl}}$, as summarized in Table III, where we give the parameter $\Phi(0)$ needed to fit the ratio of the local Hubble constant value of [13] or [14] to the CMB Hubble value quoted in [15]. This then gives a prediction of our model for the Hubble constant value extracted from baryon acoustic oscillations by the method of Alam *et al.* [18], as discussed in Appendix B, with the result given in the fourth column of Table III. The prediction coming from the Cepheid determination of the local Hubble constant [13] is 4.8 ± 2.1 percent larger than the BAO value. As argued in [19], this tension is a reflection of the fact that our model is a late time model, which generically cannot accommodate the Hubble values from both [13,18].

- (vii) When $1 + \Phi(0) = \alpha(0) < 1$, the expansion rate in our model is enhanced over that in the standard FRW cosmology for all times. To see this, take the τ derivative of the logarithm of $\psi[\tau] = \theta(t(\tau))a(t(\tau))/\theta(0)$, converting derivatives on the right to t derivatives using $d/d\tau = (1/\alpha(t))d/dt$, to get

$$\frac{d \log(\psi)}{d\tau} = \frac{1}{\alpha(t)} \left[\frac{d \log(a(t))}{dt} + \frac{d \log(\theta(t))}{dt} \right], \quad (72)$$

with t on the right hand side understood to be $t(\tau)$. Since for $\alpha(0) < 1$ we have $\frac{1}{\alpha(t)} > 1$ and $d\theta(t)/dt > 0$, Eq. (72) implies that

$$\frac{d \log(\psi)}{d\tau} - \frac{d \log(a)}{dt} > 0; \quad (73)$$

that is, in our model with $f = 1$ and negative $\Phi(0)$, the expansion rate of the universe at all times is greater than in standard FRW cosmology.

- (viii) Corresponding to the fact that expansion rate of the universe is altered in our model for $\Phi(0) \neq 0$, the age of the universe is changed. This can be calculated, to leading order in perturbations, as follows. We first note that just as the age t_0 of the FRW universe is fixed by the requirement $a(t_0) = 1$, in similar fashion the proper time age of the universe τ_0

in our model is fixed¹² by the condition $\psi[\tau_0] = 1$. Writing $t(\tau_0) = t_0 + \Delta t$ we get from $\psi(t) = a(t)\theta(t)/\theta(0)$ the equation

$$\begin{aligned} 1 &= \psi[\tau_0] = a(t(\tau_0))\theta(t(\tau_0))/\theta(0) \\ &= a(t_0 + \Delta t)[1 - \Phi(0)(\hat{\Phi}(t_0 + \Delta t) - \hat{\Phi}(0))] \\ &\simeq 1 + \Delta t \frac{da}{dt}(t_0) - \Phi(0)(\hat{\Phi}(t_0) - \hat{\Phi}(0)), \end{aligned} \quad (74)$$

which can be solved to give Δt ,

$$\begin{aligned} \Delta t &= \frac{\Phi(0)(\hat{\Phi}(t_0) - \hat{\Phi}(0))}{\frac{da}{dt}(t_0)} \\ &= \frac{\Phi(0)(\hat{\Phi}(t_0) - \hat{\Phi}(0))}{H_0} \simeq \frac{\Phi(0)(\hat{\Phi}(t_0) - 1)}{H_0^{\text{Pl}}}. \end{aligned} \quad (75)$$

From the definition of the proper time in Eq. (6) we get

$$\begin{aligned} \tau_0 &= \int_0^{t(\tau_0)} du \alpha(u) \\ &= \int_0^{t_0 + \Delta t} du [1 + \Phi(0)\hat{\Phi}(u)] \\ &= t_0 + \Delta t + \Phi(0) \int_0^{t_0} du \hat{\Phi}(u), \end{aligned} \quad (76)$$

and remembering that $t_0 = \tau_0^{\text{Pl}}/\alpha(0) \simeq \tau_0^{\text{Pl}}(1 - \Phi(0))$ this gives to first order

$$\begin{aligned} \tau_0 - \tau_0^{\text{Pl}} &= -\Phi(0)\tau_0^{\text{Pl}} + \Delta t + \Phi(0) \int_0^{t_0} du \hat{\Phi}(u) \\ &\simeq \frac{\Phi(0)}{H_0^{\text{Pl}}} \left[\hat{\Phi}(t_0) - 1 + \frac{2}{3\sqrt{\Omega_\Lambda}} \int_0^{x_0} dx \hat{\Phi}(x) \right. \\ &\quad \left. - \tau_0^{\text{Pl}} H_0^{\text{Pl}} \right]. \end{aligned} \quad (77)$$

¹²To elaborate, the age of the universe is the elapsed proper time between redshift zero and redshift infinity. When $\psi = 1$, the effective redshift z_{eff} vanishes, and at $\tau = 0$ the effective redshift is infinite since $\psi[0] = a(0) = 0$.

Using the approximation $\hat{\Phi}(x) \simeq 1 + C(x/x_0)^2$ this gives

$$\tau_0 - \tau_0^{\text{Pl}} = \frac{\Phi(0)C}{H_0^{\text{Pl}}} \left(1 + \frac{2x_0}{9\sqrt{\Omega_\Lambda}} \right) \simeq \frac{1.315\Phi(0)C}{H_0^{\text{Pl}}}, \quad (78)$$

which is used to calculate the final column of Table III, using $\tau_0^{\text{Pl}} = 13.83$ Gyr, and $1/H_0^{\text{Pl}} = \tau_0^{\text{Pl}}/0.946 = 14.62$ Gyr.

To summarize, a scale invariant but frame dependent dark energy can enhance the local Hubble constant value without spoiling the excellent CMB angular fits, with a universe that is expanding faster and hence is younger than is suggested by the standard FRW cosmology. However, our model, in which changes from the standard FRW cosmology occur only at late time, tends to give too large a value for the Hubble constant extracted by the BAO method. We look forward to future experiments to give an enlarged and improved data set against which to give a definitive test of our model.

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APPENDIX A: PERTURBATION EQUATIONS REWRITTEN IN TERMS OF Φ AND Ψ

In [4] we gave the metric perturbation equations for $f \neq 0$ in $B = 0$ gauge in terms of A , E , and F , taken as functions of both t and \vec{x} . Here we give the same equations when rewritten in terms of Ψ , Φ , and F , obtained by substituting Eq. (18) and algebraic simplification.¹³

The ij equation takes the form

$$\begin{aligned} 0 &= \delta_{ij}X + \partial_i\partial_j Y, \\ X &= 2[a\ddot{a} + 2(\dot{a})^2]\Phi + a\dot{a}\dot{\Phi} + 6a\dot{a}\dot{\Psi} + a^2\ddot{\Psi} - \nabla^2\Psi \\ &\quad + 4\pi G a^2(\delta\rho - \delta p - \nabla^2\pi^S) - 4\pi G[\dot{a}a^2 4(p + \rho) \\ &\quad + a^3(\dot{\rho}/3 + \dot{p})]F + \Lambda f a^2(t_{00}/2 - E), \\ Y &= 8\pi G a^2\pi^S + \Phi - \Psi. \end{aligned} \quad (A1)$$

The $i0$ equation becomes

$$\begin{aligned} 0 &= -2(\dot{a}/a)\partial_i\Phi - 2\partial_i\dot{\Psi} - 8\pi G(p + \rho)(\partial_i\delta u - a\partial_i F) \\ &\quad + \Lambda f t_{0i}, \end{aligned} \quad (A2)$$

and the 00 equation becomes¹⁴

$$\begin{aligned} 0 = Z &= -(1/a^2)\nabla^2\Phi - 3(\dot{a}/a)\dot{\Phi} - 6(\ddot{a}/a)\Phi - 3\ddot{\Psi} \\ &\quad - 6(\dot{a}/a)\dot{\Psi} + 4\pi G[\delta\rho + 3\delta p + \nabla^2\pi^S + a(\dot{\rho} + 3\dot{p})F] \\ &\quad + \Lambda f(t_{00}/2 + 3E). \end{aligned} \quad (A3)$$

Taking the linear combination $(1/4)a^2Z + (3/4)X + (1/4)\nabla^2Y$ gives

$$\begin{aligned} 0 &= -\nabla^2\Psi + 3a\dot{a}\dot{\Psi} + 3(\dot{a})^2\Phi + \Lambda f a^2 t_{00}/2 \\ &\quad + 4\pi G a^2[\delta\rho - 3\dot{a}(p + \rho)F], \end{aligned} \quad (A4)$$

and using this to eliminate $-\nabla^2\Psi$ from X we get

$$\begin{aligned} a^2\ddot{\Psi} + [2a\ddot{a} + (\dot{a})^2]\Phi + a\dot{a}\dot{\Phi} + 3a\dot{a}\dot{\Psi} \\ = 4\pi G a^2(\delta p + \nabla^2\pi^S) + 2\Lambda f a^2[\Phi - \dot{a}F - a\dot{F}] \\ + 4\pi G a^3\dot{p}F. \end{aligned} \quad (A5)$$

In the limit that the metric perturbations are functions only of t , using $F(t) = 0$ and $\nabla^2\pi^S(t) = 0$ this simplifies, after division by a^2 , to

$$\begin{aligned} \ddot{\Psi} + [2(\ddot{a}/a) + (\dot{a}/a)^2]\Phi + (\dot{a}/a)\dot{\Phi} + 3(\dot{a}/a)\dot{\Psi} \\ = 4\pi G\delta p + 2\Lambda f\Phi. \end{aligned} \quad (A6)$$

When $\Psi = \Phi$, this gives Eq. (20) of the text. The point of the manipulations leading to Eq. (A6) is to eliminate both energy densities $\delta\rho$ and t_{00} from the equation used to solve for Φ .

APPENDIX B: COMPARISON AT $z_{\text{eff}} = 0.51$ OF THE HUBBLE PARAMETER PREDICTED BY OUR MODEL WITH THE VALUE INFERRED FROM BAO MEASUREMENTS

Baryon acoustic oscillation measurements using three clusters of galaxies grouped around redshifts of 0.38, 0.51,

¹³We remind the reader that here we follow the notation of [4,8] and omit the normalization factor $1/\theta(0)$. This has no effect on the final result of Eq. (A6), which is independent of the normalization of a , even though Eqs. (A1)–(A3) are sensitive to the normalization of a when spatial dependences are included.

¹⁴Under the gauge changes $\Delta_g(a^2A) = 2a\dot{a}\epsilon_0$, $\Delta_g E = 2\dot{\epsilon}_0$, $\Delta_g(aF) = -\epsilon_0$, $\Delta_g\delta\rho = \dot{\rho}\epsilon_0$, $\Delta_g\delta p = \dot{p}\epsilon_0$, $\Delta_g\pi^S = 0$, $\Delta_g\delta u = -\epsilon_0$, $\Delta_g\Phi = \Delta_g\Psi = 0$, all terms of Eqs. (18) and (A1)–(A3) are invariant *except* the Λf terms, reflecting the fact that the effective action of Eq. (2) breaks four-space general coordinate invariance. When only three-space general coordinate transformations are considered [which is the most general invariance of Eq. (3)], one can take $\epsilon_0 = 0$, which is consistent with our then setting $F(t) = 0$.

and 0.61 give [18] a Hubble constant $H_0^{\text{BAO}} = 67.3 \pm 1.0 \text{ km s}^{-1} \text{ Mpc}^{-1}$, nearly identical to the CMB value [15] of $H_0^{\text{Pl}} = 67.27 \pm 0.60 \text{ km s}^{-1} \text{ Mpc}^{-1}$. To compare this with the prediction of our model, we calculate the ratio of the Hubble expansion parameters $H(t)$ evaluated at t corresponding to the central redshift 0.51. For the Hubble expansion parameter implied by the BAO measurements, we use the FRW cosmology formula of Eq. (15) with H_0 replaced by the BAO measured value, giving

$$\begin{aligned} H_{\text{BAO},0.51} &= H_0^{\text{BAO}} [\Omega_m (1.51)^3 + \Omega_\Lambda]^{1/2} \\ &= 89.9 \pm 1.3 \text{ km s}^{-1} \text{ Mpc}^{-1}. \end{aligned} \quad (\text{B1})$$

For the Hubble expansion parameter implied by our model, we use Eqs. (63) and (66) to determine the x_τ value corresponding to $1 + z_{\text{eff}} = 1.51$, and then substitute this into Eq. (69), which we expand to first order accuracy,

$$\begin{aligned} H_{\text{eff}}(t) &\simeq [1 - \Phi(t)]H(t) - \dot{\Phi}(t) \\ &\simeq H_0^{\text{Pl}} \sqrt{\Omega_\Lambda} \left\{ \coth \left[x_\tau \left(1 - \frac{\Phi(0)C}{3} \frac{x_\tau^2}{x_0^2} \right) \right] \right. \\ &\quad \left. - \frac{\Phi(0)C}{x_0^2} (x_\tau^2 \coth(x_\tau) + 3x_\tau) \right\}. \end{aligned} \quad (\text{B2})$$

This yields

$$\begin{aligned} H_{\text{eff},0.51} &= 94.2 \pm 1.3 \text{ km s}^{-1} \text{ Mpc}^{-1} \quad \text{for} \\ &\quad \Phi(0) = -0.114 \pm 0.026, \\ H_{\text{eff},0.51} &= 91.5 \pm 1.0 \pm 1.1 \text{ km s}^{-1} \text{ Mpc}^{-1} \quad \text{for} \\ &\quad \Phi(0) = -0.043 \pm 0.017 \pm 0.028, \end{aligned} \quad (\text{B3})$$

giving the numbers used to form the ratio $\frac{H_{\text{eff},0.51}}{H_{\text{BAO},0.51}}$ in Table III.

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- Correction:* Equations (63), (66), (70), (71), and (B3), related text, terms in item (vi) in Sec. V and footnote 12, and entries in Table III contained errors and have been fixed.