

Thermodynamic equilibrium in general relativity

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The thermodynamic equilibrium condition for a static self-gravitating fluid in the Einstein theory is defined by the Tolman-Ehrenfest temperature law, $T\sqrt{g_{00}(x^i)} = \text{constant}$, according to which the proper temperature depends explicitly on the position within the medium through the metric coefficient $g_{00}(x^i)$. By assuming the validity of Tolman-Ehrenfest “pocket temperature,” Klein also proved a similar relation for the chemical potential, namely, $\mu\sqrt{g_{00}(x^i)} = \text{constant}$. In this paper we prove that a more general relation uniting both quantities holds regardless of the equation of state satisfied by the medium, and that the original Tolman-Ehrenfest law form is valid only if the chemical potential vanishes identically. In the general case of equilibrium, the temperature and the chemical potential are intertwined in such a way that only a definite (position dependent) relation uniting both quantities is obeyed. As an illustration of these results, the temperature expressions for an isothermal gas (finite spherical distribution) and a neutron star are also determined.

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I. INTRODUCTION

It is usually believed, at least for nonmembers of the general relativity community, that equality of temperature is a condition for thermal equilibrium between two systems or between two parts of a single system (“zeroth” law of thermodynamics). Furthermore, the second law of thermodynamics, in one of its variants, e.g., Clausius’ one, states that heat flows from a hotter to a colder medium, till thermal equilibrium is finally restored. However, both of these basic conditions can be violated in the framework of general relativity.

Many decades ago, Tolman and Ehrenfest discussed how to determine the temperature distribution within a self-gravitating fluid that has come to thermal equilibrium. The result was a remarkable thermogravitational effect in the framework of general relativity: heat, as any other source of energy, is subjected to gravity. The preliminary results assuming spherical symmetry were obtained by Tolman in the weak field approximation [1], but a proof of the theorem valid for a more general static field was published in a subsequent paper by Tolman and Ehrenfest [2] (see also [3]). In order to discuss the equilibrium temperature to this

particular case, they assumed that the self-gravitating fluid generates a static gravitational field described by the line element

$$ds^2 = g_{00}c^2 dt^2 - g_{ij}dx^i dx^j, \quad (1)$$

where Latin indices denote spatial coordinates and the signature adopted here is $(+, -, -, -)$. The components $g_{00}(x^r)$ and $g_{ij}(x^r)$ are independent of time but depends in an arbitrary way of the spatial coordinates x^r ($r = 1, 2, 3$). Under such conditions, the “pocket temperature” Tolman-Ehrenfest (TE) theorem can be expressed as

$$T\sqrt{\bar{g}_{00}}(x^i) = \tilde{T} = \text{constant}, \quad (2)$$

where \tilde{T} is constant in all parts of the system. The interesting aspect of this relation is that the proper temperature necessarily varies from point to point within the self-gravitating fluid that has come to equilibrium, thereby violating the so-called “zeroth” law of thermodynamics [4]. This result is nowadays considered an important key in the framework of black-hole physics [5], or more generally, to compact objects in the astrophysical domain. Tolman stressed that the temperature T is directly measurable by local observers, and as such, it must be considered the fundamental quantity that we mean by temperature at a given point.

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In principle, due to its physical interest, the Tolman distribution law demands a closer scrutiny. As it appears, the proof of the TE theorem is very particular in many different aspects. To begin with, since the self-gravitating fluid may have a generic nature, they first assumed that the parts of the system whose temperature needs to be compared are in thermal contact with a small connecting tube containing blackbody radiation, or at least could be put in such contacting device without perturbing the system which should be considered a kind of reservoir. In other words, the tube works like a radiation thermometer. Second, the energy conservation law was applied to the thermometer itself not to the fluid source of curvature as it should be desirable in principle. Finally, it should be remarked that black body radiation is a very special kind of medium since its chemical potential is zero, and its basic thermodynamic quantities (temperature, pressure, and energy density) are related in a very simple way. Moreover, photons suffers gravitational redshift in the presence of a static field, a kinematic phenomenon that is not directly related with the idea of thermal equilibrium.

In literature, several have been the trials to determine or extend the TE theorem. For example Buchdahl [6] formally extended TE result to self-gravitating fluids supporting stationary spacetime through the timelike Killing vector

$$K^a = (\partial_t)^a = (1, 0, 0, 0)^a \quad (3)$$

As shown by [7], even considering that the approach of [6] looks similar to the TE result, it is incomplete because is valid only for a very specific class of 4-velocities. While in a static spacetime, as that used in the TE derivation, one has a unique candidate for the 4-velocity fields necessary to explicit the heat bath's rest frame, in the case of a stationary nonstatic spacetime (as that in [6] proof) the rest frame of the bath can be fixed by several 4-velocities fields.

A different approach to derive the TE effect, as well as Tolman-Oppenheimer-Volkoff (TOV) [8,9], Klein related result[10], and in particular the derivation of Einstein's equations from thermodynamics of the self-gravitating gas was attempted by Cocke [11]. He derived the TOV equation through a maximum entropy principle which was later on extended by [12]. A further generalization to arbitrary perfect fluids was also given by Gao [13,14]. Some time after, Roupas [15–18] specified in which thermodynamic ensemble the calculation must be performed thereby recalculating the TOV, TE, and Klein result. It is also worth mentioning that Rovelli and Smerlek [19] also obtained the TE relation by applying the equivalence principle to a property of thermal time.

In this paper, we discuss a general proof of the TE theorem, in a simpler and more general form from that discussed by [6,7], and similarly we will not use the maximum entropy principle as adopted by many authors [11–17].

As we shall see, using only thermodynamics and general relativity, it is possible to show that under given conditions (null chemical potential) the source of curvature satisfies exactly the TE law. The result is valid for any kind of fluid, not only in the radiative case as originally considered by Tolman and Ehrenfest. In the general case of equilibrium (non-null chemical potential), the temperature and the chemical potential and the metric coefficient $g_{00}(x^r)$ are entertained in such a way that only a definite (position dependent) relation uniting such quantities is obeyed. Such a result generalizes both the TE and Klein relations.

The paper is organized as follows. In Sec. III, we obtain the TE relation for a simple fluid regardless the values of its chemical potential, and in Sec. IV, we show how the temperature changes due to the TE effect in an isothermal gas, and inside neutron stars. Finally, we close with a brief summary of the main results.

II. THERMODYNAMIC STATES

As is widely known, the thermodynamic states of a relativistic simple fluid are characterized by three fundamental quantities: (i) an energy-momentum tensor $T^{\alpha\beta}$, (ii) a particle current N^α , and (iii) an entropy current S^α . In addition, the fundamental equations of motion are expressed by the conservation law of energy-momentum ($T^{\alpha\beta}{}_{;\beta} = 0$) and the number density of particles ($N^\alpha{}_{;\alpha} = 0$), where the semicolon denotes covariant derivative. Moreover, for a simple fluid, in the absence of classical dissipative mechanisms (e.g., viscosity, heat flow), the entropy flux is also conserved quantity ($S^\alpha{}_{;\alpha} = 0$). In an arbitrary hydrodynamic frame of reference, whose four-velocity obeys $u^\alpha u_\alpha = 1$, these quantities take the following forms:

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta - pg^{\alpha\beta}, \quad (4)$$

$$N^\alpha = nu^\alpha, \quad (5)$$

$$S^\alpha = n\sigma u^\alpha. \quad (6)$$

In the above relations, the variables ρ , p , n , and σ stand respectively for the energy density, thermostatic pressure, particle number density and specific entropy (per particle), and are related by the so-called Gibbs law [20–22]

$$nTd\sigma = d\rho - \frac{\rho + p}{n}dn. \quad (7)$$

The above local expression when combined with the energy conservation law for a perfect fluid ($u_\alpha T^{\alpha\beta}{}_{;\beta} = 0$) and the conservation of the particle current ($N^\alpha{}_{;\alpha} = 0$), implies that the $d\sigma/ds$ is conserved along the world lines of the fluid (see, for instance, [21]). It means that the flow is isentropic, a result also in accordance with the conserved entropy current ($S^\alpha{}_{;\alpha} = 0$).

Nevertheless, the constancy of σ along each world line, does not mean that it is a global constant within the whole

volume of the fluid. In other words, the constant may vary from world line to world line. In particular, for comoving observers at rest with the volume elements of a *static inhomogeneous* simple fluid (the case study here), $\sigma = \sigma(x^i)$. It varies from place to place thereby making sense to calculate partial space derivatives in the fluid, and, of course, the same happens with the remaining physical quantities.

On the other hand, in the frame defined by (1), $g^{00} = g_{00}^{-1}$, an observer at rest has 4-velocity $u^\alpha = \delta^\alpha_0 / \sqrt{g_{00}}$. From $u^\alpha u_\alpha = 1$ we also see that $u_\alpha = \sqrt{g_{00}} \delta_\alpha^0$ while the four-acceleration $a_\alpha = u^\beta_{;\beta} u^\alpha = \Gamma_{\alpha 0}^0 = -\partial_\alpha g_{00} / 2g_{00}$. As one may check, by using the above results valid for the static metric (1), the energy conservation law takes the following form:

$$\frac{\partial p}{\partial x^i} + \left[\frac{\rho + p}{2} \right] g_{00}^{-1} \frac{\partial g_{00}}{\partial x^i} = 0, \quad (8)$$

or, equivalently,

$$\frac{\partial \ln(\rho + p)}{\partial x^i} + \frac{\partial \ln \sqrt{g_{00}}}{\partial x^i} = \frac{1}{\rho + p} \frac{\partial \rho}{\partial x^i}. \quad (9)$$

Now, since the thermodynamic variables are related with the relativistic chemical potential (thermodynamic potential) per particle by the local Euler expression [23]

$$T\sigma = \frac{\rho + p}{n} - \mu, \quad (10)$$

there are two kind of fluids to be considered, namely those with and with no chemical potential.¹ Let us now discuss each case separately.

(i) $\mu = 0$

Particles with no chemical potential includes as particular cases the radiation blackbody fluid (massless photons) as discussed by Tolman and Ehrenfest and a spin-2 massless field described by the Fierz-Pauli Lagrangian [24] which coincides with the first order weak field approximation of general relativity (massless gravitons). It should be noticed, however, that the massless property does not imply a nullified chemical potential. For instance, in special relativity, when $g_{00} = 1$ and $g_{ij} = \delta_{ij}$ in metric (1), a kinetic theoretic approach for an effectively massless (ultra-relativistic) ideal gas yields for the fugacity (in natural units²), $\exp(\mu/T) = \frac{\pi^2 n}{T^3}$. Note also that since $n \propto T^3$ in this limit, the fugacity or equivalently, the ratio $\mu/T = \text{constant}$. Actually, for an ideal noninteracting (and nonquantum) relativistic gas such a property remains valid regardless of the temperature interval [23].

¹For an ideal relativistic gas, the chemical potential per particle includes the nonrelativistic value plus the rest mass-energy contribution, $\mu = \mu_{NR} + mc^2$.

² $\hbar = k_B = c = 1$.

Now, by using that $\rho + p = nT\sigma$, we first rewrite (9) in the form

$$\frac{\partial \ln T \sqrt{g_{00}}}{\partial x^i} = \frac{1}{\rho + p} \frac{\partial \rho}{\partial x^i} - \frac{1}{n\sigma} \frac{\partial n\sigma}{\partial x^i}, \quad (11)$$

or still,

$$\frac{\partial \ln T \sqrt{g_{00}}}{\partial x^i} = \frac{1}{\rho + p} \left[\frac{\partial \rho}{\partial x^i} - \frac{\rho + p}{n} \frac{\partial n}{\partial x^i} - nT \frac{\partial \sigma}{\partial x^i} \right]. \quad (12)$$

As discussed before, in the inhomogeneous static fluid discussed here, all quantities in the Gibbs law, namely, ρ , σ , p , T and n are local functions of the spatial coordinates alone. In this way, one may think that the differentials in (7) are just the differences between infinitely adjacent points of space.

Notice that all the local thermodynamic properties of the fluid can be expressed in terms of n and T . In particular, we have $\sigma = \sigma(n, T)$, $\rho = \rho(n, T)$, and $p = p(n, T)$. As a consequence of the second law of thermodynamics these three functions have to comply with the differential relation (7). Consequently, regarding the differentials appearing in (7) as the increments associated with neighboring points in the inhomogeneous static fluid, it follows that

$$nT \frac{\partial \sigma}{\partial x^i} = \frac{\partial \rho}{\partial x^i} - \left[\frac{\rho + p}{n} \right] \frac{\partial n}{\partial x^i}. \quad (13)$$

The above equation means³ that the right-hand side (RHS) of (12) is identically null, thereby showing that $T\sqrt{g_{00}}(x^i) = \tilde{T}$ as derived by Tolman and Ehrenfest. *The present proof is, however, more general than the original TE theorem since the fluid is not restricted to blackbody radiation, and, perhaps, more important, the introduction of a radiation thermometer connecting two parts of the medium is by no means a necessary device.* Further, since the equation of state obeyed by the fluid does not play any role in this approach, such a result strongly suggest that a general proof including a non-null chemical potential could in principle be accomplished. This case will be discussed next.

(ii) $\mu \neq 0$

Let us consider again the energy conservation law for the general static configuration. It is easy to see that Eq. (9) now leads to the following relation [compare with Eq. (12)]

$$\frac{\partial \ln T \sqrt{g_{00}}}{\partial x^i} = \frac{1}{\rho + p} \left[\frac{\partial \rho}{\partial x^i} - \frac{\rho + p}{n} \frac{\partial n}{\partial x^i} - nT \frac{\partial \sigma}{\partial x^i} - \frac{\rho + p}{\sigma + \mu/T} \frac{\partial \mu}{\partial x^i} \right], \quad (14)$$

³In this connection see also discussion above and below Eq. (16).

and since the first three terms within the bracket on the RHS sum zero due to Gibbs law (13), after some algebra the above expression reduces to:

$$\frac{\partial \ln T \sqrt{g_{00}}}{\partial x^i} + \frac{\mu}{T\sigma} \frac{\partial \ln \mu \sqrt{g_{00}}}{\partial x^i} = 0. \quad (15)$$

It thus follows that a more general relation uniting the pair of thermodynamic variables (T, μ) holds regardless of the equation of state satisfied by the medium. It also implies that the original Tolman-Ehrenfest thermodynamic theorem is valid only if the chemical potential vanishes identically or even whether the extended thermodynamic relation for the chemical potential $\mu \sqrt{g_{00}} = \text{constant}$ is assumed. In the general case of equilibrium, the temperature and the chemical potential are entertained in such a way that only a definite (position dependent) relation uniting both quantities is obeyed. Naturally, the above result is also valid in the particular case of special relativity.

At this point, it is interesting to comment on the proof of a related theorem derived long ago by Klein [10]. In his paper, the following shortcut approach was adopted under the same starting conditions, namely: the specific entropy was eliminated from Gibbs law (7) in order to recover the Gibbs-Duhem relation, $n\sigma dT = dp - n d\mu$. Further, this differential expression was combined with the energy conservation law (8) thereby obtaining (see Eq. (16) in [10]).⁴

$$n \left[\frac{\partial \mu}{\partial x^i} - \frac{\mu}{T} \frac{\partial T}{\partial x^i} \right] + (\rho + p) \left[\frac{1}{T} \frac{\partial T}{\partial x^i} + \frac{\partial \ln \sqrt{g_{00}}}{\partial x^i} \right] = 0. \quad (16)$$

From the above expression it was observed that the temperature and the chemical potential are curiously interrelated. However, Klein took for granted the general validity of the Tolman result (effectively valid only for $\mu = 0$) and concluded that the above expression (now reduced to the first term) leads to the universal relations: $\mu/T = \text{constant}$, and, subsequently (by using the Tolman law again), to the equally celebrated Klein's law:

$$\mu \sqrt{g_{00}}(x^i) = \text{constant}. \quad (17)$$

Note that our viewpoint is different by the following reason. We consider that both terms in the brackets are in principle different from zero, unless some extra simplifying condition is assumed (as the general validity of the TE law). As a matter of fact, under more general conditions, the relation μ/T also does not need to be constant. In general, this happens for an ideal gas of noninteracting particles, a very particular case of the perfect fluid description assumed here. If the fluid obeys a more general equation of state than the one valid for an ideal gas ($p = nk_B T$), as in the van der

Walls case, the ratio μ/T is different from a constant. This means that the constancy of the fugacity is by no means a general thermodynamic law. Even kinetically, it fails when interactions or even quantum effects are included in the ideal gas description. For instance, for a degenerate relativistic Fermi gas, the exact kinetic result involves special functions, but more enlightening approximate expressions can be obtained for some limits. In particular, in the almost complete degeneracy regime, $T/T_F \ll 1$ where T_F is the Fermi temperature, the chemical potential can be written as:

$$\mu = E_F \left[1 - \frac{\pi^2}{12} \left(\frac{T}{T_F} \right)^2 + \dots \right] + mc^2, \quad (18)$$

where E_F is the Fermi energy and $T_F = E_F/k_B$ (see, for instance, [25]).

The above considerations lead us to conclude that the standard TE and Klein's thermodynamic relations laws are generically valid only for the restricted class of perfect fluids satisfying the relation $\mu/T = \text{constant}$. Basically, noninteracting (ideal) relativistic gas when quantum effects are not considered [in this connection see also comment on fugacity just above Eq. (11)].

III. THE CASE OF MIXTURES

In the present work all calculations are restricted to one-component self-gravitating relativistic simple fluids, either with vanishing or with finite chemical potentials. Naturally, such results cannot be generically applied to mixtures without careful further considerations on the thermodynamic variables, as well as on the specific properties of each of the component comprising the mixture.

In general, even when locally the temperature is the same for each substance in the mixture, there are P different chemical potentials μ_i , $i = 1, 2, \dots, P$, for P independent substances. This means that the Gibbs, Euler, and the remaining relations, including the energy-momentum tensor must now be written as a sum involving all components.

In the case of a static mixture of matter and radiation, for instance, the results may also depend on the value of local temperature within the system. The local equilibrium radiation has null chemical potential and the same happens with the material component when the mass of the particles (in natural units) is much smaller than the temperature ($T \gg m$).

Naturally, our results may be extended to more complex situations by taking into account the proper extensions of the basic equations. Nevertheless, a detailed treatment involving several substances is out of the scope of the present paper and will be discussed in a forthcoming communication.

⁴Note that in Klein's notation $n \equiv C$, $\mu \equiv \alpha$, $g_{00} \equiv g_{44} = e^\nu$, and $\frac{\partial}{\partial x^i} \equiv \nabla_i$.

IV. TOLMAN-EHRENFEST LAW IN IDEAL GASES AND NEUTRON STARS

The previous calculations showed a general proof of the Tolman-Ehrenfest effect. In this section we want to show how this effect manifests itself in thermal equilibrium ideal gases (isothermal spheres, in spherical, bounded, static configurations), and in neutron stars. To this goal, we will follow [16,18].

A. Ideal gases

In GR the equation of state of the relativistic ideal gas, in a sphere of radius R , can be expressed as [16,26]

$$P(r) = \frac{1}{T_{\text{inv}}(r)[1 + L(T_{\text{inv}}(r))]} \rho(r)c^2, \quad (19)$$

where $T_{\text{inv}} = \frac{mc^2}{kT}$ and $L(T_{\text{inv}})$ is given by

$$L(T_{\text{inv}}) = \frac{K_1(T_{\text{inv}})}{K_2(T_{\text{inv}})} + \frac{3}{T_{\text{inv}}} - 1, \quad (20)$$

and K_n are the modified Bessel functions:

$$K_n(T_{\text{inv}}) = \int_0^\infty e^{-T_{\text{inv}} \cosh \theta} \cosh(n\theta) d\theta. \quad (21)$$

The thermal and dynamic equilibrium is given by the following four equations: TOV equation [9,27]

$$(i) \frac{dP}{dr} = -\left(\frac{P}{c^2} + \rho\right) \left(\frac{GM}{r^2} + 4\pi G \frac{P}{c^2} r\right) \left(1 - \frac{2GM}{rc^2}\right)^{-1}, \quad (22)$$

the mass equation

$$(ii) \frac{dM}{dr} = 4\pi r^2, \quad (23)$$

where M is the total mass (namely the sum of the rest mass, the thermal energy, and gravitational field's energy) the Tolman-Ehrenfest relation [28,29]

$$(iii) T(r)\sqrt{g_{00}} = \text{constant}, \quad (24)$$

that can be written in differential form as [14,30]

$$(iv) \frac{dT_{\text{inv}}}{dr} = -\frac{T_{\text{inv}}}{P + \rho c^2} \frac{dP}{dr}, \quad (25)$$

and to close the system we use Eq. (19). We then solve Eqs. (22), (23), (25), and (19).

The system of differential equation must be solved with the initial conditions: $\rho(0) = \rho_0$, $T_{\text{inv}}(0) = T_0$, $M(0) = 0$, with $r \in [0, R]$. The equilibrium equations can be expressed in adimensional form as shown in Eqs. (58–60) of [16], and the relative initial conditions.

In Fig. 1, we plot the result of the integration, namely the Tolman-Ehrenfest effect: the proper temperature versus the radius. The quantity $b_0 = \frac{mc^2}{kT_0}$, is fundamentally the inverse of the temperature, while the normalizing factor on x axis, $r_* = 1/\sqrt{4\pi G \rho_0 m k T_0}$, where ρ_0 is the central density. The red line represents the solution in the case $b_0 = 1$, while the black line, green, and yellow lines the case $b_0 = 2$, $b_0 = 3.23$, and $b_0 = 350$ respectively. The plot shows that the larger is the central temperature the larger is the TE effect, and that the temperature gradient decreases with distance from the system center. One can also define a rest mass as

$$M_{\text{rest}} = \int_0^R \frac{mn(r)}{\sqrt{1 - \frac{2GM}{rc^2}}} 4\pi r^2 dr, \quad (26)$$

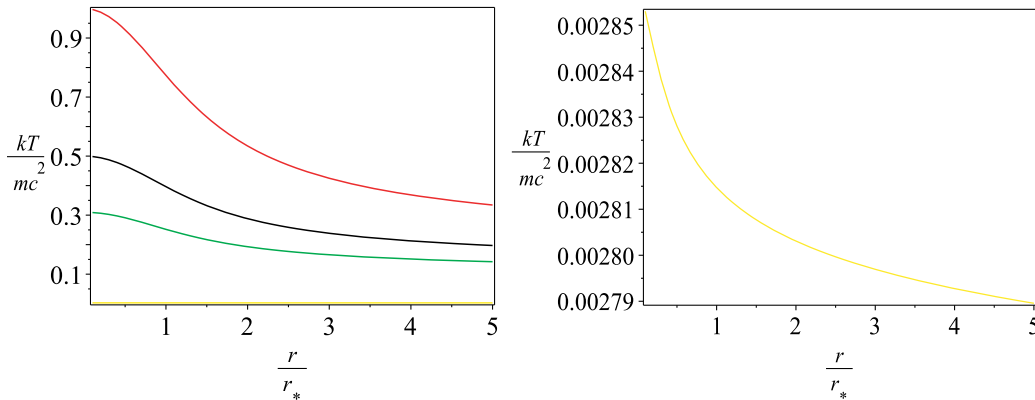


FIG. 1. The Tolman-Ehrenfest effect. The plot shows the proper temperature for different central temperature, $T_0 = \frac{mc^2}{kb_0}$. Left panel: the red line represents the solution in the case $b_0 = 1$, while the black line, green, and yellow lines the case $b_0 = 2$, $b_0 = 3.23$, and $b_0 = 350$ respectively. The quantity $b_0 = \frac{mc^2}{kT_0}$, is fundamentally the inverse of the temperature, while the normalizing factor on the x axis, $r_* = 1/\sqrt{4\pi G \rho_0 m k T_0}$, where ρ_0 is the central density. Right panel: the case $b_0 = 350$ using a different scale.

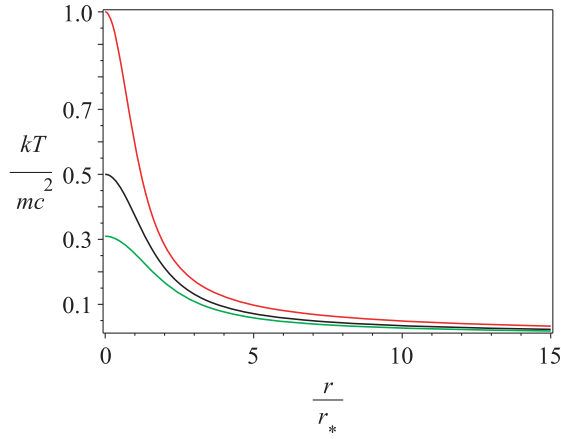


FIG. 2. The Tolman-Ehrenfest effect. The red line represents the solution in the case $b_0 = 1$, while the black line and green line the case $b_0 = 2$, and $b_0 = 3.23$ respectively. The quantity $b_0 = \frac{mc^2}{kT_0}$, is fundamentally the inverse of the temperature, while the normalizing factor on x axis, $r_* = 1/\sqrt{4\pi G\rho_0 mkT_0}$, where ρ_0 is the central density.

where $n(r)$ is the particle density, that can be expressed as [17]

$$n(r) = n_0 \frac{K_2(T_{\text{inv}})}{K_2(T_{\text{inv}}(0))} \frac{T_{\text{inv}}(0)}{T_{\text{inv}}}, \quad (27)$$

which is given, for the four cases considered, by $0.78M_S$, $0.66M_S$, $0.56M_S$, and $0.011M_S$, with $M_S = \frac{Rc^2}{2G}$.

B. Neutron stars

In this section, we discuss how the TE effect changes the inner temperature of neutron stars (NSs).

NSs are important laboratories in which extreme conditions of matter can be studied. Their equation of state (EoS) and composition are unknown at density larger than $\rho_0 = 2.8 \times 10^4 \text{ g/cm}^3$ [31], and different models predict different composition and EoS. Cooling is a powerful method to have insight on the inner structure of NSs [32].

NSs are very hot immediately after the supernovae explosion. Their temperatures is $T \simeq 10^{11} \text{ K}$ large gradient temperatures are present. In a conduction timescale, the heat flows inward, generating a cooling wave propagation from the NS center to its surface. In usual calculations [33], the TE effect is not taken into account, while in others, [34], it is claimed that the NS become isothermal in times of the order of 50–100 yrs.

In reality, neutron stars are not isothermal at all. In their Figs. 6–7, [34] are not plotting the local temperature $T(r)$, but the so called redshifted temperature $\tilde{T} = T(r, t) \exp \phi(r)$, where $\phi(r)$ is the potential. Locally the

temperature changes from one point to the other, and the system is not isothermal.

In order to find the gradients of temperature in the NS due to the TE effect, we will solve Eqs. (22), (23), and (25), coupled with an EoS. The EoS that we use is that of [35] constrained by using info coming from the high-density limit from perturbative QCD, from low-energy nuclear physics, and pulsars data. For $\rho < 3.3 \times 10^3 \text{ g/cm}^3$ the EoS used is that of [36], while for the inner and outer crust we use [37,38]. Solving the quoted equations, we get the results plotted in Fig. 2, in which the colors correspond to that of Fig. 1. In Fig. 2 the temperature falls in a steeper way than in Fig. 1, however the temperature gradient is always present, as predicted by the TE effect. The changes from 5 km to the crust of the NS becomes very small and the behavior tend to become more isothermal.

V. CONCLUSIONS

In the present paper, we advance a proof of the TE theorem that is more general than the ones proposed in [2,3,10], or than proofs based on a maximum entropy principle, such as those reported in [12] or in [13,14]. In our analysis the idea of a radiation thermometer is not necessary. The derivation presented here is as independent as possible of the properties of specific media. In that sense, it has a generality consubstantial with the robustness that a fundamental thermodynamic principle is expected to have. Other derivations of the TE law, such as the simplified ones based on gravitational redshift [39], in spite of their considerable heuristic and pedagogical values, lack the above mentioned kind of generality.

The main aspects of our proof are: (i) the fluid is not restricted to blackbody radiation, and the result within the approach followed here can be naturally extended for fluid mixtures (ii) the radiation thermometer connecting two parts of the medium is not a necessary device, and (iii) the proof is independent from the equation of state and, as such, it was also possible to provide a more general proof including a non-null chemical potential. Finally, we have solved the thermal and dynamic equilibrium equations (TOV, mass equation, TE relation) to find the relation between the temperature and radius, in the case of an isothermal gas, and neutron stars. The result shows that the temperature gradients are larger for larger central temperatures, and are larger for the NS case with respect to the isothermal gas case.

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Correction: The affiliation indicators for the first two authors were misarranged and have been fixed.