Dynamics and symmetries in chiral $SU(N)$ gauge theories

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Dynamics and symmetry realization in various chiral gauge theories in four dimensions are investigated, generalizing a recent work by Shifman and the present authors, by relying on the standard 't Hooft anomaly matching conditions and on some other general ideas. These requirements are so strong that the dynamics of the systems are severely constrained. Color-flavor or color-flavor-flavor locking, dynamical Abelianization, and combinations of these are powerful ideas which often lead to solutions of the anomaly matching conditions. Moreover, a conjecture is made on the generation of a mass hierarchy associated with symmetry breaking in chiral gauge theories, which has no analogs in vectorlike gauge theories such as QCD.

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I. INTRODUCTION

Our world has a nontrivial chiral structure. Macroscopic structures such as biological bodies often have approximately left-right symmetric forms, but not exactly. At molecular levels, $O(10^{-6}$ cm), the structure of DNA has a definite chiral spiral form. At the microscopic length scales of the fundamental interactions, $O(10^{-14}$ cm), the left- and right-handed quarks and leptons have different couplings to the $SU(3) \times SU_L(2) \times U_Y(1)$ gauge bosons. In spite of the impressive success of the standard model, and after many years of theoretical studies of four-dimensional gauge theories, our understanding today of *strongly coupled chiral* gauge theories is surprisingly limited.[1] Almost a half century of studies of vectorlike gauge theories like $SU(3)$ quantum chromodynamics (QCD), based on lattice simulations with ever more powerful computers, and roughly 25 years of beautiful theoretical developments in models with $\mathcal{N} = 2$ supersymmetries both concern vectorlike theories only. Perhaps it is not senseless to make more effort to understand this class of gauge theories, which nature might be making use of in an as yet unknown way to us.

Urged by such a motivation, we have recently revisited the physics of some chiral gauge theories [15]. In this paper, we generalize the analysis done there to a wider class of models and try to learn some general lessons from them. We use as a guiding light the standard 't Hooft anomaly matching conditions [16]. To be concrete, we shall limit ourselves to $SU(N)$ gauge theories with a set of Weyl fermions in a complex representation of the gauge group. Also, only asymptotically free types of models will be considered, as weakly coupled infrared-free theories can be reliably analyzed in perturbation theory, as in the case of the standard electroweak model. For simplicity, we shall restrict ourselves to various irreducibly chiral[2] $SU(N)$ theories, with N_ψ fermions $\psi^{\{ij\}}$ in the symmetric representation, N_χ fermions $\chi_{[ij]}$ in the antisymmetric representation, and a number of antifundamental (or fundamental) multiplets, η_i^A (or $\tilde{\eta}^{Ai}$).[3] The number of the latter is fixed by the condition that the gauge group be anomaly-free.

Figure 1 gives a schematic representation of the various irreducibly $SU(N)$ chiral theories we shall be interested in. Both N_ψ and N_χ can go up to 5 without loss of asymptotic freedom for large N. The ones we will explicitly consider are summarized in Table I with their b_0 coefficient. The gauge interactions in these models become strongly coupled in the infrared. There are no gauge-invariant bifermion condensates, no mass terms or potential terms (of renormalizable type) can be added to deform the theories, and no θ parameter exists. The main questions we would like to address, given a model of this sort, are how to solve the 't Hooft anomaly matching conditions in the IR, and whether there is more than one apparently

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[1]See, however, [1–13] for a partial list of earlier studies of these theories. See [14] for a recent work on the infrared fixed point in a class of chiral gauge theories.

[2]For example, we do not consider addition of fundamental-antifundamental pairs of fermions. Models of this type in the simplest cases, $(N_\psi, N_\chi) = (1,0), (0,1)$, were studied in [9].

[3]Let us use the indices $A, B, \ldots$ for flavor and the indices $i, j, \ldots$ for color below.

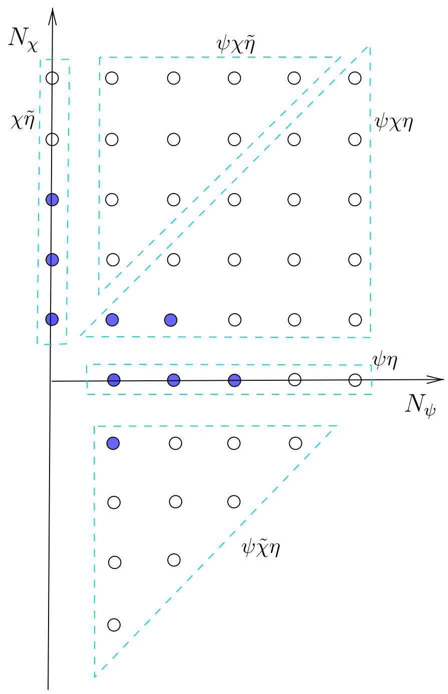


FIG. 1. A class of chiral QCD theories at large N in the plane (N_ψ, N_χ).

possible dynamical scenario that is consistent with the matching conditions.

The paper is organized as follows. In Sec. II, we revisit the $(N_\psi, N_\chi) = (1,1)$ model previously considered in [15]. In Secs. III–X, we consider, respectively, the models $(N_\psi, N_\chi) = (1,0), (2,0), (3,0), (0,1), (0,2), (0,3), (2,1), (1,-1)$. In Sec. XI, we discuss the pion-decay constant and a possible new hierarchy mechanism. We conclude in Sec. XII by trying to learn some general lessons for strongly coupled chiral gauge theories. Consistency checks of the many proposed phases with the a theorem and with the Appelquist-Cohen-Schmaltz (ACS) criterion is done in the Appendix.

TABLE I. First coefficients of the beta function.

Model	$3b_0$
(1,1)	$9N-8$
(1,0)	$9N-6$
(2,0)	$7N-12$
(3,0)	$5N-18$
(0,1)	$9N+6$
(0,2)	$7N+12$
(0,3)	$5N+18$
(2,1)	$7N-14$
(1,−1)	$7N$

A. Dynkin index and symmetric traces of the symmetric and antisymmetric representations in $SU(N)$

We recall here, for the convenience of the reader, a few well-known Dynkin indices and symmetric traces which are repeatedly used in the following analyses. The Dynkin index $T(R)$ is defined by

$$\mathrm{Tr}\, t_R^a t_R^b = T(R)\delta_{ab}. \tag{1.1}$$

Summing over $a = b$, one gets

$$D(R)C_2(R) = T(R)(N^2-1), \qquad \sum_a t_R^a t_R^a = C_2(R)\mathbb{1}_{D(R)}, \tag{1.2}$$

where $D(R)$ is the dimension of the representation and $C_2(R)$ is the quadratic Casimir. For the fundamental,

$$C_2(R) = \frac{N^2-1}{2N}, \qquad D(R) = N, \qquad \therefore \quad T(\square) = \frac{1}{2}, \tag{1.3}$$

and for the adjoint,

$$C_2(R) = N, \qquad D(R) = N^2 - 1, \qquad \therefore\ T_{adj} = N. \tag{1.4}$$

For the symmetric and antisymmetric representations,

$$\frac{T(\square\square)}{T(\square)} = N+2, \qquad \frac{T\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)}{T(\square)} = N-2. \tag{1.5}$$

The conjugate representations have the same Dynkin index.

The symmetric traces appearing in the triangle anomalies are defined by

$$d(R) = \mathrm{Tr}\, t_R^a t_R^b t_R^c + (b \leftrightarrow c). \tag{1.6}$$

For the symmetric and antisymmetric representations,

$$\frac{d(\square\square)}{d(\square)} = N+4, \qquad \frac{d\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)}{d(\square)} = N-4. \tag{1.7}$$

Furthermore, for a pair of conjugate representations,

$$d(R^*) = -d(R). \tag{1.8}$$

These are all that we need for our analysis.

II. REVISITING THE $(N_\psi, N_\chi) = (1,1)$ ("$\psi\chi\eta$") MODEL

We first review the analysis of the model with left-handed fermion matter fields

$$\psi^{\{ij\}}, \qquad \chi_{[ij]}, \qquad \eta_i^A, \qquad A = 1, 2, \ldots, 8, \tag{2.1}$$

a symmetric tensor, an antiantisymmetric tensor, and eight antifundamental multiplets of $SU(N)$, and we add a few new comments with respect to [15].[4] It is asymptotically free, with the first coefficient of the beta function being

$$b_0 = \frac{1}{3}[11N - (N+2) - (N-2) - 8] = \frac{9N-8}{3}. \tag{2.2}$$

It is a very strongly coupled theory in the infrared and unlikely to flow into an infrared-fixed point conformal field theory. A nonvanishing instanton amplitude

$$\langle \psi\psi\ldots\psi\chi\chi\ldots\chi\eta\ldots\eta\rangle \neq 0 \tag{2.3}$$

involves $N+2$ ψ's, $N-2$ χ's, and eight η's.

The model has a global $SU(8)$ symmetry. It also has three $U(1)$ symmetries, $U_\psi(1)$, $U_\chi(1)$, and $U_\eta(1)$, of which two combinations are anomaly-free. They can be taken, e.g., as

$$U_1(1)\colon\ \psi \to e^{i\frac{\alpha}{N+2}}\psi, \qquad \eta \to e^{-i\frac{\alpha}{8}}\eta,$$
$$U_2(1)\colon\ \psi \to e^{i\frac{\beta}{N+2}}\psi, \qquad \chi \to e^{-i\frac{\beta}{N-2}}\chi. \tag{2.4}$$

There are also anomaly-free discrete subgroups $\mathbb{Z}_{N+2} \otimes \mathbb{Z}_{N-2} \otimes \mathbb{Z}_8$ of $U_\psi(1)$, $U_\chi(1)$, and $U_\eta(1)$ which are not broken by the instantons. However, they are not independent of each other, in view of the nonanomalous symmetries (2.4) The global continuous symmetry of the $\psi - \chi - \eta$ model is

$$G_{\rm f} = SU(8) \times U_1(1) \times U_2(1). \tag{2.5}$$

A. Partial color-flavor locking

Possible dynamical scenarios in this model were analyzed and discussed in [15]. It was proposed that a possible phase (valid for $N \geq 12$) can be described by the nonvanishing bifermion condensates

$$\langle \phi^{iA}\rangle = \langle \psi^{ij}\eta_j^A\rangle, \qquad \langle \tilde{\phi}_j^i\rangle \equiv \langle \psi^{ik}\chi_{kj}\rangle. \tag{2.6}$$

[4]Earlier studies on this model can be found in [5,6,11].

More concretely, the proper realization of the global $SU(8)$ symmetry has led us to assume the following form for these condensates:

$$\langle \psi^{ij}\eta_j^A\rangle = \Lambda^3 \begin{pmatrix} c\mathbf{1}_8 \\ \mathbf{0}_{N-8,8} \end{pmatrix}^{iA},$$
$$\langle \psi^{ik}\chi_{kj}\rangle = \Lambda^3 \left(\begin{array}{c|ccc|c} a\mathbf{1}_8 & & & & \\ \hline & d_1 & & & \\ & & \ddots & & \\ & & & d_{N-12} & \\ \hline & & & & b\mathbf{1}_4 \end{array}\right)^{i}_{j}, \tag{2.7}$$

where

$$8a + \sum_{i=1}^{N-12} d_i + 4b = 0, \qquad a, d_i, b \sim O(1). \tag{2.8}$$

The symmetry breaking pattern is, therefore,

$$SU(N)_{\rm c} \times SU(8)_{\rm f} \times U(1)^2$$
$$\to SU(8)_{\rm cf} \times U(1)^{N-11} \times SU(4)_{\rm c}. \tag{2.9}$$

The theory dynamically Abelianizes (in part). $SU(8) \subset SU(N)$ is completely Higgsed, but due to color-flavor (partial) locking, no Nambu-Goldstone (NG) bosons appear in this sector [the would-be NG bosons make the $SU(8) \subset SU(N)$ gauge bosons massive]. Only $SU(4) \subset SU(N)$ remains unbroken and confining. The remainder of the gauge group Abelianizes. The baryons

$$\tilde{B}_j^A = \psi^{ik}\chi_{[kj]}\eta_i^A \sim \eta_j^A, \qquad (9 \leq j \leq N-4), \tag{2.10}$$

and

$$B^{\{AB\}} = \psi^{ij}\eta_i^A\eta_j^B, \tag{2.11}$$

symmetric in the flavor indices ($A \leftrightarrow B$),[5] remain massless and together saturate the 't Hooft anomaly matching condition for $SU(8)$:

$$8 + 4 + N - 12 = N. \tag{2.12}$$

Note that both nonanomalous continuous $U_{1,2}(1)$'s are broken by the two condensates. Actually, for some N, a discrete symmetry survives condensates of the form of

[5]If the massless $B^{\{AB\}}$ were antisymmetric in the flavor indices, they would contribute $8 - 4 = 4$ to the $SU(8)$ anomaly. We would then need $N-4$ massless fermions of the form $\tilde{B}_j^A \sim \eta_j^A$, but this is impossible, as the latter arises from the Abelianization of the rest of the color gauge group, $SU(N-8)$.

Eq. (2.6), and the discrete anomaly matching must be taken into account.

1. Discrete symmetries

Under the discrete symmetries, the fields transform as

$$\mathbb{Z}_{N+2} \subset U_\psi(1)\colon\ \psi \to e^{i\frac{2\pi k}{N+2}}\psi, k = 0, 1, \ldots, N+1;$$
$$\mathbb{Z}_{N-2} \subset U_\chi(1)\colon\ \chi \to e^{i\frac{2\pi \ell}{N-2}}\chi, \ell = 0, 1, \ldots, N-3;$$
$$\mathbb{Z}_8 \subset U_\eta(1)\colon\ \eta \to e^{i\frac{2\pi m}{8}}\eta, m = 0, 1, \ldots, 7. \tag{2.13}$$

A discrete subgroup survives condensates (2.6) if

$$\frac{k}{N+2} - \frac{\ell}{N-2} \in \mathbb{Z}, \qquad \frac{k}{N+2} - \frac{m}{8} \in \mathbb{Z}. \tag{2.14}$$

Clearly there is no discrete surviving symmetry for odd N. For N even, the above shows that there remains a $\mathbb{Z}_2$ symmetry, for $N = 4n$, $n \in \mathbb{Z}$, or a $\mathbb{Z}_4$ symmetry, for $N = 4n + 2$.

To be concrete, consider $N = 14$. The conditions above read, in this case,

$$\frac{k}{16} - \frac{\ell}{12} \in \mathbb{Z}, \qquad \frac{k}{16} - \frac{m}{8} \in \mathbb{Z}. \tag{2.15}$$

The transformation

$$\psi \to e^{\pi i/2}\psi, \qquad \chi \to e^{-\pi i/2}\chi, \qquad \eta \to e^{-\pi i/2}\eta \tag{2.16}$$

generates $\mathbb{Z}_4$, which is kept unbroken by $\langle\psi\chi\rangle$ and $\langle\psi\eta\rangle$. The $\mathbb{Z}_4$ charge of the (ψ, χ, η) fields is $(1, -1, -1)$ Mod 4.

Consider the discrete anomaly $SU(8)^2\mathbb{Z}_4$ [17]. In the UV, the only contribution is from the η fields, which gives

$$N \cdot 1 \cdot (-1) = -N = -14. \tag{2.17}$$

In the IR, $\eta_j^A (9 \le j \le N-4)$ gives

$$(N-12) \cdot 1 \cdot (-1) = -2, \tag{2.18}$$

whereas $B^{\{AB\}} = \psi^{ij}\eta_i^A\eta_j^B$ contributes

$$1 \cdot (8+2) \cdot (-1) = -10, \tag{2.19}$$

totalling

$$-2 - 10 = -12. \tag{2.20}$$

The difference between the UV and the IR is

$$-14 - (-12) = -2 \neq 0 \quad \text{Mod } 4. \tag{2.21}$$

Thus the discrete $SU(8)^2\mathbb{Z}_4$ anomaly does not match for $N = 14$. A similar situation is found for all N of the form $4n + 2$, $n = 3, 4, 5, \ldots$.

As for the discrete $\text{Grav}^2\mathbb{Z}_4$ anomaly, we count only the $\mathbb{Z}_4$ charges and the multiplicities: in the UV, it is

$$N \cdot 1 \cdot (-1) = N = -14, \tag{2.22}$$

whereas in the IR, the value is

$$2 \cdot 1 \cdot (-1) + \frac{8 \cdot 9}{2} \cdot (-1) = -38. \tag{2.23}$$

The difference is

$$38 - 14 = 24 = 0 \quad \text{Mod } 4, \tag{2.24}$$

so they match.

For $N = 4n$, the conditions

$$\frac{k}{4n+2} - \frac{\ell}{4n-2} \in \mathbb{Z}, \qquad \frac{k}{4n+2} - \frac{m}{8} \in \mathbb{Z} \tag{2.25}$$

leave a $\mathbb{Z}_2$ symmetry generated by the transformations with $k = 2n + 1$, $\ell = 2n - 1$, and $m = 4$. It is easy to verify that all of the discrete anomalies involving $\mathbb{Z}_2$ match in the UV and in the IR.

The fact that discrete anomaly matching does not work for $N = 4n + 2$ renders the scenario (2.6)–(2.9) unlikely to be realized for any N. There are, however, other possibilities, as discussed below.

B. Color-flavor locking and dynamical Abelianization: An alternative scenario

Another possible phase, for $N \ge 8$, which was not considered in [15], is described by condensates (2.6), but this time of the form

$$\langle\psi^{ij}\eta_j^A\rangle = \Lambda^3 \begin{pmatrix} c\mathbf{1}_8 \\ \mathbf{0}_{N-8,8} \end{pmatrix}^{iA},$$
$$\langle\psi^{ik}\chi_{kj}\rangle = \Lambda^3 \left(\begin{array}{c|cccc} \mathbf{0}_8 & & & \\ \hline & d_1 & & \\ & & \ddots & \\ & & & d_{N-8} \end{array}\right)^i_j. \tag{2.26}$$

The symmetry breaking pattern is

$$SU(N) \times SU(8) \times U(1)^2 \to SU(8)_{\text{cf}} \times U(1)^{N-8}. \tag{2.27}$$

As $U(1)^{N-8}$ is an Abelian subgroup of the color $SU(N)$, whereas both nonanomalous flavor $U(1)$ are broken by the condensates, we shall consider only the $SU(8)^3_{\text{cf}}$ anomalies.

Indicating the color indices up to 8 by i_1 or j_1 and those larger than 8 by i_2 or j_2, one has the decomposition of the fields in $SU(8)_{\rm cf}$ multiplets; see Table II. The massless baryons are shown in the lower part of Table II. The $SU(8)^3$ matching works, as in the infrared,

$$(N-8)+(8-4)+(8-4)=N. \tag{2.28}$$

As for the discrete symmetry, the surviving symmetry is either $\mathbb{Z}_2$, for $N=4n$, $n\in\mathbb{Z}$, or $\mathbb{Z}_4$ symmetry, for $N=4n+2$, under which the fields ψ, χ, and η are charged with $(1,-1,-1)$. An inspection of Table II shows that all discrete anomaly matching is also satisfied in this case, in contrast to the previous case.

TABLE II. $B^{[AB]}\sim\psi^{ij}\eta_i^{[A}\eta_j^{B]}$ and $\hat{B}^{[AB]}\sim(\psi\eta)^{A,i}\chi_{ij}(\psi\eta)^{B,j}$. $\eta^A_{j_2}$ are weakly coupled due to the Abelianization of $SU(N-8)\times U(1)\subset SU(N)$. They can be interpreted as $\tilde{B}^A_j\sim(\psi\chi\eta)^A_j$. The color indices up to 8 are indicated by i_1 or j_1 and those larger than 8 by i_2 or j_2. In this and other tables in the paper, the multiplicity, the charges, and the representation are shown for each (set of) fermion components. $(\cdot)$ stands for a singlet representation.

	Fields	$SU(8)_{\rm cf}$
UV	$\psi^{i_1j_1}$	$\overline{\square\square}$
	$\psi^{i_1j_2}$ $\psi^{i_2j_2}$	$(N-8)\cdot\bar{\square}$ $\frac{(N-8)(N-7)}{2}\cdot(\cdot)$
	χ_{i_1,j_1}	$\begin{array}{c}\square\\\square\end{array}$
	χ_{i_1,j_2} χ_{i_2,j_2}	$(N-8)\cdot\square$ $\frac{(N-8)(N-9)}{2}\cdot(\cdot)$
	$\eta^A_{j_1}$	$\square\otimes\square=\square\square\oplus\begin{array}{c}\square\\\square\end{array}$
	$\eta^A_{j_2}$	$(N-8)\cdot\square$
IR	$\tilde{B}^A_{j_2}\sim\eta^A_{j_2}$	$(N-8)\cdot\square$
	$B^{[Aj_1]}\sim\mathcal{A}(\eta^A_{j_1})$	$\begin{array}{c}\square\\\square\end{array}$
	$\hat{B}^{[i_1j_1]}\sim\chi_{i_1j_1}$	$\begin{array}{c}\square\\\square\end{array}$

C. Partial color-flavor locking for $N\leq 8$

For $N<8$, the scenario above is not viable. It is possible, however, that color-flavor locking still takes place in a different way (this possibility was not considered in [15] either). Let us assume that

$$\langle\psi^{ij}\eta^A_j\rangle=\Lambda^3\left(c\mathbf{1}_N\;\middle|\;\mathbf{0}_{N,8-N}\right)^{iA},\qquad\langle\psi^{ik}\chi_{kj}\rangle=0. \tag{2.29}$$

The symmetry breaking pattern is now

$$SU(N)\times SU(8)\times U(1)^2\to SU(N)_{\rm cf}\times SU(8-N)\times\tilde{U}(1). \tag{2.30}$$

The fermions decompose as in Table III. The massless baryons which saturate the anomalies are made of

$$B^{[A_1B_1]}=\psi^{ij}\eta_i^{A_1}\eta_j^{B_1}\sim\begin{array}{c}\square\\\square\end{array},\qquad B^{[A_1B_2]}=\psi^{ij}\eta_i^{A_1}\eta_j^{B_2}\sim\eta^{B_2},$$
$$\hat{B}^{[AB]}=\psi^{ik}\eta^A_k\chi_{ij}\psi^{j\ell}\eta^B_\ell\sim\chi_{AB}. \tag{2.31}$$

D. Full Abelianization and general N

The dynamical scenarios (2.9) assumes that $N\geq 12$, whereas the one in Eq. (2.27) requires $N\geq 8$ and Eq. (2.30) requires $N\leq 8$.

Still another option, consistent for any value of N and considered in [15], is that the gauge group dynamically Abelianizes completely, by the adjoint condensates

$$\langle\psi^{ij}\eta^A_j\rangle=0,\qquad\langle\psi^{ik}\chi_{kj}\rangle=\Lambda^3\begin{pmatrix}d_1&&\\&\ddots&\\&&d_N\end{pmatrix}^i_j, \tag{2.32}$$

with $\sum_j d_j=0$ and no other particular relations among the d_j's. No color-flavor locking takes place. The symmetry breaking occurs as

$$SU(N)_{\rm c}\times SU(8)_{\rm f}\times U(1)^2\to\prod_{\ell=1}^{N-1}U_\ell(1)\times SU(8)_{\rm f}\times\tilde{U}(1), \tag{2.33}$$

where $\tilde{U}(1)$ is an unbroken combination of the two non-anomalous $U(1)$'s, Eq. (2.4), with charges

$$\psi:\;2,\qquad\chi:\;-2,\qquad\eta:\;-1. \tag{2.34}$$

TABLE III. Partial color-flavor locking for $N \leq 8$ and the $SU(8)$ anomaly matching of Sec. II C. A_1, B_1 stand for the flavor indices up to $N(<8)$, A_2, B_2 for the rest. In this and other tables in the paper, the multiplicity, the charges, and the representation are shown for each fermion component.

	Fields	$SU(N)_{cf}$	$SU(8-N)$	$\tilde{U}(1)$
UV	ψ	$\overline{\square\square}$	$\frac{N(N+1)}{2}\cdot(\cdot)$	$N+2$
	χ	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$\frac{N(N-1)}{2}\cdot(\cdot)$	$-\frac{(N-6)(N+2)}{N-2}$
	η^{A_1}	$\square\square + \begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$N^2\cdot(\cdot)$	$-(N+2)$
	η^{A_2}	$(8-N)\cdot\square$	$N\cdot\square$	$-(N+2)$
IR	$B^{[A_1B_1]}$	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$\frac{N(N-1)}{2}\cdot(\cdot)$	$-(N+2)$
	$B^{[A_1B_2]}$	$(8-N)\cdot\square$	$N\cdot\square$	$-(N+2)$
	$\hat{B}^{[A_1B_1]}$	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$\frac{N(N-1)}{2}\cdot(\cdot)$	$-\frac{(N-6)(N+2)}{N-2}$

The fields η_i^A are all massless and weakly coupled (only to the gauge bosons from the Cartan subalgebra which we will refer to as the photons; they are infrared-free) in the infrared. Also, some of the fermions ψ^{ij} do not participate in the condensates. Owing to the fact that $\psi^{\{ij\}}$ are symmetric and $\chi_{[ij]}$ are antisymmetric, only nondiagonal elements of $\psi^{\{ij\}}$ actually condense and get mass. The diagonal fields $\psi^{\{ii\}}$, $i = 1, 2, \ldots, N$, remain massless and weakly coupled. Also, there is one NG boson. The anomaly matching works as shown in Table IV. Concretely,

(i) $SU(8)^3$: Only η_i^A contribute both in the UV and in the IR. They trivially satisfy the matching condition.

(ii) $SU(8)\tilde{U}(1)^2$: Again only η_i^A contribute both in the UV and in the IR. The matching is trivial.

(iii) $\tilde{U}(1)^3$: In the UV, ψ, χ, and η all give contributions,

$$8\cdot\frac{N(N+1)}{2} - 8\cdot\frac{N(N-1)}{2} - 8N = 0; \qquad (2.35)$$

in the IR, ψ^{ii} and η_i^A contribute,

$$8\cdot N - 8\cdot N = 0. \qquad (2.36)$$

(iv) $\tilde{U}(1)$: In the UV, ψ, χ, and η give

$$2\cdot\frac{N(N+1)}{2} - 2\cdot\frac{N(N-1)}{2} - 8\cdot N = -6N, \qquad (2.37)$$

whereas in the IR, ψ^{ii} and η_i^A give

$$2\cdot N - 8N = -6N. \qquad (2.38)$$

TABLE IV. Full dynamical Abelianization in the $\psi\chi\eta$ model in Sec. II D.

	Fields	$SU(8)$	$\tilde{U}(1)$
UV	ψ	$\frac{N(N+1)}{2}\cdot(\cdot)$	$\frac{N(N+1)}{2}\cdot(2)$
	χ	$\frac{N(N-1)}{2}\cdot(\cdot)$	$\frac{N(N-1)}{2}\cdot(-2)$
	η^A	$N\cdot\square$	$8N\cdot(-1)$
IR	$(\psi\chi\psi)^{ii}\sim\psi^{ii}$	$N\cdot(\cdot)$	$N\cdot(2)$
	$\psi\chi\eta^A\sim\eta^A$	$N\cdot\square$	$8N\cdot(-1)$

III. $(N_\psi, N_\chi) = (1, 0)$

Let us review the $(N_\psi, N_\chi) = (1, 0)$ model studied in [2,7–10,13]. The matter fermions are

$$\psi^{\{ij\}}, \qquad \eta_i^B, \qquad B = 1, 2, \ldots, N+4, \qquad (3.1)$$

or

$$\square\square + (N+4)\,\bar{\square}\,. \qquad (3.2)$$

The first coefficient of the beta function is

$$b_0 = \frac{1}{3}[11N - (N+2) - (N+4)] = \frac{9N-6}{3}. \qquad (3.3)$$

The (continuous) symmetry of this model is

$$SU(N)_c \times SU(N+4)_f \times U(1), \qquad (3.4)$$

where $U(1)$ is an anomaly-free combination of $U_\psi(1)$ and $U_\eta(1)$, with

$$Q_\psi:\ N+4, \qquad Q_\eta:\ -(N+2). \qquad (3.5)$$

There are also discrete symmetries

$$\mathbb{Z}_\psi = \mathbb{Z}_{N+2} \subset U_\psi(1), \qquad \mathbb{Z}_\eta = \mathbb{Z}_{N+4} \subset U_\eta(1). \qquad (3.6)$$

A. Chirally symmetric phase in the (1,0) model

Let us first examine the possibility that no condensates form, the system confines, and the flavor symmetry is unbroken [2]. The candidate massless baryons are

$$B^{[AB]} = \psi^{ij}\eta_i^A\eta_j^B, \qquad A, B = 1, 2, \ldots, N+4, \qquad (3.7)$$

antisymmetric in $A \leftrightarrow B$. All of the $SU(N+4)_f \times U(1)$ anomalies are saturated by $B^{[AB]}$, as can be seen by inspection of Table V. The discrete anomaly $\mathbb{Z}_\psi SU(N)^2$ is also matched, as can be easily checked.

TABLE V. Chirally symmetric phase of the (1,0) model. As in other tables in the paper, the multiplicity, the charges, and the representation are shown for each set of fermions. $(\cdot)$ stands for a singlet representation.

	Fields	$SU(N)_{\mathrm{c}}$	$SU(N+4)$	$U(1)$
UV	ψ	$\square\!\square$	$\frac{N(N+1)}{2}\cdot(\cdot)$	$N+4$
	η^A	$(N+4)\cdot\bar{\square}$	$N\cdot\square$	$-(N+2)$
IR	$B^{[AB]}$	$\frac{(N+4)(N+3)}{2}\cdot(\cdot)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$-N$

B. Color-flavor locked Higgs phase

It is also possible that a color-flavor locked phase appears [8,14], with

$$\langle\psi^{\{ij\}}\eta_i^B\rangle = c\Lambda^3\delta^{jB}, \qquad j,B=1,2,\ldots N, \tag{3.8}$$

in which the symmetry is reduced to

$$SU(N)_{\mathrm{cf}}\times SU(4)_{\mathrm{f}}\times U'(1). \tag{3.9}$$

As this forms a subgroup of the full symmetry group, Eq. (3.4), it is quite easily seen, by making the decomposition of the fields in the direct sum of representations in the subgroup, that a subset of the same baryons saturates all of the triangles associated with the reduced symmetry group; see Table VI.

The discrete anomaly $\mathbb{Z}_\psi$ is broken by the condensate $\psi\eta$. There is (for generic N) no combination between $\mathbb{Z}_\psi$ and $\mathbb{Z}_\eta$ which survives; therefore there is no discrete anomaly matching condition.

It is not known which of the possibilities, that in Sec. III A or that in Sec. III B, is realized in the (1,0) model. The low-energy degrees of freedom are $\frac{(N+4)(N+3)}{2}$ massless baryons in the former case, and $\frac{N^2+7N}{2}$ massless baryons together with $8N+1$ Nambu-Goldstone bosons in the latter. Thus the complementarity [18], as noted in [15], does not work here even though the (dynamical) Higgs scalars $\psi\eta$ are in the fundamental representation of the color.

TABLE VI. Color-flavor locked phase in the (1,0) model discussed in Sec. III B. A_1 or B_1 stand for $A,B=1,2,\ldots,N$, A_2 or B_2 the rest of the flavor indices.

	Fields	$SU(N)_{\mathrm{cf}}$	$SU(4)_{\mathrm{f}}$	$U'(1)$
UV	ψ	$\square\!\square$	$\frac{N(N+1)}{2}\cdot(\cdot)$	1
	η^{A_1}	$\overline{\square\!\square}\oplus\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$N^2\cdot(\cdot)$	-1
	η^{A_2}	$4\cdot\bar{\square}$	$N\cdot\square$	$-\frac{1}{2}$
IR	$B^{[A_1B_1]}$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\frac{N(N-1)}{2}\cdot(\cdot)$	-1
	$B^{[A_1B_2]}$	$4\cdot\bar{\square}$	$N\cdot\square$	$-\frac{1}{2}$

IV. $(N_\psi,N_\chi)=(2,0)$

This is a straightforward generalization of the $\psi\eta$ model above. The matter fermions are

$$\psi^{\{ij,m\}}, \quad \eta_i^B, \quad m=1,2, \quad B=1,2,\ldots,2(N+4), \tag{4.1}$$

or

$$2\,\square\!\square + 2(N+4)\,\bar{\square}. \tag{4.2}$$

The (continuous) symmetry of this model is

$$SU(N)_{\mathrm{c}}\times SU(2)_{\mathrm{f}}\times SU(2N+8)_{\mathrm{f}}\times U(1), \tag{4.3}$$

where $U(1)$ is an anomaly-free combination of $U_\psi(1)$ and $U_\eta(1)$,

$$U(1)\colon\ \psi\to e^{i\alpha/2(N+2)}\psi, \qquad \eta\to e^{-i\alpha/2(N+4)}\eta. \tag{4.4}$$

The first coefficient of the beta function is

$$b_0=\frac{1}{3}[11N-2(N+2)-2(N+4)]=\frac{7N-12}{3}, \tag{4.5}$$

which is positive for $N\geq 2$.

A. No chiral symmetry breaking in the (2,0) model?

Let us first assume that no condensates form and that no flavor symmetry breaking occurs. Assuming confinement, the possible massless baryons are

$$B^{m,AB}=\psi^{ij,m}\eta_i^A\eta_j^B. \tag{4.6}$$

They cannot, however, saturate the triangles associated with the flavor symmetry

$$SU(2)_{\mathrm{f}}\times SU(2N+8)_{\mathrm{f}}\times U(1). \tag{4.7}$$

For instance, the $SU(2N+8)^3$ anomaly, which is equal to N in the UV, would be at least $\sim 2N$ for any baryon like Eq. (4.6), and thus it is not reproduced in any way in the IR. We must conclude that a confinement phase with unbroken flavor symmetries cannot be realized in this system. This is in contrast to the (1,0) model which is reviewed in Sec. III A.

B. Partial color-flavor locking?

Let us consider next a partial color-flavor locking condensate

$$\langle\psi^{\{ij,1\}}\eta_i^B\rangle = c\Lambda^3\delta^{jB}, \qquad j, B = 1, 2, \ldots, N, \tag{4.8}$$

which breaks the symmetry to

$$SU(N)_{\rm cf}\times SU(N+8)\times\tilde{U}(1); \tag{4.9}$$

$SU(2)$ is broken. $\tilde{U}(1)$ is a linear combination of Eq. (4.4) and

$$U_1(1) = \begin{pmatrix} -\frac{1}{N}\mathbb{1}_N & 0 \\ 0 & \frac{1}{N+8}\mathbb{1}_{N+8}\end{pmatrix} \subset SU(2(N+4)). \tag{4.10}$$

So the unbroken $\tilde{U}(1)$ acts on the fields as

$$\psi\to e^{i\frac{\alpha}{2(N+2)}}\psi,$$
$$\eta_i^A\to e^{-i\frac{\alpha}{2(N+2)}}\eta_i^A \quad (A=1,2,\ldots,N),$$
$$\eta_i^A\to e^{-i\frac{\alpha(N+4)}{2(N+2)(N+8)}}\eta_i^A \quad (A=N+1,N+2,\ldots,2N+8). \tag{4.11}$$

The charges with respect to $\tilde{U}(1)$ are

$$\psi:\ 1, \qquad \eta^{<}:\ -1, \qquad \eta^{>}:\ -\frac{N+4}{N+8}. \tag{4.12}$$

The massless baryons are assumed to be of the form

$$B^{AB} = \psi^{ij,1}\eta_i^A\eta_j^B, \qquad A, B = 1, 2, \ldots, N, \tag{4.13}$$

and

$$\tilde{B}^{AB} = \psi^{ij,1}\eta_i^A\eta_j^B, \quad A=1,2,\ldots,N, \quad B=N+1,\ldots,2N+8. \tag{4.14}$$

Here we must choose B^{AB} in the symmetric or antisymmetric representation of the $SU(N)_{\rm cf}$ group and $\tilde{B}^{AB}$ in the $(\underline{N}, \underline{N+8})$ of $SU(N)_{\rm cf}\times SU(N+8)_{\rm flavor}$. The $\tilde{U}(1)$ charges of B^{AB} and $\tilde{B}^{AB}$ are

$$B^{AB}:\ -1, \qquad \tilde{B}^{AB}:\ -\frac{N+4}{N+8}. \tag{4.15}$$

These assumptions are made such that the $SU(N+8)_f^3$ and $\tilde{U}(1)SU(N+8)_{\rm f}^2$ anomalies are matched in the UV and IR; however, it is easy to verify that the triangles $\tilde{U}(1)^3$ and $SU(N)^3_{\rm cf}$ cannot be matched. Therefore the phase (4.8) and (4.9) cannot be realized.

C. A possible phase: A double color-flavor locking

Another possibility is to assume a double $SU(N)$ color-flavor-flavor locking

$$\langle\psi^{\{ij,1\}}\eta_j^B\rangle = c\Lambda^3\delta^{i,B}, \qquad j, B = 1, 2, \ldots, N,$$
$$\langle\psi^{\{ij,2\}}\eta_j^B\rangle = c'\Lambda^3\delta^{i,B-N}, \qquad j = 1, 2, \ldots, N,$$
$$B = N+1, \ldots, 2N. \tag{4.16}$$

The symmetry is broken down to

$$SU(N)_{\rm cf}\times\tilde{U}(1)\times U'(1)\times SU(8), \tag{4.17}$$

where $\tilde{U}(1)$ acts as before:

$$\psi:\ 1, \qquad \eta^{B\le 2N}:\ -1, \qquad \eta^{B>2N}:\ -\frac{1}{2}. \tag{4.18}$$

There are

$$3N^2+32N+3 \tag{4.19}$$

NG bosons. $U'(1)$ is a subgroup of $SU(2)_{\rm ff}$, defined below Eqs. (4.26) and (4.27), which survives condensates (4.16).

In order to saturate all of the anomalies, one assumes that somehow only

$$\hat{B}^{A,B} = \psi^{ij,1}\eta_i^A\eta_j^B, \qquad A = 1, 2, \ldots, N,$$
$$B = 2N+1, \ldots, 2N+8, \tag{4.20}$$

or

$$\tilde{B}^{A,B} = \psi^{ij,1}\eta_i^A\eta_j^B, \qquad A = N+1, N+2, \ldots, 2N,$$
$$B = 2N+1, \ldots, 2N+8 \tag{4.21}$$

(but not both) remain massless. One could write these states as

$$\hat{B}_a^B = \sum_{A=1}^{2N}\sum_{m=1,2} c_{a,m,A}\psi^{ij,m}\eta_i^A\eta_j^B, \qquad a = 1, 2, \ldots, N,$$
$$B = 2N+1, \ldots, 2N+8. \tag{4.22}$$

Furthermore, we shall need also two types of baryons,

$$B^{[AB],1} = \psi^{ij,1}\eta_i^A\eta_j^B, \quad A, B = 1, 2, \ldots, N,$$
$$B^{[AB],2} = \psi^{ij,1}\eta_i^A\eta_j^B, \quad A, B = N+1, N+2, \ldots, 2N, \tag{4.23}$$

both of which are antisymmetric in AB and all of which remain massless. It is a simple exercise to check that all anomalies, $SU(8)^3$, $SU(8)^2\tilde{U}(1)$, $\tilde{U}(1)^3$, $\tilde{U}(1)$, $SU(N)^3$, $SU(N)^2\tilde{U}(1)$, are matched.

In conclusion, the double color-flavor locking phase, with massless baryons $\hat{B}^{A,B}$ or $\tilde{B}^{A,B}$, or analogous states with $1\leftrightarrow 2$, together with $B^{AB,1}$ and $B^{AB,2}$ (both antisymmetric in AB), is consistent with anomaly matching. The asymmetric way that $\psi^{ij,1}$ and $\psi^{ij,2}$ appear in the IR baryons is consistent, as the $SU(2)$ is broken.

D. Phase with unbroken $SU(2)$

Another phase is the one with an unbroken $SU(2)$ symmetry. Assume Eq. (4.16) with the same coefficients,

$$c = c'. \tag{4.24}$$

The symmetry is broken down to

$$SU(N)_{\rm cf} \times \tilde{U}(1) \times SU(2)_{\rm ff} \times SU(8), \tag{4.25}$$

where $SU(2)_{\rm ff}$ is a linear combination of $SU(2)_{\rm f}$ and

$$SU(2) \subset SU(2N) \subset SU(2N+8), \tag{4.26}$$

which exchange the first and second N flavors. The charges of the unbroken $SU(2)$ are

$$\begin{pmatrix} \psi^{ij,1} \\ \psi^{ij,2} \end{pmatrix} \sim \underline{2}, \qquad \begin{pmatrix} \eta_i^{A\leq N} \\ \eta_i^{N\leq A\leq 2N} \end{pmatrix} \sim \underline{2}^*. \tag{4.27}$$

The $\tilde{U}(1)$ charges are as before,

$$\psi\colon\ 1, \qquad \eta^{B\leq 2N}\colon\ -1, \qquad \eta^{B>2N}\colon\ -\frac{1}{2}. \tag{4.28}$$

The baryons are

$$B^{A,C} = \sum_{i,j} (\psi^{ij,1}\eta_i^{A\leq N}\eta_j^C + \psi^{ij,2}\eta_i^{N<A\leq 2N}\eta_j^C), \qquad C > 2N, \tag{4.29}$$

which is a $SU(2)$ singlet; the others are

$$\begin{aligned} B^{[A_1B_1],1} &= \psi^{ij,1}\eta_i^{A_1}\eta_j^{B_1}, \qquad A_1, B_1 = 1, 2, \ldots, N, \\ B^{[A_2B_2],2} &= \psi^{ij,2}\eta_i^{A_2}\eta_j^{B_2}, \qquad A_2, B_2 = N+1, N+2, \ldots, 2N, \end{aligned} \tag{4.30}$$

which form a doublet. Their $\tilde{U}(1)$ charges are

$$B^{A,C}\colon\ -\frac{1}{2}, \qquad B^{[AB],m}\colon\ -1. \tag{4.31}$$

The anomaly saturation can again be seen quickly by inspecting Table VII. The discrete symmetries $\mathbb{Z}_\psi = \mathbb{Z}_{2(N+2)}$ and $\mathbb{Z}_\eta = \mathbb{Z}_{2(N+4)}$ are both broken by the condensates. Also, Witten's $SU(2)$ anomaly matches: there are

$$\frac{N(N+1)}{2} + N^2 \tag{4.32}$$

left-handed $SU(2)$ doublets in the UV, whereas the corresponding number in the IR is

$$\frac{N(N-1)}{2}; \tag{4.33}$$

the difference is

$$N(N+1), \tag{4.34}$$

which is always even.

1. Remarks on less symmetric phases

The less symmetric phases discussed in Sec. IV C can be derived from the most symmetric phase discussed here. Namely, when the bifermion condensates have no special relations, some of the global symmetries are broken, and a multiplet (irreducible representation) with respect to such a subgroup [e.g., $SU(N)_{\rm cf}$ or $SU(2)$] is replaced by a simple multiplicity of states of similar types, both for elementary fermions and for composite ones. Clearly the anomaly saturation valid in the most symmetric case implies similar results for the subset of fermions/subgroups in the less symmetric phases.

TABLE VII. An $SU(2)$ flavor-flavor locked symmetric phase in the (2,0) model, discussed in Sec. IV D. A_i and B_i ($i = 1, 2$) indicate the flavor indices up to $2N$, C the rest, $2N+1, \ldots, 2N+8$.

	Fields	$SU(N)_{\rm cf}$	$SU(8)$	$SU(2)$	$\tilde{U}(1)$
UV	ψ	$2\cdot \square\!\square$	$N(N+1)\cdot(\cdot)$	$\frac{N(N+1)}{2}\cdot \square$	1
	η^{A_i}	$2\cdot(\overline{\square\!\square} \oplus \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}})$	$2N^2\cdot(\cdot)$	$N^2\cdot \square$	-1
	η^C	$8\cdot \bar{\square}$	$N\cdot \square$	$8N\cdot(\cdot)$	$-\frac{1}{2}$
IR	$B^{A,C}$	$8\cdot \bar{\square}$	$N\cdot \square$	$8N\cdot(\cdot)$	$-\frac{1}{2}$
	$B^{[A_iB_i],m}$	$2\cdot \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$	$N(N-1)\cdot(\cdot)$	$\frac{N(N-1)}{2}\,\square$	-1

V. $(N_\psi, N_\chi) = (3,0)$

Let us consider a further generalization. The matter fermions are

$$\psi^{\{ij,m\}}, \quad \eta_i^B, \quad m = 1,2,3, \quad B = 1,2,\ldots,3(N+4), \tag{5.1}$$

or

$$3\,\square\!\square + 3(N+4)\,\bar{\square}\,. \tag{5.2}$$

The (continuous) symmetry of this model is

$$SU(N)_{\rm c} \times SU(3)_{\rm f} \times SU(3N+12)_{\rm f} \times U(1), \tag{5.3}$$

where $U(1)$ is the anomaly-free combination of $U_\psi(1)$ and $U_\eta(1)$,

$$U(1)\colon\ \psi \to e^{i\alpha/3(N+2)}\psi, \qquad \eta \to e^{-i\alpha/3(N+4)}\eta. \tag{5.4}$$

The first coefficient of the beta function is

$$b_0 = \frac{1}{3}[11N - 3(N+2) - 3(N+4)] = \frac{5N-18}{3}, \tag{5.5}$$

which is positive for $N \geq 4$. It can be seen that, as for the (2,0) model, the chiral symmetric phase and partial color-flavor locking do not provide solutions to the anomaly matching.

A. Triple color-flavor locking

Generalizing Sec. IV C, one may assume a triple color-flavor locking here:

$$\begin{aligned}
&\langle \psi^{\{ij,1\}}\eta_i^B\rangle = c\Lambda^3\delta^{jB}, \quad j,B = 1,2,\ldots,N,\\
&\langle \psi^{\{ij,2\}}\eta_i^B\rangle = c'\Lambda^3\delta^{j,B-N}, \quad j,B = N+1,N+2,\ldots,2N,\\
&\langle \psi^{\{ij,3\}}\eta_i^B\rangle = c''\Lambda^3\delta^{j,B-2N}, \quad j,B = 2N+1,2N+2,\ldots,3N.
\end{aligned} \tag{5.6}$$

The symmetry realization is

$$SU(N)_{\rm cf} \times \tilde{U}(1) \times SU(12), \tag{5.7}$$

where $\tilde{U}(1)$ acts as

$$\begin{aligned}
&\psi \to e^{i\frac{\alpha}{3(N+2)}}\psi,\\
&\eta_i^A \to e^{-i\frac{\alpha}{3(N+2)}}\eta_i^A, \quad A = 1,2,\ldots,3N,\\
&\eta_i^A \to e^{-i\frac{\alpha}{6(N+2)}}\eta_i^A, \quad A = 3N+1,3N+2,\ldots,3N+12,
\end{aligned} \tag{5.8}$$

or by renormalizing the charges:

$$\tilde{Q}_\psi\colon\ 1, \qquad \tilde{Q}_{\eta^<}\colon\ -1, \qquad \tilde{Q}_{\eta^>}\colon\ -\frac{1}{2}. \tag{5.9}$$

We now check the matching with massless baryons

$$\begin{aligned}
\hat{B}^{A,B} &= \psi^{ij,1}\eta_i^A\eta_j^B, \qquad A = 1,2,\ldots,N,\\
&\quad B = 3N+1,\ldots,3N+12,\\
B^{[AB],1} &= \psi^{ij,1}\eta_i^A\eta_j^B, \qquad A,B = 1,2,\ldots,N,\\
B^{[AB],2} &= \psi^{ij,1}\eta_i^A\eta_j^B, \qquad A,B = N+1,N+2,\ldots,2N,\\
B^{[AB],3} &= \psi^{ij,1}\eta_i^A\eta_j^B, \qquad A,B = 2N+1,2N+2,\ldots,3N.
\end{aligned} \tag{5.10}$$

The $\tilde{U}(1)$ charges of these baryons are

$$\hat{B}^{A,B}\colon\ -\frac{1}{2}, \tag{5.11}$$

$$B^{[AB],1}, B^{[AB],2}, B^{[AB],3}\colon\ -1. \tag{5.12}$$

It can be readily verified that the anomalies with respect to $SU(12)^3$, $SU(12)^2\tilde{U}(1)$, $\tilde{U}(1)^3$, $\tilde{U}(1)$, $SU(N)^3$, and $SU(N)^2\tilde{U}(1)$ agree in the UV and in the IR.

B. $SU(3)$ symmetric phase

As in Sec. IV D, one may assume a more symmetric form of condensates (5.6) with equal coefficients,

$$c = c' = c''. \tag{5.13}$$

In this case, a diagonal $SU(3)$ between $SU(3)_\psi$ and $SU(3) \subset SU(3N)$ remains unbroken. The low-energy symmetry realization is then

$$SU(N)_{\rm cf} \times SU(3) \times \tilde{U}(1) \times SU(12). \tag{5.14}$$

There are two more triangles, $SU(3)^3$ and $SU(3)^2\tilde{U}(1)$, in addition to the six types of anomalies considered in the previous subsection. The charges with respect to this $SU(3)$ are

$$\begin{pmatrix} \psi^{ij,1} \\ \psi^{ij,2} \\ \psi^{ij,3} \end{pmatrix} \sim \underline{3}, \qquad \begin{pmatrix} \eta_i^{A\leq N} \\ \eta_i^{N<A\leq 2N} \\ \eta_i^{2N<A\leq 3N} \end{pmatrix} \sim \underline{3}^*. \tag{5.15}$$

The massless baryons are

$$\begin{aligned}
B^{A,C} = \sum_{i,j} (&\psi^{ij,1}\eta_i^{A\leq N}\eta_j^C + \psi^{ij,2}\eta_i^{N<A\leq 2N}\eta_j^C\\
&+ \psi^{ij,3}\eta_i^{2N<A\leq 3N}\eta_j^C), \qquad C > 3N,
\end{aligned} \tag{5.16}$$

which is an $SU(3)$ singlet; the others are

$$\begin{aligned} B^{[AB],1} &= \psi^{ij,1}\eta_i^A\eta_j^B, \qquad A,B = 1,2,\ldots,N, \\ B^{[AB],2} &= \psi^{ij,2}\eta_i^A\eta_j^B, \qquad A,B = N+1,N+2,\ldots,2N, \\ B^{[AB],3} &= \psi^{ij,2}\eta_i^A\eta_j^B, \qquad A,B = 2N+1,2N+2,\ldots,3N, \end{aligned} \tag{5.17}$$

which form an antitriplet, $\underline{3}^*$.

Again it is convenient to have the decomposition of the fields with respect to the unbroken groups. The saturation of the anomalies $SU(12)^3$, $SU(12)^2\tilde{U}(1)$, $\tilde{U}(1)^3$, $\tilde{U}(1)$, $SU(N)^3$, $SU(N)^2\tilde{U}(1)$, $SU(3)^3$, and $SU(3)^2\tilde{U}(1)$ is seen at once by inspection of Table VIII. For illustrative purposes, let us make the equations explicit:

(i) $SU(12)^3$:

$$N = N. \tag{5.18}$$

(ii) $SU(12)^2\tilde{U}(1)$:

$$-\frac{1}{2} = -\frac{1}{2}. \tag{5.19}$$

(iii) $\tilde{U}(1)^3$: In both the UV and the IR,

$$-\frac{3N^2}{2}. \tag{5.20}$$

(iv) $\tilde{U}(1)$: In the UV,

$$3\cdot\frac{N(N+1)}{2} - 3N^2 - \frac{1}{2}\cdot 12N = -\frac{3(N^2+3N)}{2}, \tag{5.21}$$

whereas in the IR,

$$-\cdot 3\cdot\frac{N(N-1)}{2} - \frac{1}{2}\cdot 12N = -\frac{3(N^2+3N)}{2}. \tag{5.22}$$

(v) $SU(N)^3$: In the UV, it is $3N$. In the IR,

$$12 + 3(N-4) = 3N. \tag{5.23}$$

(vi) $SU(N)^2\tilde{U}(1)$: In the UV,

$$-3N, \tag{5.24}$$

while in the IR,

$$-3(N-2) - \frac{1}{2}\cdot 12 = -3N, \tag{5.25}$$

also.

As seen in the (2,0) model, Sec. IV D 1, less symmetric phases are possible in the (3,0) model as well. Condensates (5.6) are of more general forms in those cases, with unequal values, and one or both of the symmetries $SU(N)_{\rm cf}$ and $SU(3)$ can be broken spontaneously. The set of the baryons $B^{A,C}$ and $B^{[AB],m}$ will continue to saturate the anomaly triangles of the remaining symmetries.

VI. $(N_\psi, N_\chi) = (0,1)$

Let us review the $(N_\psi, N_\chi) = (0,1)$ model studied in [2,7–10,13,15]. The matter fermions are

$$\chi_{[ij]}, \qquad \tilde{\eta}^{Bj}, \qquad B = 1,2,\ldots,(N-4). \tag{6.1}$$

The first coefficient of the β function is

$$b_0 = \frac{1}{3}[11N - (N-2) - (N-4)] = \frac{9N+6}{3}. \tag{6.2}$$

The (continuous) symmetry is

$$SU(N)_{\rm c} \times SU(N-4)_{\rm f} \times U(1), \tag{6.3}$$

where the anomaly-free $U(1)$ charge is

$$\chi:\ N-4, \qquad \tilde{\eta}^{Bj}:\ -(N-2). \tag{6.4}$$

There are also discrete symmetries

$$\mathbb{Z}_\chi = \mathbb{Z}_{N-2} \subset U_\psi(1), \qquad \mathbb{Z}_\eta = \mathbb{Z}_{N-4} \subset U_\eta(1). \tag{6.5}$$

TABLE VIII. Color-favor-flavor locked $SU(3)$ symmetric phase in the (3,0) model discussed in Sec. V B.

	Fields	$SU(N)_{\rm cf}$	$SU(12)$	$SU(3)$	$\tilde{U}(1)$
UV	ψ	$3\cdot \square\!\square$	$\frac{3N(N+1)}{2}\cdot(\cdot)$	$\frac{N(N+1)}{2}\cdot\square$	1
	$\eta^{A\le 3N}$	$3\cdot(\overline{\square\!\square}\oplus\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}})$	$3N^2\cdot(\cdot)$	$N^2\cdot\bar{\square}$	-1
	$\eta^{A>3N}$	$12\cdot\bar{\square}$	$N\cdot\square$	$12N\cdot(\cdot)$	$-\frac{1}{2}$
IR	$B^{A_<,C_>}$	$12\cdot\bar{\square}$	$N\cdot\square$	$12N\cdot(\cdot)$	$-\frac{1}{2}$
	$B^{[AB],m}$	$3\cdot\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$	$\frac{3N(N-1)}{2}\cdot(\cdot)$	$\frac{N(N-1)}{2}\square$	-1

A. Chirally symmetric phase in the (0,1) model

Let us first examine the possibility that no condensates form, the system confines, and the flavor symmetry is unbroken [2]. The massless baryons are

$$B^{\{CD\}} = \chi_{[ij]}\tilde{\eta}^{iC}\tilde{\eta}^{jD}, \qquad C, D = 1, 2, \ldots, (N-4), \tag{6.6}$$

symmetric in $C \leftrightarrow D$. They have the $U(1)$ charge $-N$. The matching of the anomalies can be read off of Table IX.

B. Color-flavor locked vacuum

It was pointed out in [9] that this system may instead develop a condensate of the form

$$\langle \chi_{[ij]}\tilde{\eta}^{Bj}\rangle = \text{const}\,\Lambda^3\delta_i^B, \qquad i, B = 1, 2, \ldots, N-4, \tag{6.7}$$

namely,

$$\bar{\yng(1,1)} \otimes \yng(1) \to \bar{\yng(1)} \oplus \ldots \,. \tag{6.8}$$

The symmetry is broken down to

$$SU(N-4)_{\rm cf} \times U'(1) \times SU(4)_{\rm c}. \tag{6.9}$$

The massless baryons (6.6) saturate all of the anomalies associated with $SU(N-4)_{\rm cf} \times U'(1)$. There remain the $\chi_{i_2j_2}$ fermions which remain massless and strongly coupled to the $SU(4)_{\rm c}$. We may assume that $SU(4)_{\rm c}$ confines, that the condensate

$$\langle \chi\chi\rangle \neq 0 \tag{6.10}$$

in

$$\bar{\yng(1,1)} \otimes \bar{\yng(1,1)} \to \bar{\yng(1,1,1,1)} \oplus \ldots\,, \tag{6.11}$$

forms, and that $\chi_{i_2j_2}$ dynamically acquire mass. Assuming that the massless baryons are

$$B^{\{AB\}} = \chi_{[ij]}\tilde{\eta}^{iA}\tilde{\eta}^{jB}, \qquad A, B = 1, 2, \ldots, (N-4), \tag{6.12}$$

the saturation of all of the associated triangles can be seen in Table X. Complementarity [15,18] does work here.

VII. $(N_\psi, N_\chi) = (0,2)$

Let us now consider a generalization of the $\chi\tilde{\eta}$ model with

$$\chi^m_{[ij]}, \quad \tilde{\eta}^{Bj}, \quad m = 1, 2, \quad B = 1, 2, \ldots, 2(N-4), \tag{7.1}$$

or

$$2\,\bar{\yng(1,1)} + 2(N-4)\,\yng(1)\,. \tag{7.2}$$

The first coefficient of the β function is

$$b_0 = \frac{1}{3}[11N - 2(N-2) - 2(N-4)] = \frac{1}{3}(7N+12). \tag{7.3}$$

A. No chiral symmetry breaking in the (0, 2) model?

The symmetry is

$$SU(N)_{\rm c} \times SU(2)_{\rm f} \times SU(2N-8)_{\rm f} \times U(1), \tag{7.4}$$

where the anomaly-free $U(1)$ charge is

$$\chi\colon\ N-4; \qquad \tilde{\eta}^{Bj}\colon\ -(N-2). \tag{7.5}$$

Let us assume that the massless baryons are

$$B^{\{CD\},m} = \chi^m_{[ij]}\tilde{\eta}^{iC}\tilde{\eta}^{jD}, \qquad C, D = 1, 2, \ldots, 2(N-4), \qquad m = 1, 2, \tag{7.6}$$

TABLE IX. Confinement and unbroken symmetry in the (0,1) model.

	Fields	$SU(N)_{\rm c}$	$SU(N-4)$	$U(1)$
UV	χ	$\bar{\yng(1,1)}$	$\frac{N(N-1)}{2}\cdot(\cdot)$	$N-4$
	$\tilde{\eta}^A$	$(N-4)\cdot\yng(1)$	$N\cdot\yng(1)$	$-(N-2)$
IR	$B^{\{AB\}}$	$\frac{(N-4)(N-3)}{2}\cdot(\cdot)$	$\yng(2)$	$-N$

TABLE X. Color-flavor locking in the (0,1) model. The color index i_1 or j_1 runs up to $N-4$, and the rest is indicated by i_2 or j_2.

	Fields	$SU(N-4)_{\rm cf}$	$U'(1)$	$SU(4)_{\rm c}$
UV	$\chi_{i_1j_1}$	$\bar{\yng(1,1)}$	N	$\frac{(N-4)(N-5)}{2}\cdot(\cdot)$
	$\chi_{i_1j_2}$	$4\cdot\bar{\yng(1)}$	$\frac{N}{2}$	$(N-4)\cdot\bar{\yng(1)}$
	$\chi_{i_2j_2}$	$\frac{4\cdot 3}{2}\cdot(\cdot)$	0	$\bar{\yng(1,1)}$
	$\tilde{\eta}^{A,i_1}$	$\yng(2)\oplus\yng(1,1)$	$-N$	$(N-4)^2\cdot(\cdot)$
	$\tilde{\eta}^{A,i_2}$	$4\cdot\yng(1)$	$-\frac{N}{2}$	$(N-4)\cdot\yng(1)$
IR	$B^{\{AB\}}$	$\yng(2)$	$-N$	$\frac{(N-4)(N-3)}{2}\cdot(\cdot)$

symmetric in CD. They have the $U(1)$ charge $-N$. There is no way that $B^{\{CD\}}$ can saturate the anomalies in $SU(2N-8)_{\mathrm{f}} \times U(1)$.

One concludes that the confinement phase with unbroken chiral symmetry $SU(2) \times SU(2N-8)_{\mathrm{f}} \times U(1)$ is not possible. This is, again, in contrast to the $(N_\psi, N_\chi) = (0,1)$ model.

B. Color-flavor locking

Let us instead assume a color-flavor locked diagonal vacuum expectation value (VEV),

$$\langle \chi^1_{[ij]} \tilde{\eta}^{iB} \rangle = c\Lambda^3 \delta^B_j, \qquad j, B = 1, 2, \ldots, N-4,$$
$$\langle \chi^2_{[ij]} \tilde{\eta}^{iB} \rangle = c'\Lambda^3 \delta^{B-(N-4)}_j, \qquad j = 1, \ldots, N-4,$$
$$B = N-3, \ldots, 2N-8. \tag{7.7}$$

Then the symmetry is broken down to

$$SU(N-4)_{\mathrm{cf}} \times U'(1) \times SU(4)_{\mathrm{c}}, \tag{7.8}$$

where $U(1)'$ is the unbroken linear combination between the anomaly-free $U(1)$, Eq. (7.5), and a subgroup of the color $SU(N)$, $\mathrm{diag}(\frac{1}{N-4}\mathbb{1}_{N-4}, -\frac{1}{4}\mathbb{1}_4)$. We assume that the massless baryons are

$$B^{\{CD\},1} = \chi^1_{[ij]} \tilde{\eta}^{iC} \tilde{\eta}^{jD}, \quad C, D = 1, 2, \ldots, N-4,$$
$$\hat{B}^{\{CD\},1} = \chi^1_{[ij]} \tilde{\eta}^{iC} \tilde{\eta}^{jD}, \quad C, D = N-3, N-2, \ldots, 2(N-4). \tag{7.9}$$

The charges under $SU(N-4)_{\mathrm{cf}} \otimes U'(1)$ are given in Table XI, where the $U(1)'$ charges are appropriately renormalized by a common factor. All anomalies $SU(N-4)^3_{\mathrm{cf}}$, $U(1)'^3$, $U(1)'$, $U(1)' SU(N-4)^2_{\mathrm{cf}}$ work out fine.

TABLE XI. Color-flavor locking in the (0,2) model. The color index i_1 or j_1 runs up to $N-4$. The rest is indicated by i_2 or j_2.

	Fields	$SU(N-4)_{\mathrm{cf}}$	$U'(1)$
UV	$\chi^1_{[i_1 j_1]}$	$\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$	1
	$\chi^1_{[i_1 j_2]}$	$4 \cdot \bar{\square}$	$\frac{1}{2}$
	$\chi^1_{[i_2 j_2]}$	$(\cdot)$	0
	$\tilde{\eta}^{B,i_1}$	$4(N-4)\ \square$	-1
	$\tilde{\eta}^{B,i_2}$	$8 \cdot \square$	$-\frac{1}{2}$
IR	$B^{\{CD1\}}$	$\square\square$	-1
	$\hat{B}^{\{CD1\}}$	$\square\square$	-1

C. Phase with unbroken $SU(2)$

Assume instead that condensates (7.7) occur with with the same coefficients,

$$c = c'. \tag{7.10}$$

Then the residual symmetry is bigger:

$$SU(N-4)_{\mathrm{cf}} \times SU(2) \times U'(1) \times SU(4)_{\mathrm{c}}. \tag{7.11}$$

The baryons are

$$B^{\{CD\},1} = \chi^1_{[ij]} \tilde{\eta}^{iC} \tilde{\eta}^{jD}, \quad C, D = 1, 2, \ldots, N-4,$$
$$B^{\{CD\},2} = \chi^2_{[ij]} \tilde{\eta}^{iC} \tilde{\eta}^{jD}, \quad C, D = N-3, N-2, \ldots, 2(N-4), \tag{7.12}$$

symmetric in CD. The charges with respect to this $SU(2)$ are

$$\begin{pmatrix} \chi^1_{[ij]} \\ \chi^2_{[ij]} \end{pmatrix} \sim \underline{2}, \qquad \begin{pmatrix} \tilde{\eta}_i^{A \le N-4} \\ \tilde{\eta}_i^{N-4 < A \le 2N-8} \end{pmatrix} \sim \underline{2}^*. \tag{7.13}$$

The charges of the fields with respect to the unbroken symmetries are in Table XII. The saturation of all seven types of triangles can be seen by inspection.

$SU(2)$ has no (perturbative) triangle anomaly, but it does have a global anomaly (Witten). It can be readily checked to see that the difference of the number of the doublets in the UV and in the IR is even.

As in the (0,1) model, the fermions $\chi^m_{[i_2 j_2]}$ remain massless and coupled strongly by the unbroken color $SU(4)_{\mathrm{c}}$. It is possible that they condense as

$$\langle \epsilon^{ijk\ell} \chi^m_{ij} \chi^n_{k\ell} \rangle \neq 0, \qquad m, n = 1, 2. \tag{7.14}$$

As they are symmetric in m and n, the symmetry is broken as

$$SU(2) \to SO(2) = U(1), \tag{7.15}$$

in a scenario similar to tumbling.

So, after all, $SU(2)$ is dynamically broken. The fate of the unbroken, residual $SU(4)_{\mathrm{c}}$ is similar to what happens in the second, chiral-symmetry-breaking (XSB) scenario in Sec. VI B.

VIII. $(N_\psi, N_\chi) = (0,3)$

The model to be considered now is

$$\chi^m_{[ij]}, \quad \tilde{\eta}^{Bj}, \quad m = 1, 2, 3, \quad B = 1, 2, \ldots, 3(N-4), \tag{8.1}$$

or

TABLE XII. $SU(2)$ symmetric phase in the (0,2) model. i_1, j_1 stand for the color indices up to $N-4$, and i_2, j_2 the last four.

	Fields	$SU(N-4)_{\mathrm{cf}}$	$SU(2)$	$U'(1)$	$SU(4)_{\mathrm{c}}$
UV	$\chi^m_{[i_1 j_1]}$	$2\cdot\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\frac{(N-4)(N-5)}{2}\cdot\square$	1	$(N-4)(N-5)\cdot(\cdot)$
	$\chi^m_{[i_1 j_2]}$	$8\cdot\bar{\square}$	$4(N-4)\cdot\bar{\square}$	$\frac{1}{2}$	$2(N-4)\cdot\square$
	$\chi^m_{[i_2 j_2]}$	$12\cdot(\cdot)$	$6\cdot\square$	0	$2\cdot\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$
	$\tilde{\eta}^{Bi_1}$	$2\cdot\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}+\square\!\square\right)$	$(N-4)^2\cdot\bar{\square}$	-1	$2(N-4)^2\cdot(\cdot)$
	$\tilde{\eta}^{Bi_2}$	$8\cdot\square$	$4(N-4)\cdot\square$	$-\frac{1}{2}$	$2(N-4)\cdot\bar{\square}$
IR	$B^{CD,m}$	$2\cdot\square\!\square$	$\frac{(N-4)(N-3)}{2}\cdot\bar{\square}$	-1	$(N-4)(N-3)\cdot(\cdot)$

$$3\,\overline{\square\!\square}+3(N-4)\,\square\,. \tag{8.2}$$

The first coefficient of the β function is

$$b_0=\frac{1}{3}[11N-3(N-2)-3(N-4)]=\frac{1}{3}(5N+18). \tag{8.3}$$

The symmetry is

$$SU(N)_{\mathrm{c}}\times SU(3)\times SU(3N-12)_{\mathrm{f}}\times U(1), \tag{8.4}$$

where the anomaly-free $U(1)$ charge is

$$\chi\colon\ N-4, \qquad \tilde{\eta}^{Bj}\colon\ -(N-2). \tag{8.5}$$

Again the option of confinement with no flavor symmetry breaking is excluded.

A. Color-flavor locking

Let us try to generalize the color-flavor locking of the $(N_\psi, N_\chi)=(0,2)$ case to our $(N_\psi, N_\chi)=(0,3)$ model by assuming that

$$\begin{aligned}
&\langle\chi^1_{[ij]}\tilde{\eta}^{iB}\rangle=c\Lambda^3\delta^B_j\neq 0, \qquad j,B=1,2,\ldots,N-4,\\
&\langle\chi^2_{[ij]}\tilde{\eta}^{iB}\rangle=c\Lambda^3\delta^{B-(N-4)}_j\neq 0, \qquad j=1,2,\ldots,N-4,\\
&\qquad N-3\leq B\leq 2N-8,\\
&\langle\chi^3_{[ij]}\tilde{\eta}^{iB}\rangle=c\Lambda^3\delta^{B-(N-4)}_j\neq 0, \qquad j=1,2,\ldots,N-4,\\
&\qquad 2N-7\leq B\leq 3N-12.
\end{aligned} \tag{8.6}$$

Then the symmetry breaking pattern is

$$SU(N-4)_{\mathrm{cf}}\times U'(1)\times SU(3)\times SU(4)_{\mathrm{c}}, \tag{8.7}$$

where $U(1)'$ is the unbroken linear combination between the anomaly-free $U(1)$, Eq. (8.5), and a subgroup of the color $SU(N)$, $\mathrm{diag}(4\,\mathbb{1}_{N-4},-(N-4)\mathbb{1}_4)$. The would-be $SU(3)$ multiplets are

$$\begin{pmatrix}\chi^1_{[ij]}\\ \chi^2_{[ij]}\\ \chi^3_{[ij]}\end{pmatrix}\sim\underline{3}, \qquad \begin{pmatrix}\tilde{\eta}_i^{A\leq N-4}\\ \eta_i^{N-4<A\leq 2N-8}\\ \eta_i^{2N-8<A\leq 3N-12}\end{pmatrix}\sim\underline{3}^*. \tag{8.8}$$

We assume that the massless baryons are

$$\begin{aligned}
&B^{\{CD\},1}=\chi^1_{[ij]}\tilde{\eta}^{iC}\tilde{\eta}^{jD}, \quad C,D=1,2,\ldots,N-4,\\
&\hat{B}^{\{CD\},2}=\chi^2_{[ij]}\tilde{\eta}^{iC}\tilde{\eta}^{jD}, \quad C,D=N-3,\ldots,2(N-4),\\
&\tilde{B}^{\{CD\},3}=\chi^3_{[ij]}\tilde{\eta}^{iC}\tilde{\eta}^{jD}, \quad C,D=2N-7,\ldots,3(N-4),
\end{aligned} \tag{8.9}$$

symmetric in CD. These baryons transform as $\underline{3}^*$.

Unlike what happens to the (0,2) model, or to the (3,0) model, however, here the unbroken $SU(3)$ symmetry cannot be realized manifestly in the infrared: $SU(3)^3$ triangles do not match in the UV and IR, see Table XIII.

A possibility is that the condensates (8.6) take unequal values. With $SU(3)$ broken, the baryons $B^{\{CD\},m}$ saturate the anomalies in $SU(N-4)_{\mathrm{cf}}\times U'(1)\times SU(4)_{\mathrm{c}}$.

Another possibility is suggested by the presence of massless fermions $\chi^m_{[ij]}(i_>, j_>)$, which interact strongly with the remaining gauge group $SU(4)_{\mathrm{c}}$. It is possible that condensates

$$\langle\epsilon^{ijk\ell}\chi^m_{ij}\chi^n_{k\ell}\rangle\neq 0, \qquad m,n=1,2,3. \tag{8.10}$$

form. As they are symmetric in m, n, the symmetry is broken as

TABLE XIII. The decomposition of the fields in the (0,3) model. The color indices are divided into two groups: i_1, j_1 run up to $N-4$, and i_2, j_2 cover the rest. Moreover, the color and flavor indices are combined as in Sec. VIII A

	Fields	$SU(N-4)_{\rm cf}$	$SU(3)$	$U'(1)$	$SU(4)_{\rm c}$
UV	$\chi^m_{[i_1 j_1]}$	$3\cdot\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\frac{(N-4)(N-5)}{2}\cdot\square$	1	$\frac{3(N-4)(N-5)}{2}\cdot(\cdot)$
	$\chi^m_{[i_1 j_2]}$	$12\cdot(\bar{\square})$	$4(N-4)\cdot\bar{\square}$	$\frac{1}{2}$	$3(N-4)\cdot\bar{\square}$
	$\chi^m_{[i_2 j_2]}$	$18\cdot(\cdot)$	$6\cdot\square$	0	$3\cdot\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$
	$\tilde{\eta}^{Bi_1}$	$3\cdot\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}+\square\square\right)$	$(N-4)^2\cdot\bar{\square}$	-1	$3(N-4)^2\cdot(\cdot)$
	$\tilde{\eta}^{Bi_2}$	$12\cdot\square$	$4(N-4)\cdot\square$	$-\frac{1}{2}$	$3(N-4)\cdot\square$
IR	$B^{\{CD\},m}$	$3\cdot\square\square$	$\frac{(N-4)(N-3)}{2}\cdot\bar{\square}$	-1	$\frac{3(N-4)(N-3)}{2}\cdot(\cdot)$

$$SU(3)\to SO(3), \tag{8.11}$$

which is free of anomalies.

IX. $(N_\psi, N_\chi) = (2, 1)$

Next consider the $SU(N)$ gauge model with the chiral fermion sector

$$\begin{aligned} &\psi^{\{ij\},m}, \qquad \chi_{[ij]}, \qquad \eta_j^B, \\ &m=1,2, \qquad B=1,2,\ldots,N+12, \end{aligned} \tag{9.1}$$

or

$$2\,\square\square + \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + (N+12)\,\bar{\square}\,. \tag{9.2}$$

The symmetries of the theory are

$$SU(N)_{\rm c}\times SU(2)_{\rm f}\times SU(12+N)_{\rm f}\times U(1)^2. \tag{9.3}$$

The two $U(1)$'s are anomaly-free combinations of $U_\psi(1)$, $U_\chi(1)$, and $U_\eta(1)$, which can be taken as

$$\begin{aligned} &U_1(1)\colon\ \psi\to e^{i\frac{\alpha}{2(N+2)}}\psi,\ \eta\to e^{-i\frac{\alpha}{N+12}}\eta, \\ &U_2(1)\colon\ \psi\to e^{i\frac{\beta}{2(N+2)}}\psi,\ \chi\to e^{-i\frac{\beta}{N-2}}\chi. \end{aligned} \tag{9.4}$$

The first coefficient of the β function is

$$\begin{aligned} b_0 &= \frac{1}{3}[11N-2(N+2)-(N-2)-(12+N)] \\ &= \frac{1}{3}(7N-14). \end{aligned} \tag{9.5}$$

A. Color-flavor locking?

A possibility is that a (partial) color-flavor locking condensate

$$\langle\psi^{\{ij\},1}\eta_j^B\rangle = c\Lambda^3\delta^{iB}, \qquad i,B=1,2,\ldots,N, \tag{9.6}$$

develops, where the direction of the $SU_\psi(2)$ breaking is arbitrary. Let us assume that there is no adjoint condensate $\langle\psi\chi\rangle$. The unbroken symmetry is

$$SU(N)_{\rm cf}\times SU(12)_{\rm f}\times\tilde{U}(1), \tag{9.7}$$

where the $\tilde{U}(1)$ charges are

$$Q_\psi = 1, \qquad Q_\chi = -\frac{N-8}{N-2}, \qquad Q_\eta = -1. \tag{9.8}$$

The candidate baryons are

$$B^{CD,m} = \psi^{\{ij\},m}\eta_i^C\eta_j^D. \tag{9.9}$$

An inspection shows that these baryons do not saturate the $G_{\rm f}$ anomalies, and one concludes that phase (9.6) is not possible.

B. Color-flavor-flavor locking?

Let us assume, for $N\le 12$, condensates of the form

$$\begin{aligned} &\langle\psi^{\{ij\},1}\eta_j^{B_1}\rangle = c\Lambda^3\delta^{i,B_1}, \\ &\langle\psi^{\{ij\},2}\eta_j^{B_2}\rangle = c\Lambda^3\delta^{i,B_2-N}, \end{aligned} \tag{9.10}$$

where the flavor indices B_1 run up to N and B_2 from $N+1$ to $2N$. The symmetry is broken down to

TABLE XIV. $SU(2)$ symmetric phase in the (2,1) model. A_1, B_1 stand for the flavor indices up to N, A_2, B_2 from $N+1$ to $2N$, and A_3, B_3 the last $12-N$. Anomaly matching fails in this case.

	Fields	$SU(N)_{\rm cf}$	$SU(2)$	$SU(12-N)$	$U'(1)$
UV	$\psi^{\{ij\},m}$	$2\cdot\square\!\square$	$\frac{N(N+1)}{2}\cdot\square$	$N(N+1)\cdot(\cdot)$	1
	$\chi_{[ij]}$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\frac{N(N-1)}{2}\cdot(\cdot)$	$\frac{N(N-1)}{2}\cdot(\cdot)$	$-\frac{N-8}{N-2}$
	$\eta_i^{B_1},\eta_i^{B_2}$	$2\cdot(\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\oplus\overline{\square\!\square})$	$N^2\cdot\bar{\square}$	$2N^2\cdot(\cdot)$	-1
	$\eta_i^{B_3}$	$(12-N)\cdot\bar{\square}$	$N(12-N)\cdot(\cdot)$	$N\cdot\square$	-1
IR	$B^{[A_1B_2],m}$	$2\cdot\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\frac{N(N-1)}{2}\cdot\square$	$N(N-1)\cdot(\cdot)$	-1
	$B^{[A_1B_3],m}$	$(12-N)\cdot\bar{\square}$	$N(12-N)\cdot(\cdot)$	$N\cdot\square$	-1

$$SU(N)_{\rm cf}\times SU(2)_{\rm ff}\times SU(12-N)_{\rm f}\times U'(1). \quad (9.11)$$

The candidate baryons have the form

$$B^{AB,m}=\psi^{\{ij\},m}\eta_i^A\eta_j^B, \quad (9.12)$$

but it is not possible to achieve the anomaly matching (see Table XIV).

C. Dynamical Abelianization

Assuming that the adjoint condensate forms

$$\langle\psi^{\{ij\},1}\chi_{[ik]}\rangle=c^j\Lambda^3\delta_k^j, \qquad j,k=1,2,\ldots,N, \quad (9.13)$$

with c^j's all being different and the Cartan subgroup of $SU(N)_{\rm c}$ surviving in the infrared. $SU(2)_{\rm f}$ is broken. There is a $U(1)$ symmetry which remains unbroken, $\tilde{U}(1)$, under which

$$\psi\colon\ N+12, \qquad \chi\colon\ -(N+12), \qquad \eta\colon\ -(N+6). \quad (9.14)$$

The unbroken symmetry group is

$$SU(N+12)_{\rm f}\times\tilde{U}(1). \quad (9.15)$$

The low-energy degrees of freedom are the fermion fields η_j^B, which are unconfined and are weakly coupled to the $U(1)^{N-1}$ photons, the diagonal $\psi^{\{ii\},1}$, and all of the $\psi^{\{ij\},2}$. Also, there are $3+1=4$ NG bosons.

The anomaly equalities for $SU(12+N)_{\rm f}^3$, $\tilde{U}(1)SU(12+N)_{\rm f}^2$, $\tilde{U}(1)^3$, and $\tilde{U}(1)$ can be straightforwardly checked; see Table XV.

X. $(N_\psi,N_\chi)=(1,-1)$

Consider now a model with

$$\psi^{\{ij\}}, \qquad \tilde{\chi}^{[ij]}, \qquad \eta_i^A, \qquad A=1,2,\ldots,2N, \quad (10.1)$$

or

$$\square\!\square+\begin{smallmatrix}\square\\\square\end{smallmatrix}+2N\,\bar{\square}, \quad (10.2)$$

i.e., a symmetric tensor, an antisymmetric tensor, and $2N$ antifundamental multiplets of $SU(N)$. The first coefficient of the beta function is

$$b_0=\frac{1}{3}[11N-(N+2)-(N-2)-2N]=\frac{7N}{3}. \quad (10.3)$$

The symmetry of the system is

$$SU(N)_{\rm c}\times SU(2N)_{\rm f}\times U_1(1)\times U_2(1) \quad (10.4)$$

TABLE XV. The decomposition of the fields in the (2,1) model, assuming complete dynamical Abelianization.

	Fields	$SU(N+12)$	$\tilde{U}(1)$
UV	ψ	$2\cdot\frac{N(N+1)}{2}\cdot(\cdot)$	$N+12$
	χ	$\frac{N(N-1)}{2}\cdot(\cdot)$	$-(N+12)$
	η^A	$N\cdot\square$	$-(N+6)$
IR	$\psi^{ii,1}$	$N\cdot(\cdot)$	$N+12$
	$\psi^{ij,2}$	$\frac{N(N+1)}{2}\cdot(\cdot)$	$N+12$
	$\psi\chi\eta^A\sim\eta^A$	$N\cdot\square$	$-(N+6)$

times some discrete symmetry. The $U(1)$ charges are

$$U_1(1)\colon\ Q_\psi = \frac{1}{N+2}, \qquad Q_{\tilde{\chi}} = -\frac{1}{N-2}, \qquad Q_\eta = 0,$$
$$U_2(1)\colon\ Q_\psi = \frac{1}{N+2}, \qquad Q_{\tilde{\chi}} = 0, \qquad Q_\eta = -\frac{1}{2N}. \tag{10.5}$$

Possible baryon states are

$$B^{AB} = \psi^{\{ij\}}\eta_i^A\eta_j^B, \qquad \hat{B}^{AB} = \tilde{\chi}^{[ij]}\eta_i^A\eta_j^B, \tag{10.6}$$

both of which could form either symmetric or antisymmetric tensors in the flavor. Confinement without chiral symmetry breaking appears to be excluded: there is no way that B^{AB} or $\hat{B}^{AB}$ can match the UV $SU(2N)_\mathrm{f}$ anomaly, N.

A. Color-flavor locking

Let us try a color-flavor locking

$$\langle\psi^{\{ij\}}\eta_j^A\rangle = c\Lambda^3\delta^{iA}, \qquad i, A = 1, 2, \ldots, N,$$
$$\langle\tilde{\chi}^{[ij]}\eta_j^A\rangle = c'\Lambda^3\delta^{iA}, \qquad i, A = 1, 2, \ldots, N. \tag{10.7}$$

The symmetry is broken down to

$$SU(N)_\mathrm{cf} \times SU(N)_\mathrm{f} \times \tilde{U}(1), \tag{10.8}$$

where $\tilde{U}(1)$ is an unbroken combination of $U_{1,2}(1)$ with charges

$$\tilde{U}(1)\colon\ Q_\psi = -1, Q_{\tilde{\chi}} = -1, Q_\eta = 1. \tag{10.9}$$

Again we list the fields and their decomposition in the low-energy symmetry groups. Assuming that the only massless baryons are B^{AB}, with $A \leq N$ and $B \geq N$, the anomaly matching is obvious; see Table XVI.

TABLE XVI. The color-flavor locking scheme for the $(1,-1)$ model. The flavor indices A_1, B_1 stand for those up to N, A_2, B_2 for $N+1, \ldots, 2N$.

	Fields	$SU(N)_\mathrm{cf}$	$SU(N)_\mathrm{f}$	$\tilde{U}(1)$
UV	ψ	$\square\!\square$	$\frac{N(N+1)}{2}\cdot(\cdot)$	-1
	χ	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\frac{N(N-1)}{2}\cdot(\cdot)$	-1
	η^{A_1}	$\overline{\square\!\square} \oplus \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$N^2\cdot(\cdot)$	1
	η^{A_2}	$N\cdot\bar{\square}$	$N\cdot\square$	1
IR	$B^{A_1B_2}$	$N\cdot\bar{\square}$	$N\cdot\square$	1

XI. PION-DECAY CONSTANT IN CHIRAL THEORIES

After these exercises with various (N_ψ, N_χ) models, it would be useful to try to draw some lessons. One concerns the nature of the Nambu-Goldstone bosons (called *pions* below symbolically) and the quantity analogous to the pion-decay constant in chiral $SU(2)_L \times SU(2)_R$ QCD. As we shall see, there is a qualitative difference between the wisdom about the chiral dynamics with light quarks in QCD, which is a vectorlike theory, and what is to be expected in general chiral theories.

Consider any global continuous symmetry G_f and the associated conserved current J_μ, the field ϕ (elementary or composite) which condenses and breaks G_f, and the field $\tilde{\phi}$ which is transformed into ϕ by the G_f charge

$$Q \equiv \int d^3x J_0, \qquad [Q, \tilde{\phi}] = \phi, \qquad \langle\phi\rangle \neq 0. \tag{11.1}$$

Thus

$$\lim_{q_\mu\to 0} iq^\mu \int d^4x\, e^{-iq\cdot x}\langle 0|T\{J_\mu(x)\tilde{\phi}(0)\}|0\rangle$$
$$= \lim_{q_\mu\to 0}\int d^4x\, e^{-iq\cdot x}\partial_\mu\langle 0|T\{J_\mu(x)\tilde{\phi}(0)\}|0\rangle$$
$$= \int d^3x\langle 0|[J_0(x), \tilde{\phi}(0)]|0\rangle = \langle 0|[Q, \tilde{\phi}(0)]|0\rangle$$
$$= \langle 0|\phi(0)|0\rangle \neq 0. \tag{11.2}$$

This Ward-Takahashi-like identity implies that the two-point function

$$\int d^4x\, e^{-iq\cdot x}\langle 0|T\{J_\mu(x)\tilde{\phi}(0)\}|0\rangle \tag{11.3}$$

is singular at $q\to 0$. If the G_f symmetry is broken spontaneously, such a singularity is due to the massless NG boson, π, such that

$$\langle 0|J_\mu(q)|\pi\rangle = iq_\mu F_\pi, \qquad \langle\pi|\tilde{\phi}|0\rangle \neq 0, \tag{11.4}$$

such that the two-point function behaves as

$$q^\mu\cdot q_\mu\frac{F_\pi\langle\pi|\tilde{\phi}|0\rangle}{q^2} \sim \text{const.} \tag{11.5}$$

at $q\to 0$.

In the standard $SU(2)_L \times SU(2)_R \to SU(2)_V$ chiral symmetry breaking in QCD, the quarks are

$$\psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad \psi_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \tag{11.6}$$

and by taking

$$\phi = \bar{\psi}_R \psi_L + \text{H.c.}, \qquad \tilde{\phi} = \bar{\psi}_R t^b \psi_L - \text{H.c.},$$
$$J_\mu^{5,a} = i\bar{\psi}_L \bar{\sigma}_\mu t^a \psi_L - (L \leftrightarrow R), \tag{11.7}$$

$$t^a = \frac{\tau^a}{2}, \qquad a = 1, 2, 3. \tag{11.8}$$

It is believed that the field

$$\langle \phi \rangle = \langle \bar{u}_R u_L + \bar{d}_R d_L + \text{H.c.} \rangle \sim -\Lambda^3 \tag{11.9}$$

condenses, leaving $SU(2)_V$ unbroken; the axial $SU(2)_A$ is broken. In QCD, Λ is of the same order of the confinement mass scale, the dynamically generated mass scale of QCD,

$$\Lambda \sim 200 \text{ MeV}. \tag{11.10}$$

The pions are associated with the interpolating field

$$\pi^a \sim \tilde{\phi}^a = \bar{\psi}_R t^a \psi_L - \text{H.c.} \sim \bar{\psi}_D \gamma^5 t^a \psi_D \tag{11.11}$$

(where ψ_D is the Dirac spinor for the quarks). It is natural to expect that the pion-decay constant, the amplitude with which the current operator $J_\mu^{5,a}$ produces the pions from the vacuum, is of the same order of magnitude as Λ itself,

$$F_\pi \sim \Lambda. \tag{11.12}$$

Indeed, the best experimental estimate for F_π is

$$F_\pi \sim 130 \text{ MeV}; \tag{11.13}$$

cf. Eq. (11.10).

Now let us study the case of chiral gauge theories, as those considered in this paper. To be concrete, consider the dynamical scenarios, Sec. IV C in the (2,0) model. The symmetry breaking pattern is

$$SU(N)_\text{c} \times SU(2)_\text{f} \times SU(2N+8)_\text{f} \times U(1)$$
$$\rightarrow SU(N)_\text{cf} \times \tilde{U}(1) \times U'(1) \times SU(8). \tag{11.14}$$

The Nambu-Goldstone modes are associated with the breaking

$$SU(2)_\text{f} \times SU(2N+8)_\text{f} \rightarrow SU(8) \times U'(1). \tag{11.15}$$

There are

$$3N^2 + 32N + 3 \tag{11.16}$$

NG bosons.

To simplify the discussion, let us concentrate our attention to the two NG bosons associated with the $SU_\psi(2) \rightarrow U'(1)$ breaking.[6] The $SU_\text{f}(2)$ current is

$$J_\mu^a = i\bar{\psi}^{ij,m} \bar{\sigma}_\mu \left(\frac{\tau^a}{2}\right)_{mn} \psi^{ij,n}, \tag{11.17}$$

and the charges are

$$Q^a = \int d^3x\, J_0^a. \tag{11.18}$$

One can choose

$$\tilde{\phi}^b = \sum_{i,j,k,B} (\psi^{\{ij,m\}} \eta_i^B)^* \left(\frac{\tau^b}{2}\right)_{mn} \psi^{\{kj,n\}} \eta_k^B \tag{11.19}$$

in Eq. (11.2) so that

$$\langle [Q^a, \tilde{\phi}^b] \rangle = \delta^{ab} \left\langle \sum_{i,j,k,B} (\psi^{\{ij,m\}} \eta_i^B)^* (\psi^{\{kj,m\}} \eta_k^B) \right\rangle \neq 0. \tag{11.20}$$

An important issue here is the fact that, even though the dynamical gauge and flavor symmetry breaking is (by assumption) determined by the "dynamical Higgs scalar" condensates

$$\langle \psi^{\{ij,1\}} \eta_j^B \rangle = c\Lambda^3 \delta^{i,B}, \qquad j, B = 1, 2, \ldots, N,$$
$$\langle \psi^{\{ij,2\}} \eta_j^B \rangle = c'\Lambda^3 \delta^{i,B-N}, \quad j = 1, 2, \ldots, N,$$
$$B = N+1, \ldots, 2N, \tag{11.21}$$

at some mass scale, Λ, the pion interpolating fields appearing in the Ward-Takahashi identity must be gauge invariants such as Eq. (11.19), which are necessarily four-fermion composites. On the other hand, the *pion-decay constant* is defined as usual,

$$\langle 0 | J_\mu^a | \pi^a \rangle = i q_\mu F_\pi,$$
$$J_\mu^a = i\bar{\psi}^{ij,m} \bar{\sigma}_\mu \left(\frac{\tau^a}{2}\right)_{mn} \psi^{ij,n}, \tag{11.22}$$

as the amplitude with which the current operator produces the NG bosons from the vacuum. It is quite possible that the pion-decay constant in chiral theories is such that

$$F_\pi \ll \Lambda, \tag{11.23}$$

[6] Naturally the same discussion holds for other $3N^2 + 32N + 1$ NG bosons, but the expressions would become more clumsy.

as the bifermion current operator must produce pions, which are four-fermion composite particles, from the vacuum.[7]

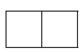
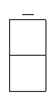
Another way of seeing the same question is to think of the pion effective action,

$$\mathcal{L}(\tilde{\phi}^a, \partial_\mu \tilde{\phi}^a) = \frac{1}{2} \partial_\mu \tilde{\phi}^a \partial^\mu \tilde{\phi}^a + \cdots, \qquad (11.24)$$

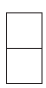
in which the interaction strength among the pions is given by F_π. The effective action involves 8-fermion, 16-fermion, etc., amplitudes, and a result such as Eq. (11.23) could well be realized using the complicated strong interaction dynamics.

XII. DISCUSSION

Let us recapitulate the class of (N_ψ, N_χ) models analyzed here. The gauge group is taken to be $SU(N)$. The numbers of Weyl fermions ψ and χ in the representations

 and  (12.1)

are indicated by N_ψ and N_χ. Let us take $N_\psi \geq 0$. In the case $N_\chi < 0$, $-N_\chi$ indicates the number of the fields $\tilde{\chi}$ in the representation

 (12.2)

instead. The number of the fermions in the antifundamental (or fundamental) representations η^a (or $\tilde{\eta}^a$) is fixed by the condition that the gauge group $SU(N)$ be anomaly-free. Also, we restrict the numbers N_ψ and N_χ such that the model is asymptotically free.

The systems considered here are rather rigid. No fermion mass terms can be added in the Lagrangian, and this also means that no gauge-invariant bifermion condensates can form. They cannot be deformed by the addition of any other renormalizable potential terms either, including the topological $\theta F_{\mu\nu} \tilde{F}^{\mu\nu}$ term. The presence of massless chiral matter fermions means that all values of θ are equivalent to $\theta = 0$. The vacuum, apart from possible symmetry breaking degeneration, is expected to be unique. The system is strongly coupled in the infrared. Our ignorance about these simple models, after more than a half century of studies of quantum field theories, certainly is severely hindering our capability of finding any application of them in a physical theory describing nature.

In the absence of other theoretical tools, we have insisted in this paper upon trying to find possible useful indications following the standard 't Hooft anomaly matching constraints (for application of some new ideas such as the generalized symmetries and higher-form gauging to these chiral gauge theories; see [19]). The main lesson to be learned is perhaps the fact that color-flavor (or color-flavor-flavor) locking and dynamical Abelianization, in various combinations, always provides natural ways to solve these consistency constraints and to find possible phases of the system.

The strategy we used in the paper, for all of the models, is summarized as follows. First we chose a set of bifermions operators that may condense. Since we do not have a gauge-invariant bifermion in our theories, we chose among the gauge-noninvariant ones, possibly guided by the maximal attractive channel (MAC) criterion. Condensation has two important effects: it breaks part or all of the color symmetry, and it breaks part or all of the flavor symmetry. The broken part of the gauge group is dynamically Higgsed. The unbroken part confines or remains in the IR if it is in the Coulomb phase (as for the dynamical Abelianization). We then have to look at the anomaly matching conditions. The part of the flavor symmetry that is broken by the condensate is saturated by massless NG boson poles. For the unbroken part instead, we need to find a set of fermions in the IR to match the computation in the UV. We then decompose the UV fermion into a direct sum of representations of the unbroken flavor subgroup that remains unbroken. Unlike the UV representation, which is chiral, the IR decomposed representations have in general vectorial subsets. All of the vectorial parts can be removed since they presumably get massive and in any case they do not contribute to the 't Hooft anomaly of the unbroken group. Other fermions remain in the IR as massless baryons and saturate the 't Hooft anomalies.

The fact that, in models (1,0) and (0,1), one can find a set of candidate massless fermions saturating the anomalies of the full unbroken flavor symmetries seems to be fortuitous, rather than being a rule. In fact, no analogous set of candidate massless baryons can be found in other (2,0), (3,0) or (0,2), (0,3) models. On the other hand, the color-flavor breaking (dynamical Higgs) phase of the (1,0) and (0,1) models finds natural generalizations in these more complicated systems.

In this sense, our proposal shares a common feature with the tumbling scheme, but it does not follow the MAC criterion literally with the multiscale chains of dynamical gauge symmetry breaking, as in the original proposal [1]. There are a few cases, however, in which the appearance of hierarchy of mass scales, for reasons entirely different from that in the tumbling mechanism, is rather natural.

The local gauge symmetries can never be "truly" spontaneously broken, and any dynamical or elementary

[7]Large N scaling would ruin this hierarchy, so N must be kept finite.

Higgs mechanism (including the case of the standard Higgs scalar in the Weinberg-Salam electroweak theory) must be reinterpreted in a gauge-invariant fashion.[8] What happens in the chiral gauge theories considered here is that the system produces a bifermion composite state such as

$$\psi(x)\eta(x), \qquad \psi(x)\chi(x), \tag{12.3}$$

which then act as an effective Higgs scalar field. As these "dynamical" Higgs fields are still strongly coupled in general, the way their condensates and consequent flavor symmetry breaking is reinterpreted in a gauge-invariant fashion may be more complicated than in the standard electroweak theory where the Higgs scalars are weakly coupled and described by perturbation theory. Even though, in strongly coupled chiral gauge theories, one does not have a simple potential describing the degenerate vacua, bifermion condensates of composite scalars such as Eq. (12.3) are just analogs of the gauge-noninvariant (and gauge dependent) Higgs scalar VEV.[9]

The proposed dynamical Higgs mechanism does, however, make a definite statement about the *flavor* symmetry breaking: the latter is described by the condensate of the composite (dynamical) Higgs fields such as those above, at the mass scale associated with them.

This brings us to a possibly relevant observation made in Sec. XI. A study of chiral Ward-Takahashi identities shows that, in contrast to what happens in vectorlike gauge theory such as QCD, the system might generate a hierarchy of mass scales between the mass scale of the condensates of the composite Higgs fields (12.3), "Λ," and the quantity corresponding to the pion-decay constant, "F_π." The latter is the amplitude that the (broken) symmetry current produces a NG boson (pion) from the vacuum. The fact that in chiral gauge theories the current is a two-fermion operator, while the pions are in general four-fermion composites, in contrast to what happens in the case of axial symmetry breaking in vectorlike theories, could imply a large hierarchy, Eq. (11.23). Such a possibility appears to be worthy of further study, both from theoretical and phenomenological points of view.

[8]As explained by 't Hooft, the Higgs VEV of the form $\langle\phi\rangle = v\binom{1}{0}$, found in all textbooks on electroweak theory, is just a gauge dependent way of describing the gauge-invariant VEV $\langle\phi^\dagger\phi\rangle$, so it is a statement such as the left-hand fermion being equal to $\psi_L = \binom{\nu_L}{e_L}$.

[9]After all, the Higgs mechanism was first discovered in the context of superconductivity: the Cooper pair condenses due to the interactions between the electrons and the lattice phonons. The Cooper pair, having charge 2, is not a gauge-invariant object. It is perhaps useful to remember that the Higgs mechanism was thus first discovered in the context of a dynamical Higgs mechanism.

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APPENDIX: *a* THEOREM AND THE ACS CRITERION

For free theory of bosons and fermions, the a and c coefficients are given by

$$a = \frac{1}{360}\left(N_S + \frac{11}{2}N_{\mathrm{f}} + 62N_V\right), \qquad c = \frac{1}{120}(N_S + 6N_{\mathrm{f}} + 12N_V), \tag{A1}$$

where N_S is the number of scalar particles, N_{f} is the number of Weyl fermions, and N_V is the number of vector bosons. The a theorem indicates that

$$a_{\mathrm{IR}} \le a_{\mathrm{UV}}. \tag{A2}$$

On the other hand, the free energy is

$$f = N_B + \frac{7}{4}N_{\mathrm{f}}, \tag{A3}$$

where N_{f} is the number of the Weyl fermions and N_B is the number of bosons. The ACS criterion is that [7,8]

$$f_{\mathrm{IR}} \le f_{\mathrm{UV}}. \tag{A4}$$

For simplicity, we shall use $\tilde{a} = 360a$. For the (N_ψ, N_χ) model,

$$\tilde{a}_{\mathrm{UV}} = 62(N^2-1) + \frac{33}{4}N_\psi N(3+N) - \frac{11}{4}N_\chi N(N-7), \tag{A5}$$

$$\tilde{a}_{\mathrm{IR}} = N_S + \frac{11}{2}N_{\mathrm{f}} + 62N_V, \tag{A6}$$

where N_V, N_S, and N_{f} are the number of vector bosons, scalars, and Weyl fermions in the infrared. For the ACS free energy,

$$f_{\mathrm{UV}} = 2(N^2-1) + \frac{7N_\psi}{8}(N^2+3N+8) + \frac{7N_\chi}{8}(N^2-3N+8), \tag{A7}$$

$$f_{\mathrm{IR}} = N_B + \frac{7}{4}N_{\mathrm{f}}. \tag{A8}$$

We put those two criteria to the test in Tables XVII and XVIII for the theories and their possible IR phases discussed in the paper. In all cases the a theorem is satisfied, whereas the ACS criterion fails only for the (3,0) and (0,3) models.

TABLE XVII. The a theorem.

Model	$\tilde{a}_{\rm UV}$	$\tilde{a}_{\rm IR}$	Status
(1,1) CFL ($N \geq 8$)	$\frac{135N^2}{2} + 44N - 62$	$106N - 538$	✓
(1,1) CFL ($N \leq 8$)	$\frac{135N^2}{2} + 44N - 62$	$-2N^2 + \frac{109N}{2} + 2$	✓
(1,1) Abelianiz.	$\frac{135N^2}{2} + 44N - 62$	$\frac{223N}{2} - 61$	✓
(1,0) No XSB	$\frac{281N^2+99N-248}{4}$	$\frac{11N^2+77N+132}{4}$	✓
(1,0)	$\frac{281N^2+99N-248}{4}$	$\frac{11N^2+109N+4}{4}$	✓
(2,0) (symm)	$\frac{1}{2}(157N^2 + 99N - 124)$	$\frac{1}{2}(17N^2 + 141N + 2)$	✓
(2,0)	$\frac{1}{2}(157N^2 + 99N - 124)$	$\frac{1}{2}(17N^2 + 141N + 8)$	✓
(3,0)	$\frac{347N^2}{4} + \frac{297N}{4} - 62$	$\frac{65N^2}{4} + \frac{519N}{4} + 1$	✓
(0,1)	$\frac{281N^2}{4} - \frac{99}{4} - 62$	$\frac{11}{4}(N-3)(N-4)$	✓
(0,2)	$\frac{1}{2}(157N^2 - 99N - 124)$	$\frac{1}{2}(17N^2 - 125N + 228)$	✓
(0,3)	$\frac{1}{4}(347N^2 - 297N - 248)$	$\frac{1}{4}(65N^2 - 487N + 876)$	✓
(2,1)	$\frac{1}{4}(303N^2 + 275N - 248)$	$\frac{1}{4}(33N^2 + 297N + 16)$	✓
$(1,-1)$	$\frac{157N^2}{2} - 62$	$\frac{15N^2}{2} + 2$	✓

TABLE XVIII. The ACS Criterion.

Model	$f_{\rm UV}$	$f_{\rm IR}$	Status
(1,1) CFL ($N \geq 8$)	$\frac{15N^2}{4} + 14N - 2$	$4(4N - 7)$	✓
(1,1) CFL ($N \leq 8$)	$\frac{15N^2}{4} + 14N - 2$	$2 + \frac{113N}{4} - 2N^2$	✓
(1,1) Abelianiz.	$\frac{15N^2}{4} + 14N - 2$	$\frac{71N}{4} - 1$	✓
(1,0) No XSB	$\frac{1}{8}(37N^2 + 63N - 16)$	$\frac{7}{8}(N^2 + 7N + 12)$	✓
(1,0)	$\frac{1}{8}(37N^2 + 63N - 16)$	$\frac{1}{8}(7N^2 + 113N + 8)$	✓
(2,0) symm	$\frac{1}{4}(29N^2 + 63N - 8)$	$\frac{1}{4}(19N^2 + 177N + 4)$	✓
(2,0)	$\frac{1}{4}(29N^2 + 63N - 8)$	$\frac{1}{4}(19N^2 + 177N + 16)$	(✓)
(3,0)	$\frac{79N^2}{8} + \frac{189N}{8} - 2$	$\frac{85N^2}{8} + \frac{723N}{8} + 8$	X
(0,1)	$\frac{37N^2}{8} - \frac{63N}{8} - 2$	$\frac{7}{8}(N-3)(N-4)$	✓
(0,2)	$\frac{1}{4}(29N^2 - 63N - 8)$	$\frac{1}{8}(19N^2 - 145N - 276)$	(✓)
(0,3)	$\frac{1}{8}(79N^2 - 189N - 16)$	$\frac{1}{8}(85N^2 - 659N + 1212)$	X
(2,1)	$\frac{1}{8}(51N^2 + 175N - 16)$	$\frac{1}{8}(21N^2 + 189N + 32)$	✓
$(1,-1)$	$\frac{29N^2}{4} - 2$	$\frac{15N^2}{4} + 2$	✓

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