Note on the Bloch theorem

C. X. Zhang[1] and M. A. Zubkov[1,*]
Physics Department, Ariel University, Ariel

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The Bloch theorem in ordinary quantum mechanics means the absence of the total electric current in equilibrium. In the present paper, we analyze the possibility that this theorem remains valid within quantum field theory relevant for the description of both high-energy physics and condensed matter physics phenomena. First of all, we prove that the total electric current in equilibrium is the topological invariant for the gapped fermions that are subject to periodical boundary conditions; i.e., it is robust to the smooth modification of such systems. This property remains valid when the interfermion interactions due to the exchange by bosonic excitations are taken into account perturbatively. We give the proof of this statement to all orders in perturbation theory. Thus, we prove the weak version of the Bloch theorem and conclude that the total current remains zero in any system, which is obtained by smooth modification of the one with the gapped charged fermions, periodical boundary conditions, and vanishing total electric current. We analyze several examples, in which the fermions are gapless. In some of them, the total electric current vanishes. At the same time, we propose the counterexamples of the equilibrium gapless systems, in which the total electric current is nonzero.

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I. INTRODUCTION

According to the conventional quantum mechanical formulation of the Bloch theorem [1] in the infinitely large equilibrium system, the total electric current is zero. The proof of this theorem is known within the framework of ordinary quantum mechanics with a fixed finite number of particles. There have been several attempts to generalize the Bloch theorem to quantum field theory (QFT).[1] However, this extension has been limited so far by the consideration of specific models. For example, a continuum model in the presence of magnetic field is considered in Ref. [2], while some lattice models are discussed in Refs. [3–5]. In Ref. [6], the attempt is made to prove the Bloch theorem for the QFT of general type. The proof presented in Ref.f [6], however, seems to us not clear enough. Moreover, below, we present the example of the QFT system, in which the Bloch theorem in its conventional formulation does not work. In the recent paper [7], the proof of the Bloch theorem has been presented for the arbitrary lattice one-dimensional model. This proof may also be extended to the higher-dimensional lattice systems, which are infinite in one particular direction and are compact in the other directions. In this setup, the Bloch theorem states the absence of a persistent current in the direction in which the system is infinite. The same form of the Bloch theorem is proposed in Ref. [8]. The possible extension of the Bloch theorem to the QFT may be important for the applications of the QFT techniques to the condensed matter physics (see, e.g., Refs. [9–31]).

In the present paper, we analyze the possible form in which Bloch theorem survives in an infinite fermionic QFT system. First of all, we demonstrate that in the conventional formulation the Bloch theorem does not hold: we present the example of an infinite system, in which there is the persistent current in equilibrium. Instead of the conventional Bloch theorem, we prove its weakened version. It states that the total electric current in the equilibrium infinite system with periodical spatial boundary conditions and gapped charged fermions is the topological invariant; i.e., it is not changed when the system is modified smoothly. The whole set of the gapped QFT systems may be divided into the homotopic classes. Within each class, the systems are connected by continuous modification. Therefore, if the total electric current vanishes in one of such systems, it vanishes in all systems that belong to the same homotopic class. On the technical side, we will use

[*]On leave of absence from Institute for Theoretical and Experimental Physics, B. Cheremushkinskaya 25, Moscow, 117259, Russia;
zubkov@itep.ru

[1]Quantum field theory represents both the mathematical basis for the description of the high-energy physics and the condensed matter physics. The difference between the corresponding models is actually in symmetry. The high-energy physics systems obey Lorentz invariance, while typical condensed matter systems do not. Aside from the Lorentz symmetry, the description of the condensed matter systems and the description of the high-energy physics systems are completely equivalent. If we consider lattice regularization of the high-energy physics models, then the analogy becomes more complete.

Wigner-Weyl formalism [32–35] adapted in Refs. [36–46] to the lattice models of solid state physics combined with the ordinary perturbation theory. In the presence of the external fields, which break the translational symmetry [47], the Wigner transformation of the Green's functions is more useful for our purposes than the Fourier transformation. Using the technique of Wigner transformation, we express the response of electric current to the external electromagnetic field. The Feynman diagrams written in terms of the Wigner-transformed Green's functions contain the same amount of integrations over momenta as the Feynman diagrams in the homogeneous theory. This facilitates the calculations considerably. At the moment, we cannot establish any definite analog of the Bloch theorem for the gapless QFT systems of general type. Instead, we analyze several particular examples, in which the Bloch theorem holds/does not hold.

The paper is organized as follows. In Sec. II A, we describe the formulation of fermionic QFT models using Wigner-Weyl formalism. In Sec. II B, we present the proof that in the noninteracting gapped fermionic systems with periodical spacial boundary conditions the total electric current is the topological invariant. In Sec. II C, we demonstrate that the conventional lattice models with noninteracting gapped fermions have a vanishing total current. Notice that the perturbative inclusion of interactions via the exchange by neutral bosons does not change this conclusion. The proof is given in Sec. II D. In Secs. III B, III C, and III D, we consider the particular gapless systems. In Sec. III B, we discuss the typical example of the system, in which the Bloch theorem holds. In Sec. III C, we consider the counterexample of the equilibrium system, in which the Bloch theorem does not hold. Notice that this system does not satisfy the additional conditions needed for the validity of Bloch theorem proposed in Refs. [8,25]. In Sec. III D, we discuss the example of the system that obeys the Bloch theorem at a certain range of the values of Fermi energy and does not obey it for another ranges of the Fermi energy. In Sec. IV, we end with conclusions.

II. GAPPED FERMIONS

A. Noninteracting fermions and Wigner-Weyl formalism

Let us consider the continuum system of noninteracting particles. In the homogeneous case, the partition function of such a system in momentum space has the form [36,37,45]

$$Z = \int D\bar{\psi} D\psi \exp\left(\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}^T(p) Q(p) \psi(p)\right). \quad (1)$$

Here, $|\mathcal{M}| = (2\pi)^D$, where D is the dimension of space-time. $\bar{\psi}$ and ψ are the anticommuting multicomponent Grassmann variables defined in momentum space. Matrix $Q(p)$ is given by

$$Q = i\omega - \hat{H}(\mathbf{p}).$$

Here, $p = (\omega, \mathbf{p})$. Introduction of the external gauge field $A(x)$ defined as a function of coordinates results in Peierls substitution (see, for example, Refs. [36,37,45]),

$$Z = \int D\bar{\psi} D\psi \exp\left(\int_{\mathcal{M}} \frac{d^3 p}{|\mathcal{M}|} \bar{\psi}^T(p) Q(p - A(i\partial_p)) \psi(p)\right), \quad (2)$$

where the products of operators inside expression $Q(p - A(i\partial_p))$ are symmetrized.

We denote for the operators $\hat{Q} = Q(p - A(i\partial_p))$ and $\hat{G} = \hat{Q}^{-1}$ their matrix elements by $\mathcal{Q}(p, q)$ and $\mathcal{G}(p, q)$ correspondingly,

$$\mathcal{Q}(p, q) = \langle p|\hat{Q}|q\rangle, \qquad \mathcal{G}(p, q) = \langle p|\hat{Q}^{-1}|q\rangle,$$

where $|p\rangle$ and $|q\rangle$ are momentum eigenstates. The basis of the Hilbert space of functions is normalized as $\langle p|q\rangle = \delta^{(D)}(p - q)$. The mentioned operators satisfy

$$\langle p|\hat{Q}\,\hat{G}\,|q\rangle = \delta^{(D)}(p - q).$$

We insert here the complete set of momentum eigenstates $\{|k\rangle\}$ and obtain

$$\int \mathcal{Q}(p, k) \mathcal{G}(k, q) dk = \delta(p - q). \quad (3)$$

Equation (2) may be rewritten as

$$Z = \int D\bar{\psi} D\psi \exp\left(\int_{\mathcal{M}} \frac{d^3 p_1}{\sqrt{|\mathcal{M}|}} \int_{\mathcal{M}} \frac{d^3 p_2}{\sqrt{|\mathcal{M}|}} \times \bar{\psi}^T(p_1) \mathcal{Q}(p_1, p_2) \psi(p_2)\right), \quad (4)$$

while the Green's function is

$$\mathcal{G}_{ab}(k_2, k_1) = \frac{1}{Z} \int D\bar{\psi} D\psi \exp\left(\int_{\mathcal{M}} \frac{d^3 p_1}{\sqrt{|\mathcal{M}|}} \int_{\mathcal{M}} \frac{d^3 p_2}{\sqrt{|\mathcal{M}|}} \times \bar{\psi}^T(p_1) \mathcal{Q}(p_1, p_2) \psi(p_2)\right) \frac{\bar{\psi}_b(k_2)}{\sqrt{|\mathcal{M}|}} \frac{\psi_a(k_1)}{\sqrt{|\mathcal{M}|}}. \quad (5)$$

Indices a and b enumerate the components of the fermionic fields, which will be omitted later for simplicity.

The Wigner transformation of $\mathcal{G}$ is defined as

$$G_W(x,p) \equiv \int dq e^{ixq} \mathcal{G}(p+q/2, p-q/2). \quad (6)$$

Here, the integral is over q that belong to momentum space. The Weyl symbol of operator $\hat{Q}$ is defined in a similar way:

$$Q_W(x,p) \equiv \int dq e^{ixq} \mathcal{Q}(p+q/2, p-q/2).$$

In the following, we will use the following identity of Wigner-Weyl formalism: if $C(p_1,p_2) = \int A(p_1,q)B(q,p_2)dq$, then the Wigner transformations of A, B, C obey $C_W(x,p) = A_W(x,p)\star B_W(x,p)$. In continuous theory, from Eq. (3), it follows that the Weyl symbol of $\hat{Q}$ and the Wigner transformation of $\mathcal{G}$ obey the Groenewold equation (see, for example, Refs. [36,37,45])

$$\begin{aligned} 1 &= G_W(x,p)\star Q_W(x,p) \\ &= G_W(x,p) e^{\frac{i}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x)} Q_W(x,p). \end{aligned} \quad (7)$$

This equation is also satisfied for the case of the lattice model with a compact Brillouin zone, provided that the fields entering $\hat{Q}$ vary slowly; i.e., their variations at the distance of the order of the lattice spacing may be neglected [44,46]. In practice, this requirement is always satisfied in the real solids if the inhomogeneity is caused by external magnetic field or by elastic deformations. In spite of the appearance of the complicated star products, the use of Wigner-transformed Green's function G_W has certain advantages compared to the use of the ordinary momentum space Green's function $G(p,q)$. Feynman diagrams written in terms of G_W are more concise. In what follows, we will see that these expressions contain the same number of integrations over momenta as the Feynman diagrams of the homogenous theory.

The Grassmann-valued Wigner function may be defined as

$$W(p,q) = \frac{\bar{\psi}(p)}{\sqrt{|\mathcal{M}|}} \frac{\psi(q)}{\sqrt{|\mathcal{M}|}}.$$

We may define operator $\hat{W}[\psi,\bar{\psi}]$, the matrix elements of which are equal to $W(p,q) = \langle p|\hat{W}[\psi,\bar{\psi}]|q\rangle$.

If the field A is slowly varying, then $Q_W(p,x) = Q_W(p - A(x))$ [45]. As a result, the partition function receives the form

$$Z = \int D\bar{\psi} D\psi e^{\sum_x \int \frac{dp}{(2\pi)^D} \mathrm{Tr} W_W[\psi,\bar{\psi}](p,x)\star Q_W(p,x)},$$

where by W_W we denote the Weyl symbol of $\hat{W}$.

B. Equilibrium current as topological invariant for the gapped noninteracting fermions

In this subsection, we consider gapped noninteracting charged fermions in the presence of periodical boundary conditions. Let us consider the variation of the partition function

$$\begin{aligned} \delta \log Z &= -\frac{1}{Z}\int D\bar{\psi}D\psi \exp\left(\sum_x \int \frac{dp}{(2\pi)^D} \mathrm{Tr} W_W[\psi,\bar{\psi}](p,x)\star Q_W(p-A(x))\right) \\ &\quad \times \sum_x \int \frac{dp}{(2\pi)^D} \mathrm{Tr} W_W[\psi,\bar{\psi}](p,x)\star \partial_{p_k} Q_W(p-A(x))\delta A_k(x) \\ &\approx -\frac{1}{Z}\int D\bar{\psi}D\psi \exp\left(\int_x \int \frac{dp}{(2\pi)^D} \mathrm{Tr} W_W[\psi,\bar{\psi}](p,x)\star Q_W(p-A(x))\right) \int dx \\ &\quad \times \int \frac{dp}{(2\pi)^D} \mathrm{Tr} W_W[\psi,\bar{\psi}](p,x)\star \partial_{p_k} Q_W(p-A(x))\delta A_k(x). \end{aligned} \quad (8)$$

The current density integrated over the whole volume of the system appears as the response to the variation of A:

$$\begin{aligned} \langle J^k \rangle &= -\frac{T}{Z}\int D\bar{\psi}D\psi \exp\left(\int_x \int \frac{dp}{(2\pi)^D} \mathrm{Tr} W_W[\psi,\bar{\psi}](p,x)\star Q_W(p-A(x))\right) \\ &\quad \times \int d^D x \int \frac{dp}{(2\pi)^D} \mathrm{Tr} W_W[\psi,\bar{\psi}](p,x) \partial_{p_k} Q_W(p-A(x)) \\ &= -T \int d^D x \int \frac{dp}{(2\pi)^D} \mathrm{Tr} G_W(p,x) \partial_{p_k} Q_W(p-A(x)). \end{aligned} \quad (9)$$

In the presence of periodic boundary conditions, the properties of the star product allow us to rewrite the last equation in the following way:

$$\langle J^k\rangle = -T\int d^Dx\int \frac{d^Dp}{(2\pi)^D}\mathrm{Tr}G_W(p,x)\star\partial_{p_k}Q_W(p-A(x)). \tag{10}$$

Provided that there are no divergencies in this expression, it is the topological invariant; i.e., it is not changed when the system is modified continuously. The proof is as follows. Let us consider an arbitrary variation $Q_W \to Q_W + \delta Q_W$, and the related variation of the Green's function, $G_W \to G_W + \delta G_W$, according to Eq. (7). The variation of the electric current receives the form

$$\begin{aligned}
\delta J^k &= -T\delta\int d^Dx\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}G_W\star\partial_{p_k}Q_W\\
&= -T\int d^Dx\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}(\delta G_W\star\partial_{p_k}Q_W + G_W\star\partial_{p_k}\delta Q_W)\\
&= -T\int d^Dx\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}(-G_W\star\delta Q_W\star G_W\star\partial_{p_k}Q_W + G_W\star\partial_{p_k}Q_W)\\
&= -T\int d^Dx\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}(\delta Q_W\star\partial_{p_k}G_W + G_W\star\partial_{p_k}\delta Q_W)\\
&= -T\int d^Dx\int\frac{d^Dp}{(2\pi)^D}\partial_{p_k}\mathrm{Tr}(\delta Q_W\star G_W) = 0,
\end{aligned} \tag{11}$$

In the last step, we assume the periodical boundary conditions in p space. This occurs, in particular, for the lattice tight-binding model with compact Brillouin zone. In the above proof, it has also been implied that the integrals are convergent. This assumes the absence of both ultraviolet and infrared divergencies. The latter are absent if neither $\hat{G}$ nor $\hat{Q}$ has poles. The ultraviolet divergencies may be eliminated if the theory on the lattice (the tight-binding model) is considered. This guarantees that the ultraviolet divergencies at large spacial momenta are absent. For the noninteracting system with $\hat{Q} = i\omega - \hat{H}$, the uncertainty in the integral over ω remains. However, if the aim is to calculate the conductivity [i.e., the response of Eq. (10) to the external electric field], then the corresponding integral in ω that follows from Eq. (10) is convergent at $\omega\to\infty$ because the expression in the integral behaves as $\frac{1}{\omega^s}$ with $s > 1$ at $\omega\to\infty$. The integral over ω is to be regularized if we are interested in the expression for the current out of the linear response to external gauge field. The standard regularization used for the calculation of various vacuum averages of the bilinear combination of operators results in the modification $G_W(p,x)\to G_W^{(\mathrm{reg})}(p,x) = e^{i\tau\omega}G_W(p,x)$, where τ is to be set to $\tau\to+0$ at the end of calculations. Correspondingly, we regularize $Q_W(p,x)\to Q_W^{(\mathrm{reg})}(p,x) = e^{-i\tau\omega}Q_W(p,x)$. This regularization allows us to save the topological invariance of the regularized Eq. (10). The integral $\int d\omega\frac{e^{i\omega\tau}}{i\omega-\mathcal{E}_n}$ entering Eq. (10) (here, $\mathcal{E}_n$ is the nth eigenvalue of the Hamiltonian) may be calculated using the residue theorem

$$\int d\omega\frac{e^{i\omega\tau}}{i\omega-\mathcal{E}_n}\bigg|_{\tau=+0} = 2\pi\theta(-\mathcal{E}_n),$$

provided that $\mathcal{E}_n \neq 0$. In the case in which the value of $\mathcal{E}_n$ vanishes, this integral remains infrared divergent.

We come to the conclusion that the total electric current is not changed if the gapped system with a compact Brillouin zone is modified smoothly. This conclusion holds also when the interactions between the fermions are taken into account (see Sec. II D).

C. Example of the system with vanishing total current

Let us discuss the case of the noninteracting fermions with Hamiltonian $\hat{H}$. Then,

$$\hat{Q} = i\omega - \hat{H}.$$

For the case of the homogeneous system with $\hat{H} = H(\hat{p})$, Eq. (10) receives the form

$$\langle J^k\rangle = V\int\frac{d\omega d^{D-1}p}{(2\pi)^D}\mathrm{Tr}\frac{1}{i\omega - H(p)}\partial_{p_k}H(p). \tag{12}$$

The integral over ω is to be regularized, $G_W(p,x)\to G_W^{(\mathrm{reg})}(p,x) = e^{i\tau\omega}G_W(p,x)$, where $\tau\to+0$. The integral entering Eq. (12) may be calculated using the residue theorem,

$$\int d\omega\frac{e^{i\omega\tau}}{i\omega-\mathcal{E}_n}\bigg|_{\tau=+0} = 2\pi\theta(-\mathcal{E}_n),$$

provided that $\mathcal{E}_n \neq 0$. We have

$$\langle J^k\rangle = V\int \frac{d^{D-1}p}{(2\pi)^{D-1}}\mathrm{Tr}\theta(-H(p))\partial_{p_k}H(p). \quad (13)$$

For the case of the periodical boundary conditions in momentum space (say, for the lattice model with a compact Brillouin zone), we obtain $\langle J^k\rangle = 0$. Any other system of gapped noninteracting fermions connected with such a homogeneous system by smooth transformation will have vanishing total electric current.

D. Introduction of interactions between the fermions

Now, let us take into account interactions between the fermions. Our consideration here, to some extent, repeats the one presented in Ref. [44]. To calculate the electric current, we consider a variation of the partition function caused by the variation of external electromagnetic field A. To introduce interaction among the fermions, we consider the system with the Euclidean action

$$S = \int dp\bar{\psi}_p\hat{Q}(p, i\partial_p)\psi_p + \alpha\int dpdqdk\bar{\psi}_{p+q}\psi_p\tilde{V}(\mathbf{q})\bar{\psi}_k\psi_{q+k}. \quad (14)$$

Here, operator $\hat{Q}$ depends on the operators of spatial coordinates $i\partial_p$ because the external field has been included [see Eq. (2)]. For definiteness, we consider the Coulomb interaction $V(\mathbf{x}) = 1/|\mathbf{x}| = 1/\sqrt{x_1^2 + x_2^2}$, for $\mathbf{x}\neq\mathbf{0}$. However, the other types of interactions that occur due to the exchange by bosonic excitations are similar. Then, the Fourier transformed Coulomb potential is $\tilde{V}(\mathbf{q}) = \sum_{\mathbf{x}}\frac{e^{i\mathbf{q}\cdot x}}{\sqrt{x_1^2+x_2^2}}$. The Coulomb interaction contributes to the self-energy of the fermions, and the leading-order contribution is proportional to α. The Green's function can be calculated through the Feynman diagrams as

$$G_\alpha(x,y) = G_0(x,y) + \int G_0(x,z_1)\Sigma(z_1,z_2)G_0(z_2,y)dz_1dz_2 + \int G_0(x,z_1)\Sigma(z_1,z_2)G_0(z_2,z_3) \times\Sigma(z_3,z_4)G_0(z_4,y)dz_1dz_2dz_3dz_4 + \cdots, \quad (15)$$

where

$$\Sigma(z_1,z_2) = \alpha G_0(z_1,z_2)\delta(\tau_1-\tau_2)V(\mathbf{z}_1-\mathbf{z}_2) + O(\alpha^2),$$

with $z_i = (\mathbf{z}_i, \tau_i)$. Using Wigner transformation, one finds that

$$G_{\alpha,W}(R,p) = G_{0,W}(R,p) + G_{0,W}(R,p)\star\Sigma_W(R,p)\star G_{0,W}(R,p) + \cdots, \quad (16)$$

where $G_{0,W}(R,p)$ satisfies $Q_{0,W}(R,p)\star G_{0,W}(R,p) = 1$, equivalent to Eq. (7), while Σ_W is Wigner transformation of Σ. It is easy to find that $G_{\alpha,W}(R,p)$ satisfies

$$Q_{\alpha,W}(R,p)\star G_{\alpha,W}(R,p) = 1, \quad (17)$$

where $Q_{\alpha,W}(R,p) = Q_{0,W}(R,p) - \Sigma_W$.

Without loss of generality, we can consider only the electric current along the x axis, J_x, i.e., $k = 1$ in Eq. (10). In what follows, we will denote $I = \langle J_x\rangle$. It is convenient to expand $G_{\alpha,W}(R,p)$ in powers of the coupling constant α as $G_{\alpha,W} = \mathcal{G}_0 + \alpha\mathcal{G}_1 + \alpha^2\mathcal{G}_2 + \ldots$ The average total electric current divided by the system volume V may also be expanded in powers of α:

$$I(\alpha) = \frac{T}{V}\int d^DR\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}G_{\alpha,W}(R,p)\star\frac{\partial}{\partial p_x}Q_{0,W}(R,p) = \frac{T}{V}\int d^DR\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}[G_{0,W} + \sum_{n=1,2,\ldots}G_{0,W}(\star\Sigma_W\star G_{0,W})^n]\star\frac{\partial}{\partial p_x}Q_{0,W}(R,p). \quad (18)$$

The corresponding Feynman diagrams are shown in Fig. 1(a). We represent $\Sigma_W = \alpha\mathcal{S}_1 + \alpha^2\mathcal{S}_2 + \ldots$, and the current ($x$ component) is given by $I = \mathcal{I}_0 + \alpha\mathcal{I}_1 + \alpha^2\mathcal{I}_2 + \ldots$, in which $\mathcal{I}_0 = \frac{T}{V}\int d^DR\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}G_{0,W}\star\partial_{p_x}Q_{0,W}$, and

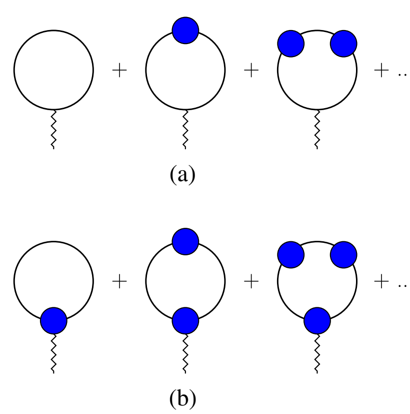

FIG. 1. (a) Feynmann diagrams for $I(\alpha) = \int\frac{Td^DR}{V}\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}G_{\alpha,W}\partial_{p_x}Q_{0,W}$ (expression for the electric current). The filled circles mark Σ_W. The external wavy line marks the position of $\partial_{p_x}Q_{0,W}$. (b) Feynmann diagrams for $\Delta I(\alpha) = \int\frac{Td^DR}{V}\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}G_{\alpha,W}\partial_{p_x}\Sigma_W$. The filled circle with the external wavy line marks $\partial_{p_x}\Sigma_W$.

$$\mathcal{I}_r = \int \frac{Td^DRd^Dp}{V(2\pi)^D}\mathrm{Tr}\sum_{l_1+\cdots+l_n=r} G_{0,W}\star\prod_{i=1}^{n}[\mathcal{S}_{l_i}\star G_{0,W}\star]\frac{\partial}{\partial p_x}\mathcal{Q}_{0,W}, \tag{19}$$

with $r \geq 1$. Let us compare the obtained expression for the total electric current with the following expression written through the interacting Green's function:

$$\tilde{I}(\alpha) = \int \frac{Td^DRd^Dp}{V(2\pi)^D}\mathrm{Tr}G_{\alpha,W}(R,p)\star\frac{\partial}{\partial p_x}Q_{\alpha,W}(R,p). \tag{20}$$

For this purpos, we calculate the difference $\Delta I(\alpha) = I(\alpha) - \tilde{I}(\alpha)$. Because $Q_{\alpha,W} = Q_{0,W} - \Sigma_W$, $\Delta I(\alpha)$ is given by

$$\begin{aligned}\Delta I &= \int\frac{Td^DR}{V}\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}G_{\alpha,W}(R,p)\star\frac{\partial}{\partial p_x}\Sigma_W(R,p)\\ &= \int\frac{Td^DR}{V}\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}(G_{0,W} + \sum_{n=1,2,\ldots}G_{0,W}(\star\Sigma_W\star G_{0,W})^n)\star\frac{\partial}{\partial p_x}\Sigma_W(R,p)\\ &= \alpha\int\frac{Td^DR}{V}\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}G_{0,W}\star\frac{\partial}{\partial p_x}\mathcal{S}_1(R,p) + \alpha^2\int\frac{Td^DR}{V}\int\frac{d^Dp}{(2\pi)^D}\bigg(\mathrm{Tr}\mathcal{S}_1\star G_{0,W}\star\frac{\partial}{\partial p_x}\mathcal{S}_1(R,p)\star G_{0,W}\\ &\quad + \mathrm{Tr}G_{0,W}\star\frac{\partial}{\partial p_x}\mathcal{S}_2(R,p)\bigg) + \cdots\end{aligned} \tag{21}$$

The Feynmann diagrams corresponding to ΔI are represented in Fig. 1(b). Let us consider the diagram with n self-energy functions Σ_W (in addition to an extra self-energy with a photon tail),

$$\Delta I^{(n)} = \int\frac{Td^DR}{V}\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}(G_{0,W}\star\Sigma_W\star)^nG_{0,W}\star\partial_{p_x}\Sigma_W, \tag{22}$$

which appeared in the third line in Eq. (21). After partial integration, we obtain

$$\begin{aligned}\Delta I^{(n)} = (n+1)\int\frac{Td^DRd^Dp}{S(2\pi)^D}\mathrm{Tr}G_{0,W}\star\partial_{p_x}Q_{0,W}\star G_{0,W}\ldots\star\Sigma_W\\ -n\int\frac{Td^DRd^Dp}{S(2\pi)^D}\mathrm{Tr}G_{0,W}\star\partial_{p_x}\Sigma_W\star\ldots\star\Sigma_W\star G_{0,W}\star\Sigma_W.\end{aligned}$$

We come to the relation

$$\begin{aligned}(n+1)\Delta I^{(n)} = (n+1)\int\frac{Td^DR}{V}\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}G_{0,W}\star\partial_{p_x}Q_{0,W}\\ \star G_{0,W}\star\ldots\star\Sigma_W\star G_{0,W}\star\Sigma_W,\end{aligned} \tag{23}$$

which gives $\Delta I^{(n)} = I^{(n+1)}$, where $I^{(n+1)}$ is the contribution to the electric current with $n+1$ insertions of Σ_W represented schematically in Fig. 1(a) (the $n+1$th term in the sum). Overall, we obtain

$$\Delta I(\alpha) = I(\alpha) - I^{(0)} = I(\alpha) - I(0).$$

We find that the total current is given by an integral of Eq. (20) as long as the value of the total current remains equal to its value without interactions. We will prove that, indeed, $I(\alpha) = I(0)$ in the region of analyticity in α, i.e., as long as the perturbation theory in α may be used.

The electric current in the absence of interactions is given by $\mathcal{I}_0 = \int\frac{Td^DR}{V}\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}G_{0,W}(R,p)\star\frac{\partial}{\partial p_x}Q_{0,W}(R,p)$. Below, we will prove that this expression does not receive corrections from interactions, i.e., for $j \geq 1$, $\mathcal{I}_j = 0$. First, let us consider $\mathcal{I}_1$ [shown in Fig. 2(a)], which can be expressed explicitly as

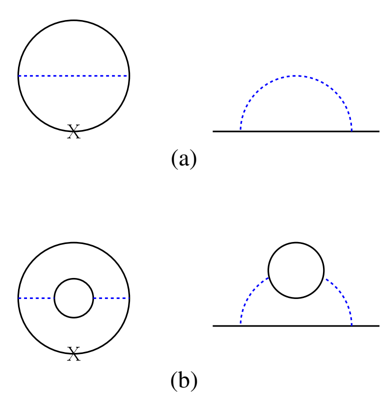

FIG. 2. Loop diagram contributions to electric current I (left side of the figure) and the corresponding diagrams of self-energy function (right side). Crosses X represent $\partial_{p_x}Q_{0,W}$. (a) The diagrams in the first order. (b) One of the second-order diagrams.

$$\mathcal{I}_1 = -\int \frac{Td^D R}{V}\int \frac{d^D p d^D q}{(2\pi)^D} \times \mathrm{Tr}(G_{0,W}(R,p-q)\mathcal{D}(q))\star \frac{\partial}{\partial p_x} G_{0,W}(R,p) = -\int \frac{Td^D R}{V}\int \frac{d^D p d^D q}{(2\pi)^D}\mathrm{Tr}(G_{0,W}(R,p-q)\mathcal{D}(q)) \times \frac{\partial}{\partial p_x} G_{0,W}(R,p). \tag{24}$$

Here, $\mathcal{D}(q)$ is the Fourier transformation of function

$$D(z_1 - z_2) = \alpha\delta(\tau_1 - \tau_2)V(\mathbf{z}_1 - \mathbf{z}_2).$$

Because $\mathcal{D}(q)$ is an even function, for each value of R, the above expression is proportional to

$$\int\int \mathcal{F}_R(p-q)\mathcal{D}_R(q)\mathcal{F}'_R(p)dpdq = 0, \tag{25}$$

where $\mathcal{F}_R(q) = G_{0,W}(R,q)$,and $\mathcal{F}'$ is the first derivative of $\mathcal{F}$. This representation allows us to prove that $\mathcal{I}_1 = 0$ (we perform the integration by parts and show that $\mathcal{I}_1 = -\mathcal{I}_1$).

Let us now consider the next-order contribution $\mathcal{I}_2$. We have

$$\mathcal{I}_2 = -\int \frac{Td^D R d^D p}{V(2\pi)^D}\mathrm{Tr}\mathcal{S}_2\star\frac{\partial}{\partial p_x}G_{0,W} - \int \frac{Td^D R d^D p}{V(2\pi)^D}\mathrm{Tr}\mathcal{S}_1\star G_{0,W}\star\mathcal{S}_1\star\frac{\partial}{\partial p_x}G_{0,W}.$$

First, similar to the proof of $\mathcal{I}_1 = 0$, the contribution of the diagram shown in Fig. 2(b) is also zero. The only necessary change in the proof is to replace $\mathcal{D}(q)$ in Eq. (24) by $\mathcal{D}(q)\Pi(q^2)\mathcal{D}(q)$, where $\Pi(q^2)$ is the vacuum polarization. Taking self-energy $\mathcal{S}_2$ in the rainbow (r.b.) approximation, we get

$$\mathcal{I}_2^{(r.b.)} = -\int \frac{Td^D R d^D p d^D k d^D q}{V(2\pi)^{2D}}\mathrm{Tr}[G_{0,W}(R,p-k) \star G_{0,W}(R,p-k-q)\mathcal{D}(q)\star G_{0,W}(R,p-k)] \times \mathcal{D}(k)\star\partial_{p_x}G_{0,W}(R,p) - \int \frac{Td^D R d^D p d^D k d^D q}{V(2\pi)^{2D}}\mathrm{Tr}G_{0,W}(R,p-q)\mathcal{D}(q) \star G_{0,W}(R,p)\star G_{0,W}(R,p-k)\mathcal{D}(k)\star\partial_{p_x}G_{0,W}(R,p)$$

In the first term, the star before ∂_{p_x} may be eliminated. It may then be inserted before $\mathcal{D}(k)$, thus giving

$$\mathcal{I}_2^{(r.b.)} = -\int \frac{Td^D R d^D p d^D k d^D q}{V(2\pi)^{2D}}\mathrm{Tr}[G_{0,W}(R,p-k)\star G_{0,W}(R,p-k-q)\mathcal{D}(q)\star G_{0,W}(R,p-k)]\star\mathcal{D}(k)\partial_{p_x}G_{0,W}(R,p) - \int \frac{Td^D R d^D p}{V(2\pi)^D}\mathrm{Tr}G_{0,W}(R,p-q)\mathcal{D}(q)\star G_{0,W}(R,p)\star G_{0,W}(R,p-k)\mathcal{D}(k)\star\partial_{p_x}G_{0,W}(R,p) = -\frac{1}{2}\int \frac{Td^D R d^D p d^D k d^D q}{V(2\pi)^{2D}}\partial_{p_x}\mathrm{Tr}[G_{0,W}(R,p-k)\star G_{0,W}(R,p-k-q)\mathcal{D}(q)\star G_{0,W}(R,p-k)]\star\mathcal{D}(k)G_{0,W}(R,p) = 0. \tag{26}$$

Notice that the last expression without a derivative with respect to p_x corresponds to the diagram similar somehow to the one called a "progenitor" in [48]. We present the form of the corresponding Feynmann diagram in Fig. 3(a) and call it the progenitor for the diagrams presented in Fig. 4. In essence, our present proof is an extension of the one given in Ref. [48]. The remaining two-loop diagrams (see Fig. 5) give the contribution that may be written as follows:

$$\mathcal{I}_2^{(\mathrm{cross})} = -\int \mathrm{Tr}[G_{0,W}(R,p-k_1)\star G_{0,W}(R,p-k_1-k_2)\star G_{0,W}(R,p-k_2)]\mathcal{D}(k_1)\mathcal{D}(k_2)\partial_{p_x}G_{0,W}(R,p)\frac{Td^D R d^D p d^D k_1 d^D k_2}{V(2\pi)^{2D}} = -\frac{1}{4}\int \frac{Td^D R d^D p d^D k d^D q}{V(2\pi)^{2D}}\partial_{p_x}\mathrm{Tr}[G_{0,W}(R,p-k_1)\star G_{0,W}(R,p-k_1-k_2)\star G_{0,W}(R,p-k_2) \star G_{0,W}(R,p)]\mathcal{D}(k_1)\mathcal{D}(k_2) = 0. \tag{27}$$

Here, the star $\star = e^{i\overleftarrow{\partial}_R\vec{\partial}_p/2 - i\overleftarrow{\partial}_p\vec{\partial}_R/2}$ acts only on G but does not act on $\mathcal{D}(k_i)$ (which does not depend on R or p). The last line of the above expression corresponds to the diagram of Fig. 3(b).

One can see that $\mathcal{I}_2 = 0$. In the same way, the higher orders may be considered. One can check that $\mathcal{I}_j = 0$ for $j > 0$ to all orders of the perturbation theory.

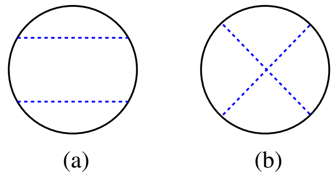

FIG. 3. (a) The progenitor diagram for the two-loop rainbow contribution to electric current. (b) The progenitor diagram for the two-loop contribution to electric current (which is beyond the rainbow approximation).

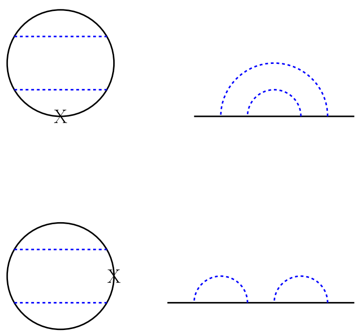

FIG. 4. Two-loop Feynmann diagrams for the self-energy Σ in the rainbow approximation (right side of the figure) and the corresponding three-loop rainbow contributions to electric current I (left side of the figure). The crosses point out the positions of the derivatives $\partial_{p_x} Q_{0,W}$.

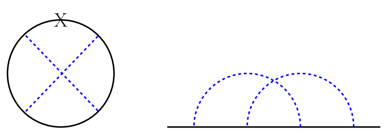

FIG. 5. Two-loop Feynmann diagrams for the self-energy Σ beyond the rainbow approximation (right side of the figure) and the corresponding three-loop contributions to electric current I (left side of the figure). The crosses point out the positions of the derivatives $\partial_{p_x} Q_{0,W}$.

The higher-order corrections may be considered in the similar way. The example of the higher-order diagram is considered in Fig. 6. The sum of the Feynman diagrams represented in Figs. 6(c), 6(d), and 6(e) contribute the Fermion self-energy that enters an expression for the total current presented in Fig. 1(a) [the diagrams in Figs. 6(d) and 6(e) are to be counted twice). The resulting contribution to the electric current is equal to the integral over the momentum of the derivative of the progenitor diagram represented in Fig. 6(a). This integral is zero for the system with compact momentum space (when lattice regularization is used). The diagrams of Fig. 6(c), 6(d), and 6(e) appear when the diagram of Fig. 6(b) is cut at the positions of the crosses.

The obtained results mean the following:

(1) The interaction corrections to the total electric current vanish.

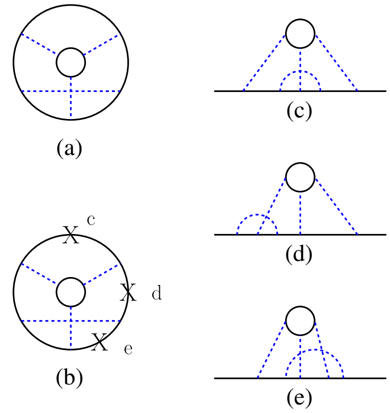

FIG. 6. An example of the high-order corrections. (a) is the progenitor diagram, and the possible contributions to the electric current appear when in (b) one of the crosses is substituted by the derivative $\partial_p Q_{0,W}$. Using each of those crosses, we form the diagram, which contributes to the electric current. Diagrams (d), (e), and (f) are the corresponding self-energy diagrams.

(2) There is the following representation for the total average electric current divided by the system volume in the considered system:

$$I(\alpha) = \int \frac{T d^D R d^D p}{V(2\pi)^D} \mathrm{Tr} G_{\alpha,W}(R,p) \star \frac{\partial}{\partial p_x} Q_{\alpha,W}(R,p). \tag{28}$$

Notice that our proof does not rely on the precise expression of the Coulomb potential. In Eq. (25), we only used that the Fourier-transformed potential is an even function of momentum. Therefore, the generalization of our result to the case of the other interactions is straightforward. In the similar way, the Yukawa interaction, the exchange by gauge bosons, and the four-fermion interactions may be considered.

III. GAPLESS FERMIONS

A. Electric current in the system of gapless noninteracting charged fermions

In this section, we discuss the case of the gapless fermions. Let us start from the consideration of the noninteracting fermions with Hamiltonian $\hat{H}$. Then,

$$\hat{Q} = i\omega - \hat{H}.$$

For the case of the homogeneous system with $\hat{H} = H(\hat{p})$, Eq. (10) receives the form

$$\langle J^k \rangle = V \int \frac{d\omega d^{D-1} p}{(2\pi)^D} \mathrm{Tr} \frac{1}{i\omega - H(p)} \partial_{p_k} H(p). \tag{29}$$

As was mentioned above, the integral over ω is to be regularized if we are interested in the expression for the current out of the linear response to the external gauge field. We modify $G_W(p,x) \to G_W^{(\mathrm{reg})}(p,x) = e^{i\tau\omega} G_W(p,x)$, where τ is set to $\tau \to +0$ at the end of the calculations. The integral $\int d\omega \frac{e^{i\omega\tau}}{i\omega - \mathcal{E}_n}$ entering Eq. (29) (here, $\mathcal{E}_n$ is the nth eigenvalue of the Hamiltonian) may be calculated using the residue theorem,

$$\int d\omega \frac{e^{i\omega\tau}}{i\omega - \mathcal{E}_n}\bigg|_{\tau=+0} = 2\pi\theta(-\mathcal{E}_n),$$

provided that $\mathcal{E}_n \neq 0$. In the case in which the value of $\mathcal{E}_n$ vanishes, this integral is divergent. This breaks the topological nature of Eq. (29) but does not mean that the whole Eq. (29) is divergent itself. Namely, we have

$$\langle J^k \rangle = V \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \mathrm{Tr}\theta(-H(p))\partial_{p_k} H(p). \quad (30)$$

For the case of the one-dimensional ($D = 2$) system with one branch of the spectrum, we get

$$\langle J \rangle = \frac{V}{2\pi} \int_{p_1}^{p_2} dp \partial_p H(p) = \frac{V}{2\pi} (H(p_2) - H(p_1)), \quad (31)$$

where p_1 and p_2 are the end points of the piece of the branch of the spectrum with $H(p) \leq 0$. If both p_1 and p_2 are finite, then $H(p_1) = H(p_2) = 0$; therefore, $\langle J^k \rangle = 0$. This occurs for the compact Brillouin zone that appears for the lattice tight-binding model. If one of the points p_1 and p_2 is placed at infinity, then the value of the total current may differ from zero. In this case, it is clear that this expression depends continuously on the smooth variation of the Fermi energy.

Thus, we are able to give another weakened version of the Bloch theorem valid for the gapless systems: the homogeneous lattice model of noninteracting fermions cannot have the nonvanishing total electric current. Since we cannot formulate at the present moment a more general version of Bloch theorem for the gapless QFT system, we consider below several particular examples. Some of them break the conventional Bloch theorem.

B. Example of the system that obeys the Bloch theorem

In this subsection, we consider the planar system placed in the (xy) plane: in the region $x < 0$, there is an infinitely high potential, while in the region $x > 0$, there is the constant electric field directed toward the positive x axis and uniform magnetic orthogonal to the (x, y) plane.

The electron in such a system satisfies the Schrödinger equation

$$-\frac{1}{2m}\partial_x^2\psi + \frac{(-i\partial_y - Bx)^2}{2m}\psi + V(x)\psi = \epsilon\psi, \quad (32)$$

where $V(x) = Ex$ for $x > 0$. Notice that we use the relativistic system of units with $\hbar = c = 1$. Separating variables $\psi(x, y) = e^{ip_y y}\phi(x)$, one obtains the equation for $\phi(x)$ as follows:

$$\phi''(x) - B^2\left(x - \frac{p_y B - mE}{B^2}\right)^2 \phi(x) + \left(2m\epsilon - \frac{2mp_y E}{B} + \frac{m^2E^2}{B^2}\right)\phi(x) = 0. \quad (33)$$

We rescale variable $x = \kappa s$, with $\kappa = 1/\sqrt{2B}$, and arrive at

$$f''(s) - \frac{1}{4}(s - s_0)^2 f(s) + \left(\nu + \frac{1}{2}\right) f(s) = 0, \quad (34)$$

where $s_0 = \sqrt{2/B}p_y - mE\sqrt{2B}/B^2$, and

$$\nu + \frac{1}{2} = \frac{m\epsilon}{B} - \frac{mp_y E}{B^2} + \frac{m^2E^2}{2B^3}. \quad (35)$$

Solution of Eq. (34) with the requirement $s \to \infty$, $f \to 0$ is the parabolic cylinder function $P(\nu, s - s_0)$. From the boundary condition $f(s = 0) = 0$, we obtain the relation between ν and s_0 (see Fig. 7).

ν linearly depends on energy ϵ and momentum p_y, while s_0 linearly depends on momentum p_y. Finally, we obtain the relation between ϵ and p_y, which is shown in Fig. 8.

The total current is equal to

$$j_y = \sum_{i=1,2} \int_{p^{(\mathrm{left})}}^{p^{(\mathrm{right})}} \frac{dp_y}{2\pi} \partial_{p_y} \mathcal{E}_i(p_y),$$

where the integral is between the two crossing points $p_y = p^{(\mathrm{left})}, p^{(\mathrm{right})}$ (of the Fermi level and the given branch of the spectrum). One can see that the total current is equal to zero. Therefore, the Bloch theorem is valid in this case.

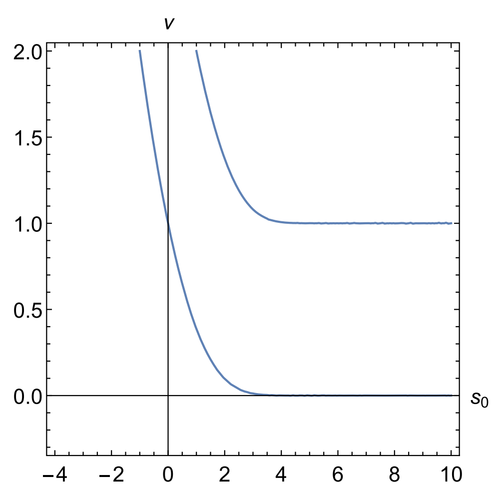

FIG. 7. ν vs s_0 in the model of Sec. III B.

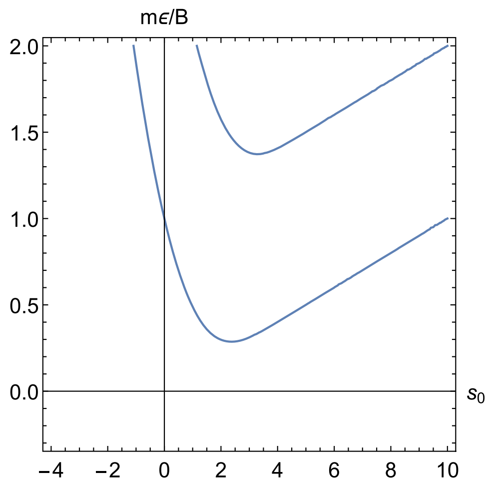

FIG. 8. Relation between energy and momentum in the model of Sec. III B. The vertical axis is $m\epsilon/(B)$, while the horizontal axis is s_0. We take the particular case of $mE/B^2 = 0.1$.

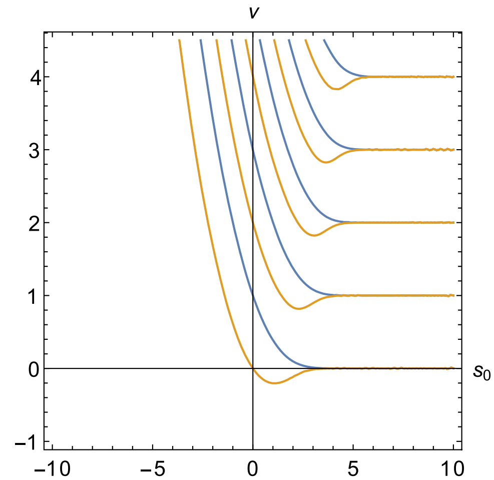

FIG. 9. ν as a function of s_0 in the model of Sec. III C. The blue lines come from the condition $P(\nu, -s_0) = 0$, while the brown lines from the condition $Q(\nu, -s_0) = 0$.

C. Counterexample

Now, let us consider another example. This is an infinite planar system in the xy plane with magnetic field B penetrating the plane: in the region $x < 0$, there is a uniform magnetic field $B_z = -B < 0$, while in the region $x > 0$, there is a uniform magnetic field in the opposite direction, i.e., $B_z = B > 0$.

An electron in such a system satisfies the following Schrödinger equation:

$$-\frac{1}{2m}\partial_x^2\psi + \frac{(-i\partial_y - Bx)^2}{2m}\psi + V(x)\psi = \epsilon\psi, \quad x > 0 \tag{36}$$

$$-\frac{1}{2m}\partial_x^2\psi + \frac{(-i\partial_y + Bx)^2}{2m}\psi + V(x)\psi = \epsilon\psi, \quad x < 0. \tag{37}$$

After the separation of variables and rescaling x via $x = \kappa s$, one obtains

$$f''(s) - \frac{1}{4}(s - s_0)^2 f(s) + \left(\nu + \frac{1}{2}\right) f(s) = 0, \tag{38}$$

for $s > 0$, where $s_0 = \sqrt{2/B}\,p_y$ and

$$\nu + 1/2 = m\epsilon/B.$$

The equation for the region $s < 0$ is similar; the only difference is a sign change in front of s in Eq. (38). The solution can be expressed in terms of parabolic cylinder function:

$$f(s) = \begin{cases} C_1 P(\nu, s - s_0) & x > 0, \\ C_2 P(\nu, -s - s_0) & x < 0. \end{cases} \tag{39}$$

The boundary condition is that $f(s)$ and $f'(s)$ should be continuous at $s = 0$. If we denote the derivative function of $P(\nu, s)$ with respect to x as $Q(\nu, s)$, the boundary condition implies $P(\nu, -s_0) = 0$ (then, $C_1 = -C_2$) or $Q(\nu, -s_0) = 0$ (then $C_1 = C_2$). We find the energy spectrum, which is shown in Fig. 9.

From this spectrum, one can see that if the Fermi level is between $(B/m)(0 + 1/2)$ and $(B/m)(1 + 1/2)$ it crosses both branches of spectrum $\mathcal{E}_{1,2}(p_y)$ corresponding to the blue and brown lines in Fig. 9. The total current is equal to

$$j_y = \sum_{i=1,2} \int_{p^{(i)}}^{\infty} \frac{dp_y}{2\pi} \partial_{p_y} \mathcal{E}_i(p_y),$$

where the integral is between the crossing point $p_y = p^{(i)}$ (of the Fermi level and the given branch of the spectrum) and $p_y = +\infty$. One can see that the total current is nonzero. Therefore, the Bloch theorem is violated in this case.

D. System with magnetic field in the quantum well

Now, let us consider the more realistic example. This is an infinite planar system in the xy plane with magnetic field B penetrating the plane as in the previous section: in the region $x < 0$, there is a uniform magnetic field $B_z = -B < 0$, while in the region $x > 0$, there is a uniform magnetic field in the opposite direction, i.e., $B_z = B > 0$. Besides, we add the potential $V(x)$ of the quantum well: $V(x) = 0$ for $x \in [-L, +L]$ and $V(x) = V_0$ for $x \in (\infty, -L) \cup (+L, +\infty)$.

An electron in such a system satisfies the following Schrödinger equation:

$$-\frac{1}{2m}\partial_x^2\psi + \frac{(-i\partial_y - Bx)^2}{2m}\psi + V(x)\psi = \epsilon\psi, \quad x > 0 \tag{40}$$

$$-\frac{1}{2m}\partial_x^2\psi + \frac{(-i\partial_y + Bx)^2}{2m}\psi + V(x)\psi = \epsilon\psi, \quad x < 0. \tag{41}$$

After a separation of variables and rescaling x via $x = \kappa s$, one obtains

$$f''(s) - \frac{1}{4}(s - s_0)^2 f(s) + \left(\nu + \frac{1}{2}\right) f(s) = 0, \tag{42}$$

for $0 < s < l = L/\kappa$, where $s_0 = \sqrt{2/B}p_y$ and

$$\nu + 1/2 = m\epsilon/B.$$

When $s > l$, $f(s)$ satisfies

$$f''(s) - \frac{1}{4}(s - s_0)^2 f(s) + \left(\nu' + \frac{1}{2}\right) f(s) = 0, \tag{43}$$

where $\nu' = \nu - mV_0/B$. The equation for the region $s < 0$ is similar; the only difference is a sign change in front of s in Eqs. (42) and (43). The solution can be expressed in terms of parabolic cylinder function and the hypergeometric function:

$$f(s) = \begin{cases} C_1 F_1(\nu, s - s_0) + C_2 F_2(\nu, s - s_0), & 0 < x < L; \\ C_3 D_\nu(s - s_0), & x > L; \\ C_1' F_1(\nu, -s - s_0) + C_2' F_2(\nu, -s - s_0), & -L < x < 0; \\ C_3' D_\nu(-s - s_0), & x < -L. \end{cases} \tag{44}$$

F_1 and F_2 are given by $F_1(\nu,x) = e^{-x^2/4}F(-\nu/2, 1/2; x^2/2)$ and $F_2(\nu,x) = xe^{-x^2/4}F(1/2 - \nu/2, 3/2; x^2/2)$. According to the boundary conditions, $f(s)$ and $f'(s)$ are continuous at $s = 0, \pm L$, which leads to six linear equations. The corresponding determinant (6×6) should be zero, which guarantees the nonzero solutions for C_i and C_i'. After linear transformations, the determinant can be decomposed into the product of two 3×3 determinants: $\mathrm{Det} = 2 \cdot \mathrm{Det}^{(1)} \cdot \mathrm{Det}^{(2)}$, with

$$\mathrm{Det}^{(1)} = \begin{vmatrix} \mathcal{A} & \mathcal{B} & -\mathcal{P} \\ \mathcal{F} & \mathcal{G} & -\mathcal{Q} \\ \mathcal{R}_1 & \mathcal{R}_2 & 0 \end{vmatrix} \qquad \mathrm{Det}^{(2)} = \begin{vmatrix} \mathcal{A} & \mathcal{B} & -\mathcal{P} \\ \mathcal{F} & \mathcal{G} & -\mathcal{Q} \\ \mathcal{S}_1 & \mathcal{S}_2 & 0 \end{vmatrix}, \tag{45}$$

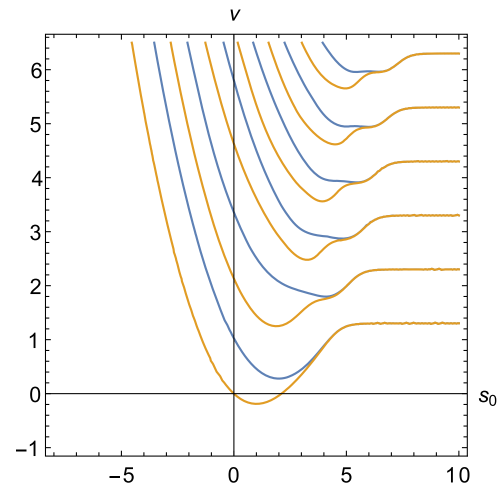

FIG. 10. ν as a function of s_0 in the model of Sec. III D. The parameters of the model are $l = 3$ and $mV_0/B = 1.3$. The blue lines come from the condition $\mathrm{Det}^{(1)} = 0$, while the brown lines come from the condition $\mathrm{Det}^{(2)} = 0$.

where $\mathcal{A} = F_1(\nu, l - s_0)$, $\mathcal{B} = F_2(\nu, l - s_0)$, $\mathcal{R}_1 = F_1(\nu, -s_0)$, $\mathcal{R}_2 = F_2(\nu, -s_0)$, $\mathcal{P} = D_{\nu'}(l - s_0)$, and $\mathcal{Q} = D'_{\nu'}(l - s_0)$. The derivatives of $F_1(\nu, x)$ and $F_2(\nu, x)$ with respect to x are denoted by $g_1(\nu, x)$ and $g_2(\nu, x)$, and then $\mathcal{F} = g_1(\nu, l - s_0)$, $\mathcal{G} = g_2(\nu, l - s_0)$, $\mathcal{S}_1 = g_1(\nu, -s_0)$, and $\mathcal{S}_2 = g_2(\nu, -s_0)$. $\mathrm{Det} = 0$ is equivalent to $\mathrm{Det}^{(1)} = 0$ or $\mathrm{Det}^{(2)} = 0$ from which we find the energy spectrum, i.e., the relation between ν and s_0, shown in Fig. 10.

From this spectrum, one can see that if the Fermi level is between $V_0 + (B/m)(0 + 1/2)$ and $V_0 + (B/m)(1 + 1/2)$ it crosses both branches of spectrum $\mathcal{E}_{1,2}(p_y)$ corresponding to the blue and brown lines in Fig. 10. The total current is equal to

$$j_y = \sum_{i=1,2} \int_{p^{(i)}}^{\infty} \frac{dp_y}{2\pi} \partial_{p_y} \mathcal{E}_i(p_y),$$

where the integral is between the crossing point $p_y = p^{(i)}$ (of the Fermi level and the given branch of the spectrum) and $p_y = +\infty$. One can see that the total current is nonzero. Therefore, the Bloch theorem is violated in this case.

IV. CONCLUSIONS

In the present paper, we consider the possibility of formulating the analog of the quantum mechanical Bloch theorem for the field theoretical systems. In the nonrelativistic quantum mechanics of fixed number of particles, the total current vanishes in equilibrium according to the conventional Bloch theorem. The essential difference from the quantum field theory is that in the latter the number of (quasi)particles is not fixed while the single-particle Hamiltonian may have the more complicated form. Moreover, the interactions with the time delay complicate the system even more. As a result, the direct analog of the Bloch theorem in the QFT has not been established, despite several attempts [7,24–28].

We consider separately the gapped and the gapless systems. Below, we list the obtained results:

(1) First of all, we demonstrate that for the gapped homogeneous noninteracting system with a compact Brillouin zone the total electric current vanishes.

(2) Next, we prove that the total electric current for the gapped noninteracting system is the topological invariant in the presence of periodical spatial boundary conditions; i.e., it is not changed when the system is modified smoothly. Therefore, any nonhomogeneous smooth modifications of the system mentioned above in item 1 also lead to a vanishing total electric current.

(3) Interactions due to exchange by bosonic excitations do not alter the total electric current for the abovementioned gapped systems as long as the interactions may be taken perturbatively. We prove this statement to all orders in the coupling constant.

(4) Considering the gapless systems, we find that the total electric current vanishes for the homogeneous ones with a compact Brillouin zone in the absence of interactions.

(5) We do not formulate any analogs of the Bloch theorem for the gapless nonhomogeneous systems. Instead, we consider several particular examples. Along with the ones in which the total current vanishes in equilibrium, we present examples in which the total electric current is nonzero. In those examples, space is divided into the pieces with different directions of magnetic field. The total current appears along the interphase between the two pieces. Notice that this setup does not satisfy conditions of the version of the Bloch theorem proposed in Refs. [7,8]. Namely, the considered system is infinite in the direction orthogonal to the persistent equilibrium current.

We conclude that the Bloch theorem in its traditional formulation (there is no total electric current in equilibrium) does not hold in quantum field theory. The examples that demonstrate this are those with gapless noninteracting fermions. At the same time, we formulate the weakened version of the Bloch theorem for the gapped interacting systems (items 1, 2, and 3 above).

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