Ratio $\rho_{\bar{p}p}^{pp}(s)$ in Froissaron and maximal odderon approach

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The ratios $\rho_{\bar{p}p}^{pp}(s)$ of the real to the imaginary part of forward elastic pp and $\bar{p}p$ scattering amplitudes at very high energies are considered in the models with rising total cross sections and its difference. It is shown from the dispersion relations for pp and $\bar{p}p$ scattering amplitudes that in the Froissaron and maximal odderon approach the ratios do not vanish asymptotically and they have the opposite signs for pp and $\bar{p}p$ scattering.

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I. INTRODUCTION

It was proved in the paper [1] that the real part of the crossing-even elastic scattering amplitude has to be positive at $s \to \infty$. Generally speaking in order to know both the ratios $\rho_{hh}^{hh}(s)$, the corresponding crossing-even and crossing-odd components must be jointly considered. We would like to remark the following points concerning the special case where hadron h is a proton.

(1) The crossing odd component, odderon, for these amplitudes plays a very important role in observed differences in pp and $\bar{p}p$ cross sections and it is lively discussed in the old and recent papers devoted to phenomenological models [2].

(2) In order to make a conclusion about possible behavior of $\rho_{\bar{p}p}^{pp}(s)$ at $s \to \infty$ we should consider the most general case for odderon contribution allowed by the known restrictions on asymptotic properties of scattering pp and $\bar{p}p$ amplitudes.

This is the goal of the present paper. Starting from the main strict results about crossing-even and crossing-odd pp and $\bar{p}p$ amplitudes we will show what we can say about total pp and $\bar{p}p$ cross sections and $\rho_{\bar{p}p}^{pp}(s)$.

We consider here the real parts of crossing-even and crossing-odd pp and $\bar{p}p$ amplitudes which are dominated at high energy by pomeron (presumably by Froissaron) and odderon contribution correspondingly. It is assumed that odderon satisfies the general bounds known from S-matrix theory.

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II. REAL PART OF THE FORWARD SCATTERING AMPLITUDE

The crossing-even and crossing-odd amplitudes $A_+(s,t=0) = A_+(s), A_-(s,t=0) = A_-$ of the forward elastic pp and $\bar{p}p$ scattering are defined as follows:

$$f_\pm(s) = \frac{1}{2}[f^{pp}(s) \pm f_{\bar{p}p}(s)], \qquad f(s) \equiv A(s,0), \quad (1)$$

where m is the mass of proton. Normalization of amplitudes is defined by the optic theorem in the following form:

$$\sigma_t(s) = \frac{\mathrm{Im}A(s)}{s\sqrt{1-4m^2/s}}. \quad (2)$$

We use here the following main facts concerning the amplitudes and cross sections under interest.

(A) Froissart-Martin-Lukazsuk bound [3]

$$\sigma_t(s) \le \frac{\pi}{m_\pi^2}\ln^2(s/s_0), \qquad s_0 = 1\ \mathrm{GeV}^2. \quad (3)$$

In what follows we consider an arbitrary rise of cross section

$$\sigma_t(s) \propto \ln^\alpha(s/s_0), \qquad 0 < \alpha \le 2. \quad (4)$$

(B) Bound on the difference of pp and $\bar{p}p$ cross sections [4]:

$$\Delta\sigma_t = |\sigma_t^{pp}(s) - \sigma_t^{\bar{p}p}(s)| = \frac{2}{s\sqrt{1-4m^2/s}}|\mathrm{Im}A_-(s,0)| \propto \ln^\beta(s/s_0), \quad (5)$$

where $\beta \le \alpha/2$.

We would like to note here that the above assumptions (4) and (5) mean that the pomeron and the odderon (at $t=0$) are located in complex angular momentum at $j=0$. It is required by unitarity for the rising with energy pomeron (in some sense we consider the unitarized pomeron and odderon). While the j-location of the odderon is in agreement with QCD results $\alpha_O(0)=1$ [5–7], a kind of QCD odderon j-singularity [either simple pole or more (less) hard singularity] is unknown theoretically. Besides that, available experimental data on $\Delta\sigma_t$ are not sufficient for any solid conclusion about difference of the pp and $\bar{p}p$ cross sections at high energy. Therefore we consider the arbitrary value of the parameter β.

(C) The amplitudes $f_\pm(s)$ are analytic functions of s in the whole complex plane. These amplitudes satisfy the twice subtracted dispersion relations because of $f_+(s)\propto s\ln^\alpha(s/4m^2)$ and $f_-(s)\propto s\ln^\beta(s/4m^2)$ at $s\gg 4m^2$, and $\alpha,\beta>0$:

$$\frac{\mathrm{Re}f_+(s)}{s}=\frac{f_+(0)}{s}+\frac{2s}{\pi}\mathcal{P}\int_{4m^2}^{\infty}\frac{ds'}{(s'^2-s^2)}\frac{\mathrm{Im}f_+(s')}{s'}, \tag{6}$$

where $\mathcal{P}$ means principal integral value,

$$\frac{\mathrm{Re}f_-(s)}{s}=f'_-(0)+\frac{2s^2}{\pi}\mathcal{P}\int_{4m^2}^{\infty}\frac{ds'}{(s'^2-s^2)}\frac{\mathrm{Im}f_-(s')}{s'^2}. \tag{7}$$

Our aim is to find an asymptotic behavior of the real part of leading terms in crossing-even and crossing-odd amplitudes making use of Eqs. (6) and (7) taking a general form of crossing-even and -odd contributions (4) and (5). It would be sufficient in the case to use the derivative dispersion relations (DDR). They were suggested in [8]. One can find more details in Refs. [9–11].

Let us consider the dispersion relations for arbitrary (but rising with energy) pomeron and odderon with the bounds (4) and (5). At $s\to\infty$ one can use the following approximation for $f_\pm(s)$:

$$\mathrm{Im}f_\pm(s)/s\approx r_\pm\begin{cases}\xi^\alpha, & \alpha\le 2,\\ \xi^\beta, & \beta\le\alpha/2,\end{cases}\quad \xi=\ln(s/4m^2). \tag{8}$$

The rigorous general constraint (see, for example, [12])

$$\frac{|\sigma_t^{pp}(s)-\sigma_t^{\bar{p}p}(s)|}{\sigma_t^{pp}(s)+\sigma_t^{\bar{p}p}(s)}\to 0\quad \text{at } s\to\infty \tag{9}$$

is satisfied for the amplitudes (8) at any $\beta\le\alpha/2$.

Making use of Eq. (8) and the method to obtain DDR for $\mathrm{Re}f_\pm(s)$ at $s\to\infty$, described in the Appendix, we can write

$$\frac{\mathrm{Re}f_\pm(s)}{s}\approx r_\pm\begin{cases}\tan\left(\frac{\pi}{2}\hat{d}\right)\xi^\alpha=\frac{\pi}{2}\hat{d}(1+\mathcal{O}(\hat{d}^2))\xi^\alpha\\ \approx\frac{\pi}{2}\alpha\xi^{\alpha-1},\\ -\cot\left(\frac{\pi}{2}\hat{d}\right)\xi^\alpha=-\frac{1-\frac{1}{2}\left(\frac{\pi\hat{d}}{2}\right)^2+\dots}{\frac{\pi\hat{d}}{2}-\frac{1}{3}\left(\frac{\pi\hat{d}}{2}\right)^3+\dots}\xi^\beta\\ \approx-\frac{2}{\pi}\hat{d}^{-1}\xi^\beta(1+\mathcal{O}(\hat{d}^2))\\ =-\frac{2}{\pi}\int d\xi\xi^\beta\approx-\frac{2}{\pi}\frac{1}{1+\beta}\xi^{\beta+1},\end{cases} \tag{10}$$

where $\hat{d}=d/d\xi$. The sign "−" in $\mathrm{Re}f_-(s)$ is originated from our definition of the amplitudes $f_\pm$ in Eq. (1).

For the leading terms at $s\to\infty$ we have

$$\frac{1}{s}\mathrm{Re}f_+(s)=r_+\frac{\pi}{2}\alpha\xi^{\alpha-1}, \tag{11}$$

$$\frac{1}{s}\mathrm{Re}f_-(s)=-r_-\frac{2}{\pi(1+\beta)}\xi^{1+\beta}, \tag{12}$$

$$\mathrm{Re}f_{\bar{p}p}^{pp}(s)=s\left[\frac{\pi}{2}r_+\alpha\xi^{\alpha-1}\mp r_-\frac{2}{\pi(1+\beta)}\xi^{1+\beta}\right]. \tag{13}$$

If we consider parameters α and β in region

$$0<\alpha\le 2\qquad 0<\beta\le\alpha/2 \tag{14}$$

which corresponds to the models with infinitely rising σ_t and $|\Delta\sigma_t|$ we find that the second term in Eq. (13) dominates at $\xi\to\infty$ because of $\beta+1>1$ and $\alpha-1\le 1$.

So, *the first conclusion* is the following.

The real part of the pp and $\bar{p}p$ scattering amplitude in the models with infinitely rising cross sections and difference of the cross sections is asymptotically dominated by the odderon contribution.

Thus, in this case

$$\rho_{\bar{p}p}^{pp}(s)=\frac{\mathrm{Re}f_{\bar{p}p}^{pp}(s)}{\mathrm{Im}f_{\bar{p}p}^{pp}(s)}\approx\mp\frac{r_-}{r_+}\frac{2/\pi}{1+\beta}\xi^{1+\beta-\alpha}. \tag{15}$$

If $0<\beta<\alpha-1$ then at $\xi\to\infty$ ratios $\rho(\xi)\to\pm 0$ (with the opposite signs for pp and $\bar{p}p$). But if $\alpha-1<\beta<\alpha/2$ then ratios $\rho(\xi)\to\pm\infty$. However, if $1+\beta\le\alpha-1$ and $\beta>-1$ the crossing-even term is dominating and $\rho(\xi)\to+0$. This is shown on Fig. 1.

In the case of the maximal odderon at fixed $0<\alpha\le 2$ we have $\beta=\alpha/2$. Then at any allowed positive α

$$\mathrm{Re}f_{\bar{p}p}^{pp}(s)=s\left[\frac{\pi}{2}r_+\alpha\xi^{\alpha-1}\mp r_-\frac{2}{\pi(1+\alpha/2)}\xi^{1+\alpha/2}\right]$$
$$\approx\mp s\frac{2r_-}{\pi(1+\alpha/2)}\xi^{1+\alpha/2}. \tag{16}$$

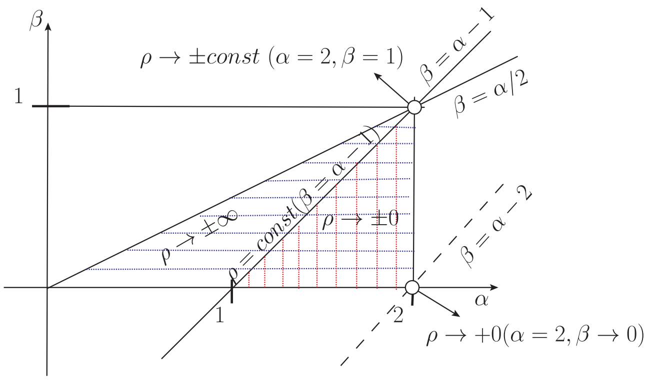

FIG. 1. Ratio $\rho_{\bar{p}p}^{pp}$ at $s \to \infty$ in different regions of the plane (α, β).

Consequently

$$\rho_{\bar{p}p}^{pp}(s) = \frac{\mathrm{Re} f_{\bar{p}p}^{pp}(s)}{\mathrm{Im} f_{\bar{p}p}^{pp}(s)} \approx \mp \frac{2r_-/\pi}{r_+} \frac{1}{1+\alpha/2} \xi^{1-\alpha/2}. \qquad (17)$$

We would like to notice that at $\beta = \alpha/2$ the $|\rho_{\bar{p}p}^{pp}(s)| \to \infty$ at $s \to \infty$ if $0 < \alpha < 2$. It is *the second conclusion*.

As well we have *the third conclusion*: $|\rho_{\bar{p}p}^{pp}(s)| \to \mathrm{const} \neq 0$ if $r_- \neq 0$ only in the case of the Froissaron and maximal odderon ($\alpha = 2$, $\beta = 1$).

In the Froissaron-maximal odderon (FMO) model [13,14] we have considered and compared with the latest data of TOTEM [15,16] the case $\alpha = 2$, $\beta = 1$. Performing fit with arbitrary values of α and β [14] we have found that α, β come back to the maximal values 2,1, correspondingly.

In [13] the leading terms were parametrized in the form

$$\frac{k}{s} f_\pm = \begin{cases} iH_1\tilde{\xi}^2 \approx iH_1\xi^2 + \pi H_1\xi, \\ O_1\tilde{\xi}^2 \approx O_1\xi^2 - iO_1\pi\xi, \end{cases} \qquad (18)$$

where $k = 0.3894, \ldots$ mbGeV^2 and $\tilde{\xi} = \xi - i\pi/2$.

Comparing Eqs. (11) and (12) with (18) we have $r_+ = H_1/k$, $r_- = -\pi O_1/k$, where $H_1 = 0.25$ mb, $O_1 = -0.05$ mb [13]. Thus

$$\lim_{s\to\infty} \rho_{\bar{p}p}^{pp}(s) = \mp \frac{r_-}{r_+\pi} = \pm \frac{O_1}{H_1} = \mp 0.2. \qquad (19)$$

In Fig. 2 we show an extrapolation of the result obtained in [13] $\rho_{\bar{p}p}^{pp}$ for higher energies. One can see that the real asymptotic regime occurs at extremely high energy. Even a change of the sign in $\rho^{pp}(s)$ is attained at $\sqrt{s} \sim 10^4$ TeV.

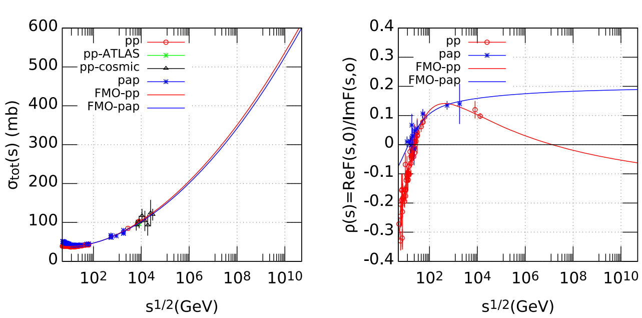

FIG. 2. Extrapolation of $\sigma_t(s)$ and $\rho(s)$ in the FMO model at $t = 0$ [13].

III. CONCLUSION

We have shown that the ratios $\rho_{\bar{p}p}^{pp}(s)$ of the real to imaginary part of forward elastic pp and $\bar{p}p$ scattering is not positive for both $pp \to pp$ and $\bar{p}p \to \bar{p}p$ processes with an odderon contribution to the amplitudes do not vanish at $s \to \infty$.

We would like to emphasize that such a regime is confirmed by comparison of the Froissaron and maximal odderon approach [13,14] with the experimental data on forward pp and $\bar{p}p$ scattering including the latest TOTEM data. The model predicts asymptotic values of the ratios $\rho_{\bar{p}p}^{pp}(s \to \infty) \approx \mp 0.2$. Of course it would be nice to have more precise data on pp and $\bar{p}p$ at various energies (for example, at 900 GeV at LHC) and not only from the TOTEM experiment.

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APPENDIX

The integrals in (6) and (7) (without unimportant at $s \to \infty$ subtracted terms) can be presented in the common form with $\nu = 1$ for $\mathrm{Im} f_+(s)$ and $\nu = 0$ for $\mathrm{Im} f_-(s)$

$$\begin{aligned} \frac{\mathrm{Re} f(s)}{s} &= \frac{2}{\pi} s^{2-\nu} \mathrm{P} \int_{4m^2}^{\infty} \frac{ds'}{s'^2 - s^2} g(s') \\ &= \frac{2}{\pi} e^{(2-\nu)\xi} \mathrm{P} \int_0^{\infty} \frac{e^{\xi'} d\xi'}{e^{2\xi'} - e^{2\xi}} g(\xi'), \end{aligned} \qquad (A1)$$

where

$$g(s') = (s')^{\nu-1} \mathrm{Im} f(s')/s'$$

or

$$g(\xi') = e^{(\nu-1)\xi'} e^{-\xi'} \mathrm{Im} f(\xi').$$

After some simple transformation one can obtain

$$e^{-\xi}\mathrm{Re}f(\xi) = \frac{1}{\pi}e^{(1-\nu)\xi}\left\{\ln\frac{e^{\xi}+1}{e^{\xi}-1}g(\xi'=0) + \int_0^\infty d\xi' \ln\frac{e^{\xi}+e^{\xi'}}{|e^{\xi}-e^{\xi'}|}(e^{(\nu-1)\xi'}g(\xi'))'\right\}. \quad \text{(A2)}$$

The logarithmic factor in the integral (A2) can be transformed as follows:

$$\begin{aligned}\ln\frac{e^{\xi'}+e^{\xi}}{|e^{\xi'}-e^{\xi}|} &= \ln\frac{e^{(\xi'-\xi)/2}+e^{-(\xi'-\xi)/2}}{|e^{(\xi'-\xi)/2}-e^{-(\xi'-\xi)/2}|} \\ &= \ln\left|\coth\frac{1}{2}(\xi'-\xi)\right| = \ln\frac{1+e^{-|x|}}{1-e^{-|x|}} \\ &= 2\sum_{p=0}^{\infty}\frac{e^{-(2p+1)|x|}}{2p+1}, \quad x=\xi'-\xi.\end{aligned}$$

All other factors in the (A2) can be expanded in powers of $\xi'-\xi$:

$$\begin{aligned}(e^{(\nu-1)\xi'}g(\xi'))' &= e^{(\nu-1)\xi'}\left(\nu-1+\frac{d}{d\xi'}\right)g(\xi'), \\ \tilde{g}(\xi') &= \left(\nu-1+\frac{d}{d\xi'}\right)g(\xi') = \sum_{k=0}^{\infty}\frac{(\xi'-\xi)^k}{k!}\hat{d}^k\tilde{g}(\xi), \\ e^{(1-\nu)\xi}e^{(\nu-1)\xi'} &= \sum_{n=0}^{\infty}\frac{(\nu-1)^n}{n!}(\xi'-\xi)^n. \\ \hat{d} &= d/d\xi. \end{aligned} \quad \text{(A3)}$$

Taking into account the above expression and omitting the first term in (A2) because it goes to 0 at $\xi\to\infty$ one can write the integral (A2) in the form

$$e^{-\xi}\mathrm{Re}f(\xi) = \frac{2}{\pi}\sum_{p=0}^{\infty}\frac{1}{2p+1}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\nu-1)^n}{k!n!} \times \mathcal{I}(\xi;p,k,n)\cdot\hat{d}^k\tilde{g}(\xi), \quad \text{(A4)}$$

where

$$\begin{aligned}\mathcal{I}(\xi;p,k,n) &= \int_0^\infty d\xi' e^{-(2p+1)|\xi'-\xi|}(\xi'-\xi)^{k+n} \\ &= (-1)^{k+n}\int_0^{\xi} d\xi' e^{-(2p+1)(\xi-\xi')}(\xi-\xi')^{k+n} \\ &\quad + \int_{\xi}^{\infty} d\xi' e^{-(2p+1)(\xi'-\xi)}(\xi'-\xi)^{k+n} \\ &= \frac{1}{(2p+1)^{k+n+1}}[\Gamma(k+n+1)) \\ &\quad + (-1)^{k+n}\gamma(k+n+1,\xi(2p+1)]\end{aligned}$$

and $\gamma(a,x)$ is incomplete gamma function. At fixed a and $x\to\infty$

$$\gamma(a,x) = \Gamma(a) - e^{-x}x^a(1+\mathcal{O}(1/x)), \quad \text{(A5)}$$

therefore we have

$$\begin{aligned}e^{-\xi}\mathrm{Re}f(\xi) &\approx \frac{2}{\pi}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(k+n)!}{k!n!} \\ &\quad \times\left(\frac{\nu-1}{2p+1}\right)^n\left(\frac{\hat{d}}{2p+1}\right)^k(1+(-1)^{k+n})\tilde{g}(\xi) \\ &= \frac{2}{\pi}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}\left\{\frac{1}{1-(\nu-1+\hat{d})/(2p+1)} + \frac{1}{1+(\nu-1+\hat{d})/(2p+1)}\right\}\tilde{g}(\xi) \\ &= \frac{4}{\pi}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2-(\nu-1+\hat{d})^2}\tilde{g}(\xi).\end{aligned}$$

The last sum in the above expression is simplified making use of the equality [17]

$$\sum_{k=0}^{\infty}\frac{1}{(2p+1)^2-a^2} = \frac{\pi}{4a}\tan(\pi a/2).$$

Thus, we have finally the asymptotic form of the derivative dispersion relations

$$\Re ef_+(s)/s \approx \tan\left(\frac{\pi}{2}\hat{d}\right)\mathrm{Im}f_+(s)/s, \quad \text{(A6)}$$

$$\begin{aligned}\mathrm{Re}f_-(s)/s &\approx \tan\left[\frac{\pi}{2}(-1+\hat{d})\right]\mathrm{Im}f_-(s)/s \\ &= -\cot\left(\frac{\pi}{2}\hat{d}\right)\mathrm{Im}f_-(s)/s.\end{aligned} \quad \text{(A7)}$$

[1] A. Martin and T. Tsun Wu, Phys. Rev. D **97**, 014011 (2018).
[2] L. Lukazsuk and B. Nicolescu, Lett. Nuovo Cimento **8**, 405 (1973); D. Joynson, E. Leader, B. Nicolescu, and C. Lopez, Nuovo Cimento A **30**, 345 (1975).
[3] M. Froissart, Phys. Rev. **123**, 1053 (1961); L. Lukaszuk and A. Martin, Nuovo Cimento A **52**, 122 (1967).
[4] R. J. Eden, Rev. Mod. Phys. **43**, 15 (1971), and the references therein; H. Cornille and R. E. Hendrick, Phys. Rev. D **10**, 3805 (1974).
[5] J. Bartels, L. N. Lipatov, and G. P. Vacca, Phys. Lett. B **477**, 178 (2000).
[6] Y. V. Kovchegov, L. Szymanowski, and S. Wallon, Phys. Lett. B **586**, 267 (2004).
[7] J. Bartels, C. Contreras, and G. Paolo Vacca, arXiv:1910.04588.
[8] J. B. Bronzan, G. L. Kane, and U. P. Sukhatme, Phys. Lett. **49B**, 272 (1974).
[9] K. Kang and B. Nicolescu, Phys. Rev. D **11**, 2461 (1975).
[10] J. R. Cudell, E. Martynov, and O. V. Selyugin, arXiv:hep-ph/0307254.
[11] R. F. Ávila and M. J. Menon, Nucl. Phys. **A744**, 249 (2004); Braz. J. Phys. **37**, 661 (2007).
[12] G. Grunderg and T. N. Truong, Phys. Rev. Lett. **31**, 63 (1973).
[13] E. Martynov and B. Nicolescu, Phys. Lett. B **778**, 414 (2018).
[14] E. Martynov and B. Nicolescu, Phys. Lett. B **786**, 207 (2018).
[15] G. Antchev *et al.*, Eur. Phys. J. C **79**, 103 (2019).
[16] G. Antchev *et al.*, Eur. Phys. J. C **79**, 785 (2019).
[17] A. P. Prudnikov, J. A. Bryčkov, and O. I. Maričev, *Integrals and series*, Elementary Functions, Vol. 1 (Gordon and Breach, New York, 1981).