

Semiclassical energy of closed Nambu-Goto strings

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(Received 5 December 2018; published 14 November 2019)

We compute semiclassical corrections to the energy of rotating closed Nambu-Goto strings. We confirm the results obtained by means of the Polchinski-Strominger action. We also show that in this semiclassical approximation, the spectrum of physical excitations contains modes that are unphysical nonperturbatively, i.e., to which no physical excitations of the covariantly quantized Nambu-Goto string correspond.

DOI: [10.1103/PhysRevD.100.106005](https://doi.org/10.1103/PhysRevD.100.106005)

I. INTRODUCTION

In the covariant quantization scheme, the Regge intercept a of the Nambu-Goto string is a free parameter, only constrained by $a \leq 1$ for $D \leq 25$ and $a = 1$ for $D = 26$, with D the dimension of the target Minkowski space [1,2]. As the Nambu-Goto string is a toy model for vortex lines in QCD, cf. [3–5] for recent works and reviews in this direction, a determination of a for noncritical dimensions is desirable. Using the Polchinski-Strominger action [6], Hellerman and Swanson [7] computed the Regge intercepts

$$a_{\text{open}} = 1, \\ a_{\text{closed}} = \frac{D-2}{24} - \frac{26-D}{48} \left(\left(\frac{J_{1,2}}{J_{3,4}} \right)^{\frac{1}{4}} - \left(\frac{J_{3,4}}{J_{1,2}} \right)^{\frac{1}{4}} \right)^2 \quad (1)$$

for open and closed Nambu-Goto strings. Here $J_{1,2}$ and $J_{3,4}$ are angular momenta in the 1–2 and 3–4 planes in which the closed string rotates. Regarding this result, two aspects are puzzling:

- (i) For the covariantly quantized string, a is a constant which is independent of the ratio of the angular momentum components, in contrast to (1) for $D \neq 26$.
- (ii) In the limit $J_{3,4} \rightarrow 0$, the closed string degenerates to two straight open strings with the same endpoints. One would thus expect $a_{\text{open}} = a_{\text{closed}}$,

which, however, only holds for $D = 26$. Even worse, for $D < 26$, a_{closed} diverges in the limit $J_{3,4} \rightarrow 0$.

Due to this seeming inconsistency in noncritical dimension, it seems desirable to verify (1) with other methods. The result for the open string, including a generalization to masses at the endpoints, was recently rederived in [8] using a semiclassical approximation, i.e., by quantizing the perturbations around classical rotating string solutions, and using a local renormalization procedure to extract quantum corrections to the energy. Here, we perform the analogous analysis for closed strings, also verifying (1).

The semiclassical quantization procedure in principle also allows us to compute the spectrum of excitations of the classical rotating solutions. In [8] it was shown that for the open string (at least for the low-lying excitations) the semiclassical approach leads to the same spectrum of physical excitations as the covariant quantization scheme. Here, we show that for the closed string the spectrum of excitations in the semi-classical theory is too large, i.e., there are semi-classical excitations that do not correspond to physical excitations of the full nonperturbative covariantly quantized theory. That such phenomena are possible in gauge theories is well known: For example, already at the classical level, Yang-Mills theory is not linearization stable [9], i.e., solutions to the linearized equations are not necessarily tangent to the manifold of solutions of the full equations, i.e., do not derive from a curve in the set of solutions of the full equations. More generally, denoting by s_0 the linearized part of the Becchi, Rouet, Stora, Tyutin (BRST) operator s , the cohomology $H(s_0)$ describing observables in the linearized theory is in general larger than the cohomology $H(s)$ of the full theory, an elementary example being again provided by Yang-Mills theory, cf. [10] for example. In the present case, the reason seems to be that the level-matching condition is not implemented at the linearized level.

Our perturbative approach is based on the finding [11] that in the sense of perturbation theory around nondegenerate

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classical solutions, the Nambu-Goto string can be quantized for any D , without anomalies. Let us sketch the argument: Let $X: \Sigma \rightarrow (\mathbb{R}^D, \eta)$ with $\eta = \text{diag}(- + \dots +)$ be the embedding of the world sheet Σ into the target Minkowski space and split X into a classical solution \bar{X} and a perturbation φ , i.e.,

$$X = \bar{X} + \gamma^{-\frac{1}{2}}\varphi, \quad (2)$$

where $\gamma^{-\frac{1}{2}}$ is seen as a formal expansion parameter. The Nambu-Goto action is

$$\mathcal{S} = -\gamma \int_{\Sigma} \sqrt{g} d^2x \quad (3)$$

where g is the determinant of the metric induced by the embedding. The full action is invariant under reparametrizations of embedding, giving rise to the BRST transformation of φ given by

$$s\varphi^a = c^\mu \partial_\mu X^a = c^\mu \partial_\mu \bar{X}^a + \gamma^{-\frac{1}{2}} c^\mu \partial_\mu \varphi^a, \quad (4)$$

with c^μ the diffeomorphism (re-parametrization) ghost vector field. Here roman (greek) indices are target space (world sheet) indices. Given a gauge condition $T_\mu[\varphi] = 0$, one supplements the action (3) by the gauge-fixing action

$$\mathcal{S}_{\text{gf}} = \int_{\Sigma} (b^\mu T_\mu[\varphi] - \bar{c}^\mu s T_\mu[\varphi]) \sqrt{\bar{g}} d^2x, \quad (5)$$

with \bar{g} the determinant of the metric $\bar{g}_{\mu\nu}$ induced by \bar{X} . This action is invariant under the BRST transformation defined by (4) and

$$s c^\mu = \gamma^{-\frac{1}{2}} c^\nu \bar{\nabla}_\nu c^\mu, \quad s \bar{c}^\mu = b^\mu, \quad s b^\mu = 0, \quad (6)$$

where $\bar{\nabla}$ is the Levi-Civita covariant derivative with respect to $\bar{g}_{\mu\nu}$.

The possible occurrence of anomalies, such as a violation of the nilpotency of the BRST charge, is governed by the cohomologies of s and $s \bmod d$. The free, i.e., $\mathcal{O}(\gamma^0)$, part of s is

$$s_0 \varphi^a = c^\mu \partial_\mu \bar{X}^a. \quad (7)$$

For a nondegenerate classical embedding \bar{X} , $\partial_\mu \bar{X}^a$ has maximal rank. It follows that the fluctuations φ^a tangential to the embedding \bar{X} form trivial pairs with the ghosts c^μ and drop out of the cohomology of s_0 . Hence, s_0 has trivial cohomology at positive ghost number. By cohomological perturbation theory [12], this carries over to the cohomology of s , and also to the cohomology of $s \bmod d$. Hence, there is a renormalization scheme in which no anomalies are present, in any order of perturbation theory.

In the following, we will truncate the action at $\mathcal{O}(\gamma^0)$, i.e., at second order in the dynamical fields φ^a , c_μ , \bar{c}_μ , b_μ . This free action is the one relevant for the semiclassical approximation. Furthermore, we use transversal gauge [13], where the absence of anomalies at the level of the free theory is manifest. For this, one chooses

$$T_\mu[\varphi] = \partial_\mu \bar{X}^a \varphi_a. \quad (8)$$

By the equations of motion derived from the free action, the unphysical fields, i.e., the auxiliary fields c_μ , \bar{c}_μ , b_μ and the pure gauge perturbations, i.e., those parallel to $\partial_\mu \bar{X}^a$, then vanish,¹ so they can be consistently set to zero at the level of the free theory.² In particular, the corresponding free BRST charge vanishes³ (and is thus also nilpotent). Hence, expanding the action to $\mathcal{O}(\gamma^0)$ and restricting to fluctuations φ^a normal to the classical embedding \bar{X} (these vanish under the action of s_0), one arrives at a free theory involving only physical fields.

The perturbation φ is naturally interpreted as a field living on the world sheet, so that quantization yields a free quantum theory on the world sheet. The equations of motion for φ only depend on the world sheet geometric data, i.e., the metric and the second fundamental form induced by the classical embedding \bar{X} . Hence, it seems natural, in line with the framework of [11], to use methods from quantum field theory on curved space-time [14,15] for the renormalization of the free world sheet Hamiltonian H^0 . This means that renormalization is performed locally, using the local geometric data. The correspondence between the world sheet Hamiltonian and the target space energy then gives corrections to the classical Regge trajectories.

Let us analyze the latter point in more detail. We recall that in the covariant quantization the ground state energies E_J for angular momentum $J_{1,2}, J_{3,4} \geq 0$ satisfy

$$E^2 = 4\pi\gamma(J_{1,2} + J_{3,4} - 2a), \quad (9)$$

with γ the string tension, a relation which we call the Regge trajectory in the following (no extension of the angular momenta to the complex plane is intended). It is important

¹At second order in the perturbation, the fluctuations parallel to the classical embedding \bar{X} drop out of the (not yet gauge fixed) Nambu-Goto action [11]. From this, one easily derives that in transversal gauge the unphysical fields vanish on-shell at the level of the free theory.

²The possibility of such a trivial gauge fixing (and also the triviality of the cohomology of s) is due to the absence of derivatives of the ghost in the BRST transformation (4). This feature in particular distinguishes the model from Yang-Mills theories and general relativity.

³As a consequence of the absence of derivatives of dynamical fields in the gauge fixing condition (8), the free BRST current, i.e., the Noether current derived from s_0 and the free part of the action, vanishes. (The Nambu-Goto action \mathcal{S} does not contribute to the free BRST current [11].)

to note that this ground state is degenerate in the sense that ground states for fixed $J = J_{1,2} + J_{3,4}$ all have the same energy E_J . We will be parameterizing the rotating string in terms of the two radii $R_{1,2}, R_{3,4}$ in the two planes. In terms of these, the classical target space energy and angular momentum are given by

$$\bar{E} = 2\pi\gamma R, \quad \bar{J}_{1,2} = \pi\gamma R_{1,2}^2, \quad \bar{J}_{3,4} = \pi\gamma R_{3,4}^2, \quad (10)$$

with

$$R = \sqrt{R_{1,2}^2 + R_{3,4}^2}. \quad (11)$$

Here the bar stands for the values corresponding to the classical (background) configuration \bar{X} , cf. (2). In our parametrization, the relation between the world sheet Hamiltonian H , the quantum correction E^q to the target space energy E , and the quantum corrections $J_{1,2}^q, J_{3,4}^q$ to the angular momenta $J_{1,2}, J_{3,4}$ is

$$E^q = \frac{1}{R}(H + J_{1,2}^q + J_{3,4}^q), \quad (12)$$

yielding

$$\begin{aligned} E^2 &= (\bar{E} + E^q)^2 \\ &= 4\pi^2\gamma^2 R^2 + 4\pi\gamma(H + J_{1,2}^q + J_{3,4}^q) + \mathcal{O}(R^{-2}) \\ &= 4\pi\gamma(J_{1,2} + J_{3,4} + H^0) + \mathcal{O}(R^{-1}). \end{aligned} \quad (13)$$

By comparison with (9), one can directly read off the intercept a from the expectation value of H^0 , i.e.,

$$a = -\frac{1}{2}\langle H^0 \rangle. \quad (14)$$

As the case of elliptic strings is notationally and computationally much more involved, we chose to first present our calculation for the case of circular strings, i.e., for $R_{1,2} = R_{3,4}$, and to discuss the modifications necessary for the treatment of elliptic strings in a separate section. The article is thus structured as follows: In the next section, we discuss the perturbations of classical rotating circular solutions, and in Sec. III the calculation of the corresponding intercept. Section IV deals with the generalization to the elliptic case. These calculations are for simplicity performed for $D = 6$. The straightforward generalization to general D is discussed in Sec. V. Finally, in Sec. VI, we compare the spectrum of physical excitations in the linearized semi-classical theory and the covariant quantization scheme. We conclude in Sec. VII. An Appendix contains some intermediate results of our treatment of the elliptic case.

II. PERTURBATIONS OF CLASSICAL ROTATING CIRCULAR STRINGS

For an embedding $X: \Sigma \rightarrow (\mathbb{R}^D, \eta)$ the Nambu-Goto action (3) yields the equations of motion

$$\square_g X = 0. \quad (15)$$

We also recall the target space momentum and angular momentum derived from the action (3):

$$P^i = \int_0^{2\pi} \frac{\delta S}{\delta \partial_0 X_i} d\sigma = -\gamma \int_0^{2\pi} \sqrt{g} g^{0\nu} \partial_\nu X^i d\sigma, \quad (16a)$$

$$\begin{aligned} L_{ij} &= \int_0^{2\pi} \left[\frac{\delta S}{\delta \partial_0 X^j} X_i - i \leftrightarrow j \right] d\sigma \\ &= \gamma \int_0^{2\pi} \sqrt{g} g^{0\nu} (X_j \partial_\nu X_i - X_i \partial_\nu X_j) d\sigma. \end{aligned} \quad (16b)$$

Here we assumed Σ to be parametrized by $(\tau, \sigma) \in \mathbb{R} \times [0, 2\pi)$. The target space energy is given by $E = P^0$.

We parametrize the rotating circular solution as

$$\bar{X}(\tau, \sigma) = \frac{R}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\tau \\ \sin \tau \cos \sigma \\ -\cos \tau \cos \sigma \\ \cos \tau \sin \sigma \\ \sin \tau \sin \sigma \\ 0 \end{pmatrix}, \quad (17)$$

where $\sigma \in [0, 2\pi)$ and R is given by (11). Note that in our parametrization, the world sheet coordinates τ, σ are dimensionless. For simplicity, we here assumed that the target space-time is six dimensional. As discussed in Sec. V, it is straightforward to add further dimensions (or to remove the sixth). The induced metric on the world sheet, in the coordinates introduced above, is a multiple of the Minkowski metric,

$$\bar{g}_{\mu\nu} = \frac{R^2}{2} \eta_{\mu\nu}. \quad (18)$$

In particular, the equation of motion (15) can be easily checked. Energy and angular momentum of the above solution were given in (10), with $R_{1,2} = R_{3,4} = \frac{R}{\sqrt{2}}$.

Our goal is to perform a (canonical) quantization of the fluctuations φ around the classical background \bar{X} , cf. (2). As discussed in the introduction, we may, at the level of the free theory, restrict to normal fluctuations. We parametrize these as

$$\varphi = f_s v_s + f_a v_a + f_b v_b + f_c v_c, \quad (19)$$

with v_s, v_a, v_b, v_c , orthonormal and normal to the world sheet. Concretely, we choose

$$v_s = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_a = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \cos \tau \cos \sigma \\ \sin \tau \cos \sigma \\ -\sin \tau \sin \sigma \\ \cos \tau \sin \sigma \\ 0 \end{pmatrix},$$

$$v_b = \begin{pmatrix} 0 \\ \sin \tau \cos \sigma \\ -\cos \tau \cos \sigma \\ \cos \tau \sin \sigma \\ \sin \tau \sin \sigma \\ 0 \end{pmatrix}, \quad v_c = \begin{pmatrix} 0 \\ \cos \tau \sin \sigma \\ \sin \tau \sin \sigma \\ \sin \tau \cos \sigma \\ -\cos \tau \cos \sigma \\ 0 \end{pmatrix}. \quad (20)$$

Here the *scalar* component f_s describes the fluctuations in the direction perpendicular to the planes of rotation. The mode f_a describes infinitesimal rotations generated by $L_{1,2} + L_{3,4}$ (v_a also has a time component, which is present to make it orthogonal to the world sheet). The mode f_b describes dilations in the subspace spanned by e_1, e_2, e_3, e_4 and f_c describes combined shears in the planes spanned by e_1, e_3 and e_2, e_4 . Subsuming the modes in the planes of rotation into $f = (f_a, f_b, f_c)$, we obtain, at $\mathcal{O}(\gamma^0)$, the free action

$$S^0 = \frac{1}{2} \int (\dot{j}^2 - f'^2 - f^T A \dot{f} - f^T B f' - f^T C f + \dot{f}_s^2 - f_s'^2) d\sigma d\tau, \quad (21)$$

where f^T is the transpose of f , derivatives with respect to τ are denoted by dots and those with respect to σ by primes and

$$A = \begin{pmatrix} 0 & \sqrt{8} & 0 \\ -\sqrt{8} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & -\sqrt{8} \\ 0 & 0 & 0 \\ \sqrt{8} & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \quad (22)$$

We see that the scalar field decouples.

From (21), we see that the scalar fluctuations decouple from the others, grouped into f , termed the *planar sector* in the following. We obtain the equations of motion

$$0 = -\ddot{f}_s + f_s'', \quad (23a)$$

$$0 = -\ddot{f} + f'' - A\dot{f} - Bf' - Cf. \quad (23b)$$

It is easily checked that the symplectic forms for the scalar and the planar sector are

$$\sigma(f_s, \tilde{f}_s) = \int_{\Sigma_\tau} (f_s \dot{\tilde{f}}_s - \dot{f}_s \tilde{f}_s) d\sigma,$$

$$\sigma(f, \tilde{f}) = \int_{\Sigma_\tau} \begin{pmatrix} f & \dot{f} \end{pmatrix} \begin{pmatrix} A & \mathbb{1}_3 \\ -\mathbb{1}_3 & 0_3 \end{pmatrix} \begin{pmatrix} \tilde{f} \\ \dot{\tilde{f}} \end{pmatrix} d\sigma \quad (24)$$

where Σ_τ is the time slice at an arbitrary time τ .

In order to prepare for a canonical quantization, we identify symplectically normalized mode solutions.⁴ For the scalar component, one finds the solutions

$$f_{s,n}^\pm = \frac{1}{\sqrt{4\pi n}} e^{\pm i n \sigma} e^{-i n \tau} \quad (25)$$

for $n \geq 1$. These are normalized with respect to their symplectic form, i.e.,

$$\sigma(\overline{f_{s,n}^\alpha}, f_{s,m}^\beta) = -i \delta_{mn} \delta^{\alpha\beta}, \quad (26)$$

where the bar stands for complex conjugation and $\alpha, \beta = \pm$. There is also a zero mode f_q which degenerates in the symplectic form and can not be normalized. It forms a canonical pair with a supplementary linearly growing mode f_p , i.e.,

$$f_q = \frac{1}{\sqrt{2\pi}}, \quad f_p = \frac{1}{\sqrt{2\pi}} \tau, \quad (27)$$

with $\sigma(f_q, f_p) = 1$. These correspond to position and momentum in the direction perpendicular to the planes of rotation, and can be seen as the Nambu-Goldstone modes for the broken invariance under translations perpendicular to the planes of rotation. In the determination of the Regge intercept, these will be discarded, as we have to fix the momentum.

For the planar sector, we find the positive energy solutions

$$f_{1,n}^\pm = \frac{1}{\sqrt{8\pi n(n^2 - 1)}} (\pm i\sqrt{2}, \mp n, n) e^{\pm i n \sigma} e^{-i n \tau}, \quad (28a)$$

⁴For an exposition of canonical quantization in the presence of single time derivatives in the action, we refer to the Appendix of [16]. Our results, in particular regarding the behavior of modes that degenerate in the symplectic form, are in complete agreement with the general properties derived there.

$$f_{2,n}^{\pm} = \frac{1}{\sqrt{16\pi(n-1)}} (\mp i\sqrt{2}, \pm 1, 1) e^{\pm i(n-2)\sigma} e^{-in\tau}, \quad (28b)$$

$$f_{3,n}^{\pm} = \frac{1}{\sqrt{16\pi(n+1)}} (\pm i\sqrt{2}, \pm 1, 1) e^{\pm i(n+2)\sigma} e^{-in\tau}. \quad (28c)$$

These are again normalized with respect to the symplectic form, analogously to (26), with a few exceptions, which we now discuss in detail.

For $n = 0$, the modes f_1^{\pm} degenerate to a single mode, which is degenerate in the symplectic form. There is another linearly growing solution complementing it, so that

$$f_{\theta} = \frac{1}{\sqrt{2\pi}} (1, 0, 0), \quad (29a)$$

$$f_{\lambda} = \frac{1}{\sqrt{2\pi}} \left[-\tau(1, 0, 0) + \frac{1}{\sqrt{2}}(0, 1, 0) \right], \quad (29b)$$

form a canonical pair,

$$\sigma(f_{\theta}, f_{\lambda}) = 1. \quad (30)$$

Recalling that f_a could be identified as an infinitesimal rotation generated by $L_{1,2} + L_{3,4}$, we can see these as Goldstone modes for the broken $L_{1,2} + L_{3,4}$ invariance. We should think of f_{θ} as an angle and f_{λ} as the corresponding angular momentum variable.⁵ This is further supported below. Furthermore, as shown in the next section, these modes are not relevant for the determination of the Regge intercept.

The modes f_2 and f_3 do not degenerate in the symplectic form for $n = 0$, so they are not conventional null modes. However, $f_{2,0}^{\pm}$ does not fulfill the normalization (26) (the sign is changed). It should thus be interpreted as a negative energy mode. Note that $\overline{f_{2,0}^{\pm}} = if_{3,0}^{\pm}$, so $f_{3,0}^{\pm}$ and $if_{2,0}^{\pm}$ are symplectic pairs of a positive and a negative energy solution. As $L_{1,3} + L_{2,4}$ and $L_{1,2} - L_{3,4}$ both commute with $L_{1,2} + L_{3,4}$, it is suggestive to regard the modes $f_{3,0}^{\pm}$ as the generators of these rotations.⁶ This is explicitly checked below. Also note that ‘‘positive energy’’ might be a misnomer, as one can easily check that these modes do not contribute to the free Hamiltonian, see below. It follows that these modes lead to a degeneracy of the ground state. This reflects the classical ground state degeneracy discussed in the Introduction.

⁵However, we draw attention to the sign of the term linear in τ in (29b) compared to (29a), which was introduced in order to have usual sign in (30). The same phenomenon occurs for the open string, and in fact also for an analogous semiclassical treatment of the hydrogen atom [8].

⁶Also $L_{1,4} - L_{2,3}$ commutes with $L_{1,2} + L_{3,4}$, but it generates reparametrizations.

For $n = 1$, the modes f_1 and f_2 degenerate in the symplectic form and can thus not be normalized. Omitting the normalization, $f_{1,1}^{\pm}$ and $f_{2,1}^{\pm}$ are solutions and $f_{1,1}^{\mp}$ and $f_{2,1}^{\pm}$ coincide. It turns out that there are linearly growing solutions complementing these. We thus have

$$f_q^{\pm} = \frac{1}{\sqrt{8\pi}} (\pm i\sqrt{2}, \mp 1, 1) e^{\pm i\sigma} e^{-i\tau}, \quad (31a)$$

$$f_p^{\pm} = \frac{1}{\sqrt{8\pi}} [\tau(\pm i\sqrt{2}, \mp 1, 1) + \frac{i}{8}(\mp i3\sqrt{2}, \mp 1, 1)] e^{\pm i\sigma} e^{-i\tau}. \quad (31b)$$

Together with their complex conjugates, these form canonical pairs,

$$\sigma(\overline{f_q^{\pm}}, f_p^{\pm}) = 1, \quad \sigma(f_q^{\pm}, \overline{f_p^{\pm}}) = 1, \quad (32)$$

with all other combinations vanishing. Hence, $f_q^{\pm}, \overline{f_q^{\pm}}$ can be interpreted as positions and $\overline{f_p^{\pm}}, f_p^{\pm}$ as the corresponding momenta. We thus have four canonical pairs of position and momenta. These have the natural interpretation as center-of-mass positions and momenta in the sub-space spanned by e_1, e_2, e_3, e_4 , i.e., as the Nambu-Goldstone modes for the broken invariance under translations in this subspace. Similarly to the scalar null modes f_q, f_p , these will be discarded in the determination of the Regge intercept.

An analogous identification of these special modes for the open string was made in [8,17]. It can also be checked explicitly: Writing

$$f = \sum_{\alpha \in \pm} \left[\sum_{r \in \{1,2,3\}} \sum_{n \in \mathbb{N}_r} (a_{r,n}^{\alpha} f_{r,n}^{\alpha}) + (q^{\alpha} f_q^{\alpha} + p^{\alpha} f_p^{\alpha}) + \text{H.c.} \right] + \theta f_{\theta} + \lambda f_{\lambda}, \quad (33a)$$

$$f_s = \sum_{n \in \mathbb{N}_s} \sum_{\alpha \in \pm} (a_{s,n}^{\alpha} f_{s,n}^{\alpha} + \text{H.c.}) + q f_q + p f_p, \quad (33b)$$

where

$$\mathbb{N}_s = \{n \geq 1\}, \quad \mathbb{N}_1 = \mathbb{N}_2 = \{n \geq 2\}, \quad \mathbb{N}_3 = \{n \geq 0\}, \quad (34)$$

and the coefficients q, p, θ, λ are real, one finds, for the expansion of the energy, angular momenta, and momenta, cf. (16),

$$E = \bar{E} + \sqrt{2\pi\gamma}\lambda + \mathcal{O}(\gamma^0), \quad (35a)$$

$$L_{1,2} + L_{3,4} = \bar{L}_{1,2} + \bar{L}_{3,4} + \sqrt{2\pi\gamma}R\lambda + \mathcal{O}(\gamma^0), \quad (35b)$$

$$L_{1,2} - L_{3,4} = \sqrt{2\pi\gamma}R\Re(a_{3,0}^{+} - a_{3,0}^{-}) + \mathcal{O}(\gamma^0), \quad (35c)$$

$$L_{1,3} + L_{2,4} = \sqrt{2\pi\gamma} R \Im(a_{3,0}^+ + a_{3,0}^-) + \mathcal{O}(\gamma^0), \quad (35d)$$

$$P^{1/3} = \sqrt{2\pi\gamma} \Im(\mp p^+ + p^-) + \mathcal{O}(\gamma^0), \quad (35e)$$

$$P^{2/4} = \sqrt{2\pi\gamma} \Re(\pm p^+ - p^-) + \mathcal{O}(\gamma^0), \quad (35f)$$

$$P^5 = \sqrt{2\pi\gamma} p + \mathcal{O}(\gamma^0). \quad (35g)$$

This supports the identification of the modes $f_p, f_\lambda, f_p^\pm, f_{3,0}^\pm$ with (angular) momenta discussed above.

Canonical quantization now proceeds as follows: One introduces annihilation and creation operators $\hat{a}_{r,n}^\pm, \hat{a}_{r,n}^{\pm*}$ for $r \in \{s, 1, 2, 3\}, n \in \mathbb{N}_r$, fulfilling

$$[\hat{a}_{r,n}^\alpha, \hat{a}_{r',n'}^{\beta*}] = \delta^{\alpha\beta} \delta_{rr'} \delta_{nn'}. \quad (36)$$

Furthermore, one introduces position operators $\hat{q}, \hat{\theta}, \hat{q}^\pm, \hat{q}^{\pm*}$ and momenta $\hat{p}, \hat{\lambda}, \hat{p}^\pm, \hat{p}^{\pm*}$ with commutation relations

$$\begin{aligned} [\hat{q}, \hat{p}] &= i, & [\hat{\theta}, \hat{\lambda}] &= i, \\ [\hat{q}^{\alpha*}, \hat{p}^\beta] &= i\delta^{\alpha\beta}, & [\hat{q}^\alpha, \hat{p}^\beta] &= 0. \end{aligned} \quad (37)$$

The complex positions \hat{q}^\pm and momenta \hat{p}^\pm can be represented on $L^2(\mathbb{R}^2)$ with canonical position and momentum operators $\hat{q}_i^\pm, \hat{p}_i^\pm$ as

$$\hat{q}^\pm = \frac{1}{\sqrt{2}}(\hat{q}_1^\pm + i\hat{q}_2^\pm), \quad \hat{p}^\pm = \frac{1}{\sqrt{2}}(\hat{p}_1^\pm + i\hat{p}_2^\pm). \quad (38)$$

In particular, with (35e), (35f), this implies

$$P^{1/3} = \sqrt{\pi\gamma}(\mp \hat{p}_2^+ + \hat{p}_2^-) + \mathcal{O}(\gamma^0), \quad (39a)$$

$$P^{2/4} = \sqrt{\pi\gamma}(\pm \hat{p}_1^+ - \hat{p}_1^-) + \mathcal{O}(\gamma^0). \quad (39b)$$

The canonically conjugate positions are thus

$$X^{1/3} = (2\sqrt{\pi\gamma})^{-1}(\mp \hat{q}_2^+ + \hat{q}_2^-) + \mathcal{O}(\gamma^{-1}), \quad (40a)$$

$$X^{2/4} = (2\sqrt{\pi\gamma})^{-1}(\pm \hat{q}_1^+ - \hat{q}_1^-) + \mathcal{O}(\gamma^{-1}). \quad (40b)$$

That these are the correct center of mass (cms) positions can indeed be explicitly checked. For example, up to a reparametrization, the combination $\sqrt{2\pi\gamma} \Im(f_q^+ - f_q^-)$ parametrizes a shift by one unit in direction e_1 .

Canonical quantization then proceeds by replacing the coefficients in (33) by the hatted corresponding operators. One can then explicitly check the fulfilment of the canonical equal time commutation relations, as dictated by the symplectic form (24). Omitting the positions and

momenta (this will be justified below) we thus have quantum fields \hat{f}_s, \hat{f} with two-point functions

$$\begin{aligned} w_s(x; x') &:= \langle \Omega | \hat{f}_s(x) \hat{f}_s(x') | \Omega \rangle \\ &= \sum_{n \in \mathbb{N}_s} \sum_{\alpha \in \pm} f_{s,n}^\alpha(x) \overline{f_{s,n}^\alpha(x')}, \end{aligned} \quad (41a)$$

$$\begin{aligned} w_{p,ij}(x; x') &:= \langle \Omega | \hat{f}_i(x) \hat{f}_j(x') | \Omega \rangle \\ &= \sum_{k=1}^3 \sum_{n \in \mathbb{N}_k} \sum_{\alpha \in \pm} f_{k,n,i}^\alpha(x) \overline{f_{k,n,j}^\alpha(x')}, \end{aligned} \quad (41b)$$

where in the second equation i, j run over the labels (a, b, c) .

III. THE WORLD SHEET HAMILTONIAN

The free Hamiltonian corresponding to the free action (21) is

$$H^0 = \frac{1}{2} \int_0^{2\pi} (\dot{f}^2 + f'^2 + f^T C f + f^T B f' + \dot{f}_s^2 + f_s'^2) d\sigma. \quad (42)$$

In terms of the creation and annihilation and position and momentum operators introduced in the previous section, this Hamiltonian formally reads as

$$\begin{aligned} H^0 &= \frac{1}{2} \left(\sum_{r \in \{s, 1, 2, 3\}} \sum_{n \in \mathbb{N}_r} \sum_{\alpha \in \pm} n (\hat{a}_{r,n}^\alpha \hat{a}_{r,n}^{\alpha*} + \hat{a}_{r,n}^{\alpha*} \hat{a}_{r,n}^\alpha) + \hat{p}^2 - \hat{\lambda}^2 \right) \\ &\quad + \sum_{\alpha \in \pm} (\hat{p}^{\alpha*} \hat{p}^\alpha + i \hat{p}^\alpha \hat{q}^{\alpha*} - i \hat{p}^{\alpha*} \hat{q}^\alpha). \end{aligned} \quad (43)$$

Rewriting the last term in (43) in terms of the position and momentum operators introduced in (38), one obtains

$$\begin{aligned} &\sum_{\alpha \in \pm} (\hat{p}^{\alpha*} \hat{p}^\alpha + i \hat{p}^\alpha \hat{q}^{\alpha*} - i \hat{p}^{\alpha*} \hat{q}^\alpha) \\ &= \sum_{\alpha \in \pm} \left[\frac{1}{2} ((\hat{p}_1^\alpha)^2 + (\hat{p}_2^\alpha)^2) + \hat{p}_1^\alpha \hat{q}_2^\alpha - \hat{p}_2^\alpha \hat{q}_1^\alpha \right]. \end{aligned} \quad (44)$$

By (39), (40), the last two terms are precisely the leading order term of the cms contribution to the angular momentum $-L_{1,2} - L_{3,4}$. As a cms contribution to the angular momentum does not change the energy (for a given momentum), this term must be present in H^0 , due to (12). For the determination of the Regge trajectory, such a cms contribution should of course be absent. Hence, we choose a state in which the expectation value of the last two terms in (43) vanishes.

To evaluate the rest mass, we should work at vanishing spatial momentum. Obviously, we may choose a state in

which the expectation values of $(\hat{p}_1^\alpha)^2 + (\hat{p}_2^\alpha)^2$ and \hat{p}^2 are arbitrarily small. We can hence neglect the momenta in (43).⁷

We now argue that also the $\hat{\lambda}^2$ term in (43) should be neglected. First, we recall that in (12), the quantum corrections $E^q, L_{1,2}^q, L_{3,4}^q$ are obtained by expanding the classical expressions (16a) for the target space energy $E = P^0$ and (16b) for the angular momentum in φ , neglecting the zeroth order (classical) terms. The world sheet Hamiltonian H generates (world sheet) time translations on the world sheet. The quantum target space energy E^q , should generate time translations in target space. However, the background world sheet is rotating in the 1-2 and the 3-4 plane and so are the vectors v_a, v_b, v_c parameterizing the fluctuations. The correct relation between H and E^q is thus (12), where the factor $\frac{1}{R}$ corrects the different scaling of target space time X^0 and world sheet time τ , and $L_{1,2}^q + L_{3,4}^q$ corrects for the missing rotation. One can easily check (12) explicitly up to $\mathcal{O}(\varphi^2)$, which is the relevant order for our purposes. A further relation between the quantum target space energy E^q and the world sheet Hamiltonian H is

$$E^q = \frac{1}{R}H \quad \text{mod} \quad \frac{1}{R}, \quad (45)$$

which follows from the fact that a world sheet translation $\tau \mapsto \tau + 2\pi$ corresponds to a target space translation $X^0 \mapsto X^0 + 2\pi R$. Comparison with (12) shows that the spectrum of $L_{1,2}^q + L_{3,4}^q$ should be discrete,

$$L_{1,2}^q + L_{3,4}^q \in \mathbb{Z}. \quad (46)$$

This of course matches the expectation for the spectrum of (the sum of) angular momentum operators. Note that this is a nonperturbative statement, which cannot be deduced from the expansion of $L_{1,2}^q + L_{3,4}^q$ in φ . Due to (35b), this means that the spectrum of $\hat{\lambda}$ is discrete. In particular, we may choose the eigenstate of eigenvalue 0, corresponding to fixing the angular momentum $L_{1,2} + L_{3,4}$ to the classical value, up to corrections of $\mathcal{O}(\gamma^0)$. By (35a), this is also an eigenstate of E , again up to corrections of $\mathcal{O}(\gamma^0)$.

We now come to the evaluation of the expectation value of H^0 (with the (angular) momenta set to zero). As in [8], we follow the proposal [11] to employ a locally covariant renormalization technique [14] developed for QFT on curved space-times. Concretely, the expectation value of Wick squares (possibly with derivatives) is determined as follows:

⁷As a consistency check, we note that the leading order contribution to E^2 of the terms quadratic in the momenta is given by $2\pi\gamma(\hat{p}^2 + \sum_{\alpha,i}(\hat{p}_i^\alpha)^2)$, cf. (13). With (35e), (35f), and (35g), we thus see that the momenta contribute $(P^i)^2$ to E^2 , up to corrections of $\mathcal{O}(\gamma^{-\frac{1}{2}})$, as expected.

$$\langle \Omega | (\nabla^\alpha \phi \nabla^\beta \phi)(x) | \Omega \rangle = \lim_{x' \rightarrow x} \nabla^\alpha \nabla'^\beta (w(x; x') - h(x; x')) \quad (47)$$

Here α, β are multiindices, w is the two-point function in the state Ω , defined as on the left-hand side (l.h.s.) of (41), and h is a distribution which is covariantly constructed out of the geometric data, the *Hadamard parametrix*. For physically reasonable states, the difference $w - h$ is smooth, so that the above coinciding point limit exists and is independent of the direction from which x' approaches x .

For our purposes, it is advantageous to perform the limit of coinciding points from the time direction, i.e., we take $x = (\tau, \sigma)$, $x' = (\tau + t, \sigma)$, and $t \rightarrow +0$. Performing the summation in (41a), we find

$$\begin{aligned} & \frac{1}{2} (\partial_0 \partial_0' + \partial_1 \partial_1') w_s(x; x') \\ &= \frac{1}{2\pi} \sum_{n=1}^{\infty} n e^{in(t+i\epsilon)} \\ &= -\frac{1}{2\pi(t+i\epsilon)^2} - \frac{1}{24\pi} + \mathcal{O}(t). \end{aligned} \quad (48)$$

For a minimally coupled scalar field with a variable mass $m^2(x)$ in two dimensional space-time, the Hadamard parametrix is given by (see, e.g., [18])

$$\begin{aligned} h(x; x') &= -\frac{1}{4\pi} \left(1 + \frac{1}{2} m^2(x) \rho(x, x') + \mathcal{O}((x - x')^3) \right) \\ &\times \log \frac{\rho_\epsilon(x, x')}{\Lambda^2}, \end{aligned} \quad (49)$$

where ρ is the *Synge world function*, i.e., $\frac{1}{2}$ times the squared (signed) geodesic distance of x and x' , cf. [19], and Λ is a length scale (the “renormalization scale”). For the local covariance, it is crucial that Λ is fixed and does not depend on any geometric data [14]. Inside of the logarithm, the world function is equipped with an $i\epsilon$ prescription as follows:

$$\rho_\epsilon(x, x') = \rho(x, x') + i\epsilon(\tau - \tau'). \quad (50)$$

For the scalar part, the mass term is absent. We thus find, for the coinciding point limit from the time direction,⁸

$$\frac{1}{2} (\partial_0 \partial_0' + \partial_1 \partial_1') h_s = -\frac{1}{2\pi(t+i\epsilon)^2} + \mathcal{O}(t). \quad (51)$$

For the scalar contribution to the energy, we thus obtain

⁸Here and in the following, $\mathcal{O}(t)$ also includes terms of the form $t \log t$.

$$\langle H_s^0 \rangle = \int_0^{2\pi} \langle H_s^0(\sigma) \rangle d\sigma = -\frac{1}{12}. \quad (52)$$

For the planar contribution to the energy density, we compute

$$\begin{aligned} & \frac{1}{2} \text{Tr}[(\partial_0 \partial_0' + \partial_1 \partial_1' + C + B \partial_1') w_p(x; x')] \\ &= \frac{1}{2\pi} \left[\sum_{n=1}^{\infty} n e^{in(t+i\epsilon)} + 2 \sum_{n=2}^{\infty} n e^{in(t+i\epsilon)} \right] \\ &= -\frac{3}{2\pi(t+i\epsilon)^2} - \frac{3}{24\pi} - \frac{1}{\pi} + \mathcal{O}(t). \end{aligned} \quad (53)$$

In order to obtain the planar parametrix, it is convenient to absorb the first order derivatives in the equation of motion (23b) by introducing suitable covariant derivatives, analogously to the procedure followed in [20] for the construction of retarded and advanced Green's functions. It is straightforward to check that with

$$\begin{aligned} A_0 &= \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ A_1 &= \sqrt{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ M^2 &= \frac{2}{R^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (54)$$

and the covariant derivative

$$\nabla_\mu = \partial_\mu + A_\mu, \quad (55)$$

the planar part of the free action can be written as

$$\mathcal{S}_p^0 = -\frac{1}{2} \int (\bar{g}^{\mu\nu} \nabla_\mu f \nabla_\nu f + f^T M^2 f) \sqrt{-\bar{g}} d^2x. \quad (56)$$

We thus can interpret M^2 as a ‘‘mass term’’. Denoting by $\mathbb{1}(x, x')$ the parallel transport from x' to x with respect to the covariant derivative, the parametrix for the planar sector is then given by

$$\begin{aligned} h_p(x, x') &= -\frac{1}{4\pi} \left[\mathbb{1}(x, x') - \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} (x - x')^2 \right. \\ &\quad \left. + \mathcal{O}((x - x')^3) \right] \log \frac{R^2(x - x')^2 + i\epsilon(\tau - \tau')^2}{4\Lambda^2}, \end{aligned} \quad (57)$$

with $(x - x')^2$ the usual Minkowski square. The parallel transport can be Taylor expanded around coinciding points, i.e., in $\Delta x = x - x'$ as

$$\mathbb{1}(x, x') = \mathbb{1} - A_\mu \Delta x^\mu + \frac{1}{2} (A_\nu A_\mu + \partial_\mu A_\nu) \Delta x^\mu \Delta x^\nu + \mathcal{O}(\Delta x^3). \quad (58)$$

Using these results, one computes

$$\begin{aligned} & \frac{1}{2} \text{Tr}[(\partial_0 \partial_0' + \partial_1 \partial_1' + C + B \partial_1') h_p(x; x')] \\ &= -\frac{3}{2\pi(t+i\epsilon)^2} - \frac{1}{\pi} + \mathcal{O}(t). \end{aligned} \quad (59)$$

To obtain this result, one notes that the last two terms on the l.h.s. are irrelevant, due to the trace and the direction from which the coinciding point limit is taken. The first two terms on the l.h.s. yield

$$\begin{aligned} & -\frac{1}{2\pi} \text{Tr} \left[\frac{\mathbb{1} + tA_0 + \frac{1}{2}(A_0 A_0 + \partial_0 A_0) t^2}{t^2} \right. \\ & \quad \left. + \frac{-A_0 - (A_0 A_0 + \partial_0 A_0) t}{t} \right], \end{aligned} \quad (60)$$

with the first term stemming from the action of all derivatives on the log, whereas the second term stems from the action of one time derivative on $\mathbb{1}$ and one on the log. For the planar contribution to the energy, we thus finally obtain

$$\langle H_p^0 \rangle = -\frac{3}{12}. \quad (61)$$

Adding this to the scalar contribution (52) and using (14), we obtain the intercept $a = \frac{1}{6}$, consistent with (1) for $D = 6$. As shown in Sec. V, this consistency persists to all dimensions and general ellipticity.

We note that there is no logarithmic divergence in the coinciding point limit of the world sheet energy density, and thus no renormalization ambiguity associated to the choice of the scale Λ . The reason is that the only renormalization ambiguity at the order considered here is a supplementary Einstein-Hilbert term, which however vanishes on our flat background (it is generically irrelevant for closed strings due to the Gauß-Bonnet theorem), cf. also the discussion at the end of the next section.

IV. GENERALIZATION TO THE ELLIPTIC CASE

To treat the case of an elliptic string, we use, instead of (17), the classical solution

$$\bar{X}(\tau, \sigma) = \begin{pmatrix} R\tau \\ R_{1,2} \sin \tau \cos \sigma \\ -R_{1,2} \cos \tau \cos \sigma \\ R_{3,4} \cos \tau \sin \sigma \\ R_{3,4} \sin \tau \sin \sigma \\ 0 \end{pmatrix}, \quad (62)$$

where $\sigma \in [0, 2\pi)$ and R is given by (11). We also introduce the angle $\theta \in [0, \frac{\pi}{2}]$ parametrizing $R_{1,2}$ and $R_{3,4}$ by

$$R_{1,2} = R \cos \theta, \quad R_{3,4} = R \sin \theta. \quad (63)$$

Two limiting cases are of particular interest: For $\theta \rightarrow \frac{\pi}{4}$, one recovers the circular case, whereas for $\theta \rightarrow 0$, one finds two straight open strings attached at the endpoints.

The induced metric on the world sheet, in the coordinates introduced above, is

$$\bar{g}_{\mu\nu} = \frac{R^2}{2} (1 - \cos 2\theta \cos 2\sigma) \eta_{\mu\nu}. \quad (64)$$

As this is conformal to the Minkowski metric, the equation of motion (15) can be easily checked. Energy and angular momentum of the above solution were given in (10). The scalar curvature is given by

$$\mathcal{R} = \frac{8 \cos^2 2\theta - \cos 2\theta \cos 2\sigma}{R^2 (1 - \cos 2\theta \cos 2\sigma)^3}. \quad (65)$$

Hence, unless $\theta = \frac{\pi}{4}$, i.e., for $R_{1,2} = R_{3,4}$, the world sheet has intrinsic curvature.

Perturbations of this classical solution are described again in the form (2), (19), but for the vectors v_a , v_b , v_c , orthonormal and normal to the world sheet, we now choose

$$v_a = \frac{1}{\mathcal{M}\mathcal{N}} \begin{pmatrix} \mathcal{N} \\ 2 \cos \theta \cos \tau \cos \sigma \\ 2 \cos \theta \sin \tau \cos \sigma \\ -2 \sin \theta \sin \tau \sin \sigma \\ 2 \sin \theta \cos \tau \sin \sigma \\ 0 \end{pmatrix},$$

$$v_b = \frac{\sqrt{2}}{\mathcal{M}} \begin{pmatrix} 0 \\ \sin \theta \sin \tau \cos \sigma \\ -\sin \theta \cos \tau \cos \sigma \\ \cos \theta \cos \tau \sin \sigma \\ \cos \theta \sin \tau \sin \sigma \\ 0 \end{pmatrix},$$

$$v_c = \frac{\sqrt{2}}{\mathcal{N}} \begin{pmatrix} 0 \\ \sin \theta \cos \tau \sin \sigma \\ \sin \theta \sin \tau \sin \sigma \\ \cos \theta \sin \tau \cos \sigma \\ -\cos \theta \cos \tau \cos \sigma \\ 0 \end{pmatrix}, \quad (66)$$

where

$$\mathcal{M} = \sqrt{1 - \cos 2\theta \cos 2\sigma}, \quad \mathcal{N} = \sqrt{1 + \cos 2\theta \cos 2\sigma}. \quad (67)$$

In the limit of the circular string, these coefficients tend to 1 and the above vectors coincide with the vectors used in Sec. II.

The free part of the action still has the form (21), but now with A , B , C replaced by

$$A = \frac{2}{\mathcal{M}^2 \mathcal{N}} \begin{pmatrix} 0 & \sqrt{2} \sin 2\theta & 0 \\ -\sqrt{2} \sin 2\theta & 0 & \mathcal{M} \cos 2\theta \sin 2\sigma \\ 0 & -\mathcal{M} \cos 2\theta \sin 2\sigma & 0 \end{pmatrix}, \quad (68a)$$

$$B = \frac{\sqrt{8} \sin 2\theta}{\mathcal{M}\mathcal{N}^2} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (68b)$$

$$C = \frac{1}{\mathcal{M}^4 \mathcal{N}^4} \begin{pmatrix} -B/4 & 0 & 3\mathcal{M}\mathcal{N}^2 \sin 4\theta \sin 2\sigma / \sqrt{2} \\ 0 & -\mathcal{N}^4 (\mathcal{M}^4 + 3\sin^2 2\theta) & 0 \\ 3\mathcal{M}\mathcal{N}^2 \sin 4\theta \sin 2\sigma / \sqrt{2} & 0 & -\mathcal{M}^4 \mathcal{N}^4 + \mathcal{M}^2 \sin^2 2\theta (4\mathcal{N}^2 + \mathcal{M}^2) \end{pmatrix}, \quad (68c)$$

where we denoted

$$\mathcal{B} = \cos 2\theta \cos 2\sigma - 5 \cos 6\theta \cos 2\sigma + 8 \cos^2 2\theta (\cos 4\sigma - 1) + 4 \cos^3 2\theta \cos 6\sigma. \quad (69)$$

The equation of motion (23a) for the scalar part is not modified, whereas the equation of motion (23b) for the planar part is replaced by

$$0 = -\ddot{f} + f'' - A\dot{f} - Bf' - (C + B'/2)f, \quad (70)$$

the modification being due to the fact that B is no longer constant. The symplectic form of the planar part is still of the form (24), with the modified A , of course. The free Hamiltonian is still given by (42), again with the modified B , C .

The symplectically normalized mode solutions for the scalar excitations are still given by (25), (27). Those for the planar sector are tedious to derive. The results are stated in the Appendix. For our purposes, the important point about these mode solutions is not their explicit form, but that they have exactly the same frequencies as in the circular case. Canonical quantization proceeds in complete analogy to the circular case. Renormalization, however, is affected by the curvature of the elliptic string. We aim to perform it in a locally covariant way, i.e., with respect to the local

geometric data. Hence, we rewrite the free action in a geometric form as in (56), now with

$$A_0 = \sqrt{2} \begin{pmatrix} 0 & \frac{\sin 2\theta}{\mathcal{M}^2 \mathcal{N}} & 0 \\ -\frac{\sin 2\theta}{\mathcal{M}^2 \mathcal{N}} & 0 & \frac{\cos 2\theta \sin 2\sigma}{\sqrt{2} \mathcal{M} \mathcal{N}} \\ 0 & -\frac{\cos 2\theta \sin 2\sigma}{\sqrt{2} \mathcal{M} \mathcal{N}} & 0 \end{pmatrix},$$

$$A_1 = \sqrt{2} \begin{pmatrix} 0 & 0 & \frac{\sin 2\theta}{\mathcal{M} \mathcal{N}^2} \\ 0 & 0 & 0 \\ -\frac{\sin 2\theta}{\mathcal{M} \mathcal{N}^2} & 0 & 0 \end{pmatrix}, \quad (71a)$$

$$M^2 = \frac{2}{R^2 \mathcal{M}^2} \begin{pmatrix} \frac{4 \cos^2 2\theta \sin^2 2\sigma}{\mathcal{M}^4 \mathcal{N}^2} & 0 & \frac{\sqrt{2} \sin 4\theta \sin 2\sigma}{\mathcal{M}^3 \mathcal{N}^2} \\ 0 & -\frac{2 \sin^2 2\theta}{\mathcal{M}^4} & 0 \\ \frac{\sqrt{2} \sin 4\theta \sin 2\sigma}{\mathcal{M}^3 \mathcal{N}^2} & 0 & \frac{2 \sin^2 2\theta}{\mathcal{M}^2 \mathcal{N}^2} \end{pmatrix}. \quad (71b)$$

For the scalar part, one proceeds analogously, with the derivative the usual one, and a vanishing ‘‘mass’’ M^2 . Furthermore, as the metric is no longer flat, but given by (64), the expression for the world function becomes more complicated. It can be Taylor expanded in coordinates around coinciding points as [21]

$$\rho(x, x') = \frac{1}{2} \bar{g}_{\mu\nu}(x) \Delta x^\mu \Delta x^\nu + A_{\mu\nu\lambda}(x) \Delta x^\mu \Delta x^\nu \Delta x^\lambda + B_{\mu\nu\lambda\rho}(x) \Delta x^\mu \Delta x^\nu \Delta x^\lambda \Delta x^\rho, \quad (72a)$$

$$A_{\mu\nu\lambda} = -\frac{1}{4} \partial_{(\mu} \bar{g}_{\nu\lambda)}, \quad (72b)$$

$$B_{\mu\nu\lambda\rho} = \frac{1}{12} \partial_{(\mu} \partial_{\nu} \bar{g}_{\lambda\rho)} - \frac{1}{24} g^{\sigma\tau} \left(\frac{1}{4} \partial_\sigma \bar{g}_{(\mu\nu} \partial_{|\tau|} \bar{g}_{\lambda\rho)} - \partial_\sigma \bar{g}_{(\mu\nu} \partial_\lambda \bar{g}_{\rho)\tau} + \partial_{(\mu} \bar{g}_{\nu|\sigma} \partial_\lambda \bar{g}_{\rho)\tau} \right), \quad (72c)$$

where $\Delta x = x - x'$. One thus finds, for a metric of the form $\bar{g}_{\mu\nu} = f(\sigma) \eta_{\mu\nu}$,

$$\rho = \frac{1}{2} f(\sigma) (-\Delta\tau^2 + \Delta\sigma^2) + \frac{1}{4} f'(\sigma) \Delta\tau^2 \Delta\sigma - \frac{1}{96} f(\sigma)^{-1} f'(\sigma)^2 \Delta\tau^4 + \left(\frac{1}{48} f(\sigma)^{-1} f'(\sigma)^2 - \frac{1}{12} f''(\sigma) \right) \Delta\tau^2 \Delta\sigma^2 + \mathcal{O}(\Delta x^5, \Delta\sigma^3), \quad (73)$$

and hence, for the coinciding point limit from the time direction,

$$\frac{1}{2} (\partial_0 \partial_0' + \partial_1 \partial_1') h_s = -\frac{1}{2\pi(t + i\varepsilon)^2} + \frac{1}{32\pi} \frac{f'^2}{f^2} - \frac{1}{48\pi} \frac{f''}{f} + \mathcal{O}(t). \quad (74)$$

For the metric (18), this yields

$$\frac{1}{2} (\partial_0 \partial_0' + \partial_1 \partial_1') h_s = -\frac{1}{2\pi(t + i\varepsilon)^2} + \frac{\cos^2 2\theta \sin^2 2\sigma}{8\pi(1 - \cos 2\theta \cos 2\sigma)^2} - \frac{\cos 2\theta \cos 2\sigma}{12\pi(1 - \cos 2\theta \cos 2\sigma)} + \mathcal{O}(t). \quad (75)$$

For the scalar contribution to the energy density, we thus obtain

$$\langle H_s^0(\sigma) \rangle = -\frac{1}{24\pi} - \frac{\cos^2 2\theta \sin^2 2\sigma}{8\pi(1 - \cos 2\theta \cos 2\sigma)^2} + \frac{\cos 2\theta \cos 2\sigma}{12\pi(1 - \cos 2\theta \cos 2\sigma)}. \quad (76)$$

In the degenerate limit $\theta \rightarrow 0$, this gives

$$\lim_{\theta \rightarrow 0} \langle H_s^0(\sigma) \rangle = -\frac{1}{12\pi \sin^2 \sigma}, \quad (77)$$

in accordance with the result obtained for the open string [8]. Integration over σ yields

$$\langle H_s^0 \rangle = -\frac{1}{12 \sin 2\theta} = -\frac{1}{24} (\tan \theta + \cot \theta). \quad (78)$$

For the planar part, the computation of the expectation value of the Hamiltonian density is challenging. However,

we are only interested in the total Hamiltonian. Noting that it can still be expressed in terms of creation and annihilation operators as in (43), the computation of the expectation value, point-split in time direction, proceeds, up to the factor 2π , as in the second line of (53). For the contribution from the two-point function, we thus obtain

$$-\frac{3}{(t+i\epsilon)^2} - \frac{3}{12} - 2. \quad (79)$$

The parametrization for the planar part is

$$h_p(x; x') = -\frac{1}{4\pi} \left(\mathbb{1}(x, x') + \frac{1}{2} M^2(x) \rho(x, x') + \mathcal{O}((x-x')^3) \right) \log \frac{\rho_\epsilon(x, x')}{\Lambda^2}, \quad (80)$$

with M^2 and the parallel transport now defined by (71). Using (58) and (73), one finds

$$\begin{aligned} \frac{1}{2} \text{Tr}[(\partial_0 \partial'_0 + \partial_1 \partial'_1 + C + B \partial'_1) h_p(x; x')] &= -\frac{3}{2\pi(t+i\epsilon)^2} - \frac{\sin^2 2\theta(17 - 5 \cos 4\sigma) + 80(\cos^4 \sigma \sin^4 \theta + \cos^4 \theta \sin^4 \sigma)}{32\pi \mathcal{M}^4} \\ &\quad - \frac{\cos 2\theta(\cos 2\theta - \cos 2\sigma)}{2\pi \mathcal{M}^4} \log \left[-\frac{R^2(t+i\epsilon)^2 \mathcal{M}^2}{4\Lambda^2} \right] + \mathcal{O}(t). \end{aligned} \quad (81)$$

Integrating this over σ , we obtain

$$-\frac{3}{(t+i\epsilon)^2} - \frac{1}{4} - \frac{7}{8} (\tan \theta + \cot \theta). \quad (82)$$

Subtracting now (82) from (79), we obtain the expression for the planar contribution to the free Hamiltonian expectation value

$$\langle H_p^0 \rangle = -2 + \frac{7}{8} (\tan \theta + \cot \theta). \quad (83)$$

As a consistency check, we note that the coefficient of the logarithm in the last term in (81) can be also expressed, using (65), as

$$-\frac{\mathcal{R}\sqrt{\bar{g}}}{8\pi}, \quad (84)$$

meaning that a change in the renormalization scale Λ corresponds to finite renormalization of the Einstein-Hilbert term. By the analysis of [14] this is the only

renormalization ambiguity for the world sheet energy density $H^0(\sigma)$ in a locally covariant renormalization scheme. For the closed string, the contribution of this term to the Hamiltonian H^0 vanishes, by the Gauß-Bonnet theorem, so that the obtained result is unambiguous.

V. THE GENERALIZATION TO ARBITRARY DIMENSIONS

For the case of general dimension D , one has to consider $D-5$ scalar fields, i.e., instead of v_s defined in (20), we consider $v_{s,i} = e_{4+i}$, for $i = 1, \dots, D-5$ and e_j the canonical basis vectors of (\mathbb{R}^D, η) . Instead of the single coefficient f_s , we also have $f_{s,i}$. With these, the free part of the action (21) is modified to

$$\begin{aligned} \mathcal{S}^0 &= \frac{1}{2} \int \left(\dot{f}^2 - f'^2 - f^T A \dot{f} - f^T B f' - f^T C f \right. \\ &\quad \left. + \sum_{i=1}^{D-5} [\dot{f}_{s,i}^2 - f'_{s,i}{}^2] \right) d\sigma d\tau. \end{aligned} \quad (85)$$

We see that at this order, the perturbations in the directions orthogonal to the rotation subspace spanned by $e_0 - e_4$ completely decouple, i.e., we are dealing with $D - 5$ independent copies of the scalar field previously considered. It follows that vacuum expectation value of the free Hamiltonian in D target space dimensions is

$$\begin{aligned} \langle H^0 \rangle &= (D - 5) \langle H_s^0 \rangle + \langle H_p^0 \rangle \\ &= -2 - \frac{D - 26}{24} (\tan \theta + \cot \theta). \end{aligned} \quad (86)$$

Using (14), we obtain the Regge intercept

$$a = 1 + \frac{D - 26}{48} (\tan \theta + \cot \theta). \quad (87)$$

Using that by (63) and (10) we have $\cot \theta = \sqrt{J_{12}/J_{34}}$, this result can be seen to coincide with the result (1) obtained in [7].

Expanding the action beyond second order in the perturbation, the different scalar modes $f_{s,i}$ will of course couple among each other and with the planar modes, so the simple linear scaling with the dimension is expected not to hold beyond the semi-classical approximation.

Let us finally comment on the origin of the divergence for $\theta \rightarrow 0, \frac{\pi}{2}$, i.e., $J_{3/4} \rightarrow 0$ or $J_{1/2} \rightarrow 0$. In this limit, the closed string is flattened to two straight open strings joined at the folds $\sigma = 0, \pi$. In particular, the scalar curvature (65) diverges at these folds. This leads to a nonintegrable divergence of the local energy density at the folds, as seen in (77). In the case of the open string [8], the same nonintegrable divergence occurred, but it could be removed using geodesic curvature boundary counterterms. This procedure can not be applied in the present case, as the closed string does not possess boundaries.⁹

VI. THE SPECTRUM OF PHYSICAL EXCITATIONS

Let us recall some ingredients of the covariant quantization of the closed Nambu-Goto string. To simplify notation, we set $\pi\gamma = 1$. We have two sets $A_m^{\pm\mu}$ of oscillators, for right- and left-moving excitations, fulfilling

$$[A_m^{\alpha\mu}, A_n^{\beta\nu}] = m\delta_{m+n}\delta^{\alpha\beta}\eta^{\mu\nu}. \quad (88)$$

One identifies

$$p^\mu = \frac{1}{2} A_0^{\pm\mu} \quad (89)$$

⁹For $\theta = 0$ one could of course define the energy to be twice that of a single open string, but that does not remedy the divergence as $\theta \rightarrow 0$.

as the center of mass momentum. There are also two sets L_m^\pm of Virasoro generators. Furthermore, there is the angular momentum operator

$$J^{\mu\nu} = -i \sum_{\alpha \in \pm} \sum_{n \geq 1} \frac{1}{n} (A_{-n}^{\alpha\mu} A_n^{\alpha\nu} - \mu \leftrightarrow \nu). \quad (90)$$

One has the commutation relations

$$\begin{aligned} [L_m^\alpha, \zeta \cdot A_{-k}^\beta] &= k\delta^{\alpha\beta}\zeta \cdot A_{m-k}^\alpha, \\ [J^{12}, \zeta \cdot A_{-k}^\alpha] &= \zeta^{12} \cdot A_{-k}^\alpha, \\ [J^{34}, \zeta \cdot A_{-k}^\alpha] &= \zeta^{34} \cdot A_{-k}^\alpha, \end{aligned} \quad (91)$$

with $\zeta = (\zeta^0, \dots, \zeta^{D-1})$ a target space vector and

$$\begin{aligned} \zeta^{12} &= (0, -i\zeta^2, i\zeta^1, 0, \dots, 0), \\ \zeta^{34} &= (0, 0, 0, -i\zeta^4, i\zeta^3, 0, \dots, 0). \end{aligned} \quad (92)$$

We also recall that physical states Φ have to fulfill the conditions

$$(L_m^\pm - \delta_m a)|\Phi\rangle = 0 \quad (93)$$

for all $m \geq 0$. In particular, we recall that

$$L_0^\pm = \frac{1}{8} p^2 + \sum_{n \geq 1} A_{-n}^\pm \cdot A_n^\pm. \quad (94)$$

The physical state of minimum energy for given angular momentum $\ell_{1,2}$ in the 1-2 plane and $\ell_{3,4}$ in the 3-4 plane, with $\ell_{1,2}, \ell_{3,4}$ positive even numbers, is given by

$$\begin{aligned} |\ell_{1,2}, \ell_{3,4}\rangle &= (\xi \cdot A_{-1}^+)^{\ell_{1,2}/2} (\tilde{\xi} \cdot A_{-1}^+)^{\ell_{3,4}/2} (\xi \cdot A_{-1}^-)^{\ell_{1,2}/2} \\ &\quad \times (\tilde{\xi} \cdot A_{-1}^-)^{\ell_{3,4}/2} |0; \sqrt{2(2\ell_{1,2} + 2\ell_{3,4} - 4a)}\rangle, \end{aligned} \quad (95)$$

with $|0; M\rangle$ denoting a ground state with vanishing spatial momentum and rest mass M and

$$\xi = \frac{1}{\sqrt{2}} (0, 1, i, 0, \dots, 0), \quad \tilde{\xi} = \frac{1}{\sqrt{2}} (0, 0, 0, 1, i, 0, \dots, 0). \quad (96)$$

Note that we have to evenly excite left- and right-movers in order to fulfill the physicality condition (93) for $m = 0$. We thus recover the Regge trajectory (9). We introduce the notation

$$M_\xi^2 = 2(2\ell_{1,2} + 2\ell_{3,4} - 4a). \quad (97)$$

Let us discuss what the analog of a scalar excitation of the state (95) would look like. Naively, we would choose a unit vector θ in the subspace orthogonal to the one spanned by $X^0 - X^4$, and act with the creation operator $\theta \cdot A_{-1}$ (and adjust the momentum, i.e., the mass accordingly).

However, in order to fulfill both equations (93) for $m = 0$ with the same a , we also have to excite a right-moving mode. It follows that minimal excitations of (95) in the directions perpendicular to X^0 — X^4 are given by

$$|\ell_{1,2}, \ell_{3,4}, \theta, \tilde{\theta}\rangle = (\theta \cdot A_{-1}^+) (\tilde{\theta} \cdot A_{-1}^-) (\xi \cdot A_{-1}^+)^{\ell_{1,2}/2} \\ \times (\tilde{\xi} \cdot A_{-1}^+)^{\ell_{3,4}/2} (\xi \cdot A_{-1}^-)^{\ell_{1,2}/2} \\ \times (\tilde{\xi} \cdot A_{-1}^-)^{\ell_{3,4}/2} |0; \sqrt{M_\rho^2 + 8}\rangle. \quad (98)$$

Here we adjusted the rest mass in order to fulfill the physicality condition (93). We see that in the covariant scheme, the minimal scalar excitation leads to a shift

$$\Delta M^2 = 8 \quad (99)$$

of the rest mass squared.

Let us now consider the situation in the semiclassically quantized string. As is obvious from (43), the minimal scalar excitation, $\hat{a}_{s,1}^{+*}$ or $\hat{a}_{s,1}^{-*}$, raises the expectation value of H^0 by 1. According to (13), and taking $\pi\gamma = 1$ into account, this leads to a shift

$$\Delta M^2 = 4 \quad (100)$$

of the rest mass squared. We thus see that, at the linearized level considered here, the semiclassically quantized string exhibits physical excitations¹⁰ that are not present in the covariantly quantized string. The point is that in the semiclassical scheme (at the linearized level) no condition on the equal excitation of left- and right-movers is present. Presently, it is unclear whether this is remedied at higher order in perturbation theory or not (and thus represents a nonperturbative effect).

VII. CONCLUSION

We computed semiclassical corrections to the energy of rotating closed Nambu-Goto strings. Special care was taken to renormalize in a local and covariant way, using techniques developed in the context of QFT on curved space-times. This calculation represents one of the few analytically tractable nontrivial applications of these techniques (needed for example for the calculation of backreaction effects in cosmological [22] or black hole spacetimes [23]). Our results agree with those obtained [7] via the Polchinski-Strominger action, which provides a mutual consistency check.

The results obtained imply a discrepancy with the covariant quantization scheme for $D < 26$, where the intercept a is only constrained by $a \leq 1$, but independent of the ellipticity. In contrast, in [7] and the present work, the intercept a is computed, but found to depend on the ellipticity for $D \neq 26$. We also showed that in our linearized approximation there are physical excitations not present in the covariant quantization scheme, even for $D = 26$. It would be interesting to learn whether this is remedied at higher order in perturbation theory, similar to linearized Yang-Mills theory containing more observables than the full nonlinear theory, or whether this constitutes a non-perturbative effect.

APPENDIX: THE PLANAR MODES IN THE ELLIPTIC CASE

The (not symplectically orthonormalized) positive frequency planar mode solutions in the elliptic case are given by

$$f_{1,n}^\pm = \frac{1}{\mathcal{M}\mathcal{N}} [\sqrt{2}(\pm\sqrt{2}i, \mp n\mathcal{N} \sin 2\theta, n\mathcal{M} \sin 2\theta) e^{\pm i n \sigma} + (\pm i(n+1) \cos 2\theta, 0, 0) e^{\pm i(n-2)\sigma} \\ + (\mp i(n-1) \cos 2\theta, 0, 0) e^{\pm i(n+2)\sigma}] e^{-in\tau}, \quad (A1a)$$

$$f_{2,n}^\pm = \frac{1}{\mathcal{M}\mathcal{N}} [2 \cos 2\theta (\pm 2i, \mp n\mathcal{N}, n\mathcal{M}) e^{\pm i n \sigma} + (\mp 2i(n-1) \cos^2 2\theta, 0, 0) e^{\pm i(n+2)\sigma} \\ + (\pm i\mathcal{A}_-, \pm 2n\mathcal{N}(1 - \sqrt{2} \sin 2\theta), 2n\mathcal{M}(1 - \sqrt{2} \sin 2\theta)) e^{\pm i(n-2)\sigma}] e^{-in\tau}, \quad (A1b)$$

$$f_{3,n}^\pm = \frac{1}{\mathcal{M}\mathcal{N}} [2 \cos 2\theta (\pm 2i, \mp n\mathcal{N}, n\mathcal{M}) e^{\pm i n \sigma} + (\pm 2i(n+1) \cos^2 2\theta, 0, 0) e^{\pm i(n-2)\sigma} \\ + (\pm i\mathcal{A}_+, \pm 2n\mathcal{N}(1 - \sqrt{2} \sin 2\theta), 2n\mathcal{M}(1 - \sqrt{2} \sin 2\theta)) e^{\pm i(n+2)\sigma}] e^{-in\tau}, \quad (A1c)$$

where $n \geq 2$ and

$$\mathcal{A}_\mp = (1 \pm 3n \mp (n \mp 1) \cos 4\theta \mp \sqrt{8}n \sin 2\theta). \quad (A2)$$

¹⁰We recall that at the order considered here, we could set all unphysical (auxiliary and pure gauge) fields to zero. The remaining ones are physical (commute with the free BRST charge) and likewise are the corresponding excitations.

The mode solution (A1c) can also be extended to $n = 1$. The zero energy mode $f_{3,0}^\pm$ is now given by

$$f_{3,0}^\pm = 4 \cos 2\theta \left(\pm \frac{\sqrt{8}i}{\mathcal{M}\mathcal{N}}, \mp \frac{1}{\mathcal{M}}, \frac{1}{\mathcal{N}} \right) + \left(\pm \sqrt{2}i \frac{3 + \cos 4\theta \mp 2 \sin 2\theta}{\mathcal{M}\mathcal{N}}, \pm 2 \frac{(\cos \theta \mp \sin \theta)^2}{\mathcal{M}}, 2 \frac{(\cos \theta \mp \sin \theta)^2}{\mathcal{N}} \right) e^{-2i\sigma} + \left(\pm \sqrt{2}i \frac{3 + \cos 4\theta \pm 2 \sin 2\theta}{\mathcal{M}\mathcal{N}}, \pm 2 \frac{(\cos \theta \pm \sin \theta)^2}{\mathcal{M}}, 2 \frac{(\cos \theta \pm \sin \theta)^2}{\mathcal{N}} \right) e^{2i\sigma}. \quad (\text{A3})$$

The modes f_q^\pm , which we interpreted as position operators in the planes of rotation, are now given by

$$f_q^\pm = \frac{1}{\mathcal{M}\mathcal{N}} [(\pm\sqrt{2}i, -\mathcal{N}, \mathcal{M})(\cos \theta \pm \sin \theta)e^{i\sigma} + (\pm\sqrt{2}i, \mathcal{N}, \mathcal{M})(\cos \theta \mp \sin \theta)e^{-i\sigma}]e^{-i\tau}. \quad (\text{A4})$$

There is also a corresponding linearly growing mode f_p^\pm , however, for its homogeneous part, i.e., the part not linearly growing in time, we were only able to derive an ODE, and not to explicitly solve it. Likewise, for the generalization of the mode f_θ describing rotations generated by $L_{1,2} + L_{3,4}$, we find

$$f_\theta = \left(\frac{2(\cos^4 \theta \cos^2 \sigma + \sin^4 \theta \sin^2 \sigma)}{\mathcal{M}\mathcal{N}}, 0, \frac{\sin 4\theta \sin 2\sigma}{4\sqrt{2}\mathcal{N}} \right), \quad (\text{A5})$$

again with an unknown homogeneous part of the corresponding linearly growing mode f_λ .

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