Covariant version of the Pauli Hamiltonian, spin-induced noncommutativity, Thomas precession, and the precession of spin

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We show that there is a manifestly covariant version of the Pauli Hamiltonian with equations of motion quadratic on spin and field strength. Relativistic covariance inevitably leads to noncommutative positions: classical brackets of the position variables are proportional to the spin. It is the spin-induced non-commutativity that is responsible for transforming the covariant Hamiltonian into the Pauli Hamiltonian, without any appeal to the Thomas precession formula. The Pauli theory can be thought to be $1/c^2$ approximation of the covariant theory written in special variables. These observations clarify the long standing question on the discrepancy between the covariant and Pauli Hamiltonians. We also discuss the transformational properties of the spin axis in the passage from the laboratory to comoving and instantaneous frames, and reveal the role of the Thomas spin vector in the covariant scheme.

I. INTRODUCTION: RELATIVISTIC SPINNING ELECTRON AND THE PROBLEM OF COVARIANT FORMALISM

Classical models of relativistic spin represent a working tool used to describe the behavior of elementary particles and rotating bodies in electromagnetic and gravitational fields. One obscure point of this approach, which has been raised for discussion already in the pioneer works [1–4] and remains under debate to this day, is the so-called problem of covariant formalism. Clarification of this issue could be of interest in various areas, including muon and electron $g - 2$ experiments [5–7], the influence of spin on the trajectory of a rotating body in general relativity [8–13], and black hole physics near horizon [8,14–16]. In the textbooks and reviews, it has become almost a tradition to discuss the problem without formulating it in an exact form.[1] Let us try to break this tradition.

Historically, the notion of a classical spinning electron [1–4] has been developed in attempts to explain the energy levels of atomic spectra. Following the ideas of Uhlenbeck and Goudsmit [1], Thomas accepted that a spinning particle can be described using its position vector $\mathbf{x}(t)$, and the vector of the spin axis $\mathbf{S}_T(t)$ attached to the particle. The position vector obeys the Lorentz-force equation

$$m\ddot{\mathbf{x}} = e\mathbf{E} + \frac{e}{c}[\dot{\mathbf{x}}, \mathbf{B}], \qquad (1)$$

[1]See p. 541 and 558 in [17], p. 87 in [18], p. 6 in [19], p. 563 in [5].

while the rate of variation of spin was initially assumed to be

$$\frac{d\mathbf{S}_T}{dt} = -\frac{e}{mc}\left\{[\mathbf{B}, \mathbf{S}_T] - \frac{1}{2mc}[[\mathbf{p}, \mathbf{E}], \mathbf{S}_T]\right\}. \qquad (2)$$

We use the notation $[\mathbf{A}, \mathbf{B}]$ and $(\mathbf{A}, \mathbf{B})$ for the vector and scalar products of three-dimensional vectors. The quantity $\mathbf{S}_T$ will be called the Thomas spin vector. Here $\mathbf{E} = \alpha\mathbf{x}/|\mathbf{x}|^3$ is Coulomb electric field and $\mathbf{B}$ is a constant magnetic field. Thomas showed [2] that these equations, together with the Bohr quantization rule of angular momentum, give a satisfactory description of atomic energy levels. A more systematic calculation of the energies was achieved in quantum mechanics constructed on the base of these equations. Pauli noticed [20] that Eqs. (1) and (2) follow from the Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + eA^0$$
$$- \frac{e}{mc}\left[(\mathbf{S}_T, \mathbf{B}) + \frac{1}{2mc}(\mathbf{S}_T, [\mathbf{E}, \mathbf{p}])\right], \qquad (3)$$

with use of canonical brackets (we show the nonvanishing brackets)

$$\{x^i, p^j\} = \delta^i{}_j, \qquad \{S_T^i, S_T^j\} = \epsilon^{ijk}S_T^k. \qquad (4)$$

So, in the Pauli formalism it is assumed that (2) represents the equation of motion of spin *in the laboratory system*. Having at hand the Hamiltonian formulation, Pauli

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constructed the quantum mechanics of the spinning electron by replacing the classical variables $z^A \equiv (x^i, p^k, S_T^j)$ by operators that, according to the Dirac quantization rule, must obey the commutators resembling the classical brackets

$$[\hat{z}^A, \hat{z}^B] = i\hbar \{z^A, z^B\}|_{z \to \hat{z}}. \qquad (5)$$

The operators are $\hat{p}^i = -i\hbar\partial_i$, $\hat{x}^i = x^i$, $\hat{S}^i = \frac{\hbar}{2}\sigma^i$, where σ^i are 2×2 matrices of Pauli. The operators act on the space of two-component wave functions $\Psi_a(t, \mathbf{x})$, $a = 1, 2$. Replacing classical variables in Eq. (3) by the operators, he obtained the quantum Hamiltonian and showed that the resulting quantum mechanics reproduces the atomic energy levels [20]. The Hamiltonian (3) can also be obtained from the Dirac equation [21,22].

Difficulties arose when trying to develop the manifestly covariant relativistic generalization of the classical theory (1)–(3). Numerous attempts (see the pioneer works [2–4,23,24] and the reviews [8,25]) lead to equations of spin and to Hamiltonians that in $1/c^2$ approximation differ from those of Pauli theory. For instance, assuming that three-dimensional spin is a spatial part of a manifestly covariant four-dimensional spin tensor of Frenkel [3,4], the covariant theory implies the following expressions (for the details, see below):

$$\frac{d\mathbf{S}}{dt} = -\frac{e}{mc}\left\{[\mathbf{B}, \mathbf{S}] + \frac{1}{mc}[\mathbf{E}, [\mathbf{p}, \mathbf{S}]]\right\}, \qquad (6)$$

$$H_{ph} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + eA^0$$
$$- \frac{e}{mc}\left[(\mathbf{S}, \mathbf{B}) + \frac{1}{mc}(\mathbf{S}, [\mathbf{E}, \mathbf{p}])\right]. \qquad (7)$$

We call the quantity $\mathbf{S}$ the Frenkel spin vector. The Hamiltonians (3) and (7) differ by the famous 1/2 factor in front of the last term, whereas the last terms in the equations of spin differ in a structure. The question, why a covariant formalism does not lead directly to the expected result, was raised already [2–4] and remain under discussion to date; see the review [8].

In attempts to explain the discrepancy, Thomas compared the variation rates of the spin axis in comoving and instantaneous frames. The resulting relation is the famous formula of Thomas precession. It should be noted that the Thomas formula itself is the object of numerous debates. A detailed analysis of controversial works on the subject can be found in [26–28]. In particular, it is widely believed [5,17–19,27] that Thomas precession is relevant to the problem of covariant formalism. In this regard, we note that the Thomas formula relates quantities of different coordinate systems, while Eqs. (1)–(3), (6), (7) are taken in the same (laboratory) system.

The aim of this work is to clarify these issues. We present a manifestly covariant formulation of a spinning particle, that in $1/c^2$ approximation implies the quantum mechanics of Pauli without any appeal to the Thomas precession formula. Then we make a detailed comparison of Thomas and Frenkel spin vectors.

The work is organized as follows. In Sec. II we start from the covariant formalism of a spinless particle and, assuming that spin in a relativistic theory can be described by the Frenkel spin tensor, we write the expected expressions for manifestly covariant and physical Hamiltonians of a spinning particle. In Sec. III we fix the brackets that guarantee the consistency of Hamiltonian equations with the supplementary condition on the Frenkel spin tensor. In particular, we show that the position variables turn out to be noncommuting in the covariant scheme. In Sec. IV we show how this spin-induced noncommutativity of positions solves the problem of covariant formalism. The calculations in these sections are valid in $1/c^2$ approximation. In Sec. V we construct the fully covariant brackets and show that there exists the manifestly covariant version of Pauli theory with the equations of motion no more than quadratic on spin and field strength. In Sec. VI we return back to the discussion of covariant theory in $1/c^2$ approximation and reveal the meaning of the Thomas spin in the covariant scheme.

II. MANIFESTLY COVARIANT AND PHYSICAL-TIME HAMILTONIANS

Here we review the known formalism of a spinless particle [29] and then discuss the most natural way to include spin in the covariant scheme. To work with manifestly covariant expressions, we describe the trajectory of the particle in a parametric form: $\mathbf{x}(t) \to x^\mu(\tau) = (ct(\tau), \mathbf{x}(\tau) \equiv \mathbf{x}(t(\tau))$. As the parameter τ we take the proper time

$$\tau(t) = \int_0^t dt\sqrt{1 - \mathbf{v}^2/c^2}, \quad \text{then} \quad \left.\frac{dx^0(\tau)}{d\tau}\right|_{\tau(t)} = c\gamma,$$
$$\left.\frac{d\mathbf{x}(\tau)}{d\tau}\right|_{\tau(t)} = \gamma\frac{d\mathbf{x}(t)}{dt}, \qquad (8)$$

where $\mathbf{v}(t) = \frac{d\mathbf{x}}{dt}$, $\gamma = (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}$, and by construction of the proper time, tangent vector $u^\mu \equiv dx^\mu(\tau)/d\tau$ to the curve $x^\mu(\tau)$ has a fixed length at any instant, $(u^\mu)^2 = -c^2$. In accordance with Eq. (8), the four- and three-dimensional vectors of velocity are related as follows: $(u^0, \mathbf{u})|_{\tau(t)} = (c\gamma, \gamma\mathbf{v})$. This is a covariant expression, so the last equation from (8) can be used to restore $\mathbf{v}(t)$ from $\mathbf{u}(\tau)$ in any inertial frame.

In the Hamiltonian formulation, to each variable $x^\mu(\tau)$ of configuration space we associate the function $p_\mu(\tau)$ called the conjugated momentum. The manifestly covariant

Hamiltonian of a spinless particle in the electromagnetic field with four-potential A^μ is

$$H = \frac{1}{2m}\left[(p^\mu - \frac{e}{c}A^\mu)^2 + (mc)^2\right]. \quad (9)$$

It is accompanied with the covariant Poisson brackets $\{x^\mu, p_\nu\} = \delta^\mu{}_\nu$, $\{x^\mu, x^\nu\} = \{p_\mu, p_\nu\} = 0$. For the canonical momentum $\mathcal{P}^\mu \equiv p^\mu - \frac{e}{c}A^\mu$, the brackets imply $\{\mathcal{P}^\mu, \mathcal{P}^\nu\} = \frac{e}{c}F^{\mu\nu}$. From the requirement of gauge invariance of the Hamiltonian it follows that the conjugated momentum p_μ is not invariant under the $U(1)$-gauge transformations

$$A_\mu = A'_\mu + \partial_\mu\lambda, \quad \text{implies}, \quad p_\mu = p'_\mu + \frac{e}{c}\partial_\mu\lambda. \quad (10)$$

In contrast, $\mathcal{P}_\mu$ is an invariant object, and we expect that the Hamiltonian equations can be presented in terms of this quantity. Computing $\dot{x}^\mu = \{x^\mu, H\}$, $\dot{p}^\mu = \{p^\mu, H\}$, the equations can be written in the form

$$m\frac{dx^\mu}{d\tau} = \mathcal{P}^\mu, \qquad \frac{d\mathcal{P}^\mu}{d\tau} = \frac{e}{mc}F^{\mu\nu}\mathcal{P}_\nu. \quad (11)$$

Computing the square of the first equation from (11), we obtain the mass-shell relation $\mathcal{P}^2 + (mc)^2 = 0$, or

$$cp^0 = c\sqrt{(mc)^2 + \mathcal{P}^2} + eA^0. \quad (12)$$

Since $x^\mu(\tau)$ and $p_\mu(\tau)$ represent the physical dynamical variables $\mathbf{x}(t)$ and $\mathbf{p}(t)$ in a parametric form, we can write $\frac{d\mathbf{x}}{dt} = c\frac{d\mathbf{x}/d\tau}{dx^0/d\tau}$, $\frac{d\mathcal{P}}{dt} = c\frac{d\mathcal{P}/d\tau}{dx^0/d\tau}$, and with use of (11) and (12) we obtain $\frac{d\mathbf{x}}{dt} = c\mathcal{P}/\sqrt{(mc)^2 + \mathcal{P}^2}$, $\frac{d\mathcal{P}}{dt} = e\mathbf{E} + e[\mathcal{P}, \mathbf{B}]/\sqrt{(mc)^2 + \mathcal{P}^2}$. Expanding them in series over $1/c$ and keeping only the terms of order $1/c$, we obtain the well-known equations[2]

$$m\frac{d\mathbf{x}}{dt} = \mathcal{P}, \qquad \frac{d\mathcal{P}}{dt} = e\mathbf{E} + \frac{e}{mc}[\mathbf{p}, \mathbf{B}]. \quad (13)$$

The first equation shows that canonical momentum $\mathcal{P}$ represents the velocity of the particle in the Hamiltonian formulation. Excluding the conjugated momentum $\mathbf{p}$ from these equations, we arrive at the Lorentz-force equation (1).

Equation (13) can also be obtained from the Hamiltonian (9), presented in terms of the physical variables $\mathbf{x}(t)$ and $\mathbf{p}(t)$. While, in general, the reparametrization-invariant theory with Dirac constraints requires some caution [30–36], the final result is very simple [37]: the physical

Hamiltonian coincides with the right-hand side of Eq. (12). Its expansion in series up to order $1/c^2$ gives the physical Hamiltonian

$$H_{ph} = cp^0 \approx mc^2 + \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + eA^0. \quad (14)$$

As it should be, this coincides with the spinless part of the Pauli Hamiltonian (3).

Concerning the spin, we introduce the vector function $\mathbf{S}(t)$ and take it in the proper-time parametrization, defining $\mathbf{S}(\tau) \equiv \mathbf{S}(t(\tau))$. Following Frenkel [3], we identify the components $S^i(\tau)$ of three-dimensional spin with the spatial part of the four-dimensional antisymmetric spin-tensor $S^{\mu\nu} = -S^{\nu\mu}$

$$S^i = \frac{1}{4}\epsilon^{ijk}S^{jk}, \quad \text{then } S^{ij} = 2\epsilon^{ijk}S^k. \quad (15)$$

We assume that the Frenkel spin S^i can be identified with the spin axis in the laboratory frame. We assume that at each instant of motion, $S^{\mu\nu}$ obeys the covariant condition[3]

$$S^{\mu\nu}\mathcal{P}_\nu = 0, \quad \text{then } S^{0i} = \frac{1}{\mathcal{P}_0}S^{ij}\mathcal{P}_j \approx -\frac{2}{mc}[\mathbf{p}, \mathbf{S}]^i + O(1/c^2). \quad (16)$$

As a consequence, the number of independent components of spin in relativistic and Pauli theories is the same. In the rest frame of the particle, where $\mathcal{P} = 0$, the extra components just vanish, $S^{0i} = 0$. Our basic variables $z^A \equiv (x^\mu, p_\nu, S^{\mu\nu})$ transform linearly under the Lorentz transformations: $x^\mu = \Lambda^\mu{}_\nu x^\nu$, $p^\mu = \Lambda^\mu{}_\nu p^\nu$, $S^{\mu\nu} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta S^{\alpha\beta}$.

In trying to include spin into the covariant Hamiltonian, we note that the only scalar function containing the desired spin-field interaction is $-\frac{e}{2c}F_{\mu\nu}S^{\mu\nu} = -\frac{2e}{c}[(\mathbf{S}, \mathbf{B}) + \frac{1}{mc}(\mathbf{S}, [\mathbf{E}, \mathcal{P}])]$. Adding this term to Eq. (9), we obtain

$$H = \frac{1}{2m}\left[\left(p^\mu - \frac{e}{c}A^\mu\right)^2 - \frac{e\mu}{2c}F_{\mu\nu}S^{\mu\nu} + (mc)^2\right]. \quad (17)$$

We added the interaction of spin with the electromagnetic field through the magnetic moment μ that corresponds to the gyromagnetic ratio $g = 2\mu$. In Secs. III, IV, and VI we will put the classical value of μ: $\mu = 1$. Comparing (17) with Eqs. (9) and (14), the expected expression for the physical Hamiltonian in $1/c^2$ approximation is

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[2]For the electromagnetic field we use the notation: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = (F_{0i} = -E_i, F_{ij} = \epsilon_{ijk}B_k)$, $E_i = -\frac{1}{c}\partial_t A_i + \partial_i A_0$, $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = \epsilon_{ijk}\partial_j A_k$.

[3]We could equally use $S^{\mu\nu}p_\nu = 0$ with the conjugated momentum p_ν instead of $\mathcal{P}_\nu$. The difference between them is of order $1/c$, and does not contribute into subsequent expressions in the $1/c^2$ approximation.

$$H_{ph} = mc^2 + \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + eA^0$$

$$- \frac{e}{mc}\left[(\mathbf{S},\mathbf{B}) + \frac{1}{mc}(\mathbf{S},[\mathbf{E},\mathbf{p}])\right]. \quad (18)$$

We confirm the validity of this expression in Sec. V.

III. NONCANONICAL BRACKETS

To obtain equations of motion, the Hamiltonian (17) should be accompanied with some brackets. The spin supplementary condition (16) should be consistent with the resulting equations of motion. This leads us to the observation that will be crucial for our explanation of the $1/2$ factor: manifestly covariant formalism inevitably leads to relativistic corrections of order $1/c^2$ to the canonical brackets (4).

The condition (16) implies $\frac{d}{d\tau}(S^{\mu\nu}\mathcal{P}_\nu) = 0$. In the Hamiltonian formalism, the variation rate of a phase-space function is equal to the bracket of this function with the Hamiltonian, so we can write $\frac{d}{d\tau}(S^{\mu\nu}\mathcal{P}_\nu) = \{S^{\mu\nu}\mathcal{P}_\nu, H\} = \{S^{\mu\nu}\mathcal{P}_\nu, z^A\}\frac{\partial H}{\partial z^A} = 0$. The latter equality certainly holds if[4]

$$\{z^A, S^{\mu\nu}\mathcal{P}_\nu\} = 0. \quad (19)$$

It is this equation that requires a modification of canonical brackets. For $z^A = x^\alpha$, Eq. (19) implies $\{x^\alpha, S^{\mu\nu}\mathcal{P}_\nu\} = S^{\mu\nu}\{x^\alpha, \mathcal{P}_\nu\} + \{x^\alpha, S^{\mu\nu}\}\mathcal{P}_\nu = 0$. This equality holds, if we take

$$\{x^\alpha, S^{\mu\nu}\} = \frac{\mathcal{P}^\mu S^{\nu\beta} - \mathcal{P}^\nu S^{\mu\beta}}{\mathcal{P}^2}\{x^\alpha, \mathcal{P}_\beta\}. \quad (20)$$

In turn, if the bracket $\{x^\alpha, \mathcal{P}_\beta\} = \{x^\alpha, p_\beta\} - \frac{e}{c}\{x^\alpha, x^\gamma\}\partial_\gamma A_\beta$ remains unmodified at the order $1/c^2$, that is $\{x^\alpha, \mathcal{P}_\beta\} = \delta^\alpha_\beta + O(1/c^3)$, Eq. (20) implies the following modification of the canonical bracket $\{x^i, S^j\}$:

$$\{x^i, S^j\} = \frac{p^i S^j - \delta^{ij}(\mathbf{p},\mathbf{S})}{(mc)^2}, \quad (21)$$

at $1/c^2$ order. For the spin tensor we impose

$$\{S^{\alpha\beta}, S^{\mu\nu}\} = 2N^{(\alpha\mu}S^{\beta\nu)}$$

$$\equiv 2(N^{\alpha\mu}S^{\beta\nu} - N^{\alpha\nu}S^{\beta\mu} - N^{\beta\mu}S^{\alpha\nu} + N^{\beta\nu}S^{\alpha\mu}), \quad (22)$$

where N is the projector on the plane orthogonal to $\mathcal{P}^\mu$: $N^{\mu\nu} = \eta^{\mu\nu} - \frac{\mathcal{P}^\mu\mathcal{P}^\nu}{\mathcal{P}^2}$, then $N^{\mu\nu}\mathcal{P}_\nu = 0$. The bracket ensures the

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[4]When this equality is satisfied, we can make the substitution (16) before computing the brackets: $\{K(z), N(z)\}|_{S^{0i} = \frac{1}{\mathcal{P}_0}S^{ij}\mathcal{P}_j} = \{K(z)|, N(z)|\}$.

validity of Eq. (19) for $z^A = S^{\alpha\beta}$. For the spatial components, Eq. (22) gives

$$\{S^i, S^j\} = \epsilon^{ijk}\left[S^k + \frac{p^k(\mathbf{p},\mathbf{S})}{(mc)^2}\right], \quad (23)$$

instead of canonical bracket (4). The Jacobi identity $\{x^i, \{x^j, S^k\}\} + \{x^j, \{S^k, x^i\}\} + \{S^k, \{x^i, x^j\}\} = 0 + O(1/c^4)$ with use of (21) requires the following modification of the position-position bracket: $\{x^i, x^j\} = \frac{1}{(mc)^2}\epsilon^{ijk}S^k$. The examination of Eq. (19) for $z^A = p_\mu$ does not imply $1/c^2$ corrections to the canonical brackets of p_i with x^j and S^k. So, in $1/c^2$ approximation, the expected nonvanishing brackets of the covariant formalism are

$$\{x^i, x^j\} = \frac{1}{(mc)^2}\epsilon^{ijk}S^k, \quad (24)$$

$$\{x^i, p^j\} = \delta^i{}_j, \quad (25)$$

$$\{x^i, S^j\} = \frac{p^i S^j - \delta^{ij}(\mathbf{p},\mathbf{S})}{(mc)^2}, \quad (26)$$

$$\{S^i, S^j\} = \epsilon^{ijk}\left[S^k + \frac{p^k(\mathbf{p},\mathbf{S})}{(mc)^2}\right]. \quad (27)$$

We point out that they coincide with $1/c^2$ approximation of Dirac brackets arising in the vector model of spin [37,38]. Together with the Hamiltonian (18), the brackets (24)–(27) imply the Frenkel equations (6) and (13).

IV. THE COVARIANT AND PAULI FORMULATIONS DETERMINE THE SAME CLASSICAL AND QUANTUM THEORY

Both H_{ph} written in Eq. (18) and the brackets (24)–(27) of the covariant theory differ from those of Pauli theory. Nevertheless, they lead to the same quantum mechanics. Indeed, we can realize our variables by Hermitian operators

$$\hat{p}^i = -i\hbar\partial_i, \qquad \hat{x}^i = x^i - \frac{\hbar}{4(mc)^2}\epsilon^{ijk}\hat{p}^j\sigma^k,$$

$$S^i = \frac{\hbar}{2}\left[\sigma^i - \frac{\hat{p}^i(\hat{\mathbf{p}},\boldsymbol{\sigma})}{2(mc)^2} + \frac{\hat{\mathbf{p}}^2\sigma^i}{2(mc)^2}\right]$$

$$\equiv \frac{\hbar}{2}\left[\sigma^i + \frac{1}{2(mc)^2}[\hat{\mathbf{p}},[\boldsymbol{\sigma},\hat{\mathbf{p}}]^i]\right]. \quad (28)$$

Their commutators are in correspondence with the classical brackets (24)–(27), as it should be in accordance with Eq. (5). We substitute the operators into the Hamiltonian (18), and expand the resulting expression in a series over

$1/c$ up to $1/c^2$ order. In this approximation we have $-\frac{e}{c}\mathbf{A}(\hat{x}^i) = -\frac{e}{c}\mathbf{A}(x^i) + O(1/c^3)$, while[5]

$$eA^0(\hat{x}^i) = eA^0(x^i) + \frac{e}{2(mc)^2}(\hat{\mathbf{S}}, [\mathbf{E}, \hat{\mathbf{p}}]). \qquad (29)$$

The last term in this expression has the same structure as the fourth term in (18), so their sum acquires the desired $1/2$ factor. In the result, the quantum Hamiltonian of the covariant formulation coincides with the Pauli Hamiltonian

$$\hat{H}_{ph} = mc^2 + \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(x^i)\right)^2 + eA^0(x^i)$$
$$- \frac{e}{mc}\left[(\hat{\mathbf{S}}, \mathbf{B}) + \frac{1}{2mc}(\hat{\mathbf{S}}, [\mathbf{E}, \hat{\mathbf{p}}])\right]. \qquad (30)$$

This solves the problem of covariant formalism.

We also can ask on the relation between covariant and Pauli formulations considered as the classical theories. Starting from the covariant formulation (18), (24)–(27), we look for the phase-space variables that obey the canonical brackets. They are

$$p_c^i = p^i, \qquad x_c^i = x^i + \frac{1}{2(mc)^2}\epsilon^{ijk}p^j S^k, \qquad (31)$$

$$S_T^i = S^i + \frac{p^i(\mathbf{p}, \mathbf{S})}{2(mc)^2} - \frac{\mathbf{p}^2 S^i}{2(mc)^2} \equiv S^i + \frac{1}{2(mc)^2}[\mathbf{p}, [\mathbf{p}, \mathbf{S}]]^i. \qquad (32)$$

The Hamiltonian (18), in terms of these variables, turns into the Hamiltonian of the Pauli theory. So, at the classical level, the covariant and Pauli formulations are related by the (noncanonical) transformation (31), (32) of the phase space, and hence describe the same theory.[6] In particular, the vector $\mathbf{S}_T$ defined by (32) should obey the Thomas equation (2). It is instructive to show this by direct computation. Using $\frac{d\mathbf{p}}{dt} = e\mathbf{E} + O(1/c)$, the Frenkel equation (6) can be rewritten as follows:

$$\frac{d}{d\tau}\left[S^i + \frac{1}{2(mc)^2}[\mathbf{p}, [\mathbf{p}, \mathbf{S}]]^i\right]$$
$$= -\frac{e}{mc}\left\{[\mathbf{B}, \mathbf{S}] - \frac{1}{2mc}[[\mathbf{p}, \mathbf{E},]\mathbf{S}]\right\}. \qquad (33)$$

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[5]Assuming the symmetric ordering of operators, we obtain the Hermitian operator $eA^0(x^i) + \frac{e}{2(mc)^2}(\hat{\mathbf{S}}, [\mathbf{E}, \hat{\mathbf{p}}]) - \frac{i\hbar e}{4(mc)^2}(\hat{\mathbf{S}}, \mathbf{rot}\,\mathbf{E})$. For the central field $\mathbf{rot}\,\mathbf{E} = 0$, and the last term vanishes.

[6]Note that $\mathbf{S}_T$ does not coincide with the spatial part of the Pauli-Lubanski four-vector $S_{PL}^\mu = \frac{1}{4\sqrt{-\mathcal{P}^2}}\epsilon^{\mu\nu\alpha\beta}\mathcal{P}_\nu S_{\alpha\beta}$. Its spatial components $S_{PL}^i = S^i + \frac{1}{(mc)^2}p^i(\mathbf{p}, \mathbf{S}) - \frac{1}{2(mc)^2}\mathbf{p}^2 S^i$ are different from (32).

Using the definition (32) on the lhs, and replacing $\mathbf{S} = \mathbf{S}_T + O(1/c^2)$ on the rhs, we obtain the Thomas equation.

A similar situation arises for a rotating body in general relativity. Here equations of motion for spin can be deduced either from the analysis of Einstein equations in multipole formalism [39], or in a geometric setting, assuming the Fermi-Walker transport of the spin vector [40]. The two spins turn out to be different, and related by the gravitational analogy of Eq. (32); compare Eq. (193) in [41] with our (33).

V. MANIFESTLY COVARIANT VERSION OF THE PAULI THEORY

As we saw above, in $1/c^2$ approximation the expression (17) leads to the Pauli theory and hence may be taken as the Hamiltonian of its manifestly covariant version. The Hamiltonian should be accompanied by the Poincare-covariant generalization of the brackets (24)–(27). Besides, the brackets should lead to $U(1)$-invariant equations of motion, so they must be invariant under the gauge transformation (10). The canonical brackets $\{x^\mu, p_\nu\} = \delta^\mu{}_\nu$, $\{p_\mu, p_\nu\} = \{p_\mu, S^{\alpha\beta}\} = 0$ are not invariant, and should be properly modified. For instance, if we substitute (10) into the bracket $\{x^\mu, p_\nu\} = \delta^\mu{}_\nu$, we obtain $\{x^\mu, p'_\nu\} = \delta^\mu{}_\nu + \frac{e}{2c\mathcal{P}^2}S^{\mu\alpha}\partial_\alpha\partial_\nu\lambda$, instead of $\{x^\mu, p'_\nu\} = \delta^\mu{}_\nu$, where the extra contribution is due to the relativistic generalization (34) of Eq. (24).

We propose the following set of Poincare-covariant and $U(1)$-invariant brackets

$$\{x^\mu, x^\nu\} = -\frac{1}{2\mathcal{P}^2}S^{\mu\nu}, \qquad (34)$$

$$\{x^\mu, p_\nu\} = \delta^\mu{}_\nu - \frac{e}{2c\mathcal{P}^2}S^{\mu\alpha}\partial_\alpha A_\nu, \qquad (35)$$

$$\{x^\alpha, S^{\mu\nu}\} = \frac{1}{\mathcal{P}^2}\mathcal{P}^{[\mu}S^{\nu]\alpha}, \qquad (36)$$

$$\{p_\alpha, S^{\mu\nu}\} = \frac{e}{c\mathcal{P}^2}\mathcal{P}^{[\mu}S^{\nu]\beta}\partial_\alpha A_\beta, \qquad (37)$$

$$\{p_\mu, p_\nu\} = -\frac{e^2}{2c^2\mathcal{P}^2}S^{\alpha\beta}\partial_\alpha A_\mu\partial_\beta A_\nu, \qquad (38)$$

$$\{S^{\alpha\beta}, S^{\mu\nu}\} = 2N^{(\alpha\mu}S^{\beta\nu)} + \frac{e}{c(\mathcal{P}^2)^2}\mathcal{P}^{[\alpha}(SFS)^{\beta][\mu}\mathcal{P}^{\nu]}. \qquad (39)$$

For the canonical momenta $\mathcal{P}_\mu = p_\mu - \frac{e}{c}A_\mu$ they imply

$$\{x^\mu, \mathcal{P}_\nu\} = \delta^\mu{}_\nu, \qquad \{\mathcal{P}^\mu, \mathcal{P}^\nu\} = \frac{e}{c}F^{\mu\nu},$$

$$\{S^{\mu\nu}, \mathcal{P}_\alpha\} = \frac{e}{c\mathcal{P}^2}\mathcal{P}^{[\mu}S^{\nu]\beta}F_{\beta\alpha}. \qquad (40)$$

Computing $\dot{z}^A = \{z^A, H\}$, we obtain the Hamiltonian equations

$$\dot{x}^\mu = \frac{1}{m}\mathcal{P}^\mu - \frac{e\mu}{2mc\mathcal{P}^2}(SF\mathcal{P})^\mu + \frac{e\mu}{8mc\mathcal{P}^2}S^{\mu\alpha}\partial_\alpha(FS),$$

(41)

$$\dot{\mathcal{P}}^\mu = \frac{e}{mc}F^{\mu\nu}\mathcal{P}_\nu - \frac{e^2\mu}{2mc^2\mathcal{P}^2}(FSF\mathcal{P})^\mu + \frac{e\mu}{4mc}\partial^\mu(FS),$$

(42)

$$\dot{S}^{\mu\nu} = \frac{e\mu}{mc}F^\mu{}_\alpha S^{\alpha\nu} + \frac{e(1-\mu)}{mc\mathcal{P}^2}\mathcal{P}^\mu(SF\mathcal{P})^\nu + \frac{e\mu}{4mc\mathcal{P}^2}\mathcal{P}^\mu S^{\nu\alpha}\partial_\alpha(FS) - \frac{e^2\mu}{2mc^2(\mathcal{P}^2)^2}\mathcal{P}^\mu(SFSF\mathcal{P})^\nu - (\mu \leftrightarrow \nu)$$

$$\equiv \frac{e\mu}{mc}F^\mu{}_\alpha S^{\alpha\nu} + 2\mathcal{P}^\mu\left[\dot{x}^\nu + \frac{e}{2mc\mathcal{P}^2}(SF\mathcal{P})^\nu - \frac{e^2\mu}{4mc^2(\mathcal{P}^2)^2}(SFSF\mathcal{P})^\nu\right] - (\mu \leftrightarrow \nu).$$

(43)

Together with the algebraic equations $S^{\mu\nu}\mathcal{P}_\nu = 0$, $H = 0$, they give a manifestly Poincare-covariant version of the Pauli theory for an arbitrary value of the magnetic moment μ. The brackets (34)–(39) obey the condition (19), so the spin supplementary condition $S^{\mu\nu}\mathcal{P}_\nu = 0$ is consistent with the dynamical equations (41)–(43). The mass-shell condition $H = 0$ is consistent by construction: $\dot{H} = \{H, H\} = 0$.

The Hamiltonian (17), being a linear function of spin and electromagnetic field strength, nevertheless gives the nonlinear equations of motion. This is due to the fact that an essential part of interaction turns out to be encoded in noncommutative brackets (34)–(40).

In $1/c^2$ approximation our equations imply Eqs. (6) and (13). As we saw above, they also represent the equations of motion of the Hamiltonian theory (18). This proves that (18) is the physical-time Hamiltonian of the covariant theory (17) in $1/c^2$ approximation.

Equation (41) in $1/c^3$ approximation implies the following expression for canonical momentum: $\mathcal{P}^\mu = m\dot{x}^\mu - \frac{e\mu}{2mc^3}(SF\dot{x})^\mu + \frac{e\mu}{8m^2c^3}S^{\mu\alpha}\partial_\alpha(FS)$. This can be used to exclude $\mathcal{P}^\mu$ from Eqs. (42) and (43), again in $1/c^3$ approximation. Keeping only the linear function on $F^{\mu\nu}$-terms (the approximation studied by Frenkel [3]), the resulting equations coincide with those of Frenkel

$$\frac{d}{d\tau}\left[\left(m - \frac{e}{4mc^3}(SF)\right)\dot{x}^\mu + \frac{e}{8m^2c^3}S^{\mu\alpha}\partial_\alpha(FS)\right]$$
$$= \frac{e}{c}(F\dot{x})^\mu + \frac{e}{4mc}\partial^\mu(FS),$$

(44)

$$\dot{S}^{\mu\nu} = \frac{e}{mc}\left[F^{[\mu}{}_\alpha S^{\alpha\nu]} - \frac{1}{4mc^2}\dot{x}^{[\mu}S^{\nu]\alpha}\partial_\alpha(FS)\right], \quad S^{\mu\nu}\dot{x}_\nu = 0.$$

(45)

By the way, we showed that the approximately covariant Frenkel equations can be made covariant by adding the terms that are no more than quadratic in spin and field strength.

VI. THE ROLE OF THOMAS SPIN VECTOR IN THE COVARIANT SCHEME

In this section we return back to the analysis of covariant theory in $1/c^2$ approximation. We assume that the vector $S^i = \frac{1}{4}\epsilon^{ijk}S^{jk}$ describes the spin axis in the laboratory system. Equations (41)–(43) at $1/c^2$ order reduce to the Frenkel equations (6) and (13). As we saw in Sec. IV, they give the Pauli quantum mechanics without any appeal to the Thomas vector or to the Thomas precession formula. To understand the role of the Thomas vector in the covariant formalism, we examine the transformational properties of the Frenkel spin in the passage from the laboratory to the comoving and instantaneous frames.

The comoving observer is a noninertial system in which the particle is always in its center. For the four-dimensional vector and tensor fields given along the particle trajectory, the transformation from laboratory to comoving frame is determined [17,42] by the matrix

$$\Lambda^\mu{}_\nu(v(\tau)) = \left(\Lambda^0{}_0 = \gamma, \Lambda^0{}_i = \Lambda^i{}_0 = -\frac{\gamma}{c}v^i,\right.$$
$$\left.\Lambda^i{}_j = \delta^i{}_j + \frac{\gamma-1}{\mathbf{v}^2}v^iv^j\right),$$

(46)

where the functions $v^i(\tau) = \frac{dx^i(t)}{dt}\big|_{t(\tau)}$ represent the velocity of the particle. Using Eq. (11), it is convenient to write the velocity through the momenta, $\mathbf{v}(\tau) = c\frac{\mathcal{P}}{\mathcal{P}^0} = \frac{\mathbf{p}(\tau)}{m} + O(1/c)$. Using the transformation law of the Frenkel spin tensor $S''^{\mu\nu}(\tau) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta S^{\alpha\beta}(\tau)$ and Eq. (16), we obtain for spatial components

$$S'^{ij} = 2\epsilon^{ijk}\left[S^k + \left(\frac{\gamma}{(\mathcal{P}^0)^2} - \frac{\gamma-1}{(\mathcal{P})^2}\right)[\mathcal{P}, [\mathcal{P}, \mathbf{S}]]^k\right]. \quad (47)$$

We assume that the functions $S'^i_{\text{comov}} = \frac{1}{4}\epsilon^{ijk}S'^{jk}$ describe the position of the spin axis in the comoving frame. At the order $1/c^2$, Eq. (47) implies the following transformation law:

$$\mathbf{S}'_{\text{comov}} = \mathbf{S} + \frac{1}{2(mc)^2} [\mathbf{p}, [\mathbf{p}, \mathbf{S}]]. \qquad (48)$$

Comparing this with the definition of the Thomas spin (32), we see that accidentally, the Thomas vector represents the spin axis in the comoving frame

$$\mathbf{S}_T = \mathbf{S}'_{\text{comov}}. \qquad (49)$$

This clarifies the meaning of the Thomas vector in the covariant theory: using the laboratory values $\mathbf{S}$ and $\mathbf{p}$, the laboratory observer can compute the vector $\mathbf{S}_T$ according to Eq. (32), thus obtaining the magnitude and direction of the spin axis as it measured in the comoving frame. From the equality (49) we expect that the Thomas equation (2) describes the evolution of the spin axis in the comoving frame. This can be confirmed by direct calculations. A derivative of Eq. (48) gives us the variation rate of the spin axis in the comoving frame through the laboratory quantities as follows[7]:

$$\frac{d\mathbf{S}'_{\text{comov}}}{d\tau} = \frac{d\mathbf{S}}{d\tau} - \frac{1}{2(mc)^2} \left[\left[\mathbf{p}, \frac{d\mathbf{p}}{dt} \right], \mathbf{S} \right]$$
$$+ \frac{1}{(mc)^2} \left[\mathbf{p}, \left[\frac{d\mathbf{p}}{dt}, \mathbf{S} \right] \right]. \qquad (50)$$

Using Eqs. (6) and (11), we obtain the equality

$$\frac{d\mathbf{S}'_{\text{comov}}}{d\tau} = -\frac{e}{mc} \left\{ [\mathbf{B}, \mathbf{S}] - \frac{1}{2mc} [[\mathbf{p}, \mathbf{E}], \mathbf{S}] \right\}. \qquad (51)$$

If the dynamics of laboratory quantities $\mathbf{p}(\tau)$ and $\mathbf{S}(\tau)$ is known, Eq. (51) acquires the form $d\mathbf{S}'_{\text{comov}}/d\tau = \mathbf{f}(\tau)$, with known function $\mathbf{f}(\tau)$. Solving this equation, the laboratory observer will obtain the dynamics of the spin axis as it seen by the comoving observer. On the rhs we can replace $\mathbf{S}$ on $\mathbf{S}'_{\text{comov}} + O(1/c^2)$, thus obtaining the equation

$$\frac{d\mathbf{S}'_{\text{comov}}}{d\tau} = -\frac{e}{mc} \left\{ [\mathbf{B}, \mathbf{S}'_{\text{comov}}] - \frac{1}{2mc} [[\mathbf{p}, \mathbf{E}], \mathbf{S}'_{\text{comov}}] \right\}.$$
$$(52)$$

If the laboratory quantities $\mathbf{B}$, $\mathbf{E}$, and $\mathbf{p}(\tau)$ are known, the laboratory observer can solve it for $\mathbf{S}'_{\text{comov}}(\tau)$.

Equation (52) just coincides with the Thomas equation (2). This explains the meaning of the Thomas equation in the covariant theory: solving Eq. (2) with use of laboratory quantities, the laboratory observer will obtain the functions $S_T^i(\tau)$ that describe the evolution of the spin

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[7]Using the conversion factor $d/d\tau = \gamma d/dt = d/dt + O(1/c^2)$, we could obtain the variation rate with respect to the laboratory time. As we work in $1/c^2$ approximation, almost all equations of this section remain valid if we just replace $\frac{d}{d\tau}$ on $\frac{d}{dt}$.

axis as seen by the comoving observer. We stress, once again, that this interpretation is valid only in $1/c^2$ approximation.

For an electromagnetic field in the comoving system we can write [17]

$$\mathbf{E} = \mathbf{E}' - \frac{1}{mc} [\mathbf{p}, \mathbf{B}'] + O(1/c^2),$$
$$\mathbf{B} = \mathbf{B}' + \frac{1}{mc} [\mathbf{p}, \mathbf{E}'] + O(1/c^2). \qquad (53)$$

Using Eqs. (53) and (11) in (51), we obtain the expression that can be thought as an equation of motion of the spin axis in the comoving frame

$$\frac{d\mathbf{S}'_{\text{comov}}}{d\tau} = -\frac{e}{mc} [\mathbf{B}', \mathbf{S}'_{\text{comov}}] + [\boldsymbol{\omega}_T, \mathbf{S}'_{\text{comov}}], \qquad (54)$$

where

$$\boldsymbol{\omega}_T = -\frac{1}{2(mc)^2} \left[\mathbf{p}, \frac{d\mathbf{p}}{dt} \right], \qquad (55)$$

is the angular velocity vector of the Thomas precession. The comoving observer will detect the torque of the spin exerted by magnetic field $\mathbf{B}'$, and an extra torque around the vector $\boldsymbol{\omega}_T$ due to a noninertial character of the comoving frame.

If $\mathbf{E} = 0$ in the laboratory frame, Eq. (11) implies $d\mathbf{p}/dt = O(1/c)$, and (54) reduces to the equation for the precession of spin in a magnetic field

$$\frac{d\mathbf{S}'_{\text{comov}}}{d\tau} = -\frac{e}{mc} [\mathbf{B}', \mathbf{S}'_{\text{comov}}]. \qquad (56)$$

To complete the analysis, we discuss the Frenkel vector in the instantaneous frame. The inertial system obtained by the Lorentz boost (46) is instantaneous at the instant τ_0 rest frame (instantaneous frame for short), where the numbers v^i are equal to the velocity of a particle at fixed instant τ_0. Using the transformation law of Frenkel spin-tensor $S''^{\mu\nu}(\tau) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta S^{\alpha\beta}(\tau)$ and Eq. (16), we obtain for spatial components

$$S''^{ij} = 2\epsilon^{ijk} \left[S^k - \frac{\gamma}{c\mathcal{P}_0} [\mathbf{v}, [\mathcal{P}, \mathbf{S}]]^k - \frac{\gamma - 1}{\mathbf{v}^2} [\mathbf{v}, [\mathbf{v}, \mathbf{S}]]^k \right]. \qquad (57)$$

At the order $1/c^2$, this implies the following transformation law of the Frenkel and Thomas spin vectors:

$$\mathbf{S}''_{\text{inst}} = \mathbf{S} + \frac{1}{mc^2} [\mathbf{v}, [\mathbf{p}, \mathbf{S}]] - \frac{1}{2c^2} [\mathbf{v}, [\mathbf{v}, \mathbf{S}]]. \qquad (58)$$

$$\mathbf{S}''_{T,\text{inst}} = \mathbf{S}_T + \frac{1}{2mc^2} [[\mathbf{v}, \mathbf{p}], \mathbf{S}_T]. \qquad (59)$$

Here **v** is the velocity of the particle at τ_0, while **p** is its momentum at the instant of observation. Both Thomas (2) and Frenkel (6) equations preserve their form under these transformations, and thus can be used as the equations of motion for these quantities by any inertial observer. Computing the derivative of these expressions, we obtain the variation rates in the instantaneous frame through the laboratory quantities

$$\frac{d\mathbf{S}''_{\text{inst}}}{d\tau} = \frac{d\mathbf{S}}{d\tau} + \frac{1}{mc^2}\left[\mathbf{v}, \left[\frac{d\mathbf{p}}{dt}, \mathbf{S}\right]\right]. \tag{60}$$

$$\frac{d\mathbf{S}''_{T,\text{inst}}}{d\tau} = \frac{d\mathbf{S}_T}{d\tau} + \frac{1}{2mc^2}\left[\left[\mathbf{v}, \frac{d\mathbf{p}}{dt}\right], \mathbf{S}_T\right]. \tag{61}$$

At the instant τ_0 the particle is instantaneously at rest, so $\mathbf{p} = \mathbf{v}$, and Eq. (58) coincides with (48), as it is expected. Equation (59) at this instant gives $\mathbf{S}''_{T,\text{inst}}(\tau_0) = \mathbf{S}_T(\tau_0)$.

Taking the difference of Eqs. (50) and (60) at the instant τ_0, and replacing in the resulting expression $\mathbf{S} = \mathbf{S}''_{\text{inst}} + O(1/c^2)$, we relate the variation rates of the spin axis in comoving and instantaneous frames, thus obtaining the Thomas precession of the Frenkel spin

$$\frac{d\mathbf{S}'_{\text{comov}}}{d\tau} - \frac{d\mathbf{S}''_{\text{inst}}}{d\tau} = [\boldsymbol{\omega}_T, \mathbf{S}''_{\text{inst}}]. \tag{62}$$

The difference at the instant τ_0 is a rotation of spin axis $\mathbf{S}''_{\text{inst}}$ around the vector (55), with the angular velocity equal to its length. So the noninertial comoving frame looks rotating in the system of the instantaneous observer. Thomas explained this kinematic effect analyzing the product of Lorentz boosts [2].

We emphasize that formally similar equations (32), (48), and (58) (the latter taken at τ_0) represent $1/c^2$ approximation of different equations, and so they have a completely different meaning. The inaccuracies (including inaccuracies in notation) made by different authors in the derivation and analysis of equations like (32), (48) and (58), represent the source of numerous confusion in the literature. A detailed analysis of the controversial works on the subject was undertaken in [26–28]. Here we clarified the meaning of these equations in the framework of the covariant theory (41)–(43).

VII. CONCLUSION

In this work, we presented the manifestly covariant version of Pauli theory for the description of a spinning electron in Hamiltonian formalism. The covariant Hamiltonian (17) is a linear function of Frenkel spin and of field strength. Covariant brackets of the theory have been obtained from the requirement of the consistency of Hamiltonian equations with the spin supplementary condition $S^{\mu\nu}\mathcal{P}_\nu = 0$. This implies rather nontrivial deformation (34)–(39) of the canonical brackets (4). In particular,

the bracket $\{x^\mu, x^\nu\} = -\frac{1}{2\mathcal{P}^2}S^{\mu\nu}$ states that position variables are noncommutative, and the spin-induced noncommutativity survives even in the noninteracting theory of a spinning electron. The nonrelativistic spinning particle has the commuting position variables; see Sec. 5 C in [38]. This also follows from the above mentioned bracket, since $1/\mathcal{P}^2 \sim 1/(mc)^2 \rightarrow 0$ as $c \rightarrow \infty$. In other words, the spin-induced noncommutativity is a relativistic effect. Hence, the manifestly covariant description of spin inevitably leads to the theory endowed with position-position noncommutative geometry. As we have shown, it is this bracket that is responsible for transforming the covariant Hamiltonian into the Hamiltonian of the Pauli theory. In this regard, we point out that spinning particles represent an exceptional example of intrinsically noncommutative and relativistic-invariant theory, with the spin-induced noncommutativity that manifests itself already at the Compton scale. The effects due to noncommutative geometry are of considerable interest in the current literature [43–51], and certainly deserve a detailed study in the relativistic-invariant context of spin-induced noncommutativity.

The brackets (34)–(39) encode an essential part of the spin-field interaction, and lead to the equations of motion (41)–(43) quadratic on spin and field strength. Hamiltonian equations consistent with a set of algebraic constraints could be obtained in a more systematic way by constructing a proper variational problem. The search for a variational problem for the spinning particle has an almost centenary history; see, for example, [38,52,53] and references therein. One possibility is to consider the spin tensor as a composite object constructed from the non-Grassmann vector and its conjugated momentum, $S^{\mu\nu} = 2(\omega^\mu \pi^\nu - \omega^\nu \pi^\mu)$. The Lagrangian that, besides the dynamical equation, implies all the necessary constraints, has been recently proposed in [54,55]. The spin supplementary condition $S^{\mu\nu}\mathcal{P}_\nu = 0$ arises here as a consequence of two second-class constraints imposed on the basic vector and its momentum. The constraints can be taken into account with help of the Dirac bracket. The point here is that this leads to higher complicated (nonpolynomial on S and F) equations of motion [54,55]. So, the existence of equations (41)–(43), that are quadratic on S and F, and are consistent with the spin supplementary condition, seem to be rather nontrivial facts that deserve further investigation.

In $1/c^2$ approximation, both the Hamiltonian (18) and brackets (24)–(27) of the covariant theory differ from that of Pauli theory (3), (4). Nevertheless, they lead to the same quantum mechanics without any appeal to the Thomas precession formula (62). The second equation from (28) shows that the position of the particle in quantum mechanics is given by the Pryce (d) operator [56]. At the classical level, Pauli theory can be thought as $1/c^2$ approximation of the covariant theory written in special variables (31) and (32). This observation explains the discrepancy between the covariant (18) and Pauli (3) Hamiltonians.

Accidentally, the relation (32) between the Thomas and Frenkel spin vectors in the laboratory frame coincides with the transformation law of the Frenkel spin vector in the passage from the laboratory to the comoving frame; see (48). This clarifies the meaning of the Thomas spin and of the Thomas equation in the covariant scheme: solving Eq. (2) with the use of the laboratory quantities, the laboratory observer will obtain the functions $S_T^i(\tau)$ that describe the evolution of the spin axis as it seen by the comoving observer.

It would be interesting to apply the developed formalism, considering the three-dimensional spin as a spatial part of the four-dimensional spin vector S^μ instead of the Frenkel spin tensor $S^{\mu\nu}$. This could give a Hamiltonian formulation for the Bargmann-Michel-Telegdi equations [24].

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