No evidence for violation of the second law in extended black hole thermodynamics

Shi-Qian Hu,[1,†] Yen Chin Ong,[1,2,‡] and Don N. Page[3,§]

[1]*Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, 180 Siwangting Road, Yangzhou City, Jiangsu Province 225002, China*
[2]*School of Aeronautics and Astronautics, Shanghai Jiao Tong University, Shanghai 200240, China*
[3]*Department of Physics, Theoretical Physics Institute, 4-181 CCIS, University of Alberta, Edmonton, Alberta T6G 2E1, Canada*

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Recently, a number of papers have claimed that the horizon area—and thus the entropy—of near extremal black holes in anti-de Sitter spacetimes can be reduced by dropping particles into them. In this paper, we point out that this is a consequence of an underlying incorrect assumption that the energy of an infalling particle changes the internal energy of the black hole by the same amount, whereas actually it is the mass or enthalpy of the black hole that increases by the energy of the particle absorbed.

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I. INTRODUCTION

In 1973 [1], Bardeen *et al.* discovered that in asymptotically flat Kerr-Newman black hole spacetimes in general relativity the black hole parameters satisfy equations that appear to be analogous to the laws of thermodynamics. Subsequently, with the discoveries of the Bekenstein-Hawking entropy and Hawking radiation, this "black hole mechanics," the term coined in Ref. [1], was gradually replaced by "black hole thermodynamics." That is to say, the viewpoint in the field has shifted from treating these properties of black holes as mere analogous curiosities to a *bona fide* thermodynamics.[1] This understanding has changed our understanding of black holes ever since, opening up vast areas of research including phase transitions and holography but at the same time also eventually leading to the infamous information paradox. For a review of black hole thermodynamics, see, e.g., Ref. [4].

For an asymptotically flat Kerr-Newman black hole, the usual first law of black hole mechanics takes the form

$$\mathrm{d}M = T\mathrm{d}S + \Phi\mathrm{d}Q + \Omega\mathrm{d}J, \tag{1}$$

where M denotes the Arnowitt-Deser-Misner (ADM) mass of the black hole, T is its Hawking temperature, S is its Bekenstein-Hawking entropy, Φ is its electrostatic potential, Q is its electrical charge, Ω is its angular velocity, and J is its angular momentum. Note that there is an absence of a $V\mathrm{d}P$ term usually found in ordinary thermodynamics. This term in the context of black hole spacetime was eventually introduced [5,6] and requires an anti-de Sitter (AdS) background, since the pressure P is related to the cosmological constant Λ. More precisely, if the cosmological constant Λ is interpreted as a perfect fluid, it has energy density $\rho = \Lambda/(8\pi G)$ and pressure $P = -\Lambda/(8\pi G)$, where G is Newton's gravitational constant and $\Lambda < 0$. The "thermodynamical volume" V is then defined as the thermodynamic conjugate $\partial M/\partial P$. In this "extended thermodynamics," which was later dubbed "black hole chemistry" [7], the first law is generalized to

$$\mathrm{d}M = T\mathrm{d}S + \Phi\mathrm{d}Q + \Omega\mathrm{d}J + V\mathrm{d}P, \tag{2}$$

where M is now *reinterpreted* as the *enthalpy*, sometimes denoted by H, of the system:

$$M \equiv H \coloneqq U + PV. \tag{3}$$

The PV term in this equation can be interpreted as the contribution to the total mass M of the black hole from the work needed to exclude the volume V.

We emphasize that V has nothing to do with any geometrical notion of volume *a priori*, though it turns out that for static charged or neutral black holes

$$V = \frac{4}{3}\pi r_+^3, \tag{4}$$

where r_+ denotes the outer horizon of the black hole, as if the black hole were a ball of radius r_+ in $\mathbb{R}^3$. This is no

†mx120170256@yzu.edu.cn
‡ycong@yzu.edu.cn
§profdonpage@gmail.com

[1]A nice discussion on why black hole thermodynamics is indeed as real as ordinary thermodynamics, but only after Hawking radiation is incorporated, can be found in Refs. [2,3].

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longer true for more complicated spacetimes such as AdS-Kerr or AdS-Taub-NUT (Taub-Newman-Unti-Tamburino); for the latter, the thermodynamical volume is even negative [8].

Given that black hole chemistry seems to give plenty of physically reasonable results (for a review, see Ref. [9]), it therefore came as a surprise when it was claimed by Gwak that by dropping a charged particle into a near-extremal black hole one could *reduce* the area of the black hole horizon [10]. Since the horizon area is interpreted as the entropy in black hole thermodynamics, this would mean that the second law has been violated. This result has received much attention lately and has been calculated to occur in a wide variety of black holes, going beyond general relativity and/or with various matter sources [11–20]. Notably, the case without electrical charge but with rotation, i.e., AdS-Kerr, was investigated [18]. The authors found that the validity of the second law depends on the spin parameter and the value of the cosmological constant.

One might be tempted to treat this as a very special property peculiar to black holes in anti-de Sitter spacetimes and argue that violations of the second law simply mean that these objects are not physically relevant. However, the utility of asymptotically AdS black holes is their wide applications in holography (now commonly also known as gauge/gravity duality, since the applicability of such correspondence has been demonstrated to go beyond the original "AdS/CFT correspondence"). Thus, any violation of the second law in the context of AdS black holes also translates to a violation of the second law in the corresponding ordinary quantum field theory, which is unacceptable.[2] Of course, in a typical holography with finite temperature field theory, the Hawking temperature is not zero, and it is the generalized second law—the total entropy of the black hole and radiation, as well as any matter in the bulk—that has to be nondecreasing. Nevertheless, it is alarming to see the claim of a violation of the second law at the classical level when Hawking radiation is not considered. Indeed, we could consider a black hole for which the outgoing Hawking radiation is balanced by ingoing radiation that was reflected back from the AdS boundary; then, the entropy content of the radiation is approximately constant, so any change to the entropy of the system corresponds to the change to the black hole entropy.

Given that there is now strong theoretical evidence in support of the notion of black hole chemistry, this putative violation of the second law *must* be explained. One possibility is that the notion of entropy must be modified; i.e., it is no longer given by the horizon of the area alone, but instead a hitherto unknown correction term exists that once it is properly included the second law can be restored. This point of view was already raised in Ref. [10]. Another possibility is that the violation can somehow be prevented when backreaction is properly taken into account, much like the way apparent violations of cosmic censorship in the attempts to overspin a black hole [24] can often be prevented by considering the "self-force" of the particle [25–28].

However, this apparent violation of the second law can be explained in a simpler manner: when a charged particle is dropped into a black hole, what changes is its enthalpy or mass,[3] *not* just the internal energy as erroneously assumed in Ref. [10] and the follow-up works.

In Sec. II, we will review Gwak's argument in Ref. [10] and explain how this changes the arguments and saves the second law. Finally, in Sec. III, we will conclude with some discussions to explain the reasons why when a particle is dropped into a black hole it is the enthalpy that increases by the particle energy instead of the internal energy.

II. CHANGE OF ENTHALPY ENSURES VALIDITY OF THE SECOND LAW

Consider a Reissner-Nordström-AdS black hole, the original case studied in Ref. [10], the metric tensor of which is given by (using units such that $G = c = 4\pi\epsilon_0 = 1$)

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \quad (5)$$

in which (with $\Lambda = -3/L^2$)

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2}. \quad (6)$$

Consider a charged particle dropped into the black hole; then, at the horizon, $r = r_+$, where $f(r_+) = 0$,

$$\mathrm{d}f = \frac{\partial f}{\partial M}\mathrm{d}M + \frac{\partial f}{\partial Q}\mathrm{d}Q + \frac{\partial f}{\partial L}\mathrm{d}L + \frac{\partial f}{\partial r_+}\mathrm{d}r_+ = 0. \quad (7)$$

For simplicity, we will only focus on the four-dimensional case, though the following naturally also generalizes to higher dimensions. It can be shown that Eq. (7) becomes

$$\mathrm{d}M - \frac{Q}{r_+}\mathrm{d}Q + \frac{r_+^3}{L^3}\mathrm{d}L = \left(\frac{r_+^2}{L^2} + \frac{M}{r_+} - \frac{Q^2}{r_+^2}\right)\mathrm{d}r_+. \quad (8)$$

In Ref. [10], Gwak considered dropping a charged particle with energy E and charge q into the black hole. He showed that a standard argument via the Hamilton-Jacobi equation yields the relationship

$$E = \frac{Q}{r_+}q + |p^r|, \quad (9)$$

[2] In the extended thermodynamics, allowing the pressure, or equivalently the cosmological constant, to change is equivalent to allowing the number of colors, N, in the corresponding boundary field theory to vary [21–23].

[3] There is also a change in the electrical charge, as we will see in Sec. II, but this is not the main point to emphasize.

where $|p^r|$ denotes the radial momentum of the particle. He then assumed that the energy of the particle changes the *internal energy* of the black hole, i.e., $E = \mathrm{d}U$, and hence

$$\mathrm{d}M - \mathrm{d}(PV) = \frac{Q}{r_+}\mathrm{d}Q + |p^r|. \tag{10}$$

Expanding $\mathrm{d}(PV) = V\mathrm{d}P + P\mathrm{d}V$, and substituting in the expressions for V and P in terms of the black hole parameters, one would obtain

$$\mathrm{d}M + \frac{r_+^3}{L^3}\mathrm{d}L - \frac{3}{2}\left(\frac{r_+^2}{L^2}\right)\mathrm{d}r_+ = \frac{Q}{r_+}\mathrm{d}Q + |p^r|. \tag{11}$$

Substituting this into Eq. (8) and simplifying, one would find that the terms involving $\mathrm{d}Q$ and $\mathrm{d}L$ cancel out, and eventually,

$$\mathrm{d}r_+ = \frac{2r_+^2|p^r|}{r_+^2 - Q^2}. \tag{12}$$

Now, from the extremal condition $f(r_+) = f'(r_+) = 0$, we can express the charge Q of an extremal black hole in terms of the horizon radius r_+ and AdS radius L as

$$Q_{\text{ext}}^2 = r_+^2 + \frac{3r_+^4}{L^2}. \tag{13}$$

(One can also write the enthalpy of an extremal black hole as $M_{\text{ext}} = r_+ + 2r_+^3/L^2$.) For an extremal black hole then, Eq. (12) can be rewritten as

$$\mathrm{d}r_+ = -\frac{2L^2|p^r|}{3r_+^2}, \tag{14}$$

which is negative. By continuity, this means that $\mathrm{d}r_+$ could be negative for *near-extremal* black holes, *if* one assumed $E = \mathrm{d}U$.

Since the entropy of the black hole is $S = A/4 = \pi r_+^2$, where A denotes the horizon area, we have $\mathrm{d}S = 2\pi r_+\mathrm{d}r_+$. It then follows that the second law could be violated. This is the argument in Ref. [10], which was then repeated in many subsequent follow-up works [11–20]. Our claim is that this putative violation of the second law is not physical and is a *an absurd conclusion* of the incorrect assumption that the infalling particle changes the internal energy by $E = \mathrm{d}U$. Instead, we argue that the correct response of the black hole is that its enthalpy should change by $E = \mathrm{d}M$. We will explain why this makes physical sense in Sec. III. For now, let us demonstrate that assuming $E = \mathrm{d}M$ when L is constant does not lead to the apparent violations of the second law that Gwak found.

Indeed, Eq. (10) would now become simply

$$\mathrm{d}M = \frac{Q}{r_+}\mathrm{d}Q + |p^r|. \tag{15}$$

Repeating essentially the same calculation as above, we would obtain the change in the horizon radius

$$\mathrm{d}r_+ = \frac{2L^2r_+^2|p^r| + 2r_+^5\mathrm{d}L/L}{3r_+^4 + L^2r_+^2 - L^2Q^2}. \tag{16}$$

The denominator is exactly equal to $L^2r_+^3f'(r_+)$. It is therefore always positive, though approaching zero in the extremal limit.

Note that in the expression for $\mathrm{d}r_+$ the term involving $\mathrm{d}L$ is no longer absent. However, for fixed cosmological constant ($\mathrm{d}L = 0$), we see that $\mathrm{d}r_+ > 0$ and thus $\mathrm{d}S > 0$ for any Reissner-Nordström-AdS black hole, always respecting the second law. If $\mathrm{d}L < -(L^3/r_+^3)|p^r|$, one would need a further increase in $\mathrm{d}M$ for the second law to be obeyed, but that is beyond the scope of this paper. Nevertheless, we will discuss in Sec. III some aspects of the physics involved when L is allowed to change and why even in that case we do not expect any violation of the second law.

III. DISCUSSION: WHY ENTHALPY IS THE CORRECT PHYSICAL VARIABLE

In the previous section, we showed that if one takes $E = \mathrm{d}M$, instead of $E = \mathrm{d}U$, then there is no violation of the second law in the extended black hole thermodynamics.

Now, let us justify *why* it is indeed $E = \mathrm{d}M$. For ordinary thermodynamics of a system at constant pressure, it would indeed be the enthalpy rather than the internal energy that is increased by the amount of heat input into the system, and for the black hole, the energy of the particle seems to be the analog of the heat. Indeed, E goes into changing M, by changing both the internal energy U and by "creating more volume" while resisting the pressure. Alternatively, one could interpret this as follows: it is because of the PV term in the formula for the internal energy, $U = M - PV$, that when the particle energy E gives an increase in M this increases V, so the $P\mathrm{d}V$ term in $\mathrm{d}U = \mathrm{d}M - P\mathrm{d}V$ at constant P decreases the internal energy. Thus, the increase in the internal energy is less than the increase in the enthalpy, $\mathrm{d}M = E$, produced by the particle absorption. Allowing P to vary changes this picture, but by fixing P, we can more readily appreciate why $E = \mathrm{d}M$ makes sense.

There are many notions of "energy" in thermodynamics; in particular, the enthalpy and the internal energy are both energy, but it is important to distinguish which is the correct one that changes under particle absorption. A violation of the second law can arise from working with the wrong mass or energy; for example, for the Kerr-(Newman)-AdS black hole, one has to be careful to use the physical conserved mass, not a mere mass parameter [29]. In the current context, the issue is similar but subtle. The enthalpy really is *the* conserved mass in the black hole spacetime. In the extended black hole thermodynamics, we write it as a sum of the internal energy, U, and what amounts to a contribution from the cosmological constant, PV. This does not change the fact that the conserved energy of the particle E should add to the conserved mass of the black hole M, since the sum is the conserved ADM mass (see Sec. IV of the review article [30]). In other words, $E = \mathrm{d}M$ is the statement of energy conservation. Indeed, if the black hole absorbs a particle, the area change will be equated to the flux of energy across the horizon, defined with respect to the horizon generating Killing field of the stationary background spacetime that is being perturbed (assuming the standard normalization procedure for the Killing vector), exactly via $E = \mathrm{d}M$. Note that this flux is insensitive to the presence of the cosmological constant.

This means that the proposal by Gwak that $E = \mathrm{d}U$ *violates energy conservation*. Since energy conservation is usually considered to be equivalent to the first law of thermodynamics, it is not surprising that the second law is then also violated. To be more specific, although the black hole does satisfy the equation

$$\mathrm{d}M = T\mathrm{d}S + \Phi\mathrm{d}Q + V\mathrm{d}P \tag{17}$$

under the erroneous assumption that $E = \mathrm{d}U$ (together with the correct equation $q = \mathrm{d}Q$), since energy conservation does not hold, the first law does not hold. This happens even with fixed cosmological constant L. Thus, our work is sufficient to refute the original claim by Gwak. Note that the correct statement $E = \mathrm{d}M$ *does* also satisfy Eq. (17), and therefore the first law holds in our case. Indeed, one can easily check that $T\mathrm{d}S = |p^r| + (r_+^3/L^3)\mathrm{d}L$ and $V\mathrm{d}P = -(r_+^3/L^3)\mathrm{d}L$, so that $\mathrm{d}M = T\mathrm{d}S + V\mathrm{d}P + \Phi\mathrm{d}Q = |p^r| + \Phi\mathrm{d}Q$, which is Eq. (15).

Lastly, let us comment on the situations in which the cosmological constant is allowed to vary, which is holographically dual to changing N in the boundary field theory (which we will refer to as "conformal field theory" (CFT) for simplicity, although in many applications, the conformal symmetry is usually broken). In standard holography, one works with the large N regime in the CFT, but this N, however large, is fixed. To change N, one could, for example, embed the field theory in a larger boundary theory with one or more scalars that determine the effective N of the CFT part. One could envision that, as the scalars evolve in the larger boundary theory, the effective N changes. Such a process is expected to involve boundary energy transfer between the scalars and the CFT, so as the CFT energy changes, so does the bulk energy of the black hole. In general, this is complicated physics. To see if or how the changes in N might affect the black hole entropy, one must know more details about the coupled scalar-CFT theory. In other words, although some change in N could naively produce a reduction in the bulk black hole entropy, it is not clear that this would represent a true violation of the second law. Whether or not the second law is violated for the bulk depends on whether or not it has all the degrees of freedom of the *total* system and, if not, what the quantum state is for the total system. This is similar to the idea of the generalized second law: although the black hole entropy decreases as it Hawking evaporates, the total entropy of the hole and the Hawking radiation is increasing. In any case, our point is that there is no implication of violations of the second law just from the erroneous argument of Gwak that does not even obey the conservation of energy (the first law).

The second law of thermodynamics plays a very fundamental role in physics, so much so that Sir Arthur Eddington once wrote the following [31]: "The law that entropy always increases holds, I think, the supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations—then so much the worse for Maxwell's equations. If it is found to be contradicted by observation—well, these experimentalists do bungle things sometimes. But if your theory is found to be against the Second Law of Thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation."

If the extended black hole thermodynamics indeed suffered from violations of the second law, it would be a warning sign that something is drastically wrong with promoting the cosmological constant to be a variable. Given that a huge amount of literature exists that shows that extended black hole thermodynamics does give rise to a lot of otherwise sensible results [9], it is satisfying to find that the apparent violation of the second law is not a sign of the underlying inconsistency of the framework but rather is due to a misinterpretation of the right physical quantity in the thought experiment.

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