

ϵ -regime of dilaton chiral perturbation theoryTaro V. Brown¹, Maarten Golterman², Svend Krøjer¹, Yigal Shamir³, and K. Splittorff¹¹*The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*²*Department of Physics and Astronomy, San Francisco State University,
San Francisco, California 94132, USA*³*Raymond and Beverly Sackler School of Physics and Astronomy, Tel Aviv University,
69978 Tel Aviv, Israel*

(Received 7 October 2019; published 24 December 2019)

The ϵ -regime of dilaton chiral perturbation theory is introduced. We compute the dilaton mass, the chiral condensate and the topological susceptibility in the ϵ -regime, as a function of the fermion mass. The microscopic spectral density of the Dirac operator is obtained from dilaton chiral perturbation theory. Our main result is that the chiral condensate and the spectral density are related to their counterparts from ordinary chiral perturbation theory via a simple scaling relation. This relation originates from the mass dependence of the dilaton potential, and is valid in both the ϵ -regime and the p -regime. In the ϵ -regime, moreover, all results agree with the universal predictions to leading order in ϵ .

DOI: [10.1103/PhysRevD.100.114515](https://doi.org/10.1103/PhysRevD.100.114515)**I. INTRODUCTION**

Lately, a large number of studies have appeared of QCD-like theories using lattice techniques originally developed for studying QCD; for reviews see Refs. [1–4]. In particular, theories where the number of fermions or their representation suggest that the theory could be in a (confining but) near-conformal phase have drawn a lot of attention. One of the interesting outcomes is the observation of a light scalar particle in addition to the usual mesonic modes (the “pions”); see Refs. [5–9] for recent examples. The possible nature of this additional mode is still an open question; an exciting possibility is that it is a pseudo-Nambu-Goldstone mode associated with approximate dilatation symmetry (see the above reviews). A fruitful way to address this question is to include the additional mode within the framework of low energy effective field theory (EFT), and match the predictions thereof with the results obtained from the underlying lattice theory. Possible low-energy effective theories which take into account both chiral and conformal symmetry go back to Refs. [10,11]. Recently, in Refs. [12–15], a systematic low energy theory was formulated as an extension of chiral perturbation theory assuming that the additional light scalar mode is a pseudo-Nambu-Goldstone mode associated with the almost restored dilatation invariance.¹ The first tests of this effective theory

against lattice results are encouraging [9,15,19–22], and we will follow this approach in the present work.

The ϵ -regime of chiral perturbation theory (χ PT) [23,24] has led to a surprising wealth of insights into QCD, and the interplay between numerical lattice QCD and analytic results from the ϵ -regime of χ PT has helped develop both lattice QCD and chiral perturbation theory. For example the analytic analysis of order a^2 effects [25–34] has explained why the width of the distribution of the smallest eigenvalues of the (massive) Hermitian Wilson-Dirac operator scales with the square root of the volume close to the continuum, as first observed on the lattice [35]. Likewise the analytic analysis for nonzero quark chemical potential [36–39] in the ϵ -regime has given substantial new insights into the sign problem in lattice QCD.

In this paper we define and perform a first study of the ϵ -regime of chiral perturbation theory extended to include the dilatonic meson ($d\chi$ PT). One technical advantage of the ϵ -regime of $d\chi$ PT as compared to the p -regime studies in Refs. [12,13,15] is (as in ordinary χ PT [23,24,40,41]) the possibility of evaluating the partition function explicitly. In order to extend this property to $d\chi$ PT we define the new power counting such that effectively the pions are in the ϵ -regime, while the dilaton is kept in the p -regime. As discussed below, other counting schemes are possible, but, because the dilaton is described by a noncompact field, they lead to integrals over the dilaton potential that include asymptotic field values which are outside the domain of our effective field theory. Our chosen counting allows us to address the influence of the dilatonic meson on the fermion mass dependence of the topological susceptibility, the chiral condensate and the average Dirac spectrum.

¹For related approaches see [16–18].

At leading order in the ϵ -counting the results all agree with the universal predictions of ordinary chiral perturbation theory. The new dependence induced by the dilaton enters at next to leading order, when the pion mass is in the ϵ -regime. We find that the chiral condensate and the spectral density are related to their counterparts from ordinary chiral perturbation theory via a simple scaling relation. This relation can be traced back to the mass dependence of the dilaton potential, and, as it turns out, is valid for both the ϵ -regime and the p -regime.

The paper is organized as follows. After a brief review of dilaton chiral perturbation theory in Sec. II we discuss the possible counting schemes in Sec. III and reanalyze the structure of the dilaton potential in Sec. IV. Section V contains the results obtained in the ϵ -regime of $d\chi$ PT for respectively the dilaton mass, the topological susceptibility and the chiral condensate. In Sec. VI we derive the average eigenvalue density for the massless Dirac operator. Finally, Sec. VII sums up the conclusions. Some technical details regarding the calculation of the topological susceptibility are relegated to Appendix A, while a few explicit results for the partition function and the generating functional are derived in Appendix B.

II. DILATON CHIRAL PERTURBATION THEORY

In addition to the pseudoscalar Nambu-Goldstone bosons associated with chiral symmetry breaking, which we will refer to as ‘‘pion’’ fields from standard χ PT, dilaton chiral perturbation theory includes a real scalar field τ representing the light dilatonic meson (‘‘dilaton,’’ for short) associated with weak breaking of dilatation invariance. This effective field theory was derived in Ref. [12] on the basis of the assumption that the dilaton can be interpreted as a pseudo-Nambu-Goldstone boson associated with a weak breaking of dilatation symmetry, postulated to occur outside, but in the vicinity of the conformal window in theories with N_f fermion flavors in the fundamental representation of the gauge group. The connection between the underlying gauge theory and the effective theory is made through the introduction of sources χ and σ for the pions and dilaton, respectively. In this construction, the sources are endowed with transformations under the relevant symmetry group, elevating them to spurions. Here we will review the transformation properties of the pion and dilaton fields as well as the spurion fields $\chi(x)$ and $\sigma(x)$. The possible terms in the $d\chi$ PT are determined by the invariance properties under these transformations.

We start off with the chiral transformations. For the pion field $U(x) \in \text{SU}(N_f)$ we have the standard $\text{SU}(N_f) \times \text{SU}(N_f)$ transformation [42]

$$U \rightarrow g_L U g_R^\dagger, \quad g_L \in \text{SU}(N_f)_L, \quad g_R \in \text{SU}(N_f)_R, \quad (1)$$

while the τ field is unaffected.

The fermion mass explicitly breaks this symmetry and to encode this into the effective theory it is promoted to the associated spurion χ , which transforms as

$$\chi \rightarrow g_L \chi g_R^\dagger. \quad (2)$$

After the construction of the EFT, the spurion field χ is set equal to the fermion mass m .

Under dilatations, the spurion σ transforms as [12]

$$\sigma(x) \rightarrow \sigma(\lambda x) + \log \lambda, \quad (3)$$

where λ is the scale factor. Just like the pion field U transforms like the source χ , the effective dilaton field τ transforms like σ , and, in addition to Eq. (3) dilatation transformations act on all fields as

$$\begin{aligned} U(x) &\rightarrow U(\lambda x), \\ \chi(x) &\rightarrow \lambda^{4-y} \chi(\lambda x), \\ \tau(x) &\rightarrow \tau(\lambda x) + \log \lambda, \end{aligned} \quad (4)$$

where $y = 3 - \gamma_*$ with γ_* the mass anomalous dimension at the nearby infrared fixed point [12]. We note that it is the renormalized mass spurion $\chi(x)$ which transforms this way [14]. It will be sufficient for our purposes to consider only a constant dilatation spurion field, $\sigma(x) = \sigma_0$, which transforms as $\sigma_0 \rightarrow \sigma_0 + \log \lambda$.

We will consider the theory in a volume $\mathcal{V} = L^4$ with periodic boundary conditions, which also breaks dilatation symmetry explicitly. However, again the symmetry can be restored formally if we let also L , the side length of the box, be a spurion transforming as

$$L \rightarrow \frac{L}{\lambda}. \quad (5)$$

Using these fields the simplest nontrivial Lagrangian \mathcal{L} for which the action $S = \int d^4x \mathcal{L}(x)$ is invariant under the stated transformations of fields and spurions is [12]

$$\mathcal{L} = \mathcal{L}_\pi + \mathcal{L}_\tau + \mathcal{L}_m + \mathcal{L}_d, \quad (6)$$

with (working in Euclidean space)

$$\mathcal{L}_\pi = \frac{f_\pi^2}{4} V_\pi (\tau - \sigma_0) e^{2\tau} \text{Tr}[\partial_\mu U^\dagger \partial^\mu U], \quad (7)$$

$$\mathcal{L}_\tau = \frac{f_\tau^2}{2} V_\tau (\tau - \sigma_0) e^{2\tau} \partial_\mu \tau \partial^\mu \tau, \quad (8)$$

$$\mathcal{L}_m = -\frac{f_\pi^2 B_\pi}{2} V_m (\tau - \sigma_0) e^{y\tau} \text{Tr}[\chi^\dagger U + U^\dagger \chi], \quad (9)$$

$$\mathcal{L}_d = f_\tau^2 B_\tau V_d (\tau - \sigma_0) e^{4\tau}, \quad (10)$$

where V_i , $i = \pi, \tau, m, d$ are invariant potentials. The low energy constants (LECs), f_π, f_τ, B_π and B_τ , are invariant under both chiral and dilatation transformations. The finite-volume action is invariant, too, provided that the linear size of the box spurion transforms as in Eq. (5).

The functional forms of V_π, V_τ, V_m and V_d are unconstrained by symmetry and call for a counting scheme to be introduced. We will turn to this in the next section.

III. ϵ -COUNTING AND LEADING ORDER LAGRANGIAN

In this section we will introduce the ϵ -counting scheme and identify the lowest order Lagrangian. As we will hold fixed the number of flavors, N_f , i.e., the “distance” to the conformal window, the amount by which dilatation symmetry is broken (in the massless theory) is also fixed. In current simulations [5–9], the dilaton mass is in the p -regime. We will thus assume that the dilaton is in the p -regime, but consider the situation in which the fermion mass m is chosen small enough that the pion mass is in the ϵ -regime. As we will explain in Sec. VI, this counting is also natural when we consider the valence and ghost quarks of the generating functional for the Dirac eigenvalues.

A. Definition of the ϵ -counting

In Ref. [12] the p -regime was considered. The small parameters in this counting are

$$m \sim n_f - n_f^* \sim p^2 \sim 1/L^2 \sim \delta \ll 1 \quad (p\text{-regime}), \quad (11)$$

$$1/N_f \sim 1/N_c \equiv 1/N \ll 1, \quad (12)$$

with

$$n_f = \lim_{N_f, N_c \rightarrow \infty} \frac{N_f}{N_c}, \quad n_f^* = \lim_{N_c \rightarrow \infty} \frac{N_f^*(N_c)}{N_c}, \quad (13)$$

where $N_f^*(N_c)$ is the largest number of fundamental-representation flavors for which the $SU(N_c)$ gauge theory still confines. The difference $n_f - n_f^* < 0$, which measures the distance from the conformal window, controls the explicit (hard) breaking of scale invariance coming from the running of the gauge coupling. This is similar to the way m controls the explicit breaking of chiral symmetry in standard χ PT [42]. (Of course, m breaks scale invariance as well.) Throughout this paper we invoke the Veneziano limit (13) in order to treat $n_f - n_f^*$ as a continuous parameter. This requires $1/N \leq |n_f - n_f^*| \sim \delta$, but otherwise we will not demand any particular relation between $1/N$ and δ .

In this paper we instead consider $d\chi$ PT in a different counting, in which the pion mass is in the ϵ -regime. There are different possible choices; we choose to define the ϵ -counting

TABLE I. The p -counting of Ref. [12] and the ϵ -counting introduced here.

Term	p -counting	ϵ -counting
$\partial_\mu, \frac{1}{L}$	$\mathcal{O}(p)$	$\mathcal{O}(\epsilon)$
m	$\mathcal{O}(p^2)$	$\mathcal{O}(\epsilon^4)$
$n_f - n_f^*$	$\mathcal{O}(p^2)$	$\mathcal{O}(\epsilon^2)$

$$m \sim p^4 \sim 1/L^4 \sim \epsilon^4 \ll 1 \quad (\epsilon\text{-regime}),$$

$$n_f - n_f^* \sim \delta \ll 1. \quad (14)$$

This corresponds loosely to treating the pion field in ϵ -counting while the dilaton field is kept in p -counting. For small m , we have $m_\pi^2 \sim m$, and so this choice implies $m_\pi L \ll 1$. Likewise, $M_\tau^2 \sim |n_f - n_f^*|$, and choosing

$$\delta \sim \epsilon^2 \sim p^2, \quad (15)$$

keeps the dilaton in the p -regime.

A comparison of the p - and ϵ -countings is shown in Table I. LECs are always order 1. As for the Veneziano limit, we have that $f_\pi \sim \sqrt{N}$, $f_\tau \sim N$, while B_π and B_τ are $\mathcal{O}(1)$, ensuring that the pion and dilaton masses are $\mathcal{O}(1)$ in the large- N counting as well.

When using ϵ -counting in standard χ PT the partition function separates into a group integral over the zero-modes and a path integral over the nonzero modes [23]. As we will show below this still holds with the added dilatonic terms, at least when the dilaton is kept in the p -regime. Table II shows the various choices one can make for the counting of both pion and dilaton degrees of freedom, along with the choice we make in this paper, in which $\delta \sim \epsilon^2$.

In principle, one might also consider a theory with the dilaton in the ϵ -regime, by choosing $\delta \sim \epsilon^4$. However, in this case, it is not clear how to proceed. An additional assumption about the behavior of V_d for $\tau \rightarrow -\infty$ would be needed, and higher order terms would have to come into play, because, with the form of V_d discussed below in Sec. IV, the integral over τ diverges in that region. This is beyond the scope of the present paper, in which we keep the dilaton in the p -regime, and thus we only need to consider small fluctuations of the τ field, including for its zero mode. This assumption is only self-consistent in the p -regime.

TABLE II. The 4 possible combinations of p - and ϵ - counting for the π and τ fields. The two combinations on the right-hand side are sensitive to the form of V_d and have yet to be explored.

$\pi \setminus \tau$	p	ϵ
p	[12,13]	Full V_d -dependence
ϵ	This paper	Full V_d -dependence

IV. THE MINIMUM OF THE DILATON POTENTIAL IN THE CHIRAL LIMIT

In the ϵ -regime of standard χ PT it is advantageous to express the pion field through its zero mode part U_0 and its nonzero mode part $\xi(x)$ [23]. The U_0 part of the standard- χ PT path integral factorizes and may then be performed independently to obtain the functional dependence of the partition function on the quark mass, to leading order in ϵ . In order to see how this works in the presence of the dilaton in $d\chi$ PT, we first need to consider the potentials V_i appearing in the Lagrangian (6), in particular, V_m and V_d . In the following we discuss the minimum of the dilaton potential in the chiral limit; we will return to the effective dilaton potential for nonzero quark mass in Sec. V. We will use the counting scheme defined in Eq. (14).

It was shown in Ref. [12] and Appendix A of Ref. [15] that the potentials can be reorganized such that, keeping the constant spurion σ_0 ,

$$V_i(\tau - \sigma_0) = \sum_{n=0}^{\infty} c_{i,n}(\tau - \sigma_0)^n, \\ c_{i,n} \sim (n_f - n_f^*)^n \sim \delta^n, \quad (16)$$

introducing an infinite number of LECs $c_{i,n}$.² For V_d , this form depends on some mild assumptions [15]. The key point here is that the n th term in the expansion is of order δ^n . This provides a power counting such that at any given order only a finite number of the $c_{i,n}$ come into play, thereby making the EFT predictive. With σ_0 transforming as in Eq. (3), the action is invariant under dilatations, treating the $c_{i,n}$ as additional invariant LECs. Setting $\sigma_0 = 0$ recovers the action of Ref. [12].³ With the power counting of Eq. (14), to lowest order we need to keep only the terms with $n = 0$ in the potentials $V_{\pi,\tau,m}$, and, moreover, we can choose $c_{i,0} = 1$ for these three potentials, as the overall normalization can be absorbed into the LECs f_π , f_τ and B_π . Terms with n larger than zero contribute at higher orders in the expansion in the small parameter δ .

The potential V_d needs to be considered more closely. The leading-order dilaton potential for zero quark mass is given by

$$V_{\text{cl}}(\tau - \sigma_0) = f_\tau^2 B_\tau e^{4\sigma_0} e^{4(\tau - \sigma_0)} V_d(\tau - \sigma_0) \\ = f_\tau^2 B_\tau e^{4\sigma_0} e^{4(\tau - \sigma_0)} \sum_{n=0}^{\infty} c_{d,n}(\tau - \sigma_0)^n, \quad (17)$$

²Each of these LECs can be expanded in powers of $(n_f - n_f^*)$, with the expansion of $c_{i,n}$ starting at $(n_f - n_f^*)^n$ [12]. However, at fixed n_f , no information about the coefficients in this expansion is available.

³The dilatation transformation of σ_0 reproduces the spurion transformation rules of the coefficients $c_{i,n}$ discussed in Appendix C of Ref. [12].

where, as in Eq. (16),

$$c_{d,n} \sim (n_f - n_f^*)^n \sim \delta^n. \quad (18)$$

The exponential $e^{4(\tau - \sigma_0)}$ dominates $V_{\text{cl}}(\tau - \sigma_0)$ for $\tau - \sigma_0 \rightarrow \pm\infty$ because of Eq. (18), and thus, to ensure that $v \equiv \langle \tau \rangle$ is finite, we assume that $\lim_{\tau - \sigma_0 \rightarrow \infty} V_d(\tau - \sigma_0) > 0$. In addition, it is assumed that the minimum of the dilaton potential is unique; at this minimum, $V_{\text{cl}}(v) < 0$ [15]. It follows that by choosing

$$\sigma_0 = -v, \quad (19)$$

we can arrange that $\langle \tau \rangle = 0$. The upshot is that, for $m = 0$, we can set $v = 0$. The choice of σ_0 defines the origin of the τ axis. Other choices will lead to exponential factors which in turn can be absorbed into the LEC's [12].

With the choice (19) for σ_0 (and a rescaling of B_τ), to order $\delta \sim n_f - n_f^*$ the classical potential for the dilaton in the chiral limit takes the form

$$V_{\text{cl}} = f_\tau^2 B_\tau e^{4\tau} c_1 \left(\tau - \frac{1}{4} \right), \quad (20)$$

where we set $c_1 \equiv c_{d,1}$, and the field τ does not have a zero mode, because $v = \langle \tau \rangle = 0$. Introducing the dynamical pion fields $\xi(x)$ through

$$U(x) = U_0 \exp(i\sqrt{2}\xi(x)), \quad (21)$$

we expand the effective Lagrangian of Eq. (6) in terms of $\xi(x)$ and $\tau(x)$ and assess the magnitude of each term through the Gaussian damping in the path integral. This is a stepping stone towards determining the order of all terms in the effective theory. In the following, the fermion mass is chosen to obey ϵ -counting and we will use $v(m = 0) = 0$ as found above. Setting $\chi = mI$, the resulting Lagrangian terms up to quadratic in the fields are

$$\mathcal{L}_\pi = \frac{f_\pi^2}{2} \text{Tr}[\partial_\mu \xi \partial^\mu \xi], \quad (22)$$

$$\mathcal{L}_\tau = \frac{f_\tau^2}{2} \partial_\mu \tau \partial^\mu \tau, \quad (23)$$

$$\mathcal{L}_m = -f_\pi^2 B_\pi m \text{ReTr}[U_0(1 - \xi^2)], \quad (24)$$

$$\mathcal{L}_d = f_\tau^2 B_\tau c_1 \left(-\frac{1}{4} + 2\tau^2 \right). \quad (25)$$

Terms linear in ξ and τ do not appear in these expressions because the zero modes have been separated out explicitly, and thus $\int d^4x \xi(x) = \int d^4x \tau(x) = 0$.

From Eq. (25), the mass of the dilaton in the chiral limit is found to be

$$M_\tau^2 = 4c_1 B_\tau. \quad (26)$$

The fluctuations in the dilaton field are limited by both the dilaton kinetic term \mathcal{L}_τ and the dilaton potential \mathcal{L}_d in the partition function to

$$\tau \sim \epsilon. \quad (27)$$

Likewise the kinetic term \mathcal{L}_π limits the fluctuations in the pion fields to

$$\xi \sim \epsilon. \quad (28)$$

We note that the ξ -dependent part of \mathcal{L}_m is of order ϵ^6 , while all other terms shown in Eqs. (22)–(25) are of order ϵ^4 , not counting the constant term in \mathcal{L}_d .

V. THE ϵ -REGIME OF $d\chi$ PT

A main result at leading order in the ϵ -regime is the possibility to evaluate the partition function explicitly. This remains possible in the ϵ -counting for $d\chi$ PT we are considering where the pions are in the ϵ -regime, while the dilaton is kept in the p -regime.

For any physical quantity accessible to our EFT, there are two types of contributions in terms of powers of ϵ : those that depend on the fermion mass m , and those that do not. In this paper, we will be focused on those that are dependent on the fermion mass m . The motivation for this is the application to lattice simulations with fixed n_f and volume, but with varying fermion mass. In order to obtain the full quark mass dependence it is essential to include the quark mass dependence of the minimum of the dilaton potential [12]. As we show below the quark mass dependence of the dilaton potential follows from the partition function which reduces to a group integral in the ϵ -regime.

A. Factorization of the leading order partition function

From the discussion in the previous sections we observe that the leading-order terms in the effective Lagrangian, Eqs. (22)–(25), do not couple U_0 to the dynamical fields $\xi(x)$ and $\tau(x)$. Since the measure for $U = U_0 \exp(i\sqrt{2}\xi)$ factorizes [23], the leading-order partition function becomes⁴

$$Z_{N_f}^{(d\chi\text{PT})}(m) = e^{-\mathcal{V}_{\text{cl}}} Z_{N_f}^{(\chi\text{PT})}(m), \quad (29)$$

$$Z_{N_f}^{(\chi\text{PT})}(m) = \int_{\text{SU}(N_f)} DU_0 e^{\frac{1}{2}\mathcal{V} f_\pi^2 B_\pi e^{i\sqrt{2}\xi} \text{Tr}[mU_0 + m^* U_0^\dagger]} \equiv e^{-\mathcal{V} V_{\text{eff}}}. \quad (30)$$

⁴As usual, the partition function for a fixed topological sector is defined by integrating over $U(N_f)$ instead of $\text{SU}(N_f)$ while inserting $\det^t(U_0)$ into the integral in Eq. (30).

Unless otherwise stated we will take $m = m^*$ real, but we wrote Eq. (30) such that it is valid for complex m as well. We have restored the dependence on the dilaton vacuum $v = v(m)$, which is now determined by minimizing the potential

$$V = V_{\text{cl}} + V_{\text{eff}}, \quad (31)$$

where V_{eff} was defined above, and V_{cl} is given by Eq. (20).

We will now use this partition function to address how a small quark mass affects the dilaton mass M_τ , and how the topological susceptibility and the chiral condensate are affected by the presence of the dilatonic meson.

B. Effective potential and τ -mass correction due to a small m

Consider the expression in the exponent of the integrand in Eq. (30),

$$x(v) \equiv m \mathcal{V} f_\pi^2 B_\pi e^{i\sqrt{2}\xi}. \quad (32)$$

In the ϵ -regime x is $\mathcal{O}(1)$. Therefore, V_{eff} defined by Eq. (30) takes on the form $-h(x)/\mathcal{V}$, where h is an $\mathcal{O}(1)$ function of x .⁵ The potential (31) then takes the form

$$V(v) = f_\tau^2 B_\tau c_1 e^{4v} \left(v - \frac{1}{4} \right) - \frac{1}{\mathcal{V}} h(x(v)), \quad (33)$$

and v solves the equation

$$0 = v e^{4v} - \frac{y}{4} \frac{x(v)}{f_\tau^2 B_\tau c_1 \mathcal{V}} h'(x(v)). \quad (34)$$

Since $f_\tau^2 B_\tau c_1 \mathcal{V}$ is of order $1/\epsilon^2$, it follows that the solution of this equation is of order ϵ^2 . This implies that we can expand in terms of v , and, to order ϵ^2 , we find

$$v(m) = \frac{y}{4} \frac{x(0)}{f_\tau^2 B_\tau c_1 \mathcal{V}} h'(x(0)) = \frac{y f_\pi^2 B_\pi m}{4 f_\tau^2 B_\tau c_1} h'(f_\pi^2 B_\pi m \mathcal{V}). \quad (35)$$

Later in this section we will compute the corrections to the topological susceptibility and the chiral condensate due to v . Since v is of order ϵ^2 these corrections enter at the NLO level in the ϵ -regime. First, however, we turn to the effect on the dilaton mass.

The dilaton mass may be expressed in terms of the function h and its derivatives:

⁵For general N_f no explicit expression for $h(x)$ is known, to the best of our knowledge.

$$\begin{aligned}
M_\tau^2 &= \left. \frac{V''(v)}{f_\tau^2 e^{2v}} \right|_{v=v(m)} = 4B_\tau c_1 e^{2v} (1 + (4-y)v) - \frac{y^2 m^2 f_\pi^4 B_\pi^2 \mathcal{V} e^{2(y-1)v}}{f_\tau^2} h''(x(v)) \\
&= 4B_\tau c_1 + \frac{y f_\pi^2 B_\pi m}{f_\tau^2} ((6-y)h'(f_\pi^2 B_\pi m \mathcal{V}) - y f_\pi^2 B_\pi m \mathcal{V} h''(f_\pi^2 B_\pi m \mathcal{V})), \tag{36}
\end{aligned}$$

where in the first line we used Eq. (34) and in the second line we expanded in $v \sim \epsilon^2$ and used Eq. (35). The m -dependent part of M_τ^2 is of order ϵ^4 , i.e., of order ϵ^2 relative to its value in the chiral limit.

If $x \ll 1$ we can express the resulting integral in terms of a power series in x [23], and we find, for $N_f > 2$ and to first nontrivial order, that

$$h(x) = \frac{1}{4} x^2, \tag{37}$$

yielding the effective dilaton potential for nonzero m ,

$$V(v) = f_\tau^2 B_\tau c_1 e^{4v} \left(v - \frac{1}{4} \right) - \frac{1}{4} \mathcal{V} f_\pi^4 B_\pi^2 m^2 e^{2yv}, \tag{38}$$

from which

$$v(m) = \frac{y f_\pi^4 B_\pi^2 m^2 \mathcal{V}}{8 f_\tau^2 B_\tau c_1}, \tag{39}$$

which is of order $\epsilon^2 x^2$. The dilaton mass for nonzero m is given by

$$\begin{aligned}
M_\tau^2 &= 4B_\tau c_1 + (6-2y) \frac{y f_\pi^4 B_\pi^2 m^2 \mathcal{V}}{2 f_\tau^2} \\
&= 4B_\tau c_1 (1 + (6-2y)v(m)). \tag{40}
\end{aligned}$$

As already mentioned, there are also $\mathcal{O}(\epsilon^2)$ corrections to these results that do not depend on the fermion mass m . For example, one may calculate the NLO correction to the potential V of Eq. (33) in the chiral limit, and this will lead to m -independent NLO corrections to v . However, such an NLO correction can be absorbed into a field redefinition, by shifting σ_0 in Eq. (19) such that $v = 0$ to NLO. In general, m -independent corrections are proportional to $n_f - n_f^*$, hence the interesting NLO corrections are those that do depend on m , because they can be probed by varying m in numerical simulations of a given theory, i.e., for a fixed n_f . We will thus restrict our discussion also in the following to m -dependent NLO corrections only.

C. Topological susceptibility

As can be seen from Eqs. (29) and (30), apart from the factor of $e^{-\mathcal{V}V_{cl}}$, the dilatonic vacuum modifies the leading-order ϵ -regime partition function by a scaling of the mass by e^{yv} . Let us promote the mass parameter to a complex

variable, $me^{i\theta}$, so that the dependence of the leading order partition function (30) on m , θ and v is now through the combination $me^{i\theta}e^{yv}$. The topological susceptibility⁶

$$\chi \equiv \frac{\langle \nu^2 \rangle}{\mathcal{V}} = -\frac{1}{\mathcal{V} N_f^2} \frac{\partial^2}{\partial \theta^2} \log Z|_{\theta=0} \tag{41}$$

may then be obtained from the expressions for $\langle \nu^2 \rangle(m)$ derived in Ref. [24] by replacing m with me^{yv} . Arriving at this result is in fact not entirely as trivial as it would seem at first, and we refer to Appendix A for more details.

For $N_f = 2$ it is possible to evaluate the partition function and the topological susceptibility in closed form [24]. For higher N_f (e.g., $N_f = 8$, relevant for the near-conformal QCD-like theories) we will content ourselves with the asymptotic form valid for $x \gg N_f, \nu$. The details may be found in Appendix A. The result [24,43]

$$\frac{Z^{(\nu, d\chi\text{PT})}}{Z^{(\nu=0, d\chi\text{PT})}} = e^{-\frac{\nu^2}{2\langle \nu^2 \rangle}}, \tag{42}$$

from which, using Eq. (A6), we obtain the asymptotic result

$$\chi = m f_\pi^2 B_\pi e^{yv(m)} / N_f. \tag{43}$$

D. Chiral condensate

Similarly we may consider the effect of including the dilaton on the chiral condensate

$$\langle \bar{\psi}\psi \rangle^{(d\chi\text{PT})}(m) = \frac{1}{\mathcal{V}} \frac{\partial}{\partial m} \log Z^{(d\chi\text{PT})}, \tag{44}$$

in the ϵ -regime. Note that $Z^{(d\chi\text{PT})}$ depends on m also through $v(m)$. The terms resulting from the m -dependence of v , however, take the form

$$\frac{\partial v}{\partial m} \frac{\partial}{\partial v} \log Z^{(d\chi\text{PT})} = -\mathcal{V} \frac{\partial v}{\partial m} \frac{\partial V}{\partial v}, \tag{45}$$

where V is given by Eq. (31). This vanishes, because $\partial V / \partial v = 0$ at the minimum of the potential. Therefore, as

⁶We trust that the use of the conventional symbol χ to denote the topological susceptibility will not lead to confusion even though the mass spurion is also conventionally denoted by χ .

for the topological susceptibility, the impact of the dilaton enters through the scaling of the mass $m \rightarrow me^{yv}$ in Eq. (30), and the chiral condensate is found to be

$$\langle \bar{\psi}\psi \rangle^{(\text{d}\chi\text{PT})}(m) = e^{yv} \langle \bar{\psi}\psi \rangle^{(\chi\text{PT})}(me^{yv}). \quad (46)$$

Here $\langle \bar{\psi}\psi \rangle^{(\chi\text{PT})}(m)$ is the result from standard χPT in the ϵ -regime, see Refs. [23,24]. Once again, $v = v(m)$ is the minimum of the potential (31).

The scaling relation (46) is in fact valid in complete generality for any mass m (in the domain of validity of $\text{d}\chi\text{PT}$), as long as the dilaton is in the p -regime. When also the pions are in the p -regime we arrive at the leading order result

$$\langle \bar{\psi}\psi \rangle^{(\text{d}\chi\text{PT})}(m) = e^{yv} N_f f_\pi^2 B_\pi. \quad (47)$$

For pions in the ϵ -regime we may expand $e^{yv} \approx 1 + yv = 1 + \mathcal{O}(\epsilon^2)$, and to leading order we recover the universal result of Refs. [23,24]. The $\mathcal{O}(\epsilon^2)$ term coming from the expansion of e^{yv} adds up to the other NLO corrections of ordinary χPT [44–47].

Note that because there are $N_c N_f$ fermionic degrees of freedom, the condensate scales like N^2 in the Veneziano limit. Remembering that f_π^2 scales like N , the correct scaling is manifest in Eq. (47) for p -regime pions. For a χPT derivation when the pions are in the ϵ -regime; see Appendix B.

VI. THE DIRAC SPECTRUM FROM $\text{d}\chi\text{PT}$

In this section we derive the microscopic limit of the spectral density of the massless Dirac operator, for fixed topological charge ν in $\text{d}\chi\text{PT}$, by connecting it to the results of Ref. [48] for standard χPT . Before considering $\text{d}\chi\text{PT}$ we briefly review the general setup for computing the eigenvalue density of the massless, anti-Hermitian Dirac operator.

The spectral density of the Dirac eigenvalues Λ_k is given by

$$\rho(\Lambda, m) = \sum_k \langle \delta(\Lambda - \Lambda_k) \rangle, \quad (48)$$

where m in the argument of ρ refers to the mass of the physical (i.e., sea) quarks entering the average. To compute the spectral density we first note that it can be written as the discontinuity across the imaginary axis in the m' plane; see e.g., Ref. [48],

$$\begin{aligned} \text{Disc}|_{m'=-i\Lambda} \Sigma_{PQ}(m', m) &= \lim_{\kappa \rightarrow 0} \Sigma_{PQ}(-i\Lambda + \kappa, m) - \Sigma_{PQ}(-i\Lambda - \kappa, m) \\ &= 2\pi\rho(\Lambda, m), \end{aligned} \quad (49)$$

where the partially quenched condensate is defined as

$$\Sigma_{PQ}(m', m) = \left\langle \sum_k \frac{1}{i\Lambda_k + m'} \right\rangle, \quad (50)$$

with m' a new parameter independent of m . The partially quenched condensate can be computed from the graded generating functional $Z_{N_f+1|1}^{(\nu)}(m, m'|m'')$,

$$\Sigma_{PQ}^{(\nu)}(m', m) = \frac{1}{\mathcal{V}} \lim_{m'' \rightarrow m'} \frac{\partial}{\partial m'} \log Z_{N_f+1|1}^{(\nu)}(m, m'|m''), \quad (51)$$

where m' and m'' are the masses of respectively the valence fermionic and bosonic (or “ghost”) quarks (for an introduction to the graded method in chiral perturbation theory; see Ref. [49]). The advantage of expressing the eigenvalue density in terms of the graded generating functional is that the latter can be evaluated directly in the effective theory.

The microscopic limit of the spectral density is defined as

$$\rho_s(u, \mu) = \lim_{\mathcal{V} \rightarrow \infty} \frac{1}{\mathcal{V}\Sigma} \rho\left(\frac{u}{\mathcal{V}\Sigma}, \frac{\mu}{\mathcal{V}\Sigma}\right), \quad (52)$$

where $u = \Lambda\mathcal{V}\Sigma$ and $\mu = m\mathcal{V}\Sigma$, with $\Sigma = f_\pi^2 B_\pi$. This limit corresponds to the thermodynamic limit in the ϵ -regime, hence if $Z_{N_f+1|1}^{(\nu)}(m, m'|m'')$ is evaluated in the ϵ -regime it is automatically ρ_s that is computed. In more detail, since the smallest eigenvalues of the Dirac operator are of order $1/\mathcal{V}$ when chiral symmetry is broken [50], and the valence-ghost sector is introduced to probe the smallest eigenvalues, we will always choose the valence and ghost quark masses m' and m'' in the generating functional to be of order $1/\mathcal{V}$. Therefore, also the associated pions are by construction in the ϵ -regime. It is important to notice that this requirement puts no restriction on the physical (“sea”) quark mass m , and the sea pions can be either in the ϵ -regime or in the p -regime.

We will now show that the partially quenched partition function obeys a scaling relation generalizing Eq. (46), namely,⁷

$$\Sigma_{PQ}^{(\text{d}\chi\text{PT})}(m', m) = e^{yv} \Sigma_{PQ}^{(\chi\text{PT})}(e^{yv} m', e^{yv} m), \quad (53)$$

and a similar scaling relation follows for the spectral density

$$\rho^{(\text{d}\chi\text{PT})}(\Lambda, m) = e^{yv} \rho^{(\chi\text{PT})}(e^{yv} \Lambda, e^{yv} m). \quad (54)$$

In both Eq. (53) and Eq. (54), v is the minimum of the potential (31) in the unquenched case. A first hint comes

⁷To avoid cumbersome notation we omit the superscript ν .

from the standard relation between the spectral density and the condensate,

$$\langle \bar{\psi}\psi \rangle(m) = \int d\Lambda \frac{\rho(\Lambda, m)}{i\Lambda + m}. \quad (55)$$

This relation holds both in χ PT and in $d\chi$ PT. Since we already know that the $d\chi$ PT condensate obeys the scaling (46), it is hard to see how this will always be compatible with Eq. (55), *unless* the spectral density scales according to Eq. (54).

In order to prove Eq. (54) we first notice that the classical solution, $v = v(m, m', m'')$, is now determined by minimizing the potential (31) for each m, m', m'' , where again V_{eff} is defined by Eq. (30), except with $Z_{N_f+1|1}^{(\chi\text{PT})}(m, m'|m'')$ replacing $Z_{N_f}^{(\chi\text{PT})}(m)$. When we apply $\partial/\partial m'$ in Eq. (51) we again encounter a term proportional to $\partial V/\partial v$, which, once again, vanishes at the minimum of the potential, and Eq. (53) follows.

Next, notice that when we now set $m'' = m'$ the valence and ghost sectors cancel each other in the partition function, and the classical solution, v , reduces to that of the unquenched case, as determined by Eq. (34). Because v no longer depends on m' , when we analytically continue in m' and finally calculate the discontinuity (49), e^{yv} plays the role of an innocuous, real multiplicative factor, and Eq. (54) readily follows.

Our reasoning leading to Eq. (53) was completely general. As an explicit check of the transition from Eq. (53) to Eq. (54), we consider the limit $x = m\mathcal{V}f_\pi^2 B_\pi e^{yv} \gg 1$. In this limit, the ϵ -regime overlaps with the p -regime [23], and all pion masses containing at least one sea quark will be large compared to pion masses containing only valence and ghost quarks, allowing us to treat those fluctuations as if they are in the p -regime. This limit is interesting, since present-day simulations of confining near-conformal theories are so far limited to sea pions in the p -regime [5–9].

Assuming $x \gg 1$ (as well as $N_f \gg 1$), we show in Appendix B that the partially quenched generating functional of ordinary χ PT factorizes,

$$Z_{N_f+1|1}^{(\nu)}(x, x'|x'') = Z_{N_f}^{(\nu)}(x) Z_{1|1}^{(\nu)}(x'|x''). \quad (56)$$

Here $x' = m'\mathcal{V}f_\pi^2 B_\pi e^{yv}$ and $x'' = m''\mathcal{V}f_\pi^2 B_\pi e^{yv}$. As a result, the dilaton potential becomes

$$V(v) = f_\tau^2 B_\tau c_1 e^{4v} \left(v - \frac{1}{4} \right) - \frac{1}{\mathcal{V}} h_{N_f}^{(\nu)}(x) - \frac{1}{\mathcal{V}} h_{1|1}^{(\nu)}(x'|x''), \quad (57)$$

where $h_{N_f}^{(\nu)}(x(v))$ replaces $h(x(v))$ in Eq. (33), and [48,51]

$$\begin{aligned} h_{1|1}^{(\nu)}(x'|x'') &\equiv \log Z_{1|1}^{(\nu)}(m'|m'') \\ &= \log [x'I_{\nu+1}(x')K_\nu(x'') + x''I_\nu(x')K_{\nu+1}(x'')]. \end{aligned} \quad (58)$$

The minimum of the dilaton potential, $v = v(m, m', m'')$, is now determined by the saddle-point equation

$$\begin{aligned} 0 &= ve^{4v} - \frac{y}{4f_\tau^2 B_\tau c_1 \mathcal{V}} \frac{x}{\partial x} h_{N_f}^{(\nu)}(x) \\ &\quad - \frac{y}{4f_\tau^2 B_\tau c_1 \mathcal{V}} \left(x' \frac{\partial}{\partial x'} + x'' \frac{\partial}{\partial x''} \right) h_{1|1}^{(\nu)}(x'|x''). \end{aligned} \quad (59)$$

Using Eq. (58), it follows that the last term in Eq. (59) vanishes for $m' = m''$. As expected, v is now determined entirely by the sea-quark sector, corroborating the transition from Eq. (53) to Eq. (54).

VII. CONCLUSIONS

In this paper we have considered chiral perturbation theory extended to include a light dilaton meson. This systematic low energy effective theory for QCD-like theories was introduced by two of us [12] and previously studied in the p -regime. Here we defined and studied the ϵ -regime of this effective theory. In this new counting the pions are in the standard ϵ -regime while the dilaton is in the p -regime, such that the dilaton potential is probed only close to its minimum. We have calculated the quark mass dependence of the effective potential, the dilaton mass, the topological susceptibility, the chiral condensate and the average spectral density of the Dirac operator. At leading order the universal results from ordinary chiral perturbation theory are recovered. The effect of the dilaton enters at the next to leading order in the ϵ -counting. In particular, we find that the $d\chi$ PT condensate and spectral density are related to their χ PT counterparts via a simple scaling of the quark mass by the quark-mass dependent classical solution of the dilaton potential. This scaling relation is valid for any sea-pion mass, including both the ϵ -regime and the p -regime.

The fact that, when the sea pions are in the ϵ -regime, the eigenvalue density of the Dirac operator at leading order is given by the universal result and that the effect of the dilaton only enters at next to leading order, could appear to be in contrast with the power law behavior expected for a scale invariant theory; see e.g., Ref. [52]. However, note that we assume that chiral symmetry is broken spontaneously and hence, by the Banks-Casher relation, the eigenvalue density at the origin is nonzero. Furthermore, we assume that the dilaton is in the p -regime, and hence it is natural that the generating functional for the eigenvalue density is dominated by the much lighter sea (and valence) pions.

It would be most interesting to contrast the findings of $d\chi$ PT against lattice studies of QCD-like theories. The advantage when considering the Dirac spectrum is that the smallest eigenvalues are naturally in the ϵ -regime, provided that the eigenvalue density is nonzero, as required through the Banks-Casher relation for spontaneous breaking of chiral symmetry. Our scaling result for the spectral density is valid for sea pions in the p -regime as well, which is where all existing lattice studies have been carried out [15]. Contrasting the results from lattice studies with the results from $d\chi$ PT may help clarify the nature of the additional light mode observed on the lattice.

ACKNOWLEDGMENTS

We would like to thank Gernot Akemann, Poul Henrik Damgaard, Thomas Rytov, Ben Svetitsky and Jac Verbaarschot for discussions. This work was supported in part by the U.S. Department of Energy under Grant No. DE-SC0013682 (SFSU) and by the Israel Science Foundation under Grant No. 491/17 (Tel Aviv).

APPENDIX A: TOPOLOGICAL SUSCEPTIBILITY: SOME DETAILS

Assuming $x \gg N_f, \nu$, and using Eqs. (29) and (30), we have

$$Z_{N_f}^{(d\chi\text{PT})}(\theta) = e^{-\nu V_{\text{cl}} + x N_f \cos \theta}, \quad (\text{A1})$$

where V_{cl} is given by Eq. (20), and x is defined in Eq. (32). The right-hand side is to be evaluated at the new minimum of the dilaton potential, which now depends on θ . Since we are only interested in small θ we may approximate $\cos \theta = 1 - \theta^2/2 + \dots$. Writing the classical solution as

$$v = v_0 + \delta v, \quad (\text{A2})$$

where $v_0 = v_0(m)$ is the solution for $\theta = 0$, the correction is $\delta v \propto \theta^2$. Finally we substitute the new classical solution back into Eq. (A1), and expand to second order in θ and to first order in δv . The linear dependence on δv vanishes, since it is proportional to saddle-point equation satisfied by $v_0(m)$, as we already saw in Sec. VD and Sec. VI. The result is

$$Z_{N_f}^{(d\chi\text{PT})}(\theta) = Z_{N_f}^{(d\chi\text{PT})}(0) e^{-x N_f \theta^2/2}, \quad (\text{A3})$$

and Eq. (43) follows.

In order to evaluate $Z^{(\nu)} \equiv Z^{(\nu, d\chi\text{PT})}$ we start from

$$Z^{(\nu, \chi\text{PT})} = \int_0^{2\pi} d\theta e^{i\nu N_f \theta} \int_{\text{SU}(N_f)} DU_0 e^{\frac{\nu}{2} \text{Tr}[e^{i\theta} U_0 + e^{-i\theta} U_0^\dagger]}. \quad (\text{A4})$$

We again assume $x \gg N_f, \nu$ and perform similar steps, keeping only the $O(\theta^2)$ part in the exponential, except that we now also perform the θ integral, finding

$$Z^{(\nu, d\chi\text{PT})} = e^{-\nu V_{\text{cl}} + x N_f - \nu^2 N_f / (2x)}. \quad (\text{A5})$$

The classical solution $v_0(m)$ for $\nu = 0$ is the same as for $\theta = 0$, and, defining δv again as in Eq. (A2), we now have a correction $\delta v \propto \nu^2$. As before, when we substitute the solution back into Eq. (A5), there is no linear dependence on δv , hence

$$Z^{(\nu, d\chi\text{PT})} = Z^{(\nu=0, d\chi\text{PT})} e^{-\nu^2 N_f / (2x)}, \quad (\text{A6})$$

and Eq. (42) follows [with χ in Eq. (43)].

APPENDIX B: SOME RESULTS FOR LARGE N_f AND/OR LARGE x

We will make use of the following representation for the partition function of the partially quenched theory in ordinary χ PT [53,54]

$$\begin{aligned} Z_{k|n}^{(\nu)}(\{z_i\}) &= \frac{\det \mathcal{A}}{\prod_{i=1}^n \prod_{j=i+1}^n (z_j^2 - z_i^2) \prod_{i=n+1}^{n+k} \prod_{j=i+1}^{n+k} (z_j^2 - z_i^2)}, \\ \mathcal{A}_{ij} &= z_i^{j-1} \mathcal{J}_{\nu+j-1}(z_i), \quad i, j = 1, \dots, n+k. \end{aligned} \quad (\text{B1})$$

Here k is the number of fermions (sea and valence), and n is the number of ghosts. We define $\mathcal{J}_\nu(z_i) = (-1)^\nu K_\nu(z_i)$ for $i = 1, \dots, n$, and $\mathcal{J}_\nu(z_i) = I_\nu(z_i)$ for $i = n+1, \dots, n+k$. In order to use this representation the z_i must be mutually different for $i = 1, \dots, n$, as well as for $i = n+1, \dots, n+k$. Degenerate limits can be taken in the end.

We show that $h_{N_f}^{(\nu)} = -\log Z_{N_f|0}^{(\nu)}$ scales like N^2 in the Veneziano limit. In this case we need Eq. (B1) for $n = 0, k = N_f$. The degenerate limit is $z_1 = \dots = z_{N_f} = x$, where x is given by Eq. (32). It follows that the z_i variables all scale like N [note the presence of f_π^2 in Eq. (32)]. Thus, for $\nu + j = \mathcal{O}(1)$ we may use the asymptotic expansion of $I_{\nu+j-1}(z_i)$ for large argument, while for $\nu + j = \mathcal{O}(N)$ we may use the uniform expansion for real variable [55]. Either way, we find that $\log \mathcal{A}_{ij}$ scales like N for any i, j , hence $\log \det \mathcal{A} = \text{tr} \log \mathcal{A}$ scales like N^2 , establishing a similar scaling for $h_{N_f}^{(\nu)}$. Finally, Eq. (44) implies the same scaling for the condensate.

We next prove that, for $x \sim N \gg 1$ the partially quenched partition function $Z_{N_f+1|1}^{(\nu)}(x, x'|x'')$ factorizes, cf. Eq. (56).

Once again, to apply Eq. (B1) we assume that the sea masses are nondegenerate, taking if desired the degenerate limit in the end. We have $z_1 = x'', z_2 = x'$, and $z_i = x_{i-2}$ for $i = 3, \dots, N_f + 2$. There are thus $k = N_f + 1$ sea and

valence quarks, and there is $n = 1$ ghost quark. We proceed by writing \mathcal{A} in a block form,

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (\text{B2})$$

where A is 2×2 , B is $2 \times N_f$, C is $N_f \times 2$, and D is $N_f \times N_f$. Using that $x'', x' = \mathcal{O}(1)$ and $x_1, \dots, x_{N_f} = \mathcal{O}(N)$ the entries of A are $\mathcal{O}(1)$. The entries of B are also $\mathcal{O}(1)$ for $j = \mathcal{O}(1)$, whereas for $j = \mathcal{O}(N)$ we may use the asymptotic expansion for large order. For the entries of C we may always use the asymptotic expansion for large argument. The same is true for the entries of D provided that $j = \mathcal{O}(1)$, while for $j = \mathcal{O}(N)$ we use the uniform expansion for real variable. Now employing Schur's determinant identity, $\det \mathcal{A} = \det A \det(D - CA^{-1}B)$, we conclude that D dominates over $CA^{-1}B$, and the determinant factorizes, $\det \mathcal{A} = \det A \det D$.

The last step is to consider the denominator in Eq. (B1). In the case of $\mathcal{Z}_{N_f+1|1}^{(\nu)}$, the first factor in the denominator trivially collapses to one. The second factor can be written more explicitly as

$$\prod_{i=1}^{N_f} (x_i^2 - x'^2) \prod_{i=1}^{N_f} \prod_{j=i+1}^{N_f} (x_j^2 - x_i^2) = \prod_{i=1}^{N_f} x_i^2 \prod_{i=1}^{N_f} \prod_{j=i+1}^{N_f} (x_j^2 - x_i^2), \quad (\text{B3})$$

where on the right-hand side we used $x_i \gg x'$. In addition, noting that D is originally the diagonal block of \mathcal{A} with index range $i, j = 3, \dots, N_f + 2$, we have

$$\det D = \det D' \prod_{i=1}^{N_f} x_i^2, \quad (\text{B4})$$

where the new matrix D' is defined by

$$D'_{ij} = x_i^{j-1} I_{\nu+j+1}(x_i), \quad i, j = 1, \dots, N_f. \quad (\text{B5})$$

Putting it together we arrive at Eq. (56) with

$$Z_{1|1}^{(\nu)}(x'|x'') = \det A, \quad (\text{B6})$$

$$Z_{N_f}^{(\nu)}(x_1, \dots, x_n) = \frac{\det D'}{\prod_{i=1}^{N_f} \prod_{j=i+1}^{N_f} (x_j^2 - x_i^2)}, \quad (\text{B7})$$

where the final identification follows from the independence of the leading order asymptotic behavior of the Bessel function on the index.

-
- [1] T. DeGrand, Lattice tests of beyond standard model dynamics, *Rev. Mod. Phys.* **88**, 015001 (2016).
- [2] D. Negradi and A. Patella, Strong dynamics, composite Higgs and the conformal window, *Int. J. Mod. Phys. A* **31**, 1643003 (2016).
- [3] C. Pica, Beyond the standard model: Charting fundamental interactions via lattice simulations, *Proc. Sci., LATTICE20162016* (2016) 015 [arXiv:1701.07782].
- [4] B. Svetitsky, Looking behind the standard model with lattice gauge theory, *EPJ Web Conf.* **175**, 01017 (2018).
- [5] Y. Aoki *et al.* (LatKMI Collaboration), Light flavor-singlet scalars and walking signals in $N_f = 8$ QCD on the lattice, *Phys. Rev. D* **96**, 014508 (2017).
- [6] T. Appelquist *et al.*, Strongly interacting dynamics and the search for new physics at the LHC, *Phys. Rev. D* **93**, 114514 (2016).
- [7] T. Appelquist *et al.* (Lattice Strong Dynamics Collaboration), Nonperturbative investigations of SU(3) gauge theory with eight dynamical flavors, *Phys. Rev. D* **99**, 014509 (2019).
- [8] Z. Fodor, K. Holland, J. Kuti, D. Negradi, and C. H. Wong, The twelve-flavor β -function and dilaton tests of the sextet scalar, *EPJ Web Conf.* **175**, 08015 (2018).
- [9] Z. Fodor, K. Holland, J. Kuti, and C. H. Wong, Tantalizing dilaton tests from a near-conformal EFT, *Proc. Sci., LATTICE2018* (2019) 196 [arXiv:1901.06324].
- [10] J. R. Ellis, Aspects of conformal symmetry and chirality, *Nucl. Phys.* **B22**, 478 (1970); Erratum, *Nucl. Phys.* **B25**, 639(E) (1971).
- [11] C. J. Isham, A. Salam, and J. A. Strathdee, Broken chiral and conformal symmetry in an effective-lagrangian formalism, *Phys. Rev. D* **2**, 685 (1970).
- [12] M. Golterman and Y. Shamir, Low-energy effective action for pions and a dilatonic meson, *Phys. Rev. D* **94**, 054502 (2016).
- [13] M. Golterman and Y. Shamir, Effective action for pions and a dilatonic meson, *Proc. Sci., LATTICE2016* (2016) 205 [arXiv:1610.01752].
- [14] M. Golterman and Y. Shamir, Effective pion mass term and the trace anomaly, *Phys. Rev. D* **95**, 016003 (2017).
- [15] M. Golterman and Y. Shamir, The large-mass regime of the dilaton-pion low-energy effective theory, *Phys. Rev. D* **98**, 056025 (2018).
- [16] C. N. Leung, S. T. Love, and W. A. Bardeen, Aspects of dynamical symmetry breaking in gauge field theories, *Nucl. Phys.* **B323**, 493 (1989).

- [17] S. Matsuzaki and K. Yamawaki, Dilaton Chiral Perturbation Theory: Determining the Mass and Decay Constant of the Technidilaton on the Lattice, *Phys. Rev. Lett.* **113**, 082002 (2014).
- [18] M. Hansen, K. Langåble, and F. Sannino, Extending chiral perturbation theory with an isosinglet scalar, *Phys. Rev. D* **95**, 036005 (2017).
- [19] T. Appelquist, J. Ingoldby, and M. Piai, Dilaton EFT framework for lattice data, *J. High Energy Phys.* **07** (2017) 035.
- [20] T. Appelquist, J. Ingoldby, and M. Piai, Analysis of a dilaton EFT for lattice data, *J. High Energy Phys.* **03** (2018) 039.
- [21] T. Appelquist, J. Ingoldby, and M. Piai, The dilaton potential and lattice data, [arXiv:1908.00895](https://arxiv.org/abs/1908.00895).
- [22] M. Golterman and Y. Shamir, *Proceedings of the Lattice*, <https://indico.cern.ch/event/764552/contributions/3428289/attachments/1864051/3067137/Lattice2019Golterman.pdf> (2019).
- [23] J. Gasser and H. Leutwyler, Light quarks at low temperatures, *Phys. Lett. B* **184**, 83 (1987); Thermodynamics of chiral symmetry, *Phys. Lett. B* **188**, 477 (1987).
- [24] H. Leutwyler and A. V. Smilga, Spectrum of Dirac operator and role of winding number in QCD, *Phys. Rev. D* **46**, 5607 (1992).
- [25] P. H. Damgaard, K. Splittorff, and J. J. M. Verbaarschot, Microscopic Spectrum of the Wilson Dirac Operator, *Phys. Rev. Lett.* **105**, 162002 (2010).
- [26] M. Kieburg, J. J. M. Verbaarschot, and S. Zafeiropoulos, Eigenvalue Density of the Non-Hermitian Wilson Dirac Operator, *Phys. Rev. Lett.* **108**, 022001 (2012).
- [27] M. Kieburg, K. Splittorff, and J. J. M. Verbaarschot, Realization of the sharpe-singleton scenario, *Phys. Rev. D* **85**, 094011 (2012).
- [28] K. Splittorff, Chiral dynamics with Wilson fermions, *Proc. Sci.*, LATTICE2012 (2012) 018 [[arXiv:1211.1803](https://arxiv.org/abs/1211.1803)].
- [29] M. Kieburg, J. J. M. Verbaarschot, and S. Zafeiropoulos, Spectral properties of the Wilson-Dirac operator and random matrix theory, *Phys. Rev. D* **88**, 094502 (2013).
- [30] S. R. Sharpe and R. L. Singleton, Spontaneous flavor and parity breaking with Wilson fermions, *Phys. Rev. D* **58**, 074501 (1998).
- [31] G. Rupak and N. Shoreh, Chiral perturbation theory for the Wilson lattice action, *Phys. Rev. D* **66**, 054503 (2002).
- [32] O. Bär, G. Rupak, and N. Shoreh, Chiral perturbation theory at $\mathcal{O}(a^2)$ for lattice QCD, *Phys. Rev. D* **70**, 034508 (2004).
- [33] O. Bär, S. Necco, and S. Schaefer, The epsilon regime with Wilson fermions, *J. High Energy Phys.* **03** (2009) 006.
- [34] A. Shindler, Observations on the Wilson fermions in the epsilon regime, *Phys. Lett. B* **672**, 82 (2009).
- [35] L. Del Debbio, L. Giusti, M. Lüscher, R. Petronzio, and N. Tantalo, Stability of lattice QCD simulations and the thermodynamic limit, *J. High Energy Phys.* **02** (2006) 011; QCD with light Wilson quarks on fine lattices. I: First experiences and physics results, *J. High Energy Phys.* **02** (2007) 056.
- [36] G. Akemann, J. C. Osborn, K. Splittorff, and J. J. M. Verbaarschot, Unquenched QCD Dirac operator spectra at nonzero baryon chemical potential, *Nucl. Phys.* **B712**, 287 (2005).
- [37] J. C. Osborn, K. Splittorff, and J. J. M. Verbaarschot, Chiral Symmetry Breaking and the Dirac Spectrum at Nonzero Chemical Potential, *Phys. Rev. Lett.* **94**, 202001 (2005).
- [38] K. Splittorff and J. J. M. Verbaarschot, Phase of the Fermion Determinant at Nonzero Chemical Potential, *Phys. Rev. Lett.* **98**, 031601 (2007); K. Splittorff and J. J. M. Verbaarschot, QCD sign problem for small chemical potential, *Phys. Rev. D* **75**, 116003 (2007).
- [39] K. Splittorff and B. Svetitsky, Sign problem via imaginary chemical potential, *Phys. Rev. D* **75**, 114504 (2007).
- [40] D. Toublan and J. J. M. Verbaarschot, Statistical properties of the spectrum of the QCD Dirac operator at low-energy, *Nucl. Phys.* **B603**, 343 (2001).
- [41] K. Splittorff and J. J. M. Verbaarschot, Factorization of correlation functions and the replica limit of the Toda lattice equation, *Nucl. Phys.* **B683**, 467 (2004).
- [42] J. Gasser and H. Leutwyler, Chiral perturbation theory to one loop, *Ann. Phys. (N.Y.)* **158**, 142 (1984); Chiral perturbation theory: Expansions in the mass of the strange quark, *Nucl. Phys.* **B250**, 465 (1985).
- [43] M. Lüscher, A semiclassical formula for the topological susceptibility in a finite space-time volume, *Nucl. Phys.* **B205**, 483 (1982).
- [44] P. H. Damgaard, T. DeGrand, and H. Fukaya, Finite-volume correction to the pion decay constant in the epsilon-regime, *J. High Energy Phys.* **12** (2007) 060.
- [45] G. Akemann, F. Basile, and L. Lellouch, Finite size scaling of meson propagators with isospin chemical potential, *J. High Energy Phys.* **12** (2008) 069.
- [46] G. Akemann and F. Pucci, Exploring the Aoki regime, *J. High Energy Phys.* **06** (2013) 059.
- [47] P. H. Damgaard and H. Fukaya, The chiral condensate in a finite volume, *J. High Energy Phys.* **01** (2009) 052.
- [48] P. H. Damgaard, J. C. Osborn, D. Toublan, and J. J. M. Verbaarschot, The microscopic spectral density of the QCD Dirac operator, *Nucl. Phys.* **B547**, 305 (1999).
- [49] M. Golterman, Applications of chiral perturbation theory to lattice QCD, [arXiv:0912.4042](https://arxiv.org/abs/0912.4042).
- [50] T. Banks and A. Casher, Chiral symmetry breaking in confining theories, *Nucl. Phys.* **B169**, 103 (1980).
- [51] M. Golterman, S. R. Sharpe, and R. L. Singleton, Jr., Effective theory for quenched lattice QCD and the Aoki phase, *Phys. Rev. D* **71**, 094503 (2005).
- [52] T. DeGrand, Volume scaling of Dirac eigenvalues in SU(3) lattice gauge theory with color sextet fermions, [arXiv:0906.4543](https://arxiv.org/abs/0906.4543).
- [53] K. Splittorff and J. J. M. Verbaarschot, Replica Limit of the Toda Lattice Equation, *Phys. Rev. Lett.* **90**, 041601 (2003).
- [54] Y. V. Fyodorov and G. Akemann, On the supersymmetric partition function in QCD inspired random matrix models, *JETP Lett.* **77**, 438 (2003).
- [55] See, e.g., <https://dlmf.nist.gov/10#PT3+>.