

Finslerian geometrization of quantum mechanics in the hydrodynamical representation

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We consider a Finslerian-type geometrization of the nonrelativistic quantum mechanics in its hydrodynamical (Madelung) formulation, by also taking into account the effects of the presence of the electromagnetic fields on the particle motion. In the Madelung representation, the Schrödinger equation can be reformulated as the classical continuity and Euler equations of classical fluid mechanics in the presence of a quantum potential, representing the quantum hydrodynamical evolution equations. The equation of particle motion can then be obtained from a Lagrangian similarly to its classical counterpart. After the reparametrization of the Lagrangian, it turns out that the Finsler metric describing the geometric properties of quantum hydrodynamics is a Kropina metric. We present and discuss in detail the metric and the geodesic equations describing the geometric properties of the quantum motion in the presence of electromagnetic fields. As an application of the obtained formalism, we consider the Zermelo navigation problem in a quantum hydrodynamical system, whose solution is given by a Kropina metric. The case of the Finsler geometrization of the quantum hydrodynamical motion of spinless particles in the absence of electromagnetic interactions is also considered in detail, and the Zermelo navigation problem for this case is also discussed.

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I. INTRODUCTION

The extremely successful geometrization of the gravitational interaction via Riemannian geometry, general relativity, and the Einstein gravitational field equations has raised the important question of whether geometrical methods could be also successfully applied for the geometrical description of other branches of physics. For Newtonian mechanics and gravity a first step in this direction was taken by Cartan [1] (see also Ref. [2]), who showed that the motion of a particle in a gravitational field can be reformulated as a geodesic equation of motion.

Hence, the geometric interpretation of classical mechanics, or, more generally, of classical physics, is a longstanding subject in both mathematical and theoretical physics, representing a very active field of research [3–24]. Two main *metric* approaches have been developed for a geometric description of classical mechanics. In the first approach, the Jacobi metric approach, in order to describe Hamiltonian mechanical dynamical systems one

introduces a metric of the form $ds_J^2 = W(q^a)\delta_{ab}dq^a dq^b$, where W represents the conformal factor given by $W(q^a) = E - V(q^a)$ [6]. This metric is called the Jacobi metric. The flow associated with a time-dependent Hamiltonian can then be reformulated as a geodesic flow in a curved, but conformally flat, manifold [6].

An alternative metric description of classical dynamics is based on the Eisenhart metric [10,12,15,16,22,23]. In this approach, one considers an ambient space with an extra dimension, $M \times \mathbf{R}^2$, with local coordinates $(q^0, q^1, \dots, q^i, \dots, q^N, q^{N+1})$. On this space we can introduce a nondegenerate pseudo-Riemannian metric, called the Eisenhart metric, given by $ds_E^2 = g_{\mu\nu}^{(E)} dq^\mu dq^\nu = \delta_{ij} dq^i dq^j - 2V(q)(dq^0)^2 + 2dq^0 dq^{N+1}$, where μ and ν run from 0 to $N + 1$, while i and j run from 1 to N . Then it can be shown that the motions of a Hamiltonian dynamical system can be obtained as the canonical projection of the geodesics of $(M \times \mathbf{R}^2, g^{(E)})$ on the configuration spacetime, $\pi: M \times \mathbf{R}^2 \mapsto M \times \mathbf{R}$ [10].

We may call the above interpretations of physical theories the *geometro-dynamical approach*. Such approaches have been known for a long time in the framework of the general geometric theory of the dynamical systems.

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Perhaps the most important field of study of theoretical physics, quantum mechanics, can also be formulated as a geometric theory, in which standard methods of differential geometry play an important role. Quantum mechanics can be cast into a classical Hamiltonian form in terms of a symplectic structure, not on the Hilbert space of state vectors but on the more physically relevant infinite-dimensional manifold of instantaneous pure states [25]. This geometrical structure can accommodate generalizations of quantum mechanics, including nonlinear relativistic models [26]. A metric tensor from the underlying Hilbert space structure for any submanifold of quantum states was defined in Ref. [27]. The case where the manifold is generated by the action of a Lie group on a fixed state vector (generalized coherent states manifold) was studied in detail. In Ref. [28], it was argued that quantum mechanics is fundamentally a geometric theory. Using the natural metric on the projective space, the Schrödinger equation for an isolated system can be reformulated in geometric terms. On the other hand, the manifold of pure quantum states can be regarded as a complex projective space endowed with the unitary-invariant Fubini-Study metric [29].

A geometrization of the nonrelativistic quantum mechanics for mixed states was introduced in Ref. [30], making use of the Uhlmann's principal fiber bundle. A formulation of the formalism of quantum mechanics in a geometrical form based on the Kähler structure of the complex projective space was proposed in Ref. [31]. A geometric framework for mixed quantum states represented by density matrices was discussed in Ref. [32]. A geometric setting of quantum variational principles and their extension to include the interaction between classical and quantum degrees of freedom was considered in Ref. [33]. The construction of a general prescription to set up a well-defined and self-consistent geometric Hamiltonian formulation of finite-dimensional quantum theories, where phase space is given by the Hilbert projective space (as a Kähler manifold) was considered in Ref. [34].

One of the important extensions of Riemannian geometry is the Finsler geometry [35–40]. A Finsler space is based on the general line element $ds = F(x^1, x^2, \dots, x^n; dx^1, dx^2, \dots, dx^n)$, where $F(x, y) > 0$ for $y \neq 0$ is a function on the tangent bundle $T(M)$ and is homogeneous of degree 1 in y . In fact, Finsler geometry is not just another generalization of Riemannian geometry; it is Riemannian geometry without the restriction that the line element must be quadratic [41]. If in Riemannian geometry we have $F^2 = g_{ij}(x)dx^i dx^j$, in a Finsler space the length of a differential line element at x depends in general on both x and y according to $\sqrt{d\vec{x} \cdot d\vec{x}} = [g_{AB}(x, y)dx^A dx^B]^{1/2}$. Finsler geometry has many important applications in electromagnetism, gravitation, and continuum mechanics [42–45]. A Finslerian approach to classical mechanics was developed in Refs. [46–49].

One of the classic problems in optimization theory is the Zermelo navigation problem [50], requiring the extremization of the travel time of an aeroplane in the presence of wind. As shown in Ref. [51], the solution of the Zermelo problem is equivalent to the determination of the minimal time travel curves in a Finsler-type geometry with a Randers metric [52]. More recently, it was shown that Kropina metrics are actually singular solutions of the Zermelo navigation problem [53]. The Zermelo navigation problem and its applications have been intensively investigated in both the mathematical and physical literature [54–58] and has also been considered in the fields of quantum mechanics and quantum computation [59–63]. The quantum navigation problem can be formulated as finding the time-optimal control Hamiltonian, which transports a given initial state to a target state in the presence of a quantum wind—that is, under the influence of external fields or potentials [62]. It is possible to obtain a universal quantum speed limit by lifting the problem from the state space to the space of unitary gates. The optimal times for implementing unitary quantum gates in a constrained finite-dimensional controlled quantum system were analyzed in Ref. [60]. A Randers metric was constructed, the geodesics of which are the time-optimal trajectories compatible with the prescribed constraint.

Recently, a Finsler-type geometrization of quantum mechanics that describes the time evolution of particles as geodesic lines in a curved space was proposed in Ref. [64], in which the curvature of the Finsler space is induced by the quantum potential. A description of the phenomenon of self-interference using Finslerian geometrical formulation was discussed in Ref. [65]. This formalism removes the need for the concept of wave function collapse in the interpretation of the act of measurement—that is, of the emergence of the classical world. The Finslerian geometrical formulation of quantum field theory in spacetime was unified with classical Einstein's general relativity in Ref. [66]. In Ref. [67] it was shown that for a system of two entangled particles, there is a dual description to the particle equations in terms of the classical theory of conformally stretched spacetime. These entangled particle equations can be interpreted via the framework of Finsler geometry. A geometrization of quantum hydrodynamics was discussed in Ref. [68].

It is the goal of the present paper to consider a Finsler-type geometrization of quantum mechanics along the lines suggested in Refs. [64,65] for a charged particle under the influence of an exterior electromagnetic field. After reformulating, with the use of the Madelung representation of the wave function [69,70], the Schrödinger equation as a quantum hydrodynamical system, it follows that the equation of motion of the quantum particles can be derived from a classical action principle, with the Lagrangian containing the kinetic energy of the particle, the electromagnetic potentials and exterior potentials, as well as the quantum

potential V_Q , which determines the quantum properties of the dynamical evolution.

With the help of a particular reparametrization of the time coordinate, one can associate with this Lagrangian a Finsler-type fundamental function, and a Finsler-type fundamental metric. It turns out that the corresponding Finsler fundamental function belongs to the class of (α, β) metrics [71,72], or, more exactly, to the Kropina metrics [73,74]. Hence, the Finsler-type representation of quantum mechanics can be discussed in the framework of a well-known geometric approach. The Finslerian approach allows us to reformulate the equation of motion of quantum particles as a geodesic equation. Moreover, the important Zermelo quantum navigation problem has an immediate solution within the quantum hydrodynamical approach and the Finslerian representation of quantum mechanics. We discuss in detail the Finsler geometric approach to quantum hydrodynamics for charged particles in the presence of electromagnetic interactions, as well as the limiting case of neutral spinless particles. In all cases, the corresponding Finsler metrics and geodesic equations are obtained.

The present paper is organized as follows: In Sec. II, we introduce the hydrodynamic formulation of quantum mechanics for charged particles in the presence of electromagnetic interactions. In Sec. III, we briefly review the basic mathematical properties and definitions of the Finsler geometries, of the (α, β) and Kropina metrics, and of the Zermelo navigation problem. The Finslerian quantization of the hydrodynamic version of quantum mechanics in the presence of electromagnetic fields is presented in Sec. IV. The solution of the Zermelo navigation problem in the hydrodynamical formulation of quantum mechanics is considered in Sec. V. The case of the Finslerian geometrization of the standard hydrodynamic representation of the Schrödinger for the case of the neutral particle is presented in detail in Sec. VI. Finally, we discuss and conclude our results in Sec. VII.

II. HYDRODYNAMICAL FORMULATION OF NONRELATIVISTIC QUANTUM MECHANICS

In the presence of an external electromagnetic field, the quantum dynamics of a spinless particle with mass m is described by the Schrödinger equation given by

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \frac{1}{2m} \left[\frac{\hbar}{i} \nabla - \frac{e}{c} \vec{A}(\vec{r}, t) \right]^2 \Psi(\vec{r}, t) + e\phi(\vec{r}, t) + V(\vec{r}, t) \Psi(\vec{r}, t), \quad (1)$$

where $\phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$ are the scalar and vector potential of the electromagnetic field, and $V = V(\vec{r}, t)$ is the potential of the nonelectromagnetic forces. The electric and magnetic fields can be obtained from the electromagnetic potentials as $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -e\nabla\phi - \dot{\vec{A}}/c$, respectively [75]. From the Schrödinger equation, one can obtain

immediately the conservation law for the probability current density as given by

$$\frac{\partial |\Psi(\vec{r}, t)|^2}{\partial t} + \nabla \cdot \vec{j} = 0, \quad (2)$$

where $\vec{j} = (\Psi^* \hat{v} \Psi + \Psi \hat{v}^* \Psi^*)/2 = \text{Re}(\Psi^* \hat{v} \Psi)$, and $\hat{v} = (\hat{p} - e\vec{A}/c)/m$.

In general, the operators $\hat{p} = (\hbar/i)\nabla$ and \vec{A} do not commute, so that $[\hat{p}, \vec{A}] = -i\hbar \partial_i A_i = -i\hbar \nabla \cdot \vec{A}$. In order to make these operators commutative, in the following we will adopt for the vector potential of the magnetic field the Coulomb gauge $\nabla \cdot \vec{A} = 0$.

As a next step, we introduce the Madelung (quantum hydrodynamical) representation of the wave function as

$$\Psi(\vec{r}, t) = R(\vec{r}, t) e^{iS(\vec{r}, t)/\hbar}, \quad (3)$$

where both R and S are real quantities. For a given wave function Ψ , this representation does not determine Ψ uniquely, since $S' = S + nh/2$ and $R' = (-1)^n R$, where $n \in \mathbb{N}$ gives the same Ψ . However, if we restrict our analysis to non-negative R , then R is uniquely determined [70]. Moreover, S is also undetermined at the nodal point $R = 0$. By substituting the wave function as given by Eq. (3) into the Schrödinger Eq. (1), after separating the real and imaginary parts we obtain [70,76–79]

$$R \left[\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\nabla S - \frac{e}{c} \vec{A} \right)^2 + e\phi + V \right] - \frac{\hbar^2}{2m} \Delta R = 0 \quad (4)$$

and

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left[\frac{R^2}{m} \left(\nabla S - \frac{e}{c} \vec{A} \right) \right] = 0, \quad (5)$$

respectively. By introducing the velocity \vec{v} of the quantum particle, defined as $m\vec{v} = \nabla S - e\vec{A}/c$, and by denoting $\rho(\vec{r}, t) = R^2(\vec{r}, t) = \Psi^*(\vec{r}, t)\Psi(\vec{r}, t)$, Eqs. (4) and (5) take the forms

$$\frac{\partial S}{\partial t} + \frac{1}{2} m \vec{v}^2 + e\phi + V + V_Q = 0, \quad (6)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad (7)$$

where we have introduced the quantum potential V_Q defined as

$$V_Q = -\frac{\hbar^2}{2m} \frac{\Delta R}{R} = -\frac{\hbar^2}{2m} \left[\frac{\Delta \rho}{\rho} - \frac{(\nabla \rho)^2}{2\rho^2} \right]. \quad (8)$$

By taking the gradient of Eq. (6), we obtain the equation of motion of the quantum particle as

$$m \frac{d\vec{v}}{dt} = \vec{F} - \nabla(V + V_Q), \quad (9)$$

where $d/dt = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$, and $\vec{F} = e\vec{E} + (e/c)\vec{v} \times \vec{B}$ is the Lorentz force acting on the particle. Equation (9) can be rewritten as

$$\frac{dv_i}{dt} = \frac{1}{m} F_i - \frac{1}{m\rho} \sum_k \frac{\partial \sigma_{ik}}{\partial x_k} - \frac{1}{m} \frac{\partial V}{\partial x_i}, \quad (10)$$

where $\sigma_{ik} = -\frac{\hbar^2}{4m} \frac{\partial^2 \ln \rho}{\partial x^i \partial x^k}$.

We introduce now the quantum trajectories via the definition $\vec{v} = d\vec{r}/dt$ [76,80,81], which allows us to write the equation of motion (9) in the form

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F} - \nabla(V + V_Q). \quad (11)$$

III. A BRIEF REVIEW OF FINSLER GEOMETRY, GENERAL (α, β) METRICS, AND ZERMELO NAVIGATION

In the present section, we will briefly review the basics of the Finsler geometry, of the (α, β) metrics, and of the Zermelo navigation problem.

Finsler geometry naturally appears in the framework of classical mechanics when dissipative effects are taken into account. Let us assume that the equations of motion of a dynamical system, defined on an n -dimensional differentiable manifold M , can be obtained from a Lagrangian L by using the Euler-Lagrange equations, given by

$$\frac{d}{dt} \frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = F_i, \quad i \in \{1, 2, \dots, n\}, \quad (12)$$

where F_i are the external forces [48]. We call the triplet (M, L, F_i) a Finslerian mechanical system [49]. If the Lagrangian L is regular, then the above Euler-Lagrange equations are equivalent to a system of second-order differential equations of the form

$$\frac{d^2 x^i}{dt^2} + 2G^i(x^j, y^j, t) = 0, \quad i \in \{1, 2, \dots, n\}, \quad (13)$$

where in a neighborhood of some initial conditions $((x)_0, (y)_0, t_0)$ each function $G^i(x^j, y^j, t)$ is C^∞ in Ω . These equations can be naturally interpreted as describing geodesic motion in a Finsler space.

A. Finsler geometry

One of the fundamental assumptions of modern theoretical physics is that spacetime can be described mathematically as a four-dimensional differentiable manifold M , endowed with a pseudo-Riemannian tensor g_{IJ} , where $I, J, K, \dots = 0, 1, 2, 3$. Then the spacetime interval between

two events x^I and $x^I + dx^I$ on the world line of a standard clock is defined by the chronological hypothesis to be $ds = (g_{IJ} dx^I dx^J)^{1/2}$ (using Einstein's summation convention) [82,83]. One of the most important metrical generalizations of the Riemannian geometry that was already hinted at by Riemann himself [84] is the geometry subsequently initiated by Finsler [35] (for a detailed description of Finsler geometry, see Refs. [36–40]).

Finsler spaces are metric spaces in which the interval ds between two infinitesimally close points $x = (x^I)$ and $x + dx = (x^I + dx^I)$ is given by

$$ds = F(x, dx), \quad (14)$$

where the Finsler metric function F is positively homogeneous of degree 1 in dx , so that

$$F(x, \lambda dx) = \lambda F(x, dx). \quad (15)$$

In order to allow the use of local coordinates in computations, the Finsler metric function F can be written in terms of $(x, y) = (x^I, y^I)$, the canonical coordinates of the tangent bundle, where $y = y^I \frac{\partial}{\partial x^I}$, is any tangent vector y at x . The Finsler tensor g_{IJ} is defined then as

$$g_{IJ}(x, y) = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^I \partial y^J}, \quad (16)$$

which allows us to write Eq. (14) as $ds^2 = g_{IJ}(x, y) y^I y^J$. Riemann spaces are special cases of Finsler spaces, corresponding to $g_{IJ}(x, y) = g_{IJ}(x)$ and $y^I = dx^I$, respectively.

With the use of the Finsler metric, we can obtain the geodesic equations in a Finsler space in the form [82,83]

$$\frac{d^2 x^I}{d\lambda^2} + 2G^I(x, \dot{x}) = 0, \quad (17)$$

or, equivalently,

$$\frac{d^2 x^I}{d\lambda^2} + \Gamma_{JK}^I(x, \dot{x}) \frac{dx^J}{d\lambda} \frac{dx^K}{d\lambda} = 0, \quad (18)$$

where

$$G^I(x, \dot{x}) = \frac{1}{2} \Gamma_{JK}^I y^J y^K = \frac{1}{4} g^{IJ} \left(\frac{\partial^2 F^2}{\partial x^K \partial y^J} y^K - \frac{\partial F^2}{\partial x^J} \right), \quad (19)$$

and Γ_{JK}^I are the analogues of the Christoffel symbols of the Riemann geometry, defined as

$$\Gamma_{JK}^I(x, \dot{x}) = \frac{1}{2} g^{IL}(x, \dot{x}) \left[\frac{\partial g_{LJ}(x, \dot{x})}{\partial x^K} + \frac{\partial g_{LK}(x, \dot{x})}{\partial x^J} - \frac{\partial g_{JK}(x, \dot{x})}{\partial x^L} \right]. \quad (20)$$

A special class of Finsler spaces, called Berwald spaces, can be obtained when $\Gamma_{JK}^I(x^A, \dot{x}^A) = \Gamma_{JK}^I(x^A)$ [36].

The functions G^I defined above are called the *coefficients of the spray* of F . They are used to define the vector field S on $TM \setminus 0$ by

$$S = y^I \frac{\partial}{\partial x^I} - 2G^I \frac{\partial}{\partial y^I}.$$

S is called the *spray* induced by F . A curve γ on M is a geodesic of F if and only if its canonical lift $\hat{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ to TM is an integral curve of S .

B. Kropina and general (α, β) metrics

A special kind of Finsler space is the Randers space [52], having

$$F = [a_{IJ}(x) dx^I dx^J]^{1/2} + b_I(x) dx^I, \quad (21)$$

where a_{IJ} is the fundamental tensor (metric) of a Riemannian space, and $b_I(x) dx^I$ is a linear one-form on the tangent bundle TM . Later on, Kropina [73,74] considered Finsler spaces equipped with metrics of the form

$$F(x, y) = \frac{a_{IJ}(x) y^I y^J}{b_I(x) y^I}. \quad (22)$$

Generalizing these results, Matsumoto [71,72] introduced the notion of the (α, β) metrics as follows: a Finsler metric function $F(x, y)$ is called an (α, β) metric when F is a positively homogeneous function $F(\alpha, \beta)$ of first degree in two variables, $\alpha(x, y) = [a_{IJ}(x) dx^I dx^J]^{1/2}$ and $\beta(x, y) = b_I(x) y^I$, respectively. In the following, we will suppose that α is a Riemannian metric—that is, it is nondegenerate (regular) and positive-definite. In the case of the Randers and Kropina metrics, we have $F = \alpha + \beta$ and $F = \frac{\alpha^2}{\beta}$, respectively. Hence, the Randers and Kropina metrics belong to the class of the (α, β) metrics. Alternatively, we can consider general (α, β) metrics as metrics of the form $F(\alpha, \beta) = \alpha \phi(\beta/\alpha) = \alpha \phi(s)$, where $s = \beta/\alpha$, and $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_o, b_o)$.

In the following, we introduce the quantities $r_{IJ} := (b_{I|J} + b_{J|I})/2$, $s_{IJ} := (b_{I|J} - b_{J|I})/2$, $s^I{}_J := a^{IK} s_{KJ}$, $s_I := b_J s^J{}_I = b^K s_{KI}$, and $e_{IJ} := r_{IJ} + b_I s_J + b_J s_I$, where “|” denotes the *covariant derivative with respect to the Levi-Civita connection of α* . In Finsler geometry, it customary to denote $r_{00} := r_{IJ} y^I y^J$, $s_0 := s_I y^I$, etc.

I. Geodesic equations for the Kropina metric

The fundamental tensor of a Kropina space with $F = \frac{\alpha^2}{\beta} = \alpha \phi(s)$, where $\phi(s) = \frac{1}{s}$, $s = \frac{\beta}{\alpha}$, reads [85]

$$g_{IJ} = \frac{2\alpha^2}{\beta^2} a_{IJ} + \frac{3\alpha^4}{\beta^4} b_I b_J - \frac{4\alpha^3}{\beta^3} (b_I \alpha_J + b_J \alpha_I) + \frac{4\alpha^2}{\beta^2} \alpha_I \alpha_J. \quad (23)$$

For the geodesic spray of a general Kropina metric, we obtain

$$G^I = \bar{G}^I - \frac{\alpha^2}{2\beta} s_0^I - \frac{\beta}{b^2 \alpha^2} \left\{ \frac{\alpha^2}{\beta} s_0 + r_{00} \right\} y^I + \frac{1}{2b^2} \left\{ \frac{\alpha^2}{\beta} s_0 + r_{00} \right\} b^I, \quad (24)$$

where the index 0 means contraction by y^I .

Then the Kropina-space unit speed geodesics' equations are given by

$$\frac{d^2 x^I}{dt^2} + 2G^I \left(x(t), \frac{dx(t)}{dt} \right) = 0. \quad (25)$$

C. The Zermelo navigation problem

We start by recalling the famous *Zermelo navigation problem* [50]. Imagine a ship sailing on the sea in the presence of a wind of speed W^i relative to the ground. The main task of the ship's captain is to minimize the travel time, by assuming that the ship sails at constant speed relative to the sea. If the ship travels a long distance, the captain must take into account the curvature of the Earth. In the absence of the wind, the optimal travel route would be a circle—that is, the geodesic of the Riemannian manifold with metric h_{ij} . In the following, we describe the sea by a Riemannian space (M, h) . The pair (h_{ij}, W^i) is referred to as the *Zermelo navigation data*. It was shown in Ref. [86] that the Zermelo problem is equivalent to finding the minimal travel-time curve in a manifold equipped with a Randers-Finsler metric with original data (a_{ij}, b_i) . The Randers geodesics are given by

$$\frac{d^2 x^I}{ds^2} + \Gamma_{JK}^I \frac{dx^J}{ds} \frac{dx^K}{ds} = a^{IJ} F_{JK} \frac{dx^K}{ds}, \quad (26)$$

where ds is the arc length of the Riemannian metric a_{IJ} , Γ_{JK}^I are the Christoffel symbols constructed from a_{IJ} , and $F_{JK} = \partial_J b_K - \partial_K b_J$ ([39]).

Suppose now that the ship can sail with a constant speed for a unit time on the calm sea. Let the speed be h -unit speed. We denote the velocity of the ship on the calm sea by a unit vector u . And let us suppose that the wind is blowing with the h -unit speed, which is the same as that of the ship

on the calm sea. We denote the wind and the velocity of the ship on the windy sea by the h -unit vector field W and the vector v , respectively. Then we have the equation $u + W = v$. From it we find $|v - W| = 1$.

The length of v is the speed of the ship on the windy sea for a unit time. The above equation means that the tip of the velocity of the ship on the windy sea lies on the W -translate of the indicatrix of the Riemannian space (M, h) . In other words, the indicatrix of the space in which we consider the velocity of the ship on the windy sea is the W -translate of the unit sphere of (M, h) .

1. Zermelo navigation and Kropina metrics

In this section, we characterize a Kropina metric $F(x, y) = \alpha^2/\beta$ on M , where $\alpha^2 = a_{IJ}(x)y^I y^J$ and $\beta = b_I(x)y^I$, by a Riemannian metric h and a unit vector field W on M . Since we suppose that the matrix $(a_{IJ}(x))$ is positive definite, it follows that the matrix $(g_{IJ}(x, y))$ is also positive definite [53].

The Kropina metrics can be described as another solution of the Zermelo navigation problem already explained above [53]. For a Kropina metric $F = \alpha^2/\beta$, we define a Riemannian metric h and a vector field W by

$$h_{IJ} = e^{\kappa(x)} a_{IJ}, \quad W_I = \frac{1}{2} e^{\kappa(x)} b_I, \quad (27)$$

where $W_I = h_{IJ} W^J$, where $\kappa(x)$ is a function of (x^I) alone and satisfies the equation

$$e^{\kappa(x)} b^2 = 4, \quad (28)$$

i.e., $|W| = 1$. Then we obtain

$$\left| \frac{y}{F(x, y)} - W \right| = 1, \quad (29)$$

and hence it follows that F is a solution of the Zermelo navigation problem. Observe that $h^{IJ} = e^{-\kappa(x)} a^{IJ}$, and $W^I = b^I/2$, respectively.

Conversely, consider the metric $F(x, y)$ defined by Eq. (29), where $|W| = 1$. Solving for F , we obtain

$$F(x, y) = \frac{|y|^2}{2h(y, W)}.$$

Defining a_{IJ} and b_I by $a_{IJ} := e^{-\kappa(x)} h_{IJ}$ and $b_I := 2e^{-\kappa(x)} W_I$, respectively, we have $F(x, y) = \alpha^2/\beta$.

Summarizing the above discussion, we obtain the following results [53,87,88]:

For a Kropina space $(M, F = \alpha^2/\beta)$, where $\alpha = \sqrt{a(y, y)}$ and $\beta = b_I(x)y^I$, we define a new Riemannian metric $h = \sqrt{h(y, y)}$ and a unit vector field $W = W^I(\partial/\partial x^I)$ by Eqs. (27) and (28). Then, the Kropina

metric F satisfies Eq. (29), and hence it is a solution of Zermelo's navigation problem.

Conversely, suppose that $h = \sqrt{h(y, y)}$ is a Riemannian metric and $W = W^I(\partial/\partial x^I)$ is a unit vector field on (M, h) . Consider the metric F defined by Eq. (29). Let $\kappa(x)$ be a function of (x^I) alone, which satisfies Eq. (28), and define a_{IJ} and b_I by $a_{IJ}(x) := e^{-\kappa(x)} h_{IJ}(x)$ and $b_I(x) := 2e^{-\kappa(x)} W_I$, respectively. Then, we have $F = \alpha^2/\beta$.

IV. FINSLERIAN GEOMETRIZATION OF QUANTUM HYDRODYNAMICS

In the present section, starting from the equivalent Lagrangian formulation of quantum hydrodynamics, we will consider a geometric formulation of the Schrödinger equation in the presence of the electromagnetic interactions. In particular, we will show that quantum mechanics can be interpreted as describing geodesic motion in a Kropina space. A similar approach for the Finsler geometrization of classical mechanics was introduced in Ref. [85].

A. Kropina space representation of quantum hydrodynamics

As a first step in our approach, we consider a general Lagrangian of the form $L(t, x^i, \dot{x}^i)$, $i = 1, 2, 3$, with which we associate the arc-length element ds given by

$$ds = L(t, x^i, \dot{x}^i) dt. \quad (30)$$

We introduce now a new evolution parameter $\tau = \tau(t)$, which allows us to rewrite the ds as

$$ds = L\left(t, x^i, \frac{dx^i}{d\tau}\right) \frac{dt}{d\tau} d\tau. \quad (31)$$

We next introduce the new coordinate $x^0 = t$, and we relabel the coordinates as $x^I = (x^0, x^i)$, $i = 1, 2, 3$. Moreover, we denote $y^I = dx^I/d\tau = (dt/d\tau, dx^i/d\tau)$. Then Eq. (31) can be written as

$$ds = F(x^I, y^I) d\tau, \quad (32)$$

where

$$F(x^I, y^I) = L\left(t, x^i, \frac{y^i}{y^0}\right) y^0. \quad (33)$$

Equation (11), giving the equation of motion of a quantum particle in the hydrodynamic representation, can be derived from the classical Lagrangian

$$L(t, x^i, \dot{x}^i) = \frac{1}{2} m_{ij}(x) \dot{x}^i \dot{x}^j + \frac{e}{c} A_i \dot{x}^i - (e\phi + V + V_Q), \quad (34)$$

where for the sake of generality we assume that the components of the mass matrix m_{ij} are functions of x^i , and that the matrix m_{ij} is positive definite. Equation (34) will be the starting point for our Finsler geometrization of the hydrodynamic version of quantum mechanics.

Therefore, we obtain for F the expression

$$F(x^I, y^I) = \frac{1}{2} m_{ij} \frac{y^i y^j}{y^0} + \frac{e}{c} A_i y^i - (e\phi + V + V_Q) y^0, \quad i, j = 1, 2, 3. \quad (35)$$

Equation (35) can be rewritten as follows:

$$F(x^I, y^I) = \frac{\frac{1}{2} m_{ij} y^i y^j + \frac{e}{c} A_i y^0 y^i - (e\phi + V + V_Q) (y^0)^2}{y^0} = \frac{\alpha^2}{\beta} (x^I, y^I), \quad (36)$$

where we have introduced the notations

$$\alpha^2 := a_{IJ}(x) y^I y^J, \quad \beta := y^0, \quad (37)$$

and $(a_{IJ})_{I,J=0,\dots,3}$ is the symmetric matrix with the entries

$$a_{00}(x^0, x^I) := -(e\phi + V + V_Q),$$

$$a_{0i}(x^I) := \frac{e}{2c} A_i, \quad a_{ij}(x^I) = \frac{1}{2} m_{ij}, \quad i \in \{1, 2, 3\}. \quad (38)$$

We will call this Riemannian metric the *associated Riemannian metric* to the Lagrangian (34).

From the definitions in Eq. (37), we observe that $\beta = b_I y^I = y^0$ —that is, the linear one-form has only one nonvanishing coefficient, $(b_0, b_1, b_2, b_3) = (1, 0, 0, 0)$. Moreover, the determinant of the matrix a_{IJ} can be explicitly written in terms of the initial data in the Lagrangian [Eq. (34)].

For $a_{ij} = (1/2)m\delta_{ij}$, where m is the mass of the particle, we find

$$\det|a_{IJ}| = -\frac{1}{8} m^3 \left[\frac{e^2}{2mc^2} (A_1^2 + A_2^2 + A_3^2) + (e\phi + V + V_Q) \right]. \quad (39)$$

For the sake of simplicity, we will assume that the matrix a_{IJ} is positive defined—that is, we have a Riemannian metric on the manifold $\mathcal{M} = \mathbb{R} \times M$, called the *extended configurations space*, with the local coordinates (x^I) . Under this assumption, it follows that the function F in Eq. (36) is a Kropina metric on \mathcal{M} , hence the theory presented in the previous sections applies. Of course, in the case when the associated metric a_{IJ} has a more general signature, the theory can be studied in a similar manner.

In the absence of the external magnetic field, $A_i = 0$, $i = 1, 2, 3$, we obtain for a^{IJ} the inverse of the matrix a_{IJ} , the simple form $a^{IJ} = -(e\phi + V + V_Q)^{-1} \times \text{diag}(1, a^{11}, a^{22}, a^{33})$, where $a^{ii} = -(2/m)(e\phi + V + V_Q)$, $i = 1, 2, 3$.

The α -norm of β is given by $b^2 = a^{IJ} b_I b_J = a^{00} = a_{11} a_{22} a_{33} / \det|a| > 0$, under the assumption that $\det|a|$ is positive. For later use, we remark that $b^I := a^{IJ} b_J = a^{I0} b_0 = a^{I0}$.

We can now summarize our findings as the following:

Theorem 1. The fundamental function $F = \frac{\alpha^2}{\beta}$, associated with the Lagrangian of the hydrodynamic representation of the quantum mechanics, in the presence of external electromagnetic fields, is a globally defined Kropina metric on the extended configurations space \mathcal{M} , where $\alpha^2 = a_{IJ} y^I y^J$ is the associated Riemannian metric to the hydrodynamic Lagrangian of the quantum mechanics, and $\beta = y^0$.

Remark.—Kropina-type Finsler metrics were associated with a time-dependent Lagrangian in several studies—like, for example, Refs. [85,89,90]. The starting point is the fundamental observation that the Euler-Lagrange equations of the initial time-dependent Lagrangian actually coincide with the Euler-Lagrange equations of the associated Kropina metric (see the references above for the proof of this result).

B. The fundamental metric tensor of the Finslerian representation of quantum mechanics

Let us consider the fundamental tensor

$$g_{IJ} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^I \partial y^J} = \frac{\partial F}{\partial y^I} \frac{\partial F}{\partial y^J} + F \frac{\partial^2 F}{\partial y^I \partial y^J} \quad (40)$$

for the globally defined Kropina metric describing the geometric properties of quantum hydrodynamics.

First, we mention here an important result on the geometry of Kropina metrics (Ref. [53]), which states that the metric tensor of a Kropina space $F = \frac{\alpha^2}{\beta}$ is positive defined, provided α is a positive-defined Riemannian metric [53].

Let us now turn back to the computation of the fundamental tensor of the Kropina metric [Eq. (36)] associated with the Schrödinger equation. For the Finsler metric tensor, we obtain the general expression

$$g_{IJ} = \left[B^2 + 2F \frac{T}{(y^0)^3} \right] \delta_I^0 \delta_J^0 + \left[\frac{B}{y^0} - \frac{F}{(y^0)^2} \right] \times (\delta_I^0 T_J + \delta_J^0 T_I) + 2B (a_{0j} \delta_I^0 \delta_J^j + a_{0i} \delta_i^0 \delta_I^j) + \left(2a_{0i} \delta_i^j + \frac{T_I}{y^0} \right) \left(2a_{0j} \delta_J^i + \frac{T_J}{y^0} \right) + F \frac{T_{IJ}}{y^0}, \quad (41)$$

where we have denoted

$$B = a_{00} - \frac{T}{(y^0)^2}, \quad T = a_{ij}(x)y^i y^j,$$

$$T_I = \frac{\partial T}{\partial y^I}, \quad T_{IJ} = \frac{\partial^2 T}{\partial y^I \partial y^J}. \quad (42)$$

Explicitly, the Finslerian metric tensor components can be given as

$$g_{00} = a_{00}^2 + 4 \frac{a_{0i} y^i}{(y^0)^3} T + 3 \frac{T^2}{(y^0)^4}, \quad (43)$$

$$g_{0i} = 2a_{0i} \left(a_{00} - \frac{T}{(y^0)^2} \right) - \frac{4a_{ij} y^j}{(y^0)^2} \left(\frac{T}{y^0} + a_{0j} y^j \right), \quad (44)$$

$$g_{ij} = 4a_{0i} a_{0j} + \frac{4(a_{ik} y^k)(a_{jl} y^l)}{(y^0)^2} + 4a_{0j} \frac{a_{il} y^l}{y^0} + 4a_{0i} \frac{a_{jl} y^l}{y^0} + 2 \left(\frac{T}{(y^0)^2} + \frac{2a_{0k} y^k}{y^0} + a_{00} \right) a_{ij}. \quad (45)$$

Alternatively, we can express the Finsler metric associated with the quantum hydrodynamical evolution as

$$g_{IJ} = 2 \left(\frac{\alpha}{\beta} \right)^2 a_{IJ} + \left(\frac{\alpha}{\beta} \right)^2 \left[4\alpha_I \alpha_J - \frac{3\alpha}{\beta} \times (\alpha_I b_J + \alpha_J b_I) + \frac{3\alpha^2}{\beta^2} b_I b_J \right]. \quad (46)$$

A simple computation shows that $\det |g_{IJ}| = 24 \left(\frac{\alpha}{\beta} \right)^8 (1 + d^2) \det |a_{IJ}|$, where $d^2 = \frac{3}{2} a^{IJ} (\alpha_I - \frac{\alpha}{\beta} b_I) (\alpha_J - \frac{\alpha}{\beta} b_J)$. Hence, the Kropina metric g_{IJ} 's positive definiteness is equivalent to the positive definiteness of a_{IJ} .

C. The local equations of the geodesics of the Kropina space

We consider now the geodesics spray G^I associated with the Kropina metric obtained in Eq. (24). In this case $r_{IJ} = -\bar{\Gamma}_{IJ}^0$, $s_{IJ} = 0$, where $\bar{\Gamma}_{IJ}^K$ are the Christoffel coefficients of the associated Riemannian metric α on \mathcal{M} .

By components, we have

$$G^0 = \bar{G}^0 - \left(\frac{\beta}{b^2 \alpha^2} y^0 - \frac{1}{2b^2} \right) r_{00}$$

$$= \bar{G}^0 + \left(\frac{\beta}{b^2 \alpha^2} y^0 - \frac{1}{2b^2} \right) \bar{\Gamma}_{IJ}^0 y^I y^J, \quad (47)$$

$$G^i = \bar{G}^i - \left(\frac{\beta}{b^2 \alpha^2} y^i \right) r_{00} = \bar{G}^i + \left(\frac{\beta}{b^2 \alpha^2} y^i \right) \bar{\Gamma}_{IJ}^0 y^I y^J, \quad (48)$$

where $\bar{G}^I = \frac{1}{2} \bar{\Gamma}_{JK}^I y^J y^K$ are the spray coefficients of the Riemannian metric a_{IJ} , obtained from the formulas (52), (54), and (56), respectively.

Hence, the Kropina metric geodesic equations are given by

$$\frac{d^2 x^0}{d\tau^2} + 2G^0 \left(x(\tau), \frac{dx}{d\tau} \right) = 0,$$

$$\frac{d^2 x^i}{d\tau^2} + 2G^i \left(x(\tau), \frac{dx}{d\tau} \right) = 0, \quad (49)$$

where τ is the F -unit parameter on the Kropina geodesics.

D. The geodesics of the associated Riemann metric

It is worth observing that the associated Riemannian metric [Eq. (38)] is a metric on the extended configuration space that includes the external potential, the electromagnetic potential, and the quantum potential generating the quantum effects during the evolution of the particle. This represents a new kind of Riemannian structure associated with such a Lagrangian function.

The Levi-Civita connection coefficients of the Riemannian metric a_{IJ} are given by

$$\bar{\Gamma}_{JK}^I = \frac{1}{2} a^{IL} \left(\frac{\partial a_{KL}}{\partial x^J} + \frac{\partial a_{JL}}{\partial x^K} - \frac{\partial a_{JK}}{\partial x^L} \right). \quad (50)$$

Generally, any second-order tensor $\partial A^i / \partial x^j$ can be decomposed into a symmetric and an antisymmetric part:

$$\frac{\partial A_i}{\partial x^j} = \frac{1}{2} \left(\frac{\partial A_i}{\partial x^j} + \frac{\partial A_j}{\partial x^i} \right) + \frac{1}{2} \left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \right)$$

$$= \frac{1}{2} D_{ij} + \frac{1}{2} F_{ij}, \quad (51)$$

where D_{ij} represents the ‘‘strain’’ associated with the potential vector, while F_{ij} can be interpreted physically as the strength of the magnetic field. The trace of D_{ij} gives the divergence of the vector potential, $D_{ii} = \sum_{i=1}^3 \partial A_i / \partial x^i$.

Thus, for the Christoffel symbols of the associated Riemann metric we obtain

$$\bar{\Gamma}_{00}^0 = \frac{1}{2} \left(a^{00} \frac{\partial a_{00}}{\partial x^0} - a^{0i} \frac{\partial a_{00}}{\partial x^i} \right), \quad (52)$$

$$\bar{\Gamma}_{00}^i = \frac{1}{2} \left(a^{i0} \frac{\partial a_{00}}{\partial x^0} - a^{ij} \frac{\partial a_{00}}{\partial x^j} \right), \quad (53)$$

$$\bar{\Gamma}_{j0}^0 = \frac{1}{2} a^{00} \frac{\partial a_{00}}{\partial x^j} + a^{0i} \frac{e}{4c} F_{ij}, \quad (54)$$

$$\bar{\Gamma}_{j0}^i = \frac{1}{2} a^{i0} \frac{\partial a_{00}}{\partial x^j} + a^{ik} \frac{e}{4c} F_{kj}, \quad (55)$$

$$\bar{\Gamma}_{jk}^0 = a^{00} \frac{e}{4c} D_{jk} - a^{00} A_i \gamma_{jk}^i, \quad (56)$$

$$\bar{\Gamma}_{jk}^i = a^{00} \frac{e}{4c} D_{jk} + \gamma_{jk}^i, \quad (57)$$

where we have denoted by γ_{jk}^i the Levi-Civita connection coefficients of the Riemannian metric a_{ij} , $i, j \in \{1, 2, 3\}$. The geodesics of the Riemannian metric a_{IJ} read now

$$\frac{d^2 x^0}{d\sigma^2} + \bar{\Gamma}_{IJ}^0 \frac{dx^I dx^J}{d\sigma d\sigma} = 0, \quad \frac{d^2 x^i}{d\sigma^2} + \bar{\Gamma}_{IJ}^i \frac{dx^I dx^J}{d\sigma d\sigma} = 0, \quad (58)$$

where σ is the α -unit parameter, and $\bar{\Gamma}_{IJ}^K$ are given above.

For a general Kropina metric $F = \frac{\alpha^2}{\beta}$, it is easy to see that F is projectively equivalent to the Riemannian metric α ; i.e., the F -geodesics coincide with the α -geodesics, if and only if β is parallel with respect to the Levi-Civita connection of α —that is, $b_{IJ} = 0$. In our case, the projectively equivalent condition is $\bar{\Gamma}_{IJ}^0 = 0$, for all $I, J \in \{0, \dots, 4\}$.

The geodesic equations [Eq. (58)] describe the motion of a quantum particle in fields generated by an external (nonelectromagnetic), electromagnetic, and quantum potential, in a purely “physical” geometry, generated by the potentials only. The derivative $d^2 x^I / d\sigma^2$ gives, from a physical point of view, the four-acceleration of the particle. Therefore, the quantity $-\bar{\Gamma}_{JK}^I \frac{dx^J dx^K}{d\sigma d\sigma}$ can be interpreted as the four-force acting on the particle as the result of the presence of the three types of potentials. Moreover, the tensor a_{IJ} plays the role of the “potential” of the total field, since its derivatives determine the total field “intensity” $\bar{\Gamma}_{JK}^I$.

V. THE ZERMELO NAVIGATION PROBLEM IN THE HYDRODYNAMICS REPRESENTATION OF QUANTUM MECHANICS

The navigation data (h, W) on \mathcal{M} corresponding to the Kropina metric geometrizing quantum hydrodynamics are

$$h_{IJ} = e^{\kappa(x)} a_{IJ}, \quad W_I = \frac{1}{2} e^{\kappa(x)} b_I, \quad (59)$$

respectively, where the conformal factor $e^{\kappa(x)}$ is obtained as

$$e^{\kappa(x)} = \frac{4}{a^{00}} = \frac{4 \det |a_{IJ}|}{a_{11} a_{22} a_{33}}, \quad (60)$$

where we assume that a_{IJ} is positive defined. More exactly, W_I has only a nonzero component, $W_0 = \frac{1}{2} e^{\kappa(x)} b_0 = \frac{1}{2} e^{\kappa(x)}$ and $W_i = 0$, for $i \in \{1, \dots, n\}$.

Taking into account that $h^{IJ} = e^{-\kappa(x)} a^{IJ}$, we find

$$W^I = h^{IJ} W_J = h^{I0} W_0 = \frac{1}{2} e^{\kappa(x)} h^{I0} = \frac{1}{2} a^{I0}, \quad (61)$$

and therefore the Kropina metric is determined by the Riemannian metric $h = h_{IJ}$ and the h -unit vector field $W = W^I \frac{\partial}{\partial x^I}$, $|W|_h = 1$, defined on the manifold \mathcal{M} .

We point out that

- (1) The Riemannian metric h_{IJ} obtained in the navigation data above is actually a conformal change of

the extended Riemannian metric a_{IJ} . Hence, one can study the geometrical and physical properties of the Riemannian metrics a_{IJ} and h_{IJ} by means of conformal transformations and conformal geometry.

- (2) The quantum wind W is actually the vector field with the components

$$W^I = \frac{1}{2 \det |a_{IJ}|} \times \left(a_{11} a_{22} a_{33}, -\frac{e}{2c} A_1 a_{22} a_{33}, -\frac{e}{2c} A_2 a_{11} a_{33}, -\frac{e}{2c} A_3 a_{11} a_{22} \right); \quad (62)$$

that is, the wind is given by a combination of the mass field a_{ij} and the electromagnetic potential A_i .

In the case of a single particle with constant mass, we have $a_{11} = a_{22} = a_{33} = m/2$, and the components of the quantum wind vector W are given by

$$W^I = \frac{(-1, \frac{e}{mc} \vec{A})}{(e^2/mc^2) |\vec{A}|^2 + 2(e\phi + V + V_Q)}, \quad (63)$$

where we have denoted by $\vec{A} = (A_1, A_2, A_3)$ the three-dimensional vector potential of the magnetic field, and $|\vec{A}|^2 = A_1^2 + A_2^2 + A_3^2$. The “timelike” component of the quantum wind is determined by $1/\det |a_{IJ}|$ only, while the spacelike components are proportional to the vector potential components, scaled by the determinant of the Riemannian metric. It is interesting to note that the wind is a function of the potentials only, and not of the fields, which are generally obtained as the gradients of the potentials. In the absence of the electromagnetic fields, the quantum wind takes the form

$$W^I = \left(-\frac{1}{2(V + V_Q)}, 0, 0, 0 \right) = \left(\frac{1}{2(-V + \frac{\hbar^2 \Delta R}{2m})}, 0, 0, 0 \right), \quad (64)$$

where the real function R is related to the wave function $\Psi(\vec{r}, t)$ by the relation $|\Psi(\vec{r}, t)| = R^2(\vec{r}, t)$. It is interesting to point out that the quantum wind is generated by the quantum potential V_Q , together with the external potential. When the magnitude of the quantum potential is much larger than that of the external potential, we obtain

$$W^I \approx \frac{m}{\hbar^2} \left(\frac{R}{\Delta R}, 0, 0, 0 \right). \quad (65)$$

On the other hand, if the vector potential of the magnetic field satisfies the condition $(e^2/mc^2) |\vec{A}|^2 \gg 2(e\phi + V + V_Q)$, then the quantum wind is a purely classical quantity, which can be approximated as

$$W^I \approx \left(-\frac{mc^2}{e^2 |\vec{A}|^2}, \frac{c}{e} \frac{\vec{A}}{|\vec{A}|^2} \right). \quad (66)$$

Hence, the wind acting on the quantum particle is determined by the magnetic field potential only. The conformal factor $e^{\kappa(x)}$ is explicitly given by

$$e^{\kappa(x)} = -4 \left[\frac{e^2}{2mc^2} |\vec{A}|^2 + (e\phi + V + V_Q) \right], \quad (67)$$

and it must satisfy the condition $e^{\kappa(x)} > 0$. Such a conformal factor can always be obtained by taking into account that the vector potential of the magnetic field is not uniquely fixed, since if \vec{A} is a vector potential, then $\vec{A} + \nabla\chi$ is also a vector potential, with χ being an arbitrary function of the coordinates. Similarly, the electric field potential and the external potential can be rescaled. Therefore, one can always choose a particular gauge for which the conformal factor is positive. For the components of the navigation metric, we obtain the explicit components

$$h_{00}(x^0, x^i) = 4 \left[a \frac{e^2}{2mc^2} |\vec{A}|^2 + (e\phi + V + V_Q) \right] \times (e\phi + V + V_Q), \quad (68)$$

$$h_{0i}(x^0, x^i) = -2 \frac{2}{c} A_i \left[\frac{e^2}{2mc^2} |\vec{A}|^2 + (e\phi + V + V_Q) \right], \quad (69)$$

$$h_{ij}(x^0, x^i) = -2m_{ij} \left[\frac{e^2}{2mc^2} |\vec{A}|^2 + (e\phi + V + V_Q) \right]. \quad (70)$$

Even if the components of the metric a_{IJ} are functions of the spacelike coordinates only, or constants, the conformally transformed navigation metric is a function of both space and time coordinates. In the particular case of a neutral particle or in the absence of the electromagnetic interactions, $\phi \equiv 0$ and $\vec{A} \equiv 0$, and thus the conformal factor and the navigation metric take the simple forms

$$e^{\kappa(x)} = -4(V + V_Q) = -4 \left(V - \frac{\hbar^2}{2m} \frac{\Delta R}{R} \right), \quad (71)$$

$$h_{00}(x^0, x^i) = 4(V + V_Q)^2 = 4 \left(V - \frac{\hbar^2}{2m} \frac{\Delta R}{R} \right)^2, \quad (72)$$

$$h_{0i}(x^0, x^i) = 0, \quad i = 1, 2, 3, \quad (73)$$

$$h_{ij}(x^0, x^i) = -2m\delta_{ij}(V + V_Q) = -2m\delta_{ij} \times \left(V - \frac{\hbar^2}{2m} \frac{\Delta R}{R} \right), \quad i, j = 1, 2, 3. \quad (74)$$

The navigation metric is diagonal, and it is completely determined by the sum of the external potential, and of the quantum potential. Since $V + V_Q$ is the potential energy E_p of the quantum hydrodynamic system, it turns out that all the components of the navigation metric are expressed in terms of the potential energy in the energy space. Similarly, the conformal factor $e^{\kappa(x)}$ is basically the potential energy of the quantum system. On the other hand, in the case of the metric a_{IJ} , only the component $a_{00} = -(V + V_Q)$ gives the potential energy of the quantum system, with the components a_{ij} being proportional to the mass (or rest-mass energy mc^2) of the system, assumed to be a constant. But all the components of the conformal metric h_{IJ} are expressed in terms of the same physical variable, the potential energy of the quantum system.

Observe that the vector field W has no zeros on \mathcal{M} . Indeed, if W had zeros, then at such a point, the functions a^{I0} would vanish, for all $I \in \{0, 1, \dots, n\}$, but this is not possible, because it would imply that at such a point $\det |a_{IJ}| = 0$. Also, at a zero point of W , the potential $e\phi + V + V_Q$ and A_i must vanish simultaneously, and this is not possible from a physical point of view, since during the quantum evolution $V_Q \neq 0$.

Moreover, it easily follows that $a_{IJ}b^Ib^J = a_{IJ}a^{I0}a^{J0} = a^{00} = b^2$, and therefore $h_{IJ}W^IW^J = 1$. Hence, our results are consistent with the general theory.

We can formulate now our findings by means of a theorem as follows.

Theorem 2. The fundamental function $F = \frac{\alpha}{\beta} : T(\mathbb{R} \times M) \rightarrow \mathbb{R}$ given by Eq. (36), associated with the Lagrangian [Eq. (34)] describing the quantum hydrodynamical motion of a particle in the presence of external electromagnetic and nonelectromagnetic fields, is the solution of the Zermelo navigation problem on the Riemannian manifold $(\mathbb{R} \times M, h_{IJ})$ under the influence of the h -unit wind W .

In other words, the time-minimizing trajectories of a point on the space $\mathbb{R} \times M$ are not the Riemannian geodesics of a_{IJ} , but the geodesics of the globally defined Kropina metric $(\mathbb{R} \times M, F)$.

The Kropina metric obtained from the quantum Lagrangian in Eq. (34) is therefore obtained by a rigid translation of the Riemannian metric h_{IJ} by the h -unit vector field W . Hence, the quantum Lagrangian is equivalent to a Kropina type-1 homogeneous Lagrangian, and this is obtained by the deformation of a Riemannian conformal metric by means of W .

VI. FINSLERIAN GEOMETRIZATION OF THE SCHRÖDINGER EQUATION FOR A SPINLESS UNCHARGED PARTICLE

Let us consider the simple Lagrangian describing the evolution of a single uncharged particle in the hydrodynamic representation of the Schrödinger equation, given by

$$L(t, x, \dot{x}) = \frac{1}{2} \sum_{i=1}^3 m(\dot{x}^i)^2 - Q(t, x), \quad (75)$$

where $Q(t, x) = V(t, x) + V_Q(t, x)$, with V denoting the external potential, while V_Q is the quantum potential, responsible for the intrinsic quantum features of the dynamic evolution. The associated Kropina metric is given by $F = \frac{\alpha^2}{\beta}$, where $\alpha^2 = a_{IJ}y^Iy^J$ has the entries

$$a_{00}(x^0, x^i) := -Q, \quad a_{0i} := 0, \quad a_{ij} := \frac{1}{2}m\delta_{ij}, \quad i, j \in \{1, 2, 3\}. \quad (76)$$

As for β , it is given by $\beta = b_Iy^I = y^0$ —that is, the linear one-form has only one nonvanishing coefficient, $(b_0, b_1, b_2, b_3) = (1, 0, 0, 0)$.

The determinant of a_{IJ} is given by $\det|a_{IJ}| = -m^3Q/8$. The associated Riemannian metric is positive definite if and only if $Q < 0$ —that is, we need to be in the setting where the potential energy is positive. This can always be achieved by changing the reference system.

We can conclude therefore our preliminary investigations of the geometric description of the quantum hydrodynamics of a single particle by stating the following.

Theorem 3. The fundamental function $F = \frac{\alpha^2}{\beta} : T(\mathbb{R} \times M) \rightarrow \mathbb{R}$, associated with the Lagrangian (75) describing the quantum motion of a neutral spinless particle, is a globally defined Kropina metric on the extended configurations space \mathcal{M} , provided $Q < 0$.

The inverse of the matrix a_{IJ} is given by $a^{IJ} = \text{diag}(-1/Q, 2/m, 2/m, 2/m)$.

A. Geodesic evolution in the Finsler space

The Finsler metric function F associated with the Lagrangian (75) is given by

$$F(x^I, y^I) = \frac{\alpha^2}{\beta} = \frac{m\delta_{ij}y^iy^j/2 - Q(x^0, x^i)y^0}{y^0} = \frac{a_{00}(x^0, x^i)(y^0)^2 + T}{y^0} = a_{00}(x^0, x^i)y^0 + \frac{T}{y^0}, \quad (77)$$

where we have denoted $\alpha^2 = a_{00}(y^0)^2 + T$, $T = m\delta_{ij}y^iy^j/2$, and $\beta = y^0$.

Then, with the use of Eq. (40), the fundamental tensor of the Kropina space describing the geometric structure of the Schrödinger equation for a spinless particle is obtained as

$$g_{IJ} = \left[a_{00}^2(x^I) + \frac{3T^2}{(y^0)^4} \right] \delta_I^0 \delta_J^0 - \frac{2T}{(y^0)^3} (T_I \delta_J^0 + T_J \delta_I^0) + \frac{1}{(y^0)^2} T_I T_J + T_{IJ} \left[a_{00}(x^I) + \frac{T}{(y^0)^2} \right]. \quad (78)$$

Explicitly, the components of the Finslerian metric tensor are given by

$$g_{00} = Q^2 + 3 \frac{T^2}{(y^0)^4}, \quad g_{0i} = -2m \frac{\delta_{ij}y^j}{(y^0)^3} T, \quad (79)$$

$$g_{ij} = m\delta_{ij} \left(a_{00} + \frac{T}{(y^0)^2} \right) + m^2 \frac{\delta_{il}y^l \delta_{jk}y^k}{(y^0)^2}. \quad (80)$$

The spray coefficients describing the evolution of a neutral spinless quantum particle in the associated Kropina space are defined as

$$G^I(x, y) = \frac{1}{2} \Gamma_{JK}^I y^J y^K = \frac{1}{4} g^{IJ} \left(\frac{\partial^2 F^2}{\partial x^K \partial y^J} y^K - \frac{\partial F^2}{\partial x^J} \right), \quad (81)$$

where Γ_{JK}^I are the analogues of the Christoffel symbols of the Riemann geometry. By taking into account the explicit form of the Finslerian metric tensor coefficients as given by Eq. (78), we obtain

$$\Gamma_{JK}^I(x, y) = g^{IL} a_{00} \left\{ \frac{\partial}{\partial x^K} [a_{00} \delta_L^0 \delta_J^0 + \ln \sqrt{|a_{00}| T_{LL}}] + \frac{\partial}{\partial x^J} [a_{00} \delta_L^0 \delta_K^0 + \ln \sqrt{|a_{00}| T_{LK}}] - \frac{\partial}{\partial x^L} [|a_{00}| \delta_J^0 \delta_K^0 + \ln \sqrt{|a_{00}| T_{JK}}] \right\}. \quad (82)$$

By denoting

$$\tilde{g}_{IJ} = a_{00}(X^I) \delta_I^0 \delta_J^0 + \ln \sqrt{a_{00}(X^I) T_{IJ}}, \quad (83)$$

we find for the spray coefficients the expressions

$$G^I(x, y) = \frac{1}{2} g^{IL} a_{00} \left[\frac{\partial \tilde{g}_{LJ}}{\partial x^K} + \frac{\partial \tilde{g}_{LK}}{\partial x^J} - \frac{\partial \tilde{g}_{JK}}{\partial x^L} \right] y^J y^K, \quad (84)$$

which allows us to write the equations of the geodesics of the Finslerian geometric formulation of the quantum hydrodynamics as

$$\frac{d^2 x^I}{d\lambda^2} + 2G^I(x, y) = 0. \quad (85)$$

B. Geodesic equations in the associated Riemann space

Let us observe that the Levi-Civita connection coefficients of the Riemannian metric a_{IJ} given by Eq. (50) can be written as

$$\bar{\Gamma}_{00}^0 = \frac{1}{2} a^{00} \frac{\partial a_{00}}{\partial x^0}, \quad \bar{\Gamma}_{00}^i = -\frac{1}{2} a^{ij} \frac{\partial a_{00}}{\partial x^j}, \quad \bar{\Gamma}_{j0}^0 = \frac{1}{2} a^{00} \frac{\partial a_{00}}{\partial x^j}, \quad \bar{\Gamma}_{j0}^i = 0, \quad \bar{\Gamma}_{jk}^0 = 0, \quad \bar{\Gamma}_{jk}^i = \gamma_{jk}^i = 0, \quad (86)$$

where we have denoted by γ_{jk}^i the Levi-Civita connection coefficients of the Riemannian metric a_{ij} (obviously $\gamma_{jk}^i = 0$ because a_{ij} are constants).

The geodesics of the Riemannian metric a_{IJ} are given by

$$\frac{d^2 x^0}{d\sigma^2} + \frac{1}{2Q} \left(\frac{\partial Q}{\partial x^0} \frac{dx^0}{d\sigma} + 2 \frac{\partial Q}{\partial x^i} \frac{dx^i}{d\sigma} \right) \frac{dx^0}{d\sigma} = 0, \quad (87)$$

$$\frac{d^2 x^i}{d\sigma^2} + \frac{1}{m} \frac{\partial Q}{\partial x^j} \delta^{ij} \left(\frac{dx^0}{d\sigma} \right)^2 = 0, \quad (88)$$

where σ is the α -unit parameter. Equation (87) can be rewritten as

$$\frac{d}{d\sigma} \ln \left(\frac{dx^0}{d\sigma} \sqrt{Q} \right) = - \frac{1}{2} \left(\frac{1}{Q} \frac{\partial Q}{\partial x^i} \right) \Big|_{(x^0(\sigma), x^i(\sigma))} \frac{dx^i}{d\sigma}. \quad (89)$$

By integrating the above equation from 0 to σ and by substituting the resulting expression of $dx^0/d\sigma$ in Eq. (88), it follows that the equation of evolution of the spacelike coordinates x^i of the spinless neutral quantum particle in the associated Riemann space is described by the equation

$$\frac{d^2 x^i}{d\sigma^2} + \frac{C^2}{m} \frac{\partial Q}{\partial x^j} \delta^{ij} \exp \left[- \int_0^\sigma \left(\frac{1}{Q} \frac{\partial Q}{\partial x^i} \right) \Big|_{(x^0(\sigma), x^i(\sigma))} \frac{dx^i}{d\sigma} d\sigma \right] = 0, \quad (90)$$

where C is an arbitrary integration constant. Once the explicit form of the potential Q is known, the solutions of the geodesic equations in the associated Riemann space can be obtained by using either analytical or numerical methods.

C. The Zermelo navigation problem

We also observe that we can formulate the Zermelo navigation problem for this simple Lagrangian. Indeed, for the navigation data (H, W) , where $h_{IJ} = e^{k(x)} a_{IJ} = -4Q \text{diag}(-Q, m/2, m/2, m/2)$ and $W_I = (W_0, 0, 0, 0) = (-Q/2, 0, 0, 0)$, we have the following theorem. Here we have used $e^{k(x)} = 4/a^{00} = -4Q$:

Theorem 4. The fundamental function $F = \frac{\alpha^2}{\beta} : T(\mathbb{R} \times M) \rightarrow \mathbb{R}$ associated with the Lagrangian (75) is the solution of the Zermelo navigation problem on the Riemannian manifold $(\mathbb{R} \times M, h_{IJ})$ under the influence of the h -unit wind W . In other words, the time-minimizing trajectories of a point on the space $\mathbb{R} \times M$ are not the Riemannian geodesics of a_{IJ} , but the geodesics of the globally defined Kropina metric $(\mathbb{R} \times M, F)$.

For the Christoffel symbols in the h space, we obtain

$$\Gamma_{00}^{(h)0} = \frac{5}{2} \frac{1}{Q} \frac{\partial Q}{\partial t}, \quad \Gamma_{ij}^{(h)0} = 0, \quad \Gamma_{0j}^{(h)0} = \frac{3}{2} \frac{1}{Q} \frac{\partial Q}{\partial x^j}. \quad (91)$$

Then the Killing conditions read

$$\frac{\partial Q}{\partial t} = 0, \quad \frac{\partial Q}{\partial x^j} = 0; \quad (92)$$

that is, the wind W is Killing for the globally defined Kropina metric induced by the Lagrangian (75) if and only if Q is a constant. For the sake of convenience, we will consider this constant to be negative: $Q < 0$.

In this case, we can apply the general theory presented in Ref. [88] to this Kropina metric; however, the following peculiarities appear:

- (1) Since the potential Q is constant, the Lagrangian (75) is a time-independent Lagrangian, so the new coordinates (x^0, y^0) are no longer geometrically intrinsic.
- (2) Since all the entries of the matrix a_{IJ} are constants, it follows that the Riemannian metric α is Euclidean, the Levi-Civita connection coefficients $\bar{\Gamma}_{IJ}^K$ all vanish, and the one-form β is parallel. This means that the F -geodesics coincide with the α -geodesics. In other words, the F -geodesics are straight lines.

Even though the Riemannian metric and the unit Killing vector field W are both complete, this does not mean that we can join any two points by an F -geodesic.

VII. DISCUSSIONS AND FINAL REMARKS

In the present paper, we have considered an alternative geometric perspective of one of the fundamental fields of theoretical physics, quantum mechanics. Even that quantum mechanics is intrinsically a geometric theory, formulated in a Hilbert space, and its structure is completely different from the spacetime geometrical approach that was so successful in the description of gravitational interaction. The geometrizations of quantum mechanics in its standard formulation, even if they use some of the theoretical tools of Riemannian geometry, still present fundamental differences as compared to the spacetime geometry of special and general relativity. In our work, we have adopted a different approach to the geometrization of quantum mechanics: namely, we have adopted as a starting point the so-called hydrodynamical (or Madelung) formulation of quantum mechanics. In this formulation, extensively used in many fields of physics, the quantum motion of the particle is described by the standard equations of classical fluid dynamics (continuity and Euler equations), in the presence of a quantum potential, which essentially determines the quantum properties of the motion. The hydrodynamical interpretation is a standard theoretical tool in the physics of quantum gases, superfluidity, and Bose-Einstein condensation theory, and it allows the realization of a deeper connection between experiment and theory. This formulation is also at the basis of the de Broglie-Bohm pilot-wave (deterministic) interpretation of quantum mechanics (see Ref. [76] for an in-depth analysis of the

causal interpretation of the quantum theory). Hence, in the hydrodynamic interpretation of the Schrödinger equation, the evolution of the quantum particle is described by the motion of a classical fluid in the presence of a specific potential. However, the Madelung representation of the wave mechanics can be seen as a mathematical tool, which does not change the physical interpretation of the quantum mechanics.

On the other hand, any spacetime-type geometrization of quantum mechanics requires the introduction of the concept of quantum trajectory [91]. In the Bohmian interpretation of quantum mechanics, the quantum trajectory plays the role of a hidden variable, with the path being the fundamental dynamical variable of the theory [80,81], while the wave function evolves in time along the Lagrangian motion of the path [68]. However, it is important to point out that the particle dynamics along the Bohmian trajectories is not equivalent to the point-particle evolution along classical Newtonian trajectories [68]. One reason is that in principle, the classical behavior should be obtained by adopting a pointlike initial density of the Dirac delta function type for the density, and then integrating the Euler-Lagrange equation over a reference density. However, the structure of the quantum potential does not allow this type of initial condition [68].

Once the definition of the quantum trajectory is introduced, the equations of motion of the quantum particle can be derived from an action principle, which, in the presence of electromagnetic fields, contain the standard classical terms plus the quantum potential, responsible for the nonclassical features of the motion. The Lagrangian function can be taken as the starting point for the geometrization of quantum hydrodynamics. After performing the homogenization procedure of the Lagrangian introduced in Ref. [85], it turns out that the Finsler function associated with the Lagrangian of the quantum mechanics takes the form of the Kropina function [73,74], which is a particular form of the general (α, β) metrics [71,72], whose geometric properties have been extensively studied in the mathematical literature. The Kropina metric can also be expressed in terms of a Riemann metric a_{IJ} associated with the quantum Lagrangian, and which is entirely determined by the components of the external potentials (including the quantum potential and the electromagnetic potentials). Once the Finsler metric function is known, the Finslerian metric

tensor components can be obtained in a straightforward way, as well as the geodesic equations, which are equivalent to the quantum equations of motion of the particle. It is interesting that similar equations of motion, fully determined by the external potentials, can also be written down in the associated Riemann space, with metric a_{IJ} . They describe the motion of the quantum particle in the presence of a particular force generated via the Christoffel symbols associated with the Riemann metric.

The geometrization of the hydrodynamic formulation of the quantum mechanics also opens some new perspectives on the quantum Zermelo navigation problem, a subject of major interest in quantum computation. Once the geometric Finsler-type structure of the quantum hydrodynamic flow is known, the Zermelo navigation data can be easily constructed by means of a conformal transformation of the associated Riemannian metric a_{IJ} , and of the vector b_I . The quantum wind can be obtained as a function of the external and quantum potentials, respectively. The Kropina fundamental function of quantum hydrodynamics is a solution of the Zermelo problem, a result that may find some useful applications in the study of quantum speed limits and quantum information transfer.

One of the main advantages of the Finslerian geometrization of quantum mechanics is that the entire formalism is constructed in the ordinary spacetime geometry, with all geometrical quantities being functions of the external physical variables (field potentials). Even though the expressions of the Finslerian metric tensors and of the geodesic sprays are relatively complicated, they can be handled from a mathematical point of view in a much easier way than their purely quantum counterparts defined on a Hilbert space. Hence, the results obtained in the present paper may provide some deeper insights into the complex relation between the classical and quantum worlds, as well as on the geometrical structures underlying the quantum mechanical dynamics.

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