Higher curvature Bianchi identities, generalized geometry, and L_∞ algebras

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The Bianchi identities for bosonic fluxes in supergravity can receive higher derivative quantum and string corrections, the most well known being that of heterotic theory $\mathrm{d}H = \frac{1}{4}\alpha'(\mathrm{tr}\,F^2 - \mathrm{tr}\,R^2)$. Less studied are the modifications at order R^4 that may arise, for example, in the Bianchi identity for the seven-form flux of M theory compactifications. We argue that such corrections appear to be incompatible with the exceptional generalized geometry description of the lower order supergravity, and seem to imply a gauge algebra for the bosonic potentials that cannot be written in terms of an (exceptional) Courant bracket. However, we show that this algebra retains the form of an L_∞ gauge field theory, which terminates at a level ten multibracket for the case involving just the seven-form flux.

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I. GENERALIZED GEOMETRY AND BIANCHI IDENTITIES

The formalism of generalized geometry has proven to be a very powerful tool to tackle problems in string theory and supergravity. By looking at structures on a generalized tangent space which has “baked in” the much richer gauge field content of these theories, it provides a unified language for the bosonic sector that brings within reach previously intractable problems. However, precisely because the gauge fields are built into the definition of the generalized geometry, their Bianchi identities are assumed by construction and any modification of them requires a change in the formalism. As we will review shortly, including the first α' correction due to the Green-Schwarz mechanism in heterotic theory requires relaxing the exactness condition of the Courant algebroid in “base” generalized geometry, and adding the Ramond-Ramond fields (or moving to M theory) requires the introduction of an exceptional Courant algebroid. In this paper, we will argue that further considering R^4 corrections—which would be highly desirable as it could provide a path to finally obtain their supersymmetric completion and would have applications to phenomenological models that rely on perturbative effects to fix moduli in flux compactifications—implies again expanding the exceptional generalized geometry, and also that the gauge algebra can no longer be captured by a bracket acting just on elements of the generalized tangent space. Finally we will show that there is nonetheless an L_∞ algebra structure remaining, which we compute explicitly for a particular case.

A. Generalized geometry

Generalized geometry was originally introduced in [1,2] as a way of combining complex and symplectic geometry, by considering structures on the generalized tangent bundle E

$$T^* \to E \to T, \tag{1}$$

so that E is locally isomorphic to the sum of the tangent and cotangent bundle, $E \sim T \oplus T^*$. The generalized tangent bundle is naturally equipped with a Dorfman bracket or, equivalently, its antisymmetrization a Courant bracket [3,4]

$$[X_1, X_2] = [x_1, x_2] + \mathcal{L}_{x_1}\lambda_2 - \mathcal{L}_{x_2}\lambda_1 - \frac{1}{2}\mathrm{d}(i_{x_1}\lambda_2 - i_{x_2}\lambda_1), \tag{2}$$

where $X_i = x_i + \lambda_i \in E$. The generalized tangent bundle is in fact an example of an exact Courant algebroid [5] and it possesses a three-form H that is closed [6],

$$\mathrm{d}H = 0, \tag{3}$$

which can be thought of as the curvature of a “gerbe” B [7], i.e., $H = \mathrm{d}B$ locally, which specifies a splitting of the sequence (1). Physicists quickly realized that this formalism provides a way of geometrizing the Neveu-Schwarz–Neveu-Schwarz (NSNS) sector of type II supergravity [8–11], B being identified with the Kalb-Ramond field

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and H being its flux. The Dorfman bracket along a generalized vector L_X, $X \in E$ then generates the combined (infinitesimal) bosonic symmetries of the theory: diffeomorphisms $\mathcal{L}_x$ by taking the Lie derivative along a vector field $x \in T$, and gauge B shifts by $d\lambda$, exact two-forms parametrized by one-forms $\lambda \in T^*$. Introducing also a metric, it is possible to unify all the NSNS fields into a single object, and rewrite all the supergravity equations as a generalized geometry equivalent of Einstein gravity [12] (see also [13] for an overview of the closely related subject of double field theory that often implies many of these results).

B. Heterotic generalized geometry

In heterotic theory, however, the field strength H is no longer closed. Supersymmetry and the Green-Schwarz anomaly cancelation mechanism [14] require that H satisfy a more complicated Bianchi identity. This can be handled in the generalized geometry formalism by enlarging the generalized tangent space. The resulting 'heterotic generalized geometry" [15–20] is given in terms of a bundle which is a transitive, but not exact, Courant algebroid E, that can be built as a result of two extensions

$$\begin{aligned} \mathfrak{g} \to \mathcal{A} \to T, \\ T^* \to E \to \mathcal{A}. \end{aligned} \tag{4}$$

The first sequence defines a Lie algebroid $\mathcal{A}$ known as the Atiyah algebroid for the quadratic Lie algebra $\mathfrak{g}$, which replaces the role of the tangent bundle T in the original generalized geometry. Writing $X_i = x_i + \Lambda_i + \lambda_i \in E \sim T \oplus \mathfrak{g} \oplus T^*$, the Courant bracket takes the form

$$\begin{aligned} [X_1, X_2] = [x_1, x_2] + \mathcal{L}_{x_1}\Lambda_2 - \mathcal{L}_{x_2}\Lambda_1 + [\Lambda_1, \Lambda_2] \\ + \mathcal{L}_{x_1}\lambda_2 - \mathcal{L}_{x_2}\lambda_1 - \frac{1}{2}d(i_{x_1}\lambda_2 - i_{x_2}\lambda_1) \\ + \mathrm{tr}\,(\Lambda_2 d\Lambda_1 - \Lambda_1 d\Lambda_2). \end{aligned} \tag{5}$$

The bundle E then encodes the information for local gauge fields: a two-form[1] B and a Yang-Mills one-form A taking values in $\mathfrak{g}$. These fields are not independent, they satisfy the global condition in terms of their respective field strengths

$$dH = \mathrm{tr}\,F^2. \tag{6}$$

By considering a Lie group G (with algebra $\mathfrak{g}$) composed of two factors, a "gravitational" Lorentz group, and the usual $SO(32)$ or $E_8 \times E_8$ (and choosing the correct normalization of the metric in $\mathfrak{g}$), one obtains the heterotic Bianchi identity:

$$dH = \frac{1}{4}\alpha'(\mathrm{tr}\,F^2 - \mathrm{tr}\,R^2), \tag{7}$$

where R, the field strength of the Lorentz factor, is now identified with the gravitational curvature. Once more, the Courant bracket in E precisely reproduces the physical infinitesimal bosonic symmetries: diffeomorphisms $\mathcal{L}_x$, B shifts by $d\lambda$, and now also non-Abelian gauge transformations by some parameter $\Lambda \in \mathfrak{g}$. It is then possible to show that formulating the generalized equivalent of Einstein gravity in E precisely reproduces the known heterotic supergravity to order α' [15]. The "trick" of treating the gravitational term in (7) as if it were a Yang-Mills factor goes back to [22], though, as shown there, supersymmetry requires that the tr R^2 be given by the curvature of a specific torsionful connection $\nabla - \frac{1}{2}H$. In [19] it was shown that this is entirely consistent with the generalized geometry setup.

C. M theory and $E_{7(7)} \times \mathbb{R}^+$ generalized geometry

In M theory, the equation of motion for the four-form flux $\mathcal{F}$ in eleven-dimensional supergravity [23] is corrected by higher order terms, starting with eight derivatives [24,25]

$$d * \mathcal{F} = -\frac{1}{2}\mathcal{F}^2 + \kappa(\mathrm{tr}\,R^4 - \frac{1}{4}(\mathrm{tr}\,R^2)^2), \tag{8}$$

where κ is some constant which will be set to 1 as it will not influence the rest of our discussion, and with further terms which are functions of the flux expected to appear at the same order in derivatives but whose complete form is not yet known.

In order to find four-dimensional Minkowski backgrounds of M theory, one considers field *Ansätze* that are compatible with the external global Lorentz symmetry. This means decomposing the eleven-dimensional manifold $\mathcal{M}_{11}$ as a warped product

$$ds^2_{11}(\mathcal{M}_{11}) = e^{\Delta} ds^2_4(\mathbb{R}^{3,1}) + ds^2_7(M), \tag{9}$$

where M is some seven-dimensional internal space and Δ the warp factor, demanding that the fields depend only on internal coordinates, and keeping the components of the $\mathcal{F}$ flux which are external scalars, i.e., the purely internal four-form F and seven-form $\tilde{F}$. Their components are set in terms of the eleven-dimensional $\mathcal{F}$ simply by restricting

$$F = \mathcal{F}|_M, \qquad \tilde{F} = (*\mathcal{F})|_M, \tag{10}$$

where $*\mathcal{F}$ is the eleven-dimensional Hodge dual. All other components of $\mathcal{F}$ are set to zero. The fact that $\mathcal{F}$ is closed in eleven dimensions together with the equation of motion (8) then implies the Bianchi identities for the internal fluxes

[1]The heterotic B field we are considering here is not gauge invariant under Yang-Mills transformations and is not a gerbe connection, it is a rather more complicated object [21].

$$\mathrm{d}F = 0, \tag{11}$$

$$\mathrm{d}\tilde{F} = -\frac{1}{2}F^2 + \mathrm{tr}\,R^4 - \frac{1}{4}(\mathrm{tr}\,R^2)^2, \tag{12}$$

where the second equation should be taken as purely formal, since it vanishes identically in the seven-dimensional M. These induce internal local potentials, a three-form C and a six-form $\tilde{C}$, which together with a Riemannian metric for M and a warp factor Δ make up the bosonic degrees of freedom (d.o.f.) of the theory.

Ignoring the higher curvature terms, it was shown in [26] that this supergravity setup (together with the fermionic sector) has a very natural interpretation as the analogue of Einstein gravity when formulated in $E_{7(7)} \times \mathbb{R}^+$ generalized geometry, also known as exceptional generalized geometry [27–29]. One introduces a generalized tangent bundle again as a series of extensions, such that it has a local form

$$E \sim T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus (T^* \otimes \Lambda^7 T^*), \tag{13}$$

which encodes the bosonic symmetries of the theory, namely, a diffeomorphism generated by vector fields x and shifts by two-forms ω and five-forms σ of the gauge fields C and $\tilde{C}$, respectively. The peculiar one-form-tensor-seven-form term $\tau \in T^* \otimes \Lambda^7 T^*$ would be a charge for a "dual graviton" field which happens to vanish identically in seven-dimensional compactifications and so, while it is implied by the higher-dimensional M theoretic geometry, it has no immediate physical meaning given this setup (though see [30], for example, where τ and other mixed-symmetry charges become crucial to formulate higher exceptional geometries). By construction, the generalized tangent bundle defines a global closed four-form F that can locally be expressed in terms of the potential

$$F = \mathrm{d}C, \tag{14}$$

and a seven-form such that

$$\tilde{F} = \mathrm{d}\tilde{C} - \frac{1}{2}CF. \tag{15}$$

The supergravity Bianchi identities inherited from eleven dimensions are thus automatically satisfied. The gauge algebra is then given by the natural differential structure over E, the (exceptional) Courant bracket of two generalized vector fields, which takes the form[2]

$$\begin{aligned}[X_1, X_2] = {} & \mathcal{L}_{x_1}x_2 + \mathcal{L}_{x_1}\omega_2 - \mathcal{L}_{x_2}\omega_1 - \frac{1}{2}\mathrm{d}(i_{x_1}\omega_2 - i_{x_2}\omega_1) \\ & + \mathcal{L}_{x_1}\sigma_2 - \mathcal{L}_{x_2}\sigma_1 - \frac{1}{2}\mathrm{d}(i_{x_1}\sigma_2 - i_{x_2}\sigma_1) + \frac{1}{2}\sigma_1\mathrm{d}\sigma_2 - \frac{1}{2}\sigma_2\mathrm{d}\sigma_1 \\ & + \frac{1}{2}\mathcal{L}_{x_1}\tau_2 - \frac{1}{2}\mathcal{L}_{x_2}\tau_1 + \frac{1}{2}(j\sigma_1 \wedge \mathrm{d}\omega_2 - j\sigma_2 \wedge \mathrm{d}\omega_1) - \frac{1}{2}(j\sigma_2 \wedge \mathrm{d}\omega_1 - j\sigma_1 \wedge \mathrm{d}\omega_2),\end{aligned} \tag{16}$$

where we write $X_i = x_i + \omega_i + \sigma_i + \tau_i \in E$. The bundle E has a natural $E_{7(7)} \times \mathbb{R}^+$ structure and the bracket is compatible with this structure. The bosonic d.o.f. turn out to simply be the components of a generalized metric for the generalized tangent space, reducing the structure group to its maximal compact subgroup $SU(8)/\mathbb{Z}_2$, and the corresponding generalized Ricci scalar precisely reproduces the supergravity bosonic action. That eleven-dimensional supergravity admitted this larger symmetry had already been proven in [31]. This efficient rewriting has made it possible to tackle several physical problems in full generality (without needing to restrict to some subsector of the fluxes, for example), such as classifying supersymmetric backgrounds [32–40] or describing their moduli spaces and holographic duals [41,42]. It would thus seem promising to apply the same techniques with the higher derivative corrections included [43].

[2] The j notation corresponds to a projection to the $T^* \otimes \Lambda^7 T^*$ space; see [28,29] for its precise definition, though it will not be needed for what follows.

D. M theory corrections

So now let us consider adding back the higher curvature terms originating in eleven dimensions (8). These are incompatible with the $E_{7(7)} \times \mathbb{R}^+$ generalized tangent bundle previously introduced, since by construction it forces $\tilde{F} = \mathrm{d}\tilde{C} - \frac{1}{2}CF$. On the contrary, the corrected Bianchi identity (12) implies the local form for the flux $\tilde{F}$

$$\tilde{F} = \mathrm{d}\tilde{C} - \frac{1}{2}CF + \omega_7(A) - \frac{1}{4}\omega_3(A)\,\mathrm{tr}\,R^2, \tag{17}$$

where A is the spin connection for the Riemann curvature R and $\omega_n(A)$ denotes the Chern-Simons n-form for A such that $\mathrm{d}\omega_{2n-1}(A) = \mathrm{tr}\,R^n$; see the Appendix for their explicit form.

Following the same trick as for the heterotic case, we may treat at first the curvature R simply as the field strength for a generic Yang-Mills gauge field A taking values in some algebra $\mathfrak{g}$, though naturally it will eventually be necessary to identify $\mathfrak{g}$ with $spin(7)$ and express A in terms of gravitational d.o.f.[3] The heterotic generalized geometric prescription would then lead us to consider structures over a generalized tangent space of the form

$$T \oplus \mathfrak{g} \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus T^* \otimes \Lambda^7 T^*, \tag{18}$$

in other words, replacing the tangent bundle component of (13) with the Atiyah algebroid.

In what follows, however, we will restrict ourselves to simpler versions of this problem, which will still suffice to show that the situation is more complex than the one of heterotic generalized geometry. In particular, we will find gauge algebras that are best described in terms of higher order L_∞ algebras.

In Sec. II we will first look at a Bianchi identity

$$\mathrm{d}\tilde{F}_5 = \mathrm{tr}\, R^3, \tag{19}$$

where $\tilde{F}_5$ is a five-form which, even though it has no immediate physical motivation, is easier to handle and already displays the important features we wish to demonstrate. The corresponding generalized tangent space will be of the form

$$T \oplus \mathfrak{g} \oplus \Lambda^3 T^*. \tag{20}$$

We will then move on in Sec. III to the case

$$\mathrm{d}\tilde{F}_7 = \mathrm{tr}\, R^4, \tag{21}$$

where now $\tilde{F}_7$ is genuinely a seven-form, and so this corresponds to a special case of (17). The generalized tangent space is then

$$T \oplus \mathfrak{g} \oplus \Lambda^5 T^*. \tag{22}$$

In both cases we will find that the Bianchi identities imply a gauge algebra which cannot be expressed in terms of simply a Courant bracket. Instead it is of the type of the L_∞ field theory formalism of [44]. The analysis of the complete Bianchi identity implied by the corrected eleven-dimensional supergravity will be left for future work.

As an aside, we expect that similar conclusions would hold for $(2n-1)$-form fluxes $\tilde{F}_{(2n-1)}$ satisfying

$$\mathrm{d}\tilde{F}_{(2n-1)} = \mathrm{tr}\, R^n, \tag{23}$$

based on generalized tangent spaces

$$E \sim T \oplus \mathfrak{g} \oplus \Lambda^{(2n-3)} T^*, \tag{24}$$

though we will not attempt to prove this here. Note as well that in all these cases the Bianchi identities, when viewed in cohomology classes, correspond to obstructions to this construction, namely, the requirement that nth Chern character of the gauge vector bundle is trivial.

We also remark that the fact that the gauge algebras we are examining fit into the L_∞ setting is not surprising. It has already been shown that the "higher Courant algebroids" of the type $T \oplus \Lambda^p T^*$ have an associated L_∞ algebra [45], and the extra terms we are considering arise from adding an invariant polynomial to the Bianchi, which in the context of chiral anomalies lead to the well-known "descent equations" derived from the extended Cartan homotopy [46], with many of the terms in the brackets we present here being directly related to the extra homotopy operator.

E. L_∞ algebras and field theory

L_∞ algebras or strong homotopy Lie algebras, introduced in [47,48] to the physics context, have found numerous applications in both mathematics and physics; see [49] for a recent review of the field. In particular, they can be found in the theories of Courant algebroids and generalized geometry. Courant algebroids were shown to have an L_3 algebra in [50]. In the case of heterotic Courant algebroids, it has recently been proven that this algebra is directly connected to the physical problem of finding the moduli of finite deformations of the Strominger-Hull system [51]. Higher Courant algebroids over a space $T \oplus \Lambda^p T^*$ were proven to have L_∞ algebras for arbitrary p in [45] using a derived bracket construction, and in [52] a large class of "Leibniz algebroids" (a Leibniz bracket being a generalization of the Lie derivative that still satisfies the Leibniz identity but is not necessarily antisymmetric), of which exceptional generalized geometries are examples, were likewise shown to admit L_∞ algebras. This later point was further explored in [53,54], where the correct L_∞ algebra was demonstrated to follow from interpreting the M theory geometries as dg-symplectic manifolds. More broadly speaking, the higher structures that feature in string and M theory are known to be classified by super homotopy theory; see the review [55] and references therein. In particular, note that the heterotic generalized geometry that we described earlier corresponds to a "string Lie 2-algebra" [56], while anomaly cancelation in M theory was examined in this formalism in [57]. There has also been much current work showing how such structures appear in the related fields of double/exceptional field theory, for example, in [58–70].

Recently, in [44] (see also [71]) many of these ideas were systematized in a manner to be more immediately applicable to physics, by introducing the notion of "L_∞ gauge field theories." It is this approach that we will be following,

[3]Though an intriguing possibility is to consider a larger gauge group that could accommodate the flux d.o.f., such as taking $\mathfrak{g} = su(8)$ and relating A to the $SU(8)$ connections implied by supersymmetry. This could naturally give rise to a Bianchi identity which includes higher derivative flux terms.

and we start by quickly reviewing some of the concepts that will be relevant here.

There are a few alternative ways of defining an L_∞ algebra. Following the conventions of [44], we will be working with the "ℓ picture" in terms of graded-antisymmetric multilinear brackets. Given a $\mathbb{Z}$-graded vector space

$$V = \oplus_{i\in\mathbb{Z}} V_i, \tag{25}$$

where the subscript denotes the degree, one defines an L_∞ algebra by endowing it with a series of multilinear products $\ell_n\colon \Lambda^n V \mapsto V$. These brackets are of degree $n-2$, i.e., for inputs $v_i \in V$, the total degree of $\ell_n(v_1, \ldots, v_n)$ is

$$\deg \ell_n(v_1, \ldots, v_n) = n - 2 + \sum_{i=1}^{n} \deg v_i. \tag{26}$$

They are also graded antisymmetric,

$$\ell_n(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = (-1)^{|\sigma|}\epsilon(\sigma)\ell_n(v_1, \ldots, v_n), \tag{27}$$

for some permutation σ and where ϵ is the Koszul sign for the given permutation and grading of V. Crucially, for each n the brackets must also satisfy a Jacobi identity "up to higher homotopies," namely, the generalized Jacobi identities

$$\sum_{i+j=n+1}(-1)^{i(j-1)}\sum_{\sigma}(-1)^{|\sigma|}\epsilon(\sigma)\ell_j(\ell_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0, \tag{28}$$

or in abbreviated form

$$\sum_{i+j=n+1}(-1)^{(j-1)i}\ell_j\ell_i = 0. \tag{29}$$

Explicitly, this gives at level one

$$\ell_1(\ell_1(v)) = 0, \tag{30}$$

which shows that the graded vector space V of an L_∞ algebra forms a chain complex with the operator ℓ_1. Level two establishes ℓ_1 as a derivation on ℓ_2,

$$\ell_1(\ell_2(v_1, v_2)) = \ell_2(\ell_1(v_1), v_2) + (-1)^{|v_1|}\ell_2(v_1, \ell_1(v_2)). \tag{31}$$

Level three would be the "traditional" Jacobi identity, where the L_∞ algebra starts to diverge from the normal graded Lie algebras

$$\begin{aligned}
&\ell_1(\ell_3(v_1, v_2, v_3)) + \ell_3(\ell_1(v_1), v_2, v_3)\\
&\quad + (-1)^{|v_1|}\ell_3(v_1, \ell_1(v_2), v_3)\\
&\quad + (-1)^{|v_1|+|v_2|}\ell_3(v_1, v_2, \ell_1(v_3)) + \ell_2(\ell_2(v_1, v_2), v_3)\\
&\quad + (-1)^{(|v_2|+|v_3|)|v_1|}\ell_2(\ell_2(v_2, v_3), v_1)\\
&\quad + (-1)^{(|v_1|+|v_2|)|v_3|}\ell_2(\ell_2(v_3, v_1), v_2) = 0,
\end{aligned} \tag{32}$$

and so on.

Proceeding with the proposal of [44] for a gauge field theory, one considers spaces of type[4]

$$V = \oplus_{i>0} V_i \oplus V_0 \oplus V_{-1}. \tag{33}$$

An important point here is that, since one allows a space with negative grading, there is *a priori* no guarantee that the L_∞ algebra will ever terminate even for a finite number of V_i. This is in contrast to L_n algebras, defined such that the graded vector space is concentrated in degrees 0 to $n-1$ and therefore all brackets of degree higher than $n+1$ vanish trivially as a consequence of (26). However, we will see that the cases we consider in the next sections do indeed truncate and there is a finite number of brackets to consider.

In order to find the physical meaning of (33), one identifies elements $X \in V_0$ with gauge parameters and $\Psi \in V_{-1}$ are taken to be the gauge fields. Elements of $\oplus_{i>0} V_i$ are to be thought of as making up a tower of trivial gauge parameters. An L_∞ gauge field theory may then be defined with the symmetries given by

$$\delta_X \Psi = \Sigma_n \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\ell_{n+1}(X, \Psi^n), \tag{34}$$

satisfying a gauge algebra

$$\begin{aligned}
&[\delta_{X_1}, \delta_{X_2}]\Psi = \delta_{[X_1,X_2]}\Psi,\\
&\quad [X_1, X_2] = \Sigma_n \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\ell_{n+2}(X_1, X_2, \Psi^n).
\end{aligned} \tag{35}$$

Note, in particular, that in this formalism the gauge algebra of the parameters is permitted to depend explicitly on the fields. In what follows we will show how the higher curvature problem we are considering fits precisely into this picture.

We also remark that another way the gauge fields may appear in Courant brackets is via their "twisting." One can think of the fields as defining connections that split the exact sequences that define the generalized tangent bundle, i.e., they make explicit the isomorphism $E \sim T \oplus \ldots$. Under that map, the Courant bracket then becomes twisted by the curvature of the connection—for example, for the exact Courant algebroid case the B field splits the defining

[4] In [44] an extra subspace V_{-2} is also allowed, corresponding to the equations of motion, but we will not make use of it here.

sequence $T^* \to E \to T$ through a map $e^B\colon x+\lambda \mapsto x + \lambda + i_x B$ (see [2]), and the bracket (2) then gets modified by $[e^B X_1, e^B X_2] = e^B [X_1, X_2] + i_{x_1} i_{x_2} H$. Note also that if B is closed, physically "pure gauge," then it is a symmetry of the bracket, extending the usual diffeomorphism invariance of the Lie bracket. One would expect that something similar for the higher order brackets we define in the following sections, it should be possible to twist them with the field strengths of the gauge fields, and pure gauge finite transformations should leave them invariant. However, we do not attempt to verify this here; a full study of the twisted structure of the higher order generalized tangent bundles, their patching rules (which would involve computing finite gauge transformations of the Chern-Simons forms), the automorphisms of these L_∞ structures, etc. will be left for future work.

II. $\mathrm{d}\tilde{F}_5 = \mathrm{tr}\, R^3$

We begin by considering a theory with a globally defined five-form flux $\tilde{F}$ and a Yang-Mills $\mathfrak{g}$-valued potential A with corresponding field strength R such that

$$\mathrm{d}_A R = 0, \tag{36}$$

$$\mathrm{d}\tilde{F}_5 = \mathrm{tr}\, R^3. \tag{37}$$

We can thus define a four-form potential $\tilde{C}$ for the flux by

$$\tilde{F}_5 = \mathrm{d}\tilde{C}_4 + \omega_5(A). \tag{38}$$

Much like the B field in heterotic theory, we find that since $\tilde{F}_5$ is gauge invariant, $\tilde{C}_4$ must transform to compensate for a variation of the Chern-Simons five-form $\omega_5(A)$. That is, if $\Lambda \in \mathfrak{g}$ parametrizes an infinitesimal gauge transformation, we must have that locally

$$\begin{aligned} \mathrm{d}\delta_\Lambda \tilde{C}_4 &= -\delta_\Lambda \omega_5(A) = -\mathrm{d}\omega_4^1(\Lambda, A) \\ &= -\mathrm{d}\,\mathrm{tr}\,\mathrm{d}\Lambda\left(A\mathrm{d}A + \frac{1}{2}A^3\right), \end{aligned} \tag{39}$$

from the properties of the Chern-Simons forms (see the Appendix). It is also clear that $\tilde{F}$ remains invariant under shifts of $\tilde{C}_4$ by a closed four-form, locally parametrized by the exterior derivative of some three-form σ. Together with a diffeomorphism symmetry parametrized by some vector field x, we have that the potentials obey the infinitesimal gauge transformations

$$\begin{aligned} \delta_X A &= \mathcal{L}_x A - \mathrm{d}\Lambda - [A, \Lambda], \\ \delta_X \tilde{C}_4 &= \mathcal{L}_x \tilde{C}_4 - \mathrm{d}\sigma - \mathrm{tr}\,\mathrm{d}\Lambda\left(A\mathrm{d}A + \frac{1}{2}A^3\right), \end{aligned} \tag{40}$$

where δ_X denotes the combined infinitesimal diffs, gauge, and shifts in terms of parameters $X = x + \Lambda + \sigma$. This therefore suggests a generalized tangent space

$$E = T \oplus \mathfrak{g} \oplus \Lambda^3 T^*. \tag{41}$$

So far, this precisely matches the procedure for constructing the heterotic generalized geometry; see, for example, [19]. However, let us look at how the algebra of transformations δ_X closes when acting on the fields. Taking two parameters $X_1, X_2 \in E$, we find that

$$\begin{aligned} [\delta_{X_1}, \delta_{X_2}]A &= \mathcal{L}_{[x_1,x_2]}A - \mathrm{d}([\Lambda_1, \Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1) \\ &\quad - [A, [\Lambda_1, \Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1], \\ [\delta_{X_1}, \delta_{X_2}]\tilde{C}_4 &= \mathcal{L}_{[x_1,x_2]}\tilde{C}_4 \\ &\quad - \mathrm{d}\left(i_{x_1}\mathrm{d}\sigma_2 - i_{x_2}\mathrm{d}\sigma_1 + \frac{1}{2}\mathrm{d}i_{x_1}\sigma_2 - \frac{1}{2}\mathrm{d}i_{x_2}\sigma_1\right) \\ &\quad - \mathrm{tr}\,\mathrm{d}([\Lambda_1, \Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1) \\ &\quad \times \left(A(\mathrm{d}A)^2 + \frac{1}{2}A^3\right) \\ &\quad + \mathrm{d}\,\mathrm{tr}(\Lambda_1\mathrm{d}\Lambda_2\mathrm{d}A - \Lambda_2\mathrm{d}\Lambda_1\mathrm{d}A), \end{aligned} \tag{42}$$

so we have that indeed the algebra closes on a parameter X_3 given by

$$\begin{aligned} [\delta_{X_1}, \delta_{X_2}](A + \tilde{C}_4) &= \delta_{X_3}(A + \tilde{C}_4), \\ X_3 &= [x_1, x_2] + [\Lambda_1, \Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1 \\ &\quad + i_{x_1}\mathrm{d}\sigma_2 - i_{x_2}\mathrm{d}\sigma_1 + \frac{1}{2}\mathrm{d}i_{x_1}\sigma_2 - \frac{1}{2}\mathrm{d}i_{x_2}\sigma_1 \\ &\quad - \mathrm{tr}(\Lambda_1\mathrm{d}\Lambda_2\mathrm{d}A - \Lambda_2\mathrm{d}\Lambda_1\mathrm{d}A) \in E, \end{aligned} \tag{43}$$

but note that this depends not just on X_1 and X_2 but also explicitly on the fields. Therefore, unlike the previous examples in generalized geometry, the gauge algebra does not define for us a bracket over just the space E. It does, nonetheless, fit into the L_∞ field theory setting.

A. An L_∞ gauge algebra for R^3

Let us then introduce the graded vector space:

$$V = V_3 \oplus V_2 \oplus V_1 \oplus V_0 \oplus V_{-1}, \tag{44}$$

where[5]

[5]A more "generalized" treatment in the sense of [29] would presumably involve introducing a space of "generalized frames" for E [that is a subspace of $\mathrm{End}(E)$ that preserves the defining generalized structures—$O(d,d)$ in NSNS generalized geometry, $E_{7(7)}$ in exceptional generalized geometry, etc.] and identifying its "geometric subspace" with V_{-1}, which is used to construct the physical brackets.

$$V_3 = C^\infty(M), \qquad V_2 = T^*, \qquad V_1 = \Lambda^2 T^*,$$
$$V_0 = E = T \oplus \mathfrak{g} \oplus \Lambda^3 T^*, \qquad V_{-1} = T^* \otimes \mathfrak{g} \oplus \Lambda^4 T^*, \tag{45}$$

and we label elements in the subspaces as

$$\xi \in V_3 \oplus V_2 \oplus V_1, \qquad X = x + \Lambda + \sigma \in V_0,$$
$$\Psi = A + \tilde{C} \in V_{-1}. \tag{46}$$

We will then endow V with a series of multilinear brackets to define an L_∞ algebra that will realize the gauge algebra (43). Terms in the brackets involving only elements in $V_{i>0}$ or the vector + three–form part of V_0 will be necessarily the ones in [45], but we must introduce new products for terms involving the Lie algebra $\mathfrak{g}$. Comparing with (35), we can directly read off some of the multi-brackets, since we must insist that picking particular elements $A + \tilde{C}_4 = \Psi \in V_{-1}$ corresponds to specifying the data for the supergravity gauge fields, i.e., that they satisfy a gauge algebra

$$\delta_X \Psi = \ell_1(X) + \ell_2(X, \Psi) - \frac{1}{2}\ell_3(X, \Psi, \Psi) - \frac{1}{6}\ell_4(X, \Psi, \Psi, \Psi), \tag{47}$$

with

$$[\delta_{X_1}, \delta_{X_2}]\Psi = \delta_{X_3}\Psi, \quad X_3 = \ell_2(X_1, X_2) + \ell_3(X_1, X_2, \Psi), \tag{48}$$

such that it precisely matches (40) and (43), respectively.

Several more brackets are necessary to complete the algebra, which can be obtained from the requirement that they satisfy the generalized Jacobi identities (28). It is possible to do this exhaustively term-by-term since, due to both the grading of the vector space V and the subdivisions inside V_0 and V_{-1}, many will vanish trivially. For example, we will see that all brackets ℓ_n of level $n > 2$ whose image is in V_{-1} actually only map to the four-form subspace, i.e., they are $\tilde{C}$-type objects. On the other hand, the brackets of level $n > 2$ that take an object in V_{-1} as input are all independent of $\tilde{C}$. So chaining together those sets of brackets is trivial.

Note that due to the grading and symmetry properties of the ℓ_n brackets (27), products involving multiple factors of X_i will always have to be antisymmetrized, and products involving products of Ψ_i will always have to be symmetrized. We denote this explicitly using the typical index notation of symmetrizers and antisymmetrizers.

We find that the (nontrivial) L_∞ products are then:

at level one

$$\ell_1(\xi) = \mathrm{d}\xi, \quad \ell_1(X) = -\mathrm{d}\Lambda - \mathrm{d}\sigma, \quad \ell_1(\Psi) = 0, \tag{49}$$

at level two

$$\ell_2(X, \xi) = \frac{1}{2}\mathcal{L}_x \xi, \tag{50a}$$

$$\ell_2(X_1, X_2) = [x_1, x_2] + [\Lambda_1, \Lambda_2] + \mathcal{L}_{x_1}\Lambda_2 - \mathcal{L}_{x_2}\Lambda_1 + \mathcal{L}_{x_1}\sigma_2 - \mathcal{L}_{x_2}\sigma_1 - \frac{1}{2}\mathrm{d}i_{x_1}\sigma_2 + \frac{1}{2}\mathrm{d}i_{x_2}\sigma_1, \tag{50b}$$

$$\ell_2(X, \Psi) = \mathcal{L}_x \Psi - [A, \Lambda], \tag{50c}$$

at level three

$$\ell_3(\xi, X_1, X_2) = -\frac{1}{6}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1, x_2]})\xi, \tag{51a}$$

$$\ell_3(X_1, X_2, X_3) = -\frac{1}{2}(i_{x_{[1}}\mathcal{L}_{x_2} + i_{[x_{[1}, x_2]} + i_{x_{[1}}i_{x_2}\mathrm{d})\sigma_{3]}, \tag{51b}$$

$$\ell_3(X_1, X_2, \Psi) = -2\,\mathrm{tr}\,\Lambda_{[1}\mathrm{d}\Lambda_{2]}\mathrm{d}A, \tag{51c}$$

$$\ell_3(X, \Psi_1, \Psi_2) = 2\,\mathrm{tr}\,\mathrm{d}\Lambda A_{(1}\mathrm{d}A_{2)}, \tag{51d}$$

at level four

$$\ell_4(X_1, X_2, X_3, X_4) = -12\,\mathrm{tr}\Lambda_{[1}\Lambda_2\Lambda_3\mathrm{d}\Lambda_{4]}, \tag{52a}$$

$$\ell_4(X_1, X_2, X_3, \Psi) = 6\,\mathrm{tr}\Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_{3]}A - \frac{3}{2}i_{x_{[1}}\ell_3(X_2, X_{3]}, \Psi), \tag{52b}$$

$$\ell_4(X, \Psi_1, \Psi_2, \Psi_3) = 3\,\mathrm{tr}\,\mathrm{d}\Lambda A_{(1}A_2 A_{3)}, \tag{52c}$$

at level five

$$\ell_5(X_1, \ldots, X_5) = -12\,\mathrm{tr}\,\Lambda_{[1}\Lambda_2\Lambda_3\Lambda_4\Lambda_{5]} - \frac{5}{2}i_{x_{[1}}\ell_4(X_2, X_3, X_4, X_{4]}) - \frac{1}{3}i_{x_{[1}}i_{x_2}\ell_3(X_3, X_4, X_{5]}), \tag{53a}$$

$$\ell_5(X_1, X_2, X_3, X_4, \Psi) = -2i_{x_{[1}}\ell_4(X_2, X_3, X_{4]}, \Psi) - 2i_{x_{[1}}i_{x_2}\ell_3(X_3, X_{4]}, \Psi), \tag{53b}$$

and finally at level six

$$\ell_6(X_1, \ldots, X_5, \Psi) = \frac{5}{3}i_{x_{[1}}i_{x_2}\ell_4(X_3, X_4, X_{5]}, \Psi) + \frac{5}{2}i_{x_{[1}}i_{x_2}i_{x_3}\ell_3(X_4, X_{5]}, \Psi). \tag{54}$$

All other brackets vanish. Even though there is a large number of them, it should be clear from the form of the nontrivial brackets that they are straightforward to obtain by iterating through the generalized Jacobi identities. As an example, consider the fourth level identity when the inputs are in $(V_0)^3 \otimes V_{-1}$. This will involve the bracket $\ell_4(X_1, X_2, X_3, \Psi)$ given in (52b), which cannot be read directly from the gauge algebra (and has not been previously derived in the literature); instead we infer it from the generalized Jacobi identity. We compute from the lower order brackets

$$\begin{aligned}\ell_3(\ell_2(X_{[1}, X_2), X_{3]}, \Psi) &= -2\,\mathrm{tr}\,\mathcal{L}_{x_{[1}}(\Lambda_2 \mathrm{d}\Lambda_3)\mathrm{d}A + 2\,\mathrm{tr}\,\Lambda_{[1}\mathrm{d}\Lambda_2\Lambda_{3]}\mathrm{d}A,\\ \ell_3(X_{[1}, X_2, \ell_2(X_{3]}, \Psi)) &= -2\,\mathrm{tr}\,\Lambda_{[1}\mathrm{d}\Lambda_2(\mathcal{L}_{x_{3]}}\mathrm{d}A - \mathrm{d}[A, \Lambda_{3]}]),\\ \ell_2(X_{[1}, \ell_3(X_2, X_{3]}, \Psi)) &= -2\mathcal{L}_{x_{[1}}\,\mathrm{tr}\,\Lambda_2\mathrm{d}\Lambda_{3]}\mathrm{d}A + \mathrm{d}i_{x_{[1}}\,\mathrm{tr}\,\Lambda_2\mathrm{d}\Lambda_{3]}\mathrm{d}A,\end{aligned} \tag{55}$$

and noting that $\ell_4(\ell_1(X_1), X_2, X_3, \Psi)$ and $\ell_4(X_1, X_2, X_3, \ell_1(\Psi))$ vanish identically, the generalized Jacobi identity then reads

$$\begin{aligned}\ell_1(\ell_4(X_1, X_2, X_3, \Psi)) &= 3\ell_4(\ell_1(X_{[1}), X_2, X_{3]}, \Psi) + 3\ell_2(X_{[1}, \ell_3(X_2, X_{3]}, \Psi))\\ &\quad - 3\ell_3(X_{[1}, X_2, \ell_2(X_{3]}, \Psi)) - 3\ell_3(\ell_2(X_{[1}, X_2), X_{3]}, \Psi)\\ &= 6\mathrm{d}\,\mathrm{tr}\,\Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_{3]}A + 3\mathrm{d}i_{x_{[1}}\,\mathrm{tr}\,\Lambda_2\mathrm{d}\Lambda_{3]}\mathrm{d}A\\ &= 6\mathrm{d}\,\mathrm{tr}\,\Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_{3]}A - \frac{3}{2}\mathrm{d}i_{x_{[1}}\ell_3(X_2, X_{3]}, \Psi),\end{aligned} \tag{56}$$

which is consistent with (52b).

III. $\mathrm{d}\tilde{F}_7 = \mathrm{tr}\,R^4$

The analysis for a seven-form flux follows in much the same way as the five-form case we just considered, it is simply more computationally intensive. We again introduce a $\mathfrak{g}$-valued one-form potential A with field strength R, and a globally defined seven-form $\tilde{F}$ such that

$$\mathrm{d}\tilde{F}_7 = \mathrm{tr}\,R^4, \tag{57}$$

and so locally we define a six-form potential $\tilde{C}$ by

$$\tilde{F}_7 = \mathrm{d}\tilde{C}_6 + \omega_7(A). \tag{58}$$

As previously remarked, this is a toy example for the supergravity theory of Sec. I C when one truncates Eq. (17). Now, the gauge invariance of $\tilde{F}_7$ once again implies that $\tilde{C}_6$ must vary as

$$\begin{aligned}\mathrm{d}\delta_\Lambda\tilde{C}_6 &= -\delta_\Lambda\omega_7(A) = -\mathrm{d}\omega_6^1(\Lambda, A)\\ &= -\mathrm{d}\,\mathrm{tr}\,\mathrm{d}\Lambda\bigg(A(\mathrm{d}A)^2 + \frac{2}{5}(A^3\mathrm{d}A + \mathrm{d}AA^3 + A^5)\\ &\quad + \frac{1}{5}(A^2\mathrm{d}AA + A\mathrm{d}AA^2)\bigg),\end{aligned} \tag{59}$$

for some gauge parameter $\Lambda \in \mathfrak{g}$ [from Eq. (A8) in the Appendix]. We also have the usual diffeomorphism $\mathcal{L}_x$ and shift symmetries $\mathrm{d}\sigma$ generated by, respectively, a vector field $x \in T$ and a five-form $\sigma \in \Lambda^5 T^*$. We are thus led to consider a generalized tangent space

$$\begin{aligned}E &= T \oplus \mathfrak{g} \oplus \Lambda^5 T^*,\\ X &= x + \Lambda + \sigma \in E.\end{aligned} \tag{60}$$

This is a close cousin of the $SL(8,\mathbb{R}) \times \mathbb{R}^+$ "half-exceptional" generalized geometry obtained by truncating the $E_{7(7)} \times \mathbb{R}^+$ case, as was described in [72]. We can then group the infinitesimal symmetries as

$$\begin{aligned}\delta_X &= \text{infinitesimal diffs, gauge and shifts}\\ \delta_X A &= \mathcal{L}_x A - \mathrm{d}\Lambda - [A, \Lambda]\\ \delta_X\tilde{C}_6 &= \mathcal{L}_x\tilde{C}_6 - \mathrm{d}\sigma\\ &\quad - \mathrm{tr}\,\mathrm{d}\Lambda\bigg(A(\mathrm{d}A)^2 + \frac{2}{5}(A^3\mathrm{d}A + \mathrm{d}AA^3 + A^5)\\ &\quad + \frac{1}{5}(A^2\mathrm{d}AA + A\mathrm{d}AA^2)\bigg).\end{aligned} \tag{61}$$

As in the R^3 case, we find that the gauge algebra closes on terms that explicitly depend on the gauge fields. Taking two parameters X_1, $X_2 \in E$, we have

$$
\begin{aligned}
[\delta_{X_1},\delta_{X_2}]A &= \mathcal{L}_{[x_1,x_2]}A - \mathrm{d}([\Lambda_1,\Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1) - [A,[\Lambda_1,\Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1],\\
[\delta_{X_1},\delta_{X_2}]\tilde{C}_6 &= \mathcal{L}_{[x_1,x_2]}\tilde{C}_6 - \mathrm{d}\left(i_{x_1}\mathrm{d}\sigma_2 - i_{x_2}\mathrm{d}\sigma_1 + \frac{1}{2}\mathrm{d}i_{x_1}\sigma_2 - \frac{1}{2}\mathrm{d}i_{x_2}\sigma_1\right)\\
&\quad - \mathrm{tr}\,\mathrm{d}([\Lambda_1,\Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1)\left(A(\mathrm{d}A)^2 + \frac{2}{5}(A^3\mathrm{d}A + \mathrm{d}AA^3 + A^5) + \frac{1}{5}(A^2\mathrm{d}AA + A\mathrm{d}AA^2)\right)\\
&\quad + \mathrm{d}\,\mathrm{tr}\left(\Lambda_1\left(\mathrm{d}\Lambda_2\mathrm{d}A\mathrm{d}A + \frac{3}{5}\mathrm{d}\Lambda_2\mathrm{d}(A^3) + \frac{1}{5}\mathrm{d}(A^2\mathrm{d}\Lambda_2 A)\right) - \Lambda_2\left(\mathrm{d}\Lambda_1\mathrm{d}A\mathrm{d}A + \frac{3}{5}\mathrm{d}\Lambda_1\mathrm{d}(A^3) + \frac{1}{5}\mathrm{d}(A^2\mathrm{d}\Lambda_1 A)\right)\right),
\end{aligned}
\tag{62}
$$

and therefore, the algebra of the gauge parameters is

$$
\begin{aligned}
[X_1,X_2] = [x_1,x_2] &+ [\Lambda_1,\Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1\\
&+ i_{x_1}\mathrm{d}\sigma_2 - i_{x_2}\mathrm{d}\sigma_1 + \frac{1}{2}\mathrm{d}i_{x_1}\sigma_2 - \frac{1}{2}\mathrm{d}i_{x_2}\sigma_1\\
&- \mathrm{tr}\left(\Lambda_1\left(\mathrm{d}\Lambda_2\mathrm{d}A\mathrm{d}A + \frac{3}{5}\mathrm{d}\Lambda_2\mathrm{d}(A^3) + \frac{1}{5}\mathrm{d}(A^2)\mathrm{d}\Lambda_2 A - \frac{1}{5}A^2\mathrm{d}\Lambda_2\mathrm{d}A\right)\right.\\
&\left. - \Lambda_2\left(\mathrm{d}\Lambda_1\mathrm{d}A\mathrm{d}A + \frac{3}{5}\mathrm{d}\Lambda_1\mathrm{d}(A^3) + \frac{1}{5}\mathrm{d}(A^2)\mathrm{d}\Lambda_1 A - \frac{1}{5}A^2\mathrm{d}\Lambda_1\mathrm{d}A\right)\right) \in E.
\end{aligned}
\tag{63}
$$

Let us then see how this fits with the L_∞ formalism.

A. An L_∞ gauge algebra for R^4

We start by building a seven-term graded vector space

$$V = V_5 \oplus V_4 \oplus V_3 \oplus V_2 \oplus V_1 \oplus V_0 \oplus V_{-1}, \tag{64}$$

where

$$
\begin{aligned}
&V_5 = C^\infty(M), \qquad V_4 = T^*, \qquad V_3 = \Lambda^2 T^*,\\
&V_2 = \Lambda^3 T^*, \qquad V_1 = \Lambda^4 T^*,\\
&V_0 = E = T \oplus \mathfrak{g} \oplus \Lambda^5 T^*, \qquad V_{-1} = T^* \otimes \mathfrak{g} \oplus \Lambda^6 T^*,
\end{aligned}
\tag{65}
$$

whose elements we will generically label as

$$
\begin{aligned}
&\xi \in V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5, \quad X = x + \Lambda + \sigma \in V_0,\\
&\Psi = A + \tilde{C} \in V_{-1}.
\end{aligned}
\tag{66}
$$

We now construct the L_∞ products as before. The terms in the products which are independent of V_{-1} or the $\mathfrak{g}$ part of V_0 must reproduce the results of [45]. Again we read off some of the brackets by comparing (61) and (63) with (34) and (35), respectively. Then picking a specific point Ψ in the space V_{-1} will correspond to specifying the supergravity data by demanding that the gauge algebra obeyed by Ψ is

$$
\begin{aligned}
\delta_X\Psi = \ell_1(X) + \ell_2(X,\Psi) - \frac{1}{2}\ell_3(X,\Psi^2) - \frac{1}{6}\ell_4(X,\Psi^3)\\
+ \frac{1}{24}\ell_5(X,\Psi^4) + \frac{1}{120}\ell_6(X,\Psi^5),
\end{aligned}
\tag{67}
$$

and

$$
\begin{aligned}
[X_1,X_2] = \ell_2(X_1,X_2) + \ell_3(X_1,X_2,\Psi) - \frac{1}{2}\ell_4(X_1,X_2,\Psi^2)\\
- \frac{1}{6}\ell_5(X_1,X_2,\Psi^3),
\end{aligned}
\tag{68}
$$

such that its components $\Psi = A + \tilde{C}_6$ match (61) and (63) by construction.

We can then use the generalized Jacobi conditions to complete the algebra. As in the previous section, we can verify the relations by exhaustively going through every term of (28), for each level n and for each possible set of inputs for the brackets, since the extra substructure of the vector spaces V_0 and V_{-1} means that many of those terms vanish trivially and thus the method becomes tractable. The full list of nonvanishing multilinear brackets is nonetheless still rather long. We find the following:

at level one (these show that V is a differential chain complex)

$$\ell_1(\xi) = \mathrm{d}\xi, \quad \ell_1(X) = -\mathrm{d}\Lambda - \mathrm{d}\sigma, \quad \ell_1(\Psi) = 0, \tag{69}$$

at level two (these include the normal gauge transformations)

$$\ell_2(X,\xi) = \frac{1}{2}\mathcal{L}_x\xi, \tag{70a}$$

$$\ell_2(X_1,X_2) = [x_1,x_2] + [\Lambda_1,\Lambda_2] + \mathcal{L}_{x_1}\Lambda_2 - \mathcal{L}_{x_2}\Lambda_1 + \mathcal{L}_{x_1}\sigma_2 - \mathcal{L}_{x_2}\sigma_1 - \frac{1}{2}\mathrm{d}i_{x_1}\sigma_2 + \frac{1}{2}\mathrm{d}i_{x_2}\sigma_1, \tag{70b}$$

$$\ell_2(X,\Psi) = \mathcal{L}_x\Psi - [A,\Lambda], \tag{70c}$$

at level three (this is the level where the usual Jacobi identity breaks and one is led to use the higher formalism)

$$\ell_3(\xi,X_1,X_2) = -\frac{1}{6}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1,x_2]})\xi, \tag{71a}$$

$$\ell_3(X_1,X_2,X_3) = -\frac{1}{2}(i_{x_{[1}}\mathcal{L}_{x_2} + i_{[x_{[1},x_2]} + i_{x_{[1}}i_{x_2}\mathrm{d})\sigma_{3]}, \tag{71b}$$

at level four

$$\ell_4(X_1,X_2,\Psi_1,\Psi_2) = 4\,\mathrm{tr}\,\Lambda_{[1}\mathrm{d}\Lambda_{2]}\mathrm{d}A_{(1}\mathrm{d}A_{2)}, \tag{72a}$$

$$\ell_4(X,\Psi_1,\Psi_2,\Psi_3) = 6\,\mathrm{tr}\,\mathrm{d}\Lambda A_{(1}\mathrm{d}A_2\mathrm{d}A_{3)}, \tag{72b}$$

at level five

$$\ell_5(\xi,X_1,X_2,X_3,X_4) = -\frac{1}{5}i_{x_{[1}}i_{x_2}\ell_3(\xi,X_3,X_{4]}), \tag{73a}$$

$$\ell_5(X_1,\ldots,X_5) = -\frac{1}{3}i_{x_{[1}}i_{x_2}\ell_3(X_3,X_4,X_{5]}), \tag{73b}$$

$$\ell_5(X_1,X_2,X_3,X_4,\Psi) = -\frac{24}{5}\,\mathrm{tr}\,(2\Lambda_{[1}\Lambda_2\Lambda_3\mathrm{d}\Lambda_{4]} - \Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_3\Lambda_{4]})\mathrm{d}A, \tag{73c}$$

$$\ell_5(X_1,X_2,X_3,\Psi_1,\Psi_2) = -\frac{3}{2}i_{x_{[1}}\ell_4(X_2,X_{3]},\Psi_1,\Psi_2) - \frac{12}{5}\,\mathrm{tr}\,(2\Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_{3]}\mathrm{d}A_{(1}A_{2)} + 3\Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_{3]}A_{(1}\mathrm{d}A_{2)} - \Lambda_{[1}\mathrm{d}\Lambda_2\Lambda_{3]}\mathrm{d}(A_{(1}A_{2)}) - \Lambda_{[1}\mathrm{d}\Lambda_2\mathrm{d}A_{(1}\Lambda_{3]}A_{2)} + \Lambda_{[1}\mathrm{d}\Lambda_2A_{(1}\Lambda_{3]}\mathrm{d}A_{2)}), \tag{73d}$$

$$\ell_5(X_1,X_2,\Psi_1,\Psi_2,\Psi_3) = \frac{12}{5}\,\mathrm{tr}\,(3\Lambda_{[1}\mathrm{d}\Lambda_{2]}\mathrm{d}(A_{(1}A_2A_{3)}) + \Lambda_{[1}\mathrm{d}(A_{(1}A_2\mathrm{d}\Lambda_{2]}A_{3)})), \tag{73e}$$

$$\ell_5(X,\Psi_1,\Psi_2,\Psi_3,\Psi_4) = -\frac{24}{5}\,\mathrm{tr}\,\mathrm{d}\Lambda(2A_{(1}A_2A_3\mathrm{d}A_{4)} + A_{(1}A_2\mathrm{d}A_3A_{4)} + A_{(1}\mathrm{d}A_2A_3A_{4)} + 2\mathrm{d}A_{(1}A_2A_3A_{4)}), \tag{73f}$$

at level six

$$\ell_6(X_1,\ldots,X_6) = -144\,\mathrm{tr}\,\Lambda_{[1}\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_{6]}, \tag{74a}$$

$$\ell_6(X_1,\ldots,X_5,\Psi) = 24\,\mathrm{tr}\,(2\Lambda_{[1}\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_{5]} + \Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\Lambda_{5]})A - \frac{5}{2}i_{x_{[1}}\ell_5(X_2,X_3,X_4,X_{5]},\Psi), \tag{74b}$$

$$\ell_6(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2) = -\frac{48}{5}\,\mathrm{tr}\,(\Lambda_{[1}\mathrm{d}(\Lambda_2\Lambda_3)\Lambda_{4]}A_{(1}A_{2)} + \Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_3A_{(1}\Lambda_{4]}A_{2)} + \Lambda_{[1}\mathrm{d}\Lambda_2A_{(1}\Lambda_3\Lambda_{4]}A_{2)}) - 2i_{x_{[1}}\ell_5(X_2,X_3,X_{4]},\Psi_1,\Psi_2) - 2i_{x_{[1}}i_{x_2}\ell_4(X_3,X_{4]},\Psi_1,\Psi_2), \tag{74c}$$

$$\ell_6(X_1,X_2,X_3,\Psi_1,\Psi_2,\Psi_3) = -\frac{3}{2}i_{x_{[1}}\ell_5(X_2,X_{3]},\Psi_1,\Psi_2,\Psi_3) - \frac{36}{5}\,\mathrm{tr}\,(3\Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_{3]}A_{(1}A_2A_{3)} - \Lambda_{[1}\Lambda_2A_{(1}A_2\mathrm{d}\Lambda_{3]}A_{3)}), \tag{74d}$$

$$\ell_6(X,\Psi_1,\ldots,\Psi_5) = -48\,\mathrm{tr}\,\mathrm{d}\Lambda A_{(1}A_2A_3A_4A_{5)}, \tag{74e}$$

at level seven (last level with a bracket acting just on the generalized tangent space E, the corresponding higher Courant algebroid of [45] would terminate here)

$$\ell_7(X_1,\ldots,X_7) = -144\,\mathrm{tr}\,\Lambda_{[1}\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\Lambda_{7]} - \frac{7}{2} i_{x_{[1}}\ell_6(X_2,\ldots,X_{7]}) + \frac{1}{3} i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4}\ell_3(X_5,X_6,X_{7]}), \tag{75a}$$

$$\ell_7(X_1,\ldots,X_6,\Psi) = -3 i_{x_{[1}}\ell_6(X_2,\ldots,X_{6]},\Psi) - 5 i_{x_{[1}} i_{x_2}\ell_5(X_3,X_4,X_5,X_{6]},\Psi), \tag{75b}$$

$$\ell_7(X_1,\ldots,X_5,\Psi_1,\Psi_2) = -\frac{5}{2} i_{x_{[1}}\ell_6(X_2,X_3,X_4,X_{5]},\Psi_1,\Psi_2) - \frac{10}{3} i_{x_{[1}} i_{x_2}\ell_5(X_3,X_4,X_{5]},\Psi_1,\Psi_2) - \frac{5}{2} i_{x_{[1}} i_{x_2} i_{x_3}\ell_4(X_4,X_{5]},\Psi_1,\Psi_2), \tag{75c}$$

$$\ell_7(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2,\Psi_3) = -2 i_{x_{[1}}\ell_6(X_2,X_3,X_{4]},\Psi_1,\Psi_2,\Psi_3) - 2 i_{x_{[1}} i_{x_2}\ell_5(X_3,X_{4]},\Psi_1,\Psi_2,\Psi_3), \tag{75d}$$

at level eight

$$\ell_8(X_1,\ldots,X_7,\Psi) = \frac{7}{2} i_{x_{[1}} i_{x_2}\ell_6(X_3,\ldots,X_{7]},\Psi) + \frac{35}{4} i_{x_{[1}} i_{x_2} i_{x_3}\ell_5(X_4,X_5,X_6,X_{7]},\Psi), \tag{76a}$$

$$\ell_8(X_1,\ldots,X_6,\Psi_1,\Psi_2) = \frac{5}{2} i_{x_{[1}} i_{x_2}\ell_6(X_4,X_5,X_{6]},\Psi_1,\Psi_2) + 5 i_{x_{[1}} i_{x_2} i_{x_3}\ell_5(X_4,X_5,X_{6]},\Psi_1,\Psi_2) + \frac{9}{2} i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4}\ell_4(X_5,X_{6]},\Psi_1,\Psi_2), \tag{76b}$$

$$\ell_8(X_1,\ldots,X_5,\Psi_1,\Psi_2,\Psi_3) = \frac{5}{3} i_{x_{[1}} i_{x_2}\ell_6(X_3,X_4,X_{5]},\Psi_1,\Psi_2,\Psi_3) + \frac{5}{2} i_{x_{[1}} i_{x_2} i_{x_3}\ell_5(X_4,X_{5]},\Psi_1,\Psi_2,\Psi_3), \tag{76c}$$

at level nine

$$\ell_9(X_1,\ldots,X_7,\Psi_1,\Psi_2) = -\frac{7}{6} i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4}\ell_5(X_5,X_6,X_{7]},\Psi_1,\Psi_2) - \frac{7}{4} i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4} i_{x_5}\ell_4(X_6,X_{7]},\Psi_1,\Psi_2), \tag{77a}$$

$$\ell_9(X_1,\ldots,X_6,\Psi_1,\Psi_2,\Psi_3) = -\frac{1}{2} i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4}\ell_5(X_5,X_{6]},\Psi_1,\Psi_2,\Psi_3), \tag{77b}$$

and finally at level ten

$$\ell_{10}(X_1,\ldots,X_7,\Psi_1,\Psi_2,\Psi_3) = -\frac{7}{6} i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4}\ell_6(X_5,X_6,X_{7]},\Psi_1,\Psi_2,\Psi_3) - \frac{7}{4} i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4} i_{x_5}\ell_5(X_6,X_{7]},\Psi_1,\Psi_2,\Psi_3). \tag{78}$$

All other brackets vanish. As in the previous section, we observe that most of terms in the brackets can be expressed recursively, which is to be expected since they were built by explicitly iterating through the generalized Jacobi identities. Note also that, as mentioned earlier, the terms that depend only on elements Λ and A are simply reproducing the (polarized) p-forms that result from the descent equations of the anomaly polynomial. For example, we have that

$$\omega_6^1(\Lambda, A) = \frac{1}{3!}\ell_4(\Lambda, A^3) - \frac{1}{4!}\ell_5(\Lambda, A^4) - \frac{1}{5!}\ell_6(\Lambda, A^5). \tag{79}$$

Despite no longer being able to describe these higher order gauge algebras in terms of just a Leibniz bracket on the generalized tangent space, we thus have that the extra structure of E is still enough to ensure that we can find an L_∞ algebra, and that this algebra has a finite number of brackets. And while there should not be much difficulty in adding the extra geometrical data that make up the physical d.o.f. such as the Riemannian metric, it will require further study to see whether this weaker differential structure will be enough to give a natural geometric description of the dynamics of higher-derivative-corrected supergravity.

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APPENDIX: CONVENTIONS AND CHERN-SIMONS FORMS

We mostly follow the conventions of [19], though we generally omit the wedge symbol for the product of differential forms.

Given a Lie algebra-valued one-form A we define its curvature by

$$R(A) = \mathrm{d}A + A^2, \tag{A1}$$

which satisfies

$$\mathrm{d}_A R = \mathrm{d}R + [A, R] = 0, \tag{A2}$$

and is invariant under the infinitesimal gauge transformations of the potential

$$\delta_\Lambda A = -\mathrm{d}_A\Lambda = -\mathrm{d}\Lambda - [A, \Lambda]. \tag{A3}$$

As is well known from the study of anomalies [46,73–75], taking the trace of powers of the curvature one can define invariant polynomials

$$\mathrm{d}\,\mathrm{tr}\,R^n = \delta\,\mathrm{tr}\,R^n = 0, \tag{A4}$$

from the nth Chern character $\mathrm{tr}\,R^n$ of the gauge vector bundle. Poincaré's lemma then implies that one can locally define the Chern-Simons forms $\omega_{(2n-1)}$

$$\mathrm{d}\omega_{(2n-1)}(A) = \mathrm{tr}\,R^n, \tag{A5}$$

and applying the lemma once again, now for δ, gives

$$\mathrm{d}\omega^1_{(2n-2)}(\Lambda, A) = \delta_\Lambda\omega_{(2n-1)}(A), \tag{A6}$$

where the superscript denotes the powers of the gauge parameter, since, in principle, one can continue "descending" along this chain.

We can list (up to exact terms) some of the Chern-Simons forms that will be important for us explicitly

$$\omega_7(A) = \mathrm{tr}\left(A(\mathrm{d}A)^3 + \frac{8}{5}A^3(\mathrm{d}A)^2 + \frac{4}{5}A\mathrm{d}AA^2\mathrm{d}A + 2A^5\mathrm{d}A + \frac{4}{7}A^7\right), \tag{A7}$$

$$\omega_6^1(\Lambda, A) = \mathrm{tr}\,\mathrm{d}\Lambda\left(A(\mathrm{d}A)^2 + \frac{2}{5}(A^3\mathrm{d}A + \mathrm{d}AA^3 + A^5) + \frac{1}{5}(A^2\mathrm{d}AA + A\mathrm{d}AA^2)\right), \tag{A8}$$

$$\omega_5(A) = \mathrm{tr}\left(A(\mathrm{d}A)^2 + \frac{3}{2}A^3\mathrm{d}A + \frac{3}{5}A^5\right), \tag{A9}$$

$$\omega_4^1(\Lambda, A) = \mathrm{tr}\,\mathrm{d}\Lambda\left(A\mathrm{d}A + \frac{1}{2}A^3\right), \tag{A10}$$

$$\omega_3(A) = \mathrm{tr}\left(A\mathrm{d}A + \frac{2}{3}A^3\right), \tag{A11}$$

$$\omega_2^1(\Lambda, A) = \mathrm{tr}\,\mathrm{d}\Lambda A. \tag{A12}$$

The last two are not used in this work but are the ones that are featured in heterotic generalized geometry. These agree with the usual ones in the literature [74] up to exact terms corresponding to our convention choice of having the differential acting on the parameter Λ.

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